Fundamental Inequalities on Higher Order Metrics in Several Complex Variables

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By making use of the Bergman kernel K. H. Look gave a useful fundamental theorem with respect to the estimation of $|\partial_u f(t)|$ for any bounded holomorphic mapping f on a bounded domain in C^n.

The main object of this paper is to generalize and sharpen the fundamental theorem of Look [8] and the theorem of Hahn-Burbea [2] with respect to some higher order metrics on a bounded domain. As applications, the Cauchy type estimation formulas in the L_2-bounded class on a bounded domain and the generalized Schwarz-Pick type lemma between a bounded homogeneous domain and a bounded (or bounded homogeneous) domain are given.

As an example the best possible inequality with the generalized Schwarz constant between the Bergman metrics of the classical Cartan domain of the first type $M_{I(m\times n)}$ in C^{mn} and the unit ball in C^k is given.

The main tools are the L_2-minimum problem and the properties of the Bergman kernel and the Carathéodory metric.

1. L_2-minimum problem, higher order metrics.

Throughout this paper let $D\equiv D_n$ be a bounded domain in C^n. $\mathrm{Hol}(D, G_k)$ (resp. ${}_k\mathrm{Hol}(D)$) denotes the class of holomorphic mappings of D into another bounded domain G_k in C^k (resp. the class of k-tuple holomorphic column vector functions in D). Further let ${}_kL_2(D)$ be the class $\{f\in{}_k\mathrm{Hol}(D)\,|\,(f, f)_D<\infty\}$, where

$$(f,f)_D=||f||_D{}^2=\int_D|f(z)|^2\omega_z=\sigma\int_D f(z)f^*(z)\,\omega_z.$$

Hereafter ω_z, σ and $*$ denote the Euclidean volume element, the trace symbol and the adjoint symbol, respectively.

${}_k\mathrm{Hol}(D:R)$ (resp. ${}_kL_2(D:R)$) denotes a bounded class $\{f\in{}_k\mathrm{Hol}(D)\,|\,|f|\leqq R\}$ (resp. $\{g\in{}_kL_2(D)\,|\,||g||_D\leqq R\}$).

For $\partial/\partial z=(\partial/\partial z_1, \cdot\cdot, \partial/\partial z_n)(z={}^t(z_1, \cdot\cdot, z_n)\in D)$ and $u={}^t(u_1, \cdot\cdot, u_n)\subset C^n-\{0\}$ set $\partial_u{}^j\bullet=(\partial_u\times\cdot\cdot\times\partial_u)\bullet=\{(\partial/\partial z\times\cdot\cdot\times\partial/\partial z)\bullet\}(u\times\cdot\cdot\times u)$ (j-times kronecker product), then we may consider bounded linear functional vectors

(1.1) $L_{J(r)tu}\equiv L_{J(r),t,u}=(\partial_u{}^{j_1}, \cdot\cdot, \partial_u{}^{j_r})_{z=t}$,
$J(r)=(j_1, \cdot\cdot, j_r)$, $0\leqq j_1<j_2<\cdot\cdot<j_r$ $(r\geqq 1)$.

Let $\phi(z)={}^t(\phi_1(z), \phi_2(z), \cdots)$ be a complete orthonormal system in D, then we have the Bergman kernel $K_D(z, t)=\phi^*(t)\phi(z)$ for $(z, t)\in D\times D$ and the Bergman tensor

$T_D(z,t)=\partial^2\log K_D(z,t)/\partial z^*\partial z$,
$\partial/\partial z^*=(\partial/\partial z)^*$.

Lemma 1. 1. *Let ${}_k\lambda_{P(r)tu}(D)$ be the L_2-minimum value for the class $\{g\in{}_kL_2(D)\,|\,L_{J(r)tu}g=(P_1, \cdot\cdot, P_r)\equiv P(r)\}$, then we have*

(1.2) ${}_k\lambda_{P(r)tu}(D)$
$=\sigma\ \{P(r)(L_{J(r)tu}{}^*L_{J(r)tu}K_D)^{-1}P^*(r)\}$,

where $K_D\equiv K_D(z, \zeta)=\phi^*(\zeta)\phi(z)$ denotes the Bergman kernel and $L_{J(r)tu}{}^*L_{J(r)tu}K_D=(L_{J(r)tu}\phi)^*(L_{J(r)tu}\phi)$ (see details in [11]).

