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# Resultants and discriminants of the multiplication polynomials of Jacobi elliptic functions

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## Abstract

We compute the resultants and discriminants of the multiplication polynomials of Jacobi elliptic functions by using Fourier expansions.

*Keywords:*

Jacobi elliptic function, multiplication polynomial, resultant, discriminant, Chebyshev polynomial

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## 1. Introduction

Resultants and discriminants have been calculated for various kinds of division (sometimes called multiplication) polynomials; cyclotomic polynomials (several different proofs are known; see for example [1], [3], [6]), real cyclotomic polynomials ([7], [11]), Chebyshev polynomials ([4], [8], [11]), multiplication polynomials of the Weierstrass  $\wp$ -function ([9]). It seems that no one has ever calculated the resultants of multiplication polynomials of Jacobi elliptic functions.

Consider the Jacobi elliptic functions  $\operatorname{sn}$ ,  $\operatorname{cn}$ ,  $\operatorname{dn}$  with modulus  $k$  ( $k^2 \neq 0, 1$ ;  $k$  may be complex). Let  $x = \operatorname{sn} u$ ,  $y = \operatorname{cn} u$ ,  $z = \operatorname{dn} u$ . For each positive

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integer  $n$ , there exist polynomials  $A_n, B_n, C_n, D_n$  in  $x$  with the following property:

$$(\operatorname{sn} nu, \operatorname{cn} nu, \operatorname{dn} nu) = \begin{cases} \left( \frac{x A_n(x)}{D_n(x)}, \frac{y B_n(x)}{D_n(x)}, \frac{z C_n(x)}{D_n(x)} \right) & \text{if } n \text{ is odd,} \\ \left( \frac{xyz A_n(x)}{D_n(x)}, \frac{B_n(x)}{D_n(x)}, \frac{C_n(x)}{D_n(x)} \right) & \text{if } n \text{ is even.} \end{cases} \quad (1.1)$$

See Section 2 for a precise description. The coefficients of  $A_n, B_n, C_n, D_n$  belong to  $\mathbb{Z}[k^2]$ .

The main result of this paper is the following

**Theorem 1.1.** *Let  $X, Y \in \{A, B, C, D\}, X \neq Y$  and  $n \geq 1$ . We have*

$$\operatorname{res}(X_n, Y_n) = \kappa_n(X, Y) k^{2l_n(X, Y)} (1 - k^2)^{m_n(X, Y)}, \quad (1.2)$$

where

$$\begin{aligned} \kappa_n(X, Y) &= 2^{\frac{n^2(n^2-1)}{3}}, \quad l_n(X, Y) = l_n(Y, X), \quad m_n(X, Y) = m_n(Y, X), \\ l_n(A, B) = m_n(A, D) &= \begin{cases} \frac{n^2(n^2-1)}{6} & \text{if } n \text{ is odd,} \\ \frac{n^2(n^2-4)}{6} & \text{if } n \text{ is even,} \end{cases} \\ l_n(A, C) = l_n(A, D) = m_n(A, B) = m_n(A, C) &= \begin{cases} \frac{(n^2-1)(2n^2-3)}{6} & \text{if } n \text{ is odd,} \\ \frac{n^2(n^2-4)}{6} & \text{if } n \text{ is even,} \end{cases} \\ l_n(B, C) = l_n(B, D) = m_n(B, D) = m_n(C, D) &= \begin{cases} \frac{(n^2-1)(2n^2-3)}{6} & \text{if } n \text{ is odd,} \\ \frac{n^2(n^2-1)}{6} & \text{if } n \text{ is even,} \end{cases} \\ l_n(C, D) = m_n(B, C) &= \begin{cases} \frac{n^2(n^2-1)}{6} & \text{if } n \text{ is odd,} \\ \frac{n^2(n^2+2)}{6} & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

The proof goes along the lines of Schmidt [9]. First we show the existence of integers  $\kappa_n(X, Y) > 0$ ,  $l_n(X, Y) \geq 0$ , and  $m_n(X, Y) \geq 0$  not depending on  $k$  such that (1.2) holds (Lemma 3.2). Then, comparing the  $q$ -expansions of both sides of (1.2), we determine the three constants. To be more precise, a comparison of the degrees of the leading terms determines  $l_n(X, Y)$ . Changing  $k$  to its complementary modulus  $k'$  (i.e.,  $k^2 + k'^2 = 1$ ), we get  $m_n(X, Y)$  (Corollary 3.3). A comparison of the leading constants finally yields the determination of  $\kappa_n(X, Y)$ .

The organization of this paper is as follows. In Section 2 we review on Jacobi elliptic functions and basic properties of their multiplication polynomials  $A_n, B_n, C_n, D_n$ . In Section 3 we show the general shape of  $\text{res}(X_n, Y_n)$  as given in (1.2). After preparing some  $q$ -expansions in Section 4, we give a proof of Theorem 1.1 in Section 5. As an application, we also calculate the discriminants of  $A_n, B_n, C_n, D_n$  in Section 6. In the final Section 7 we mention the degenerate cases  $k^2 = 0, 1$ .