Definition 1. 1. For a bounded domain D we can define the m-th order metrics $(m\geqq 1)$:

(1.3) ${}_k\tilde{C}_{D,m}(t, u)=\sup\{|\partial_u{}^m f(t)|\,|\,f\in{}_k\mathrm{Hol}(D:1)\}$,

(1.4) ${}_k\tilde{S}_{D,m}(t, u)={}_k\hat{S}_{D,m}(t, u)K_D{}^{-1/2}(t, t)$
$=\sup\{|\partial_u{}^m g(t)|K_D{}^{-1/2}(t, t)\,|\,g\in{}_kL_2(D:1)\}$,

the (k, m)-Carathéodory-Reiffen metric

(1.5) ${}_kC_{D,m}(t, u)$
$=\sup\{|\partial_u{}^m f(t)|\,|\,f\in{}_k\mathrm{Hol}_{mtu}(D:1)\}$

(cf.[2]) and the (k, m)-Bergman metric

(1.6) ${}_kS_{D,m}(t, u)$
$=\sup\{|\partial_u{}^m g(t)|K_D{}^{-1/2}(t, t)\,|\,g\in{}_kL_{2:mtu}(D:1)\}$

(cf.[2]), where for $L_{mtu}=(1,\partial_u,\cdot\cdot, \partial_u{}^{m-1})_{z=t}$

(1.7) ${}_k\mathrm{Hol}_{mtu}(D : 1)$
$= \{f \in {}_k\mathrm{Hol}(D : 1) \mid L_{mtu}f = (0, \cdot\cdot, 0)\}$

and

(1.8) ${}_kL_{2\,:\,mtu}(D : 1)$
$= \{g \in {}_kL_2(D : 1) \mid L_{mtu}g = (0, \cdot\cdot, 0)\}$.

Lemma 1. 2. *For each $m(\geqq 1)$ ${}_kC_{D,m}(z, u)$, ${}_kS_{D,m}(z, u)$, ${}_k\tilde{C}_{D,m}(z, u)$ and ${}_k\tilde{S}_{D,m}(z, u)$ coincide with them for $k=1$, respectively.*

${}_1C_{D,1}(z, u)$ and ${}_1S_{D,1}(z, u)$ coincide with the usual Carathéodory and Bergman metrics (we denote them as $C_D(z, u)$ and $S_D(z, u)$), respectively.

Proof. The similar procedure as in [4] for $m=1$ enables us to prove this lemma. Therefore we have only to treat the case of ${}_kC_{D,m}$.

Suppose that $f_1 \in {}_1\mathrm{Hol}_{mtu}(D : 1)$, then $f = {}^t(f_1, 0, \cdot\cdot, 0)$ belongs to ${}_k\mathrm{Hol}_{mtu}(D : 1)$ with $|\partial_u{}^m f(t)| = |\partial_u{}^m f_1(t)|$. This gives ${}_kC_{D,m}(t, u) \geqq {}_1C_{D,m}(t, u)$. On the other hand set $f_1 = \nu f$ for any $f = {}^t(f_1, \cdot\cdot, f_k) \in {}_k\mathrm{Hol}_{mtu}(D : 1)$ and $\nu = (\nu_1, \cdot\cdot, \nu_k) \in \mathbf{C}^k$ with $|\nu| = 1$, then we have $f_1 \in {}_1\mathrm{Hol}_{mtu}(D : 1)$. Therefore we have

$${}_1C_{D,m}(t, u) \geqq \sup\{|\partial_u{}^m f_1(t)| \mid f_1 \in {}_1\mathrm{Hol}_{mtu}(D : 1)\}$$
$$= \sup\{|\nu\partial_u{}^m f(t)| \mid f \in {}_k\mathrm{Hol}_{mtu}(D : 1), \nu \in C^k(|\nu| = 1)\}$$
$$\geqq \sup_f \sup_\nu |\nu\partial_u{}^m f(t)| = \sup|\partial_u{}^m f(t)| = {}_kC_{D,m}(t, u).$$

This completes the proof. (See also Lemma 1. 3.)

Lemma 1. 3. *For each $m(\geqq 1)$ and $k(\geqq 1)$ we have*

(1.9) ${}_kS_{D,m}{}^2(t, u) = J_{D,m}(t, u)/K_D(t, t)J_{D,m-1}(t, u)$

and

(1.10) ${}_k\tilde{S}_{D,m}{}^2(t, u) = {}_k\hat{S}_{D,m}{}^2(t, u)/K_D(t, t)$
$= \partial_u{}^{*m}\partial_u{}^m K_D(t, t)/K_D(t, t)$,

where $J_{D,r}(t, u) = \det(L_{(r+1)tu}{}^*L_{(r+1)tu}K_D)$ for $L_{(r+1)tu} = (1, \partial_u, \cdot\cdot, \partial_u{}^r)_{z=t}$.

Proof. Using Lemma 1. 1 for $g \in {}_kL_{2\,:\,mtu}(D : 1)$ we have

$$1 \geqq (g, g)_D \geqq {}_k\lambda_{P(m+1)tu}(D)$$
$$= |\partial_u{}^m g(t)|^2 J_{D,m-1}(t, u)/J_{D,m}(t, u),$$

where $P(m+1) = L_{(m+1)tu}g = (0, \cdot\cdot, 0, \partial_u{}^m g(t))$. Therefore we have

$$|\partial_u{}^m g(t)|^2/K_D(t, t)$$
$$\leqq J_{D,m}(t, u)/K_D(t, t)J_{D,m-1}(t, u).$$

The normal family argument on ${}_kL_{2\,:\,mtu}(D : 1)$ shows that the supremum of the left hand side of the above inequality for each (t, u) is attained by some $g_0 \in {}_kL_{2\,:\,mtu}(D : 1)$ (see [1], [2]). The extremal function g_0 is unique up to unitary transformations by a Hilbert space argument. Now we have (1.9). (1.10) is similarly obtained by Lemma 1. 1.

Theorem 1. 1. *For each $m(\geqq 1)$ and arbitrary positive integers k and r we have*

(1.11) ${}_kC_{D,m}(t, u) < {}_rS_{D,m}(t, u)$

(cf. [2], [4], [6]) *and*

(1.12) ${}_k\tilde{C}_{D,m}(t, u) < \mathrm{vol}^{1/2}(D){}_r\hat{S}_{D,m}(t, u)$,

$t \in D$, $u \in C^n - \{0\}$.

Proof. By Lemma 1. 2 it is sufficient to prove this theorem for $k=r=1$. (1.11) has been proved for $k=r=1$ by Burbea [2].

Set $F(z) = \mathrm{vol}^{-1/2}(D)f(z)$ for $f \in {}_1\mathrm{Hol}(D : 1)$, then we have

$$1 \geqq (F, F)_D \geqq |\partial_u{}^m f(t)|^2(\mathrm{vol}(D){}_1\hat{S}_{D,m}{}^2(t, u))^{-1}$$

from (1.2), i. e.,

$${}_1\tilde{C}_{D,m}{}^2(t, u) \leqq \mathrm{vol}(D){}_1\hat{S}_{D,m}{}^2(t, u).$$

If the equality of this holds for some (t, u) and $f_0 \in {}_1\mathrm{Hol}(D : 1)$, then we have $(F_0, F_0)_D = 1$ for $F_0 = \mathrm{vol}^{-1/2}(D)f_0(z)$, i. e., $\int_D (1-|f_0(z)|^2)\omega_z = 0$, which shows $|f_0(z)| = 1$ in D. By the maximum principle we have $f_0(z) = $ constant in D. This is a contradiction.

(1.11) and (1.12) give us various estimations of higher order differentials of $f \in {}_k\mathrm{Hol}(D : R)$ or ${}_kL_2(D : R)$ in terms of the Bergman kernel.

Remark 1. 1.

(1) ${}_kC_{D,m}$ and ${}_kS_{D,m}$ are biholomorphically invariant [2] but ${}_k\tilde{C}_{D,m}$, ${}_k\tilde{S}_{D,m}$ and ${}_k\hat{S}_{D,m}$ are not so.

(2) From Definition 1. 1 we have

(1.13) ${}_kC_{D,m}(t, u) \leqq {}_k\tilde{C}_{D,m}(t, u)$

and

(1.14) ${}_kS_{D,m}{}^2(t, u) \leqq {}_k\tilde{S}_{D,m}{}^2(t, u)$
$= {}_k\hat{S}_{D,m}{}^2(t, u)/K_D(t, t)$

since for $R = {}_k\hat{S}_{D,m}{}^2(t, u) = \partial_u{}^{*m}\partial_u{}^m K_D(t, t)$ we have

$$J_{D,m}(t, u) = \det\begin{pmatrix} P & Q \\ Q^* & R \end{pmatrix} = J_{D,m-1}(t, u)(R - Q^*P^{-1}Q)$$
$$\geqq J_{D,m-1}(t, u)R$$

in (1.9),

where P is a positive Hermitian matrix.

In particular, if D is a bounded complete circular domain with center at 0, say a classical domain, then we have

(1.15) ${}_kS_{D,m}(0, u) = {}_k\tilde{S}_{D,m}(0, u)$
$= \mathrm{vol}^{1/2}(D){}_k\hat{S}_{D,m}(0, u)$,

$u \in C^n$-$\{0\}$, since $\partial_u^{*i}\partial_u^{j}K_D(0,\ 0)=0$ $(i \neq j)$ and $K_D(0,\ 0)=\mathrm{vol}^{-1}(D)$ hold (see [10]).