## 2. Jacobi elliptic functions

Since the assertion of Theorem 1.1 is an identity in  $k$ , we may assume  $0 < k < 1$  for the proof. Following the traditional notation as in [2] or [10], we write Jacobi elliptic functions  $\text{sn}(u, k), \text{cn}(u, k), \text{dn}(u, k)$  simply as  $\text{sn } u, \text{cn } u, \text{dn } u$ , respectively. The periods of  $\text{sn}, \text{cn}, \text{dn}$  are

$$4mK + 2niK', \quad 2mK + 2n(K + iK'), \quad 2mK + 4niK' \quad (m, n \in \mathbb{Z})$$

respectively, where

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad K' = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}$$

are complete elliptic integrals of the first kind and  $k' = \sqrt{1-k^2}$  is the complementary modulus.

The following are basic properties of the polynomials  $A_n, B_n, C_n, D_n$  referred to in Introduction.

### Proposition 2.1.

1.  $A_n, B_n, C_n, D_n$  are determined by

$$A_1 = B_1 = C_1 = D_1 = 1$$

and the recurrence relations

$$\begin{aligned}
A_{2n} &= 2A_n B_n C_n D_n, \\
B_{2n} &= \begin{cases} y^2 B_n^2 D_n^2 - x^2 z^2 A_n^2 C_n^2 & \text{if } n \text{ is odd,} \\ B_n^2 D_n^2 - x^2 y^2 z^2 A_n^2 C_n^2 & \text{if } n \text{ is even,} \end{cases} \\
C_{2n} &= \begin{cases} z^2 C_n^2 D_n^2 - k^2 x^2 y^2 A_n^2 B_n^2 & \text{if } n \text{ is odd,} \\ C_n^2 D_n^2 - k^2 x^2 y^2 z^2 A_n^2 B_n^2 & \text{if } n \text{ is even,} \end{cases} \\
D_{2n} &= \begin{cases} D_n^4 - k^2 x^4 A_n^4 & \text{if } n \text{ is odd,} \\ D_n^4 - k^2 x^4 y^4 z^4 A_n^4 & \text{if } n \text{ is even,} \end{cases} \\
A_{2n+1} &= \begin{cases} A_n B_{n+1} C_{n+1} D_n + y^2 z^2 A_{n+1} B_n C_n D_{n+1} & \text{if } n \text{ is odd,} \\ y^2 z^2 A_n B_{n+1} C_{n+1} D_n + A_{n+1} B_n C_n D_{n+1} & \text{if } n \text{ is even,} \end{cases} \\
B_{2n+1} &= B_n B_{n+1} D_n D_{n+1} - x^2 z^2 A_n A_{n+1} C_n C_{n+1}, \\
C_{2n+1} &= C_n C_{n+1} D_n D_{n+1} - k^2 x^2 y^2 A_n A_{n+1} B_n B_{n+1}, \\
D_{2n+1} &= D_n^2 D_{n+1}^2 - k^2 x^4 y^2 z^2 A_n^2 A_{n+1}^2.
\end{aligned}$$

2. The coefficients of  $A_n, B_n, C_n, D_n$  belong to  $\mathbb{Z}[k^2]$ .
3.  $A_n, B_n, C_n, D_n$  are even polynomials, i.e., polynomials in  $x^2$ .
4. The leading terms are as follows.

$$\begin{aligned}
A_n(x) &= \begin{cases} (-1)^{(n-1)/2} (\sqrt{k} x)^{n^2-1} + \dots & \text{if } n \text{ is odd,} \\ (-1)^{(n-2)/2} n (\sqrt{k} x)^{n^2-4} + \dots & \text{if } n \text{ is even,} \end{cases} \\
B_n(x) &= \begin{cases} (\sqrt{k} x)^{n^2-1} + \dots & \text{if } n \text{ is odd,} \\ (\sqrt{k} x)^{n^2} + \dots & \text{if } n \text{ is even,} \end{cases} \\
C_n(x) &= \begin{cases} (\sqrt{k} x)^{n^2-1} + \dots & \text{if } n \text{ is odd,} \\ (\sqrt{k} x)^{n^2} + \dots & \text{if } n \text{ is even,} \end{cases} \\
D_n(x) &= \begin{cases} (-1)^{(n-1)/2} n (\sqrt{k} x)^{n^2-1} + \dots & \text{if } n \text{ is odd,} \\ (-1)^{n/2} (\sqrt{k} x)^{n^2} + \dots & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

5.  $A_n(0) = n, B_n(0) = C_n(0) = D_n(0) = 1$ .