(3) From Definition 1.1 ${}_kC_{D,m}$, ${}_k\tilde{C}_{D,m}$ and ${}_k\hat{S}_{D,m}$ have the antimonotonicities, say

(1.16) ${}_kC_{G,m}(t,\ u) \leqq {}_kC_{D,m}(t,\ u)$,
$t \in D \subset G \subset\subset C^n$ (cf. [3]).

(4) From Definition 1.1 (1.5) ${}_kC_{D,m}$ has the decreasing property :

(1.17) ${}_kC_{M,m}(f(t),\ \partial_u f(t)) \leqq {}_kC_{D,m}(t,\ u)$,
$f \in \mathrm{Hol}(D,\ M)$ (cf. [3]).

(5) We easily have biholomorphic invariances:

(1.18) $\Pi_{j=1}^{N}\ {}_kS_{D,j}^2(t,\ u) = J_{D,N}(t,\ u)/K_D^N(t,\ t),\ N \geqq 1.$

2. Cauchy type estimation formulas.

Lemma 2.1. *Let $B \equiv B_n(0,\ r)$ be a ball of radius r with center at 0, then we have*

(2.1) $${}_kS_{B,m}^2(t,\ u) = \binom{m+n}{m}(m!)^2(n+1)^{-m}\cdot {}_kS_{B,1}^{2m}(t,\ u),$$

(2.2) $${}_kC_{B,m}(t,\ u) = m!\,{}_kC_{B,1}^{m}(t,\ u)$$

and a generalization of Graham [3, Proposition 1] :

(2.3) $${}_kS_{B,m}^2(t,\ u) = \binom{m+n}{m}{}_kC_{B,m}^2(t,\ u),$$

$t \in B,\ u \in C^n$-$\{0\}$. *In particular we have*

(2.4) $${}_kC_{B,m}(0,\ u) = {}_k\tilde{C}_{B,m}(0,\ u) = m!(|u|/r)^m < {}_kS_{B,m}(0,\ u) = {}_k\tilde{S}_{B,m}(0,\ u) = \mathrm{vol}^{1/2}(B)\cdot {}_k\hat{S}_{B,m}(0,\ u) = \binom{m+n}{m}^{1/2} m!(|u|/r)^m.$$

Proof. By direct calculations and (1.15) we have

(2.5) $${}_kS_{B,m}^2(0,\ u) = {}_k\tilde{S}_{B,m}^2(0,\ u) = \partial_u^{*m}\partial_u^{m}K_B(0,\ 0)/K_B(0,\ 0) = \binom{m+n}{m}(m!)^2(|u|/r)^{2m} = \binom{m+n}{m}(m!)^2(n+1)^{-m}{}_kS_{B,1}^{2m}(0,\ u)$$

since ${}_kS_{B,1}^2(0,\ u) = {}_1S_{B,1}^2(0,\ u) = (n+1)(|u|/r)^2$. From the biholomorphic invariancies of ${}_kS_{B,m}(t,\ u)$ $(m \geqq 1)$ we have (2.1).

For $f_0(z) = (u^*z)^m/(r|u|)^m$ we have $f_0 \in {}_1\mathrm{Hol}_{mtu}(B : 1)$ with $\partial_u^m f_0(0) = m!(|u|/r)^m$. Therefore from (1.13) we have

$${}_k\tilde{C}_{B,m}^2(0,u) \geqq {}_kC_{B,m}^2(0,\ u) \geqq |\partial_u^m f_0(0)|^2 = (m!)^2(|u|/r)^{2m}.$$

On the other hand set $F(s) = f(us/|u|)$ $(s \in C)$ for any $f \in {}_k\mathrm{Hol}(B : 1)$, then we have $F^{(m)}(0) = m! \cdot (2\pi i)^{-1}\int_{|s|=r-\varepsilon} F(s)s^{-(m+1)}ds,\ 0<\varepsilon<r,$ and thus $|F^{(m)}(0)| = |\partial_u^m f(0)|/|u|^m \leqq m!/(r-\varepsilon)^m$. Therefore we have

(2.6) $${}_kC_{B,m}(0,\ u) = {}_k\tilde{C}_{B,m}(0,\ u) = m!(|u|/r)^m = m!\,{}_kC_{B,1}^{m}(0,\ u).$$

Noting the biholomorphic invariancies of ${}_kC_{B,m}$ $(m \geqq 1)$ we have (2.2). From (2.5) and (2.6) we have (2.3). (2.4) is clear from (2.5), (2.6), (1.11) and $K_B(0,\ 0) = \mathrm{vol}^{-1}(B)$.

Theorem 2.1. *At any $t \in D$ with $r = \mathrm{dist}(t,\ \partial D)$ we have*

(1) *the generalized Cauchy's estimation formulas for a bounded mapping $f \in {}_k\mathrm{Hol}(D : R)$:*

(2.7) $|\partial_u^m f(t)| \leqq R(m!)(|u|/r)^m$,
$u \in C^n$-$\{0\},\ m \geqq 0$ [5], *and*

(2) *the generalized Cauchy type estimation formulas for a L_2-bounded mapping $g \in {}_kL_2(D : R)$:*

(2.8) $|\partial_u^m g(t)| \leqq R((m+n)!m!/\pi^n)^{1/2}|u|^m/r^{m+n},\ u \in C^n$-$\{0\}$, $m \geqq 1$.

Proof. Since from (1.16), (2.2) and (2.4) we have

$${}_k\tilde{C}_{D,m}(t,\ u) \leqq {}_k\tilde{C}_{B(t,r),m}(t,\ u) = {}_kC_{B(t,r),m}(t,\ u) = m!(|u|/r)^m$$

and thus (2.7), where $B(t,\ r) = B_n(t,\ r)$ denotes the maximum ball in D with center at $t \in D$. From the antimonotonicity of ${}_k\hat{S}_{D,m}(t,\ u)$ (see Remark 1.1 (3)) and (2.4) we have

$$|\partial_u^m g(t)|^2 \leqq R^2{}_k\hat{S}_{D,m}^2(t,\ u) \leqq R^2{}_k\hat{S}_{B(t,r),m}^2(t,\ u) = R^2K_{B(t,r)}(t,\ t){}_k\tilde{S}_{B(t,r),m}^2(t,\ u) = \binom{m+n}{m}\mathrm{vol}^{-1}(B(t,\ r))R^2(m!)^2(|u|/r)^{2m}$$

for $g \in {}_kL_2(D : R)$, where $\mathrm{vol}(B(t,\ r)) = \pi^n r^{2n}/n!$.

Lemma 2.2. *For any $f \in F \equiv \{f \in {}_k\mathrm{Hol}(D : R)\ |\ L_{mtu}f = (\xi,\ 0,\ \cdots,\ 0)\}$ $(\xi = f(t))$ we have*

(2.9) $|\partial_u^m f(t)|^2 \leqq (R^2 - |f(t)|^2){}_kS_{D,m}^2(t,\ u)$,
$t \in D,\ u \in C^n$-$\{0\}$, *where the equality holds iff $f(z) = \xi$ (constant vector with $|\xi| = R$) in D.*

Proof. From the reproducing property of the Bergman kernel we have

$$0 \leqq \int_D |(f(z)-\xi)K_D(z,\ t)|^2\omega_z = \int_D (|f(z)|^2 - |\xi|^2)|K_D(z,\ t)|^2\omega_z$$

$\leqq (R^2 - |\xi|^2) K_D(t, t) \equiv L^2,\ f \in F.$

Therefore $F(z) = (f(z) - \xi) K_D(z, t)$ $(f \in F)$ belongs to ${}_kL_{2\,:\,mtu}(D : L)$ with $P(m+1) = L_{(m+1)tu}F = (1, \partial_u, \cdots, \partial_u{}^m)_{z=t}F = (0, \cdots, 0, \partial_u{}^m f(t) K_D(t, t))$. Hence by Lemma 1. 1 we have

$$L^2 = (R^2 - |\xi|^2) K_D(t, t) \geqq (F, F)_D \geqq {}_k\lambda_{P(m+1)tu}(D) = |\partial_u{}^m f(t)|^2 K_D(t, t) / {}_kS_{D,m}{}^2(t, u),$$

which gives (2 . 9).

If the equality of (2 . 9) holds for $f_0 \in F$, then we have $(R^2 - |\xi|^2) K_D(t, t) = (F, F)_D$, i. e., $\int_D (R^2 - |f_0(z)|^2) |K_D(z, t)|^2 \omega_z = 0$. Therefore we have $|f_0(z)| = R$ a. e. in D with $|f_0(t)| = |\xi| = R$. Set $h(z) = \xi^* f_0(z) / R$, then we have $h(z) \in {}_1\mathrm{Hol}(D : R)$ with $h(t) = R$. By the maximum principle of holomorphic functions in $D \subset C^n$ we have $h(z) = \text{constant}$, i. e., $\xi^* f_0(z) = R^2$ in D. Hence we have

$$|f_0(z) - \xi|^2 = (f_0(z) - \xi)^* (f_0(z) - \xi) = |f_0(z)|^2 - 2\mathrm{Re}(\xi^* f_0(z)) + |\xi|^2 = R^2 - 2R^2 + R^2 = 0,\ z \in D,$$

which gives $f_0(z) = \xi$ $(|\xi| = R)$ in D. Converse is true. (2 . 9) includes Theorem 1. 1 (1 .11).