6.

$$A_n(1) = \begin{cases} (-1)^{(n-1)/2} (k'^2)^{(n^2-1)/4} & \text{if } n \text{ is odd,} \\ n(-k'^2)^{n^2/4-1} & \text{if } n \text{ is even,} \end{cases}$$

$$B_n(1) = \begin{cases} (-1)^{(n-1)/2} n (k'^2)^{(n^2-1)/4} & \text{if } n \text{ is odd,} \\ (-k'^2)^{n^2/4} & \text{if } n \text{ is even,} \end{cases}$$

$$C_n(1) = D_n(1) = (k'^2)^{\lfloor n^2/4 \rfloor}.$$

7.

$$A_n(1/k) = \begin{cases} (-1)^{(n-1)/2} (k'^2 k^{-2})^{(n^2-1)/4} & \text{if } n \text{ is odd,} \\ n(k'^2 k^{-2})^{n^2/4-1} & \text{if } n \text{ is even,} \end{cases}$$

$$C_n(1/k) = \begin{cases} (-1)^{(n-1)/2} n (k'^2 k^{-2})^{(n^2-1)/4} & \text{if } n \text{ is odd,} \\ (k'^2 k^{-2})^{n^2/4} & \text{if } n \text{ is even,} \end{cases}$$

$$B_n(1/k) = D_n(1/k) = (-k'^2 k^{-2})^{\lfloor n^2/4 \rfloor}.$$

8. We have the following factorization:

$$A_n(x) = a_n \prod_{(r,s) \in R_n, (r,s) \equiv (0,0)} \left( x^2 - \operatorname{sn}^2 \frac{rK + siK'}{n} \right),$$

$$B_n(x) = b_n \prod_{(r,s) \in R_n, (r,s) \equiv (1,0)} \left( x^2 - \operatorname{sn}^2 \frac{rK + siK'}{n} \right),$$

$$C_n(x) = c_n \prod_{(r,s) \in R_n, (r,s) \equiv (1,1)} \left( x^2 - \operatorname{sn}^2 \frac{rK + siK'}{n} \right),$$

$$D_n(x) = d_n \prod_{(r,s) \in R_n, (r,s) \equiv (0,1)} \left( x^2 - \operatorname{sn}^2 \frac{rK + siK'}{n} \right),$$

where

$$R_n = (\{1, 2, \dots, n-1\} \times \{0, n\}) \sqcup (\{0, 1, \dots, 2n-1\} \times \{1, 2, \dots, n-1\}),$$

the leading coefficients  $a_n, b_n, c_n, d_n$  are as given in part 4, and the congruences are taken modulo 2.

9.  $A_n, B_n, C_n, D_n$  are pairwise prime to each other as polynomials in  $x$ .

*Proof.* First we note that the polynomials  $A_n, B_n, C_n, D_n$  in our notation are the ones denoted by  $A'_n, B'_n, C'_n, D'_n$  in [2, p.87]. With this translation in mind, part 1 is found in [2, p.79]. Parts 2–7 are then deduced from the recurrence relations. Part 8 is essentially found in [2, p.92]. The last claim follows from the factorization.  $\square$

We describe the effect of changing  $k$  to  $k'$ . To this end, we introduce a notation:

$$f^*(x) = \sqrt{1-x^2}^{\deg f} f\left(\frac{ix}{\sqrt{1-x^2}}\right). \quad (2.1)$$

**Lemma 2.2.** *Let  $f(x)$  be an even polynomial of degree  $2n$ .*

1.  $f^*(x)$  is also an even polynomial.
2. The coefficient of  $x^{2n}$  in  $f^*(x)$  is  $(-1)^n f(1)$ . In particular,  $\deg f^* = \deg f$  if and only if  $f(1) \neq 0$ .
3.  $f^{**}(x) = f(x)$  if  $f(1) \neq 0$ .

Recall that  $A_n, B_n, C_n, D_n$  are even polynomials and do not vanish at  $x = 1$ .

**Proposition 2.3.** *If we write  $A_n(x, k)$  etc., to indicate the dependency on  $k$ , then we have*

$$\begin{aligned} A_n(x, k') &= A_n^*(x, k), & B_n(x, k') &= D_n^*(x, k), \\ C_n(x, k') &= C_n^*(x, k), & D_n(x, k') &= B_n^*(x, k). \end{aligned}$$

*Proof.* If the zeros of an even polynomial  $f(x)$  with  $f(1) \neq 0$  are  $\pm\alpha_j$  ( $j = 1, 2, \dots, n$ ), then those of  $f^*(x)$  are

$$\pm \frac{i\alpha_j}{\sqrt{1-\alpha_j^2}} \quad (j = 1, 2, \dots, n).$$

Using this observation, the identity

$$\operatorname{sn}(iu, k) = \frac{i \operatorname{sn}(u, k')}{\operatorname{cn}(u, k')}$$

(cf. [10, 22.4]), and part 8 of Proposition 2.1, we see that both sides have exactly the same set of zeros in each case. We also see that the leading coefficients coincide by using parts 4 and 6 of Proposition 2.1. This completes the proof.  $\square$

### 3. Resultants

The resultant of two polynomials

$$\begin{aligned} f(x) &= a(x - \alpha_1) \cdots (x - \alpha_m), \quad a \neq 0, \\ g(x) &= b(x - \beta_1) \cdots (x - \beta_n), \quad b \neq 0 \end{aligned}$$

is defined as

$$\text{res}(f, g) = a^n b^m \prod_{i=1}^m \prod_{j=1}^n (\alpha_i - \beta_j) = a^n \prod_{i=1}^m g(\alpha_i).$$