Theorem 2. 2. *Let $D \equiv D_n$ be a bounded domain in C^n and $H \equiv H_k$ be a bounded homogeneous domain in C^k. Let $B_k(c, R_c)$ be the least ball containing H_k with center at an arbitrary point $c \in C^k$ and $\mu(\alpha, H)$ be the maximum characteristic value of the Bergman tensor $T_H(\alpha, \alpha)$. Suppose that f belongs to* Hol(D, H) *and α is an arbitrary point in H, then we have a generalization of the Schwarz-Pick type lemma of Look* [8] :

(2 .10) $S_H(f(t), \partial_u f(t)) \leqq K_0(H) S_D(t, u)$, *$t \in D$, $u \in C^n - \{0\}$, where*

$$K_0{}^2(H) = \inf\{\mu(\alpha, H)(R_c{}^2 - |\alpha - c|^2) \mid \alpha \in H,\ c \in C^k\}\ .$$

Proof. If f belongs to $\mathrm{Hol}(D, \bar{B}_k(c, R_c))$ with $L_{2tu} f = (f(t), \partial_u f(t))$, then we have

(2 .11) $|\partial_u f(t)|^2 \leqq (R_c{}^2 - |f(t) - c|^2)\,{}_kS_D{}^2(t, u)$.

Indeed, for $g \in \mathrm{Hol}(D, \bar{B}_k(c, R_c))$ with $L_{(m+1)tu} g = (g(t), 0, \cdots, 0, \partial_u{}^m g(t))$ we have, from (2 . 9),

$$|\partial_u{}^m g(t)|^2 \leqq (R_c{}^2 - |g(t) - c|^2)\,{}_kS_{D,m}{}^2(t, u),\ t \in D.$$

Set $m = 1$, then we have (2 .11).

Now, for $f \in \mathrm{Hol}(D, H) \subset \mathrm{Hol}(D, \bar{B}_k(c, R_c))$ with $f(t) = \xi$, set $F = h_\xi \circ f$ with $F(t) = \alpha \in H$, where h_ξ is a transitive mapping of H with $h_\xi(\xi) = \alpha$, then we have $F \in \mathrm{Hol}(D, H) \subset \mathrm{Hol}(D, \bar{B}_k(c, R_c))$ $(c \in C^k)$, $L_{2tu}F = (1, \partial_u)_{z=t} F = (\alpha, (dh_\xi(\xi)/dw) \partial_u f(t))$ and $|F(z) - c| \leqq R_c$ for $z \in D$. Therefore from (1 .11) we have

$$|(dh_\xi(\xi)/dw) \partial_u f(t)|^2 \leqq (R_c{}^2 - |F(t) - c|^2)\,{}_kS_D{}^2(t, u),$$

$F(t) = \alpha$. On the other hand, since

$$T_H(\xi, \xi) = (dh_\xi(\xi)/dw)^* T_H(\alpha, \alpha)(dh_\xi(\xi)/dw) \leqq \mu(\alpha, H)(dh_\xi(\xi)/dw)^*(dh_\xi(\xi)/dw)$$

holds, then we have (2 .10) by making use of

$$S_H{}^2(f(t), \partial_u f(t)) = (\partial_u f(t))^* T_H(\xi, \xi) \partial_u f(t),$$

$\xi = f(t)$, and Lemma 1. 2.

Remark 2. 1. Let $B_k(\gamma, R_\gamma)$ be the least ball containing H, then we have

$$K_0{}^2(H) = \inf\{\mu(\alpha, H)(R_c{}^2 - |\alpha - c|^2) \mid \alpha \in H,\ c \in C^k\} \leqq \inf\{\mu(\alpha, H) R_\gamma{}^2 \mid \alpha \in H\} = K^2(H)$$

since $R_\gamma{}^2 - |\alpha - \gamma|^2 \leqq R_\gamma{}^2$ holds. Therefore $K_0(H)$ is a sharper bound than $K(H)$ given by Look [8], because $R_\gamma{}^2 - |\alpha - \gamma|^2 < R_\gamma{}^2$ holds when $\gamma \notin H$.

3. Generalized Schwarz constants.

Let $H \equiv H_n$ be a bounded homogeneous domain in C^n, then from Theorem 2. 2 there exists a least positive constant $K(H)$ depending only on H such that

(3 . 1) $S_H(f(z), \partial_u f(z)) \leqq K(H) S_H(z, u)$, $z \in H$, $u \in C^n - \{0\}$, holds for any $f \in \mathrm{Hol}(H, H)$, where S_H denotes the Bergman metric of H. $K(H)$ is called the Schwarz constant of H and is biholomorphically invariant. K. H. Look [8] has given the Schwarz constants of the classical domains.

Now we shall treat the generalized Schwarz-Pick lemmas between arbitrary two sorts of bounded homogeneous domains which are not necessarily biholomorphically equivalent each other.

Theorem 3. 1. *Let $H \equiv H_n \subset C^n$ and $H' \equiv H_k' \subset C^k$ be arbitrary bounded homogeneous domains, then we have the generalized Schwarz-Pick lemma*

(3 . 2) $S_{H'}(f(z), \partial_u f(z)) \leqq K(H, H') S_H(z, u)$, *$z \in H$, $u \in C^n - \{0\}$, where*

(3 . 3) $$K^2(H, H') = \inf\{R^2(\beta, H')\mu(\beta, H') \cdot (r^2(\alpha, H)\nu(\alpha, H))^{-1} \mid \alpha \in H,\ \beta \in H'\}.$$

Here $\mu(a, D)$ (resp. $\nu(a, D)$) denotes the maximum (resp. minimum) characteristic value of the Bergman tensor $T_D(a, a)$ for a bounded domain D and $R(a, D)$ (resp. $r(a, D)$) denotes the radius of the circum-

sphere (resp. inscribed sphere) of D with center at a.

Proof. Noting $B(\alpha, r)\equiv B_n(\alpha, r(\alpha, H))\subset H\subset B_n(\alpha, R(\alpha, H))\equiv B(\alpha, R)$ for any $\alpha\in H$, we have

$$C_{B(\alpha,R)}(\alpha, u)\leqq C_H(\alpha, u)\leqq C_{B(\alpha,r)}(\alpha, u)$$

from the antimonotonicity (1.16). Since $C_{B(\alpha,\rho)}(\alpha, u)=|u|/\rho$ from (2.4) and

$$\nu(\alpha, H)|u|^2\leqq S_H{}^2(\alpha, u)=u^*T_H(\alpha, \alpha)u \leqq\mu(\alpha, H)|u|^2,$$

then we have

$$(R^2(\alpha, H)\mu(\alpha, H))^{-1}S_H{}^2(\alpha, u)\leqq C_H{}^2(\alpha, u) \leqq(r^2(\alpha, H)\nu(\alpha, H))^{-1}S_H{}^2(\alpha, u),\ u\in C^n-\{0\}.$$

Making use of the biholomorphic invariancies of $C_H(\alpha, u)$ and $S_H(\alpha, u)$ and the homogeneity of H we have

$$(3.4)\quad (R^2(\alpha,H)\mu(\alpha,H))^{-1}S_H{}^2(z,u)\leqq C_H{}^2(z,u) \leqq(r^2(\alpha, H)\nu(\alpha, H))^{-1}S_H{}^2(z, u),\ \alpha\in H.$$

From the left and right hand sides of (3.4) and the decreasing property (1.17) for $f\in \mathrm{Hol}(H, H')$ we obtain (3.2).

Definition 3.1. If a bounded domain D satisfies

$$(3.5)\quad T_D(\alpha, \alpha)=cE_n,\ c>0,\ \alpha\in D,$$

then it is said that D has "*Prop(A)*" at $\alpha\in D$, where E_n denotes the unit matrix of order n (see [10]).

Corollary 3.1. *If $H\equiv H_n$ is a bounded homogeneous domain with* Prop(A) *at $0\in H$, then we have*

$$(3.6)\quad S_H(f(z), \partial_u f(z))\leqq K(H, H)S_H(z, u),$$

$f\in \mathrm{Hol}(H, H)$, where $K(H, H)=R(0, H)/r(0, H)$.

The classical (Cartan) domains M_j $(j=I, II, III, IV)$ with Prop(A) at $0\in M_j$ are defined as follows:

$M_I\equiv M_{I(m\times n)}=\{z\in C^{mn}|\ E_n-z^*z>0,\ z=(z_{ij}) : m\times n$ matrix, $m\geqq n\}$,

$M_{II}\equiv M_{II(n)}=\{z\in C^{n(n+1)/2}|\ E_n-x^*x>0,\ x=(x_{ij}) : n\times n$ symmetric matrix, $z=(z_{ij}),\ x_{ii}=z_{ii},\ x_{ij}=2^{1/2}z_{ij}\ (i\neq j)\}$,

$M_{III}\equiv M_{III(n)}=\{z\in C^{n(n-1)/2}|\ E_n-z^*z>0,\ z=(z_{ij}) : n\times n$ skew symmetric matrix$\}$ and

$M_{IV}\equiv M_{IV(n)}=\{z\in C^n|\ 1+|{}^tzz|^2-2z^*z>0,\ 1-|{}^tzz|>0,\ z={}^t(z_1, \cdots, z_n)\}$.