**Lemma 3.1.** *Let  $h$  be another polynomial.*

1.  $\text{res}(g, f) = (-1)^{mn} \text{res}(f, g)$ .
2.  $\text{res}(f, gh) = \text{res}(f, g) \text{res}(f, h)$ ,  $\text{res}(fg, h) = \text{res}(f, h) \text{res}(g, h)$ .
3.  $\text{res}(f, g + fh) = a^{s-n} \text{res}(f, g)$ , where  $s = \deg(g + fh)$ .
4. *Suppose further that  $f, g$  are even polynomials which do not vanish at  $x = 1$ . Then with the notation in (2.1), we have*

$$\text{res}(f^*, g^*) = \text{res}(f, g).$$

*Proof.* The first three are easily seen. The proof of the last assertion is reduced to the case  $f(x) = x^2 - p, g(x) = x^2 - q$ , since both the resultant and the operation  $*$  are multiplicative. In this case we have  $f^*(x) = (p - 1)x^2 - p, g^*(x) = (q - 1)x^2 - q$ , so that

$$\text{res}(f^*, g^*) = (p - q)^2 = \text{res}(f, g)$$

as desired. □

Let  $X, Y \in \{A, B, C, D\}, X \neq Y$  and  $n \geq 1$ . Our object is the calculation of  $\text{res}(X_n, Y_n)$ . By a well known Sylvester determinant expression of the resultant and by part 2 of Proposition 2.1, we see that  $\text{res}(X_n, Y_n) \in \mathbb{Z}[k^2]$ . Also, we have  $\text{res}(X_n, Y_n) = \text{res}(Y_n, X_n)$  since both  $\deg X_n$  and  $\deg Y_n$  are even.

The general shape of  $\text{res}(X_n, Y_n)$  is given as follows.

**Lemma 3.2.** *Let  $X, Y \in \{A, B, C, D\}, X \neq Y$ . For each positive integer  $n$ , there exist integers  $\kappa_n(X, Y) > 0$ ,  $l_n(X, Y) \geq 0$ , and  $m_n(X, Y) \geq 0$  not depending on  $k$  such that*

$$\text{res}(X_n, Y_n) = \kappa_n(X, Y) k^{2l_n(X, Y)} (1 - k^2)^{m_n(X, Y)}.$$



*Proof.* We regard  $\text{res}(X_n, Y_n)$  as a polynomial in  $k^2$ . Part 9 of Proposition 2.1 implies that  $\text{res}(X_n, Y_n) \neq 0$  if  $k^2 \neq 0, 1$ . Hence by the Hilbert Nullstellensatz (cf. [5, IX §1 Theorem 1.5]), there exist a positive integer  $t$  and a polynomial  $Q \in \mathbb{Q}[k^2]$  such that  $(k^2(1 - k^2))^t = Q \text{res}(X_n, Y_n)$ . Since  $\mathbb{Q}[k^2]$  is a unique factorization domain, it follows that  $\text{res}(X_n, Y_n) = \kappa k^{2l}(1 - k^2)^m$  for some  $\kappa \in \mathbb{Q}$  and some non-negative integers  $l, m$ . Since  $\text{res}(X_n, Y_n) \in \mathbb{Z}[k^2]$  and both  $X_n$  and  $Y_n$  are polynomials in  $x^2$ , we see that  $\kappa$  is a positive integer.  $\square$

By Proposition 2.3 and Lemma 3.1, we have

**Corollary 3.3.** *With the notation of Lemma 3.2,*

$$\begin{aligned} l_n(A, B) &= m_n(A, D), & l_n(A, D) &= m_n(A, B), \\ l_n(A, C) &= m_n(A, C), \\ l_n(B, C) &= m_n(C, D), & l_n(C, D) &= m_n(B, C), \\ l_n(B, D) &= m_n(B, D), \\ \kappa_n(A, B) &= \kappa_n(A, D), \\ \kappa_n(B, C) &= \kappa_n(C, D). \end{aligned}$$

On the other hand, by part 8 of Proposition 2.1, we get

**Proposition 3.4.** *Let  $X, Y \in \{A, B, C, D\}$ ,  $X \neq Y$ . Then we have*

$$\text{res}(X_n, Y_n) = x_n^{\deg Y_n} y_n^{\deg X_n} \prod_{(r,s) \in R_n^X} \prod_{(r',s') \in R_n^Y} f(r, s, r', s')^2,$$

where  $x_n, y_n$  denote the leading coefficients of  $X_n, Y_n$  respectively,

$$\mathbf{v}_A = (0, 0), \quad \mathbf{v}_B = (1, 0), \quad \mathbf{v}_C = (1, 1), \quad \mathbf{v}_D = (0, 1),$$

$$R_n = (\{1, 2, \dots, n-1\} \times \{0, n\}) \sqcup (\{0, 1, \dots, 2n-1\} \times \{1, 2, \dots, n-1\}),$$

$$R_n^X = \{(r, s) \in R_n \mid (r, s) \equiv \mathbf{v}_X \pmod{2}\},$$

and

$$f(r, s, r', s') = \text{sn}^2 \frac{rK + siK'}{n} - \text{sn}^2 \frac{r'K + s'iK'}{n}.$$

So we need a closer look at  $f(r, s, r', s')^2$ .

#### 4. $q$ -expansions

Let  $\tau = iK'/K$  and  $q = e^{\pi i\tau}$ . By our assumption  $0 < k < 1$ , we see that  $K$  and  $K'$  are positive real numbers, so that  $\text{Im}(\tau) > 0$  and  $0 < |q| < 1$ . We use the following  $q$ -expansions ([10, 21.61]):

$$k^{\frac{1}{2}} = \frac{\vartheta_2(0)}{\vartheta_3(0)} = \frac{2 \sum_{j=0}^{\infty} q^{(j+\frac{1}{2})^2}}{1 + 2 \sum_{j=0}^{\infty} q^{j^2}} = 2q^{\frac{1}{4}} + \dots, \quad (4.1)$$

$$2K = \pi \vartheta_3(0)^2 = \pi \left( 1 + \sum_{j=0}^{\infty} q^{j^2} \right)^2.$$

Let  $u = 2Kx/\pi$ . We also use the following ([10, 22.6,22.61]):

$$\text{sn } u = \frac{2\pi}{Kk} \sum_{j=0}^{\infty} \frac{q^{j+\frac{1}{2}} \sin(2j+1)x}{1 - q^{2j+1}} \quad (4.2)$$

valid throughout the strip  $|\text{Im}(x)| < \frac{\pi}{2} \text{Im}(\tau)$ , and

$$\text{ns } u = \frac{1}{\text{sn } u} = \frac{\pi}{2K} \text{cosec } x + \frac{2\pi}{K} \sum_{j=0}^{\infty} \frac{q^{2j+1} \sin(2j+1)x}{1 - q^{2j+1}} \quad (4.3)$$

valid throughout  $|\text{Im}(x)| < \pi \text{Im}(\tau)$ , except at the points  $x \in \pi\mathbb{Z}$ .

Now let

$$u = \frac{rK + siK'}{n}, \quad (r, s) \in R_n.$$

We can apply (4.2) if  $0 \leq s < n$  and (4.3) if  $s = n$ . By using  $\exp ix = \zeta^r q^{\frac{s}{2n}}$ , where  $\zeta = \exp \frac{\pi i}{2n}$ , we find that the complex number

$$-4 \text{sn}^2 \frac{rK + siK'}{n}$$

is expressed as a Laurent series in  $q^{\frac{1}{2n}}$  whose leading term has degree  $q^{-\frac{s}{n}}$  and coefficient

$$\begin{cases} (\zeta^r - \zeta^{-r})^2 & \text{if } s = 0, \\ \zeta^{-2r} & \text{if } 0 < s < n, \\ (\zeta^r - \zeta^{-r})^{-2} & \text{if } s = n. \end{cases}$$

Thus we get the leading term of the  $q$ -expansion of  $f(r, s, r', s')^2$ . We may suppose  $s \geq s'$  since  $f(r', s', r, s)^2 = f(r, s, r', s')^2$ .

**Lemma 4.1.** *For  $s \geq s'$ , the leading term of the  $q$ -expansion of  $16f(r, s, r', s')^2$  has degree  $q^{-\frac{2s}{n}}$  and coefficient  $L(r, s, r', s')$ , where*

$$L(r, s, r', s') = \begin{cases} ((\zeta^r - \zeta^{-r})^2 - (\zeta^{r'} - \zeta^{-r'})^2)^2 & \text{if } s' = s = 0, \\ \zeta^{-4r} & \text{if } s' < s < n, \\ (\zeta^{-2r} - \zeta^{-2r'})^2 & \text{if } s' = s < n, \\ (\zeta^r - \zeta^{-r})^{-4} & \text{if } s' < s = n, \\ ((\zeta^r - \zeta^{-r})^{-2} - (\zeta^{r'} - \zeta^{-r'})^{-2})^2 & \text{if } s' = s = n, \end{cases}$$

and  $\zeta = \exp \frac{\pi i}{2n}$ .

## 5. Proof of Theorem 1.1

Let  $X, Y \in \{A, B, C, D\}$ ,  $X \neq Y$ . We have  $\text{res}(X_n, Y_n) = \text{res}(Y_n, X_n)$  as noticed earlier, so we have only to consider the six cases

$$(X, Y) \in \{(A, B), (A, C), (A, D), (B, C), (B, D), (C, D)\}.$$

As in Proposition 3.4, let  $x_n, y_n$  denote the leading coefficients (with respect to the variable  $x$ ) of  $X_n, Y_n$ , respectively. By part 4 of Proposition 2.1 and (4.1), we can consider the  $q$ -expansions of  $x_n$  and  $y_n$ , so let  $x'_n$  and  $y'_n$  denote the leading terms of these  $q$ -expansions, respectively. Then by Proposition 3.4 and Lemma 4.1, the leading term of the  $q$ -expansion of  $\text{res}(X_n, Y_n)$  is

$$x'_n{}^{\deg Y_n} y'_n{}^{\deg X_n} \prod_{(r,s) \in R_n^X} \prod_{(r',s') \in R_n^Y} \frac{L(r, s, r', s')}{16} q^{-\frac{2}{n} \max\{s, s'\}}. \quad (5.1)$$

First we observe the degree in  $q$ . By Lemma 3.2 and (4.1), the leading term of the  $q$ -expansion of  $\text{res}(X_n, Y_n)$  has degree  $q^{l_n(X, Y)}$ . On the other hand, it follows from part 4 of Proposition 2.1 and (4.1) that  $x'_n{}^{\deg Y_n} y'_n{}^{\deg X_n}$  has degree  $q^{\frac{1}{2} \deg X_n \deg Y_n}$ , hence by (5.1) we find that

$$l_n(X, Y) = \frac{1}{2} \deg X_n \deg Y_n - \frac{2}{n} \sum_{(r,s) \in R_n^X} \sum_{(r',s') \in R_n^Y} \max\{s, s'\}.$$

The authors are not aware of any clever method to compute the last double sum, but anyway an elementary counting argument gives the value of  $l_n(X, Y)$ , and hence that of  $m_n(X, Y)$  by Corollary 3.3, as stated in Theorem 1.1.

Next we observe the leading coefficient of the  $q$ -expansion of  $\text{res}(X_n, Y_n)$ . By Lemma 3.2 and (4.1), it is  $\kappa_n(X, Y)16^{l_n(X, Y)}$ . Since  $\kappa_n(X, Y)$  is known to be positive, it suffices to determine the absolute value of the coefficient of (5.1). By Lemma 3.2, we may further assume that

$$(X, Y) \in \{(A, C), (A, D), (B, C), (B, D)\}. \quad (5.2)$$

By part 4 of Proposition 2.1 and (4.1), the leading coefficient of  $x'_n{}^{\deg Y_n} y'_n{}^{\deg X_n}$  is

$$\begin{cases} 2^{2 \deg X_n \deg Y_n} n^{n^2} & \text{if } X = A, n \text{ is even,} \\ 2^{2 \deg X_n \deg Y_n} n^{n^2-1} & \text{if } Y = D, n \text{ is odd,} \\ 2^{2 \deg X_n \deg Y_n} & \text{otherwise.} \end{cases} \quad (5.3)$$

The contribution of 16's in (5.1) gives

$$16^{-\#R_n^X \#R_n^Y} = 2^{-\deg X_n \deg Y_n}. \quad (5.4)$$

Now let

$$P = \prod_{(r,s) \in R_n^X} \prod_{(r',s') \in R_n^Y} |L(r, s, r', s')|$$

and decompose  $P$  into the product  $P_1 P_2 P_3 P_4$ , where

$$P_1 = \prod_{\substack{(r,s) \in R_n^X, \\ s \notin \{0, n\}}} \prod_{\substack{(r',s') \in R_n^Y, \\ s' \notin \{0, n\}}} |L(r, s, r', s')|,$$

$$P_2 = \prod_{\substack{(r,s) \in R_n^X, \\ s \in \{0, n\}}} \prod_{\substack{(r',s') \in R_n^Y, \\ s' \notin \{0, n\}}} |L(r, s, r', s')|,$$

$$P_3 = \prod_{\substack{(r,s) \in R_n^X, \\ s \notin \{0, n\}}} \prod_{\substack{(r',s') \in R_n^Y, \\ s' \in \{0, n\}}} |L(r, s, r', s')|,$$

$$P_4 = \prod_{\substack{(r,s) \in R_n^X, \\ s \in \{0, n\}}} \prod_{\substack{(r',s') \in R_n^Y, \\ s' \in \{0, n\}}} |L(r, s, r', s')|.$$

If we write  $\mathbf{v}_X = (v_X, w_X)$  and  $\mathbf{v}_Y = (v_Y, w_Y)$ , then by the assumption (5.2), we have  $w_X = 0$  and  $w_Y = 1$ . So, as far as we are concerned here,  $s$  is always even and  $s'$  is always odd. By Lemma 4.1, we have

$$|L(r, s, r', s')| = \begin{cases} |1 - \zeta_{2n}^{r'}|^{-4} & \text{if } s < s' = n \text{ and } n \text{ is odd,} \\ |1 - \zeta_{2n}^r|^{-4} & \text{if } s' < s = n \text{ and } n \text{ is even,} \\ 1 & \text{otherwise,} \end{cases}$$

where we put  $\zeta_m = \exp \frac{2\pi i}{m}$ , so that

$$\begin{aligned} P_1 &= 1, \\ P_2 &= \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \prod_{0 < r < n, r \equiv v_X} |1 - \zeta_{2n}^r|^{-2n^2} & \text{if } n \text{ is even,} \end{cases} \\ P_3 &= \begin{cases} \prod_{0 < r' < n, r' \equiv v_Y} |1 - \zeta_{2n}^{r'}|^{-2n(n-1)} & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even,} \end{cases} \\ P_4 &= \begin{cases} \prod_{0 < r' < n, r' \equiv v_Y} |1 - \zeta_{2n}^{r'}|^{-2(n-1)} & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

By Lemma 5.1 below, we find that

$$P = \begin{cases} 1 & \text{if } n \text{ is odd and } Y = C, \\ n^{-(n^2-1)} & \text{if } n \text{ is odd and } Y = D, \\ 2^{-n^2} & \text{if } n \text{ is even and } X = B, \\ (n/2)^{-n^2} & \text{if } n \text{ is even and } X = A. \end{cases} \quad (5.5)$$

Putting (5.1), (5.3), (5.4), and (5.5) together and using the value of  $l_n(X, Y)$ , we complete the proof of Theorem 1.1.  $\square$

**Lemma 5.1.** *Let  $\zeta_m = \exp \frac{2\pi i}{m}$ . For any positive integer  $n$ , we have*

$$\prod_{0 < r < n, r: \text{odd}} |1 - \zeta_{2n}^r|^2 = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even,} \end{cases} \quad (5.6)$$

$$\prod_{0 < r < n, r: \text{even}} |1 - \zeta_{2n}^r|^2 = \begin{cases} n & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases} \quad (5.7)$$

*Proof.* Using  $\prod_{r=1}^{n-1} (1 - \zeta_n^r) = n$  and  $|1 - \zeta_n^r| = |1 - \zeta_n^{n-r}|$ , we have

$$\prod_{0 < r < n/2} |1 - \zeta_n^r|^2 = \begin{cases} n & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even,} \end{cases} \quad (5.8)$$

which is nothing but (5.7). Replacing  $n$  with  $2n$  in (5.8) and then dividing it by (5.7), we get (5.6).  $\square$

## 6. Discriminants

The discriminant of a polynomial  $f$  with degree  $n$  and leading coefficient  $a$  is defined as

$$\text{disc}(f) = (-1)^{n(n-1)/2} a^{-1} \text{res}(f, f')$$

(see [5, IV §8]). As an application of Theorem 1.1, we get the following

**Theorem 6.1.** *Let  $X \in \{A, B, C, D\}$  and  $n \geq 1$ . We have*

$$\text{disc}(X_n) = \kappa_n(X) k^{2l_n(X)} (1 - k^2)^{m_n(X)},$$

where:

*If  $n$  is odd, then*

$$\begin{aligned} \kappa_n(A) &= \kappa_n(D) = (-1)^{\frac{n-1}{2}} 2^{\frac{n^2(n^2-1)}{3}} n^{n^2-2}, \\ \kappa_n(B) &= \kappa_n(C) = 2^{\frac{n^2(n^2-1)}{3}} n^{n^2-3}, \\ l_n(A) &= l_n(B) = l_n(C) = l_n(D) = \frac{(n^2-1)(2n^2-3)}{12}, \\ m_n(A) &= m_n(B) = m_n(C) = m_n(D) = \frac{(n^2-1)(n^2-3)}{6}. \end{aligned}$$

If  $n$  is even, then

$$\begin{aligned}
\kappa_n(A) &= (-1)^{\frac{n-2}{2}} 2^{\frac{n^2(n^2-1)}{3}} n^{n^2-10}, \\
\kappa_n(B) = \kappa_n(C) = \kappa_n(D) &= (-1)^{\frac{n}{2}} \kappa_n(D) = 2^{\frac{n^2(n^2-1)}{3}} n^{n^2}, \\
l_n(A) &= \frac{(n^2-4)(2n^2-9)}{12}, \\
l_n(B) &= \frac{n^2(2n^2-5)}{12}, \\
l_n(C) = l_n(D) &= \frac{n^2(2n^2+1)}{12}, \\
m_n(A) &= \frac{(n^2-4)(n^2-6)}{6}, \\
m_n(B) = m_n(C) &= \frac{n^2(n^2-1)}{6}, \\
m_n(D) &= \frac{n^2(n^2-4)}{6}.
\end{aligned}$$

*Proof.* First we recall that

$$\begin{aligned}
\frac{d}{du} \operatorname{sn} u &= \operatorname{cn} u \operatorname{dn} u, \\
\frac{d}{du} \operatorname{cn} u &= -\operatorname{sn} u \operatorname{dn} u, \\
\frac{d}{du} \operatorname{dn} u &= -k^2 \operatorname{sn} u \operatorname{cn} u.
\end{aligned}$$

On differentiating both sides of (1.1) with respect to  $x$ , we get

$$nB_n C_n = \begin{cases} A_n D_n + x(A'_n D_n - A_n D'_n) & \text{if } n \text{ is odd,} \\ (y^2 z^2 - x^2 z^2 - k^2 x^2 y^2) A_n D_n + x y^2 z^2 (A'_n D_n - A_n D'_n) & \text{if } n \text{ is even,} \end{cases} \quad (6.1)$$

$$-nx A_n C_n = \begin{cases} -xB_n D_n + y^2(B'_n D_n - B_n D'_n) & \text{if } n \text{ is odd,} \\ B'_n D_n - B_n D'_n & \text{if } n \text{ is even,} \end{cases} \quad (6.2)$$

$$-nk^2 x A_n B_n = \begin{cases} -k^2 x C_n D_n + z^2(C'_n D_n - C_n D'_n) & \text{if } n \text{ is odd,} \\ C'_n D_n - C_n D'_n & \text{if } n \text{ is even.} \end{cases} \quad (6.3)$$

We illustrate the proof in the case of  $B_n$ ,  $n$ :even. Taking resultants of both sides of (6.2) with  $B_n$ , we get, by Lemma 3.1,

$$\operatorname{res}(-nxA_nC_n, B_n) = b_n^{-2} \operatorname{res}(B_n'D_n, B_n),$$

where  $b_n = k^{(n^2-1)/2}$  is the leading coefficient of  $B_n(x)$  (cf. part 4 of Proposition 2.1). By the multiplicativity of resultant and by the facts  $\deg B_n = n^2 - 1$  and  $\operatorname{res}(x, B_n) = B_n(0) = 1$ , we get

$$n^{n^2-1} \operatorname{res}(A_n, B_n) \operatorname{res}(B_n, C_n) = \operatorname{res}(B_n, D_n) b_n^{-1} \operatorname{disc}(B_n).$$

Substituting the result of Theorem 1.1, we get  $\operatorname{disc}(B_n)$ .

The computation in the remaining cases is similar; Theorem 1.1 and Proposition 2.1 contain all the information we need; for example,

$$\operatorname{res}(z^2, A_n) = \operatorname{res}(1 - k^2x^2, A_n) = (-k^2)^{\deg A_n} A_n(1/k)^2.$$

□

## 7. Degenerate Case

In this final section, we state without proof what happens in the “degenerate” cases  $k^2 = 0, 1$ , if we formally define the polynomials  $A_n, B_n, C_n, D_n$  by using the same recurrence relations in part 1 of Proposition 2.1. The proof can be carried out, for example, as in [11].

In the case  $k^2 = 0$ , we have

$$A_n(x) = \begin{cases} \mathcal{U}_n(\sqrt{1-x^2}) & \text{if } n \text{ is odd,} \\ \frac{\mathcal{U}_n(\sqrt{1-x^2})}{\sqrt{1-x^2}} & \text{if } n \text{ is even,} \end{cases}$$

$$B_n(x) = \begin{cases} \frac{T_n(\sqrt{1-x^2})}{\sqrt{1-x^2}} & \text{if } n \text{ is odd,} \\ T(\sqrt{1-x^2}) & \text{if } n \text{ is even,} \end{cases}$$

$$C_n(x) = D_n(x) = 1,$$

where  $T_n$  and  $\mathcal{U}_n$  are the Chebyshev polynomials of the first and second kind, of degree  $n$  and  $n - 1$ , respectively.



In the case  $k^2 = 1$ , we have

$$A_n(x) = \begin{cases} (1-x^2)^{n(n-1)/2} \tilde{A}_n(x) & \text{if } n \text{ is odd,} \\ (1-x^2)^{n(n-1)/2-1} \tilde{A}_n(x) & \text{if } n \text{ is even,} \end{cases}$$

$$B_n(x) = C_n(x) = (1-x^2)^{\lfloor n^2/2 \rfloor},$$

$$D_n(x) = (1-x^2)^{n(n-1)/2} \tilde{D}_n(x),$$

where

$$\tilde{A}_n(x) = \sqrt{1-x^2}^{n-1} \mathcal{U}_n \left( \frac{1}{\sqrt{1-x^2}} \right),$$

$$\tilde{D}_n(x) = \sqrt{1-x^2}^n T_n \left( \frac{1}{\sqrt{1-x^2}} \right).$$

Note that  $\tilde{A}_n, \tilde{D}_n$  are polynomials in  $x^2$ .

The identity (1.1) holds true also in these degenerate cases, as

$$\operatorname{sn} u = \sin u, \quad \operatorname{cn} u = \cos u, \quad \operatorname{dn} u = 1$$

if  $k^2 = 0$ , and

$$\operatorname{sn} u = \tanh u, \quad \operatorname{cn} u = \operatorname{dn} u = \frac{1}{\cosh u}$$

if  $k^2 = 1$ .

**Proposition 7.1.**

1. If  $k^2 = 0$ , then  $\operatorname{res}(A_n, B_n) = 2^{n(n-1)}$ .
2. If  $k^2 = 1$ , then  $\operatorname{res}(\tilde{A}_n, \tilde{D}_n) = 2^{n(n-1)}$ .

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