Further we set

$M_V\equiv M_{V(n)}=\{z\in C^n|\ z={}^t(z_1, \cdots, z_n),\ |z_i|<1\ (i=1, \cdots, n)\}$. (See [8].)

These are complete (Carathéodory) circular domains with center at 0 and multicanonical domains, i. e., the Bergman minimal, Bergman representative and also Mitchell moment minimal domains with the same center at 0 (see [10]). Prop(A) is related to the Mitchell moment minimal domains.

Lemma 3.1. *For M_j $(j=I, II, III, IV, V)$ we have the table*

j	I	II	III	IV	V
$\mu(0, M_j)=\nu(0, M_j)$	$m+n$	$n+1$	$n-1$	$2n$	2
$R(0, M_j)$	$n^{1/2}$	$n^{1/2}$	$[n/2]^{1/2}$	1	$n^{1/2}$
$r(0, M_j)$	1	1	1	$2^{-1/2}$	1

(see [7], [8]).

Corollary 3.2. *For M_j $(j=I, II, III, IV, V)$ we have*

$$(3.7)\quad S_{M_j}(f(z), \partial_u f(z))\leqq K(M_j, M_j)S_{M_j}(z, u),$$

$z\in M_j$, $u\in C^{\dim(M_j)}-\{0\}$, where

j	I	II	III	IV	V
$K(M_j, M_j)$	$n^{1/2}$	$n^{1/2}$	$[n/2]^{1/2}$	$2^{1/2}$	$n^{1/2}$

and these coincide with the Schwarz constants of M_j for $f\in \mathrm{Hol}(M_j, M_j)$ $(j=I, II, III, IV, V)$ given by Look [8].

We may set the conjecture such that for the classical domains M_j with Prop(A) $K(M_i, M_j)$ $(i, j=I, II, III, IV, V)$ for $\alpha=\beta=0$ in Theorem 3.1 give the generalized Schwarz constants.

We shall treat this elsewhere. Here we only give an affirmative example of this conjecture, which gives a generalization of the theorem of Ozaki-Matsuno.

Example 3.1. Set $M\equiv M_{I(m\times n)}$ and $B\equiv B_k(0, 1)$, then from (3.2) we have

$$(3.8)\quad S_B(f(z), \partial_u f(z))\leqq K(M, B)S_M(z, u),$$

$z\in M$, $u\in C^{mn}-\{0\}$, for $f\in \mathrm{Hol}(M, B)$, where $K(M, B)=(k+1)/(m+n)$ since $R(0, B)=r(0, M)=1$, $\mu(0, M)=\nu(0, M)=m+n$ and $\mu(0, B)=\nu(0, B)=k+1$ hold.

We should like to emphasize that (3.8) is best possible, that is, $K(M, B)$ gives the generalized Schwarz constant for $\mathrm{Hol}(M, B)$. Owing to show this, it is sufficient to prove the equality of (3.8) at $\dot{t}=0$, namely, (3.9) $|\partial_u f(0)| = R(0,B) r^{-1}(0,M)|u| = |u|$

for some u and some $f \in \mathrm{Hol}(M, B)$ since the Bergman metrics of M and B are invariant under transitive mappings of M and B, respectively.

Take $u = u_0 = {}^t(1, 0, \cdots, 0)$ ($mn \times 1$ type) and $f(z) = f_0(z) = (\zeta_0, 0)\tilde{z}$, where $\zeta_0 = {}^t(1, 0, \cdots, 0)$ ($k \times 1$ type), $(\zeta_0, 0)$ *is a* $k \times mn$ matrix and $\tilde{z} = (\widetilde{z_{ij}}) = {}^t(z_{11}, z_{21}, \cdots, z_{m1}; z_{12}, \cdots, z_{m2}; \cdots; z_{1n}, \cdots, z_{mn})$ for $z = (z_{ij}) \in M$, then f_0 belongs to $\mathrm{Hol}(M, B)$ and satisfies (3.9): $|\partial_{u_0} f_0(0)| = |u_0| = 1$.

By the way we show that (3.8) gives a generalization of the theorem of Ozaki-Matsuno [9] for Hol $(B, B)(B \equiv B_k(0, 1))$ which is a direct generalization of the usual Schwarz-Pick lemma for the unit disc Δ. Since

$$S_M^2(z, u) = \mu(z, M)|u|^2 = (m+n)(1-||z||^2)^{-2}|u|^2$$

and

$$S_B^2(f(z), \partial_u f(z)) = \nu(f(z), B)|\partial_u f(z)|^2 = (k+1)(1-|f(z)|^2)^{-N}|\partial_u f(z)|^2,$$

where $N=2$ for $k=1$ and $N=1$ for $k \geqq 2$, then from (3.8) we have

(3.10) $$||df(z)/dz||^2 = \sup\{|\partial_u f(z)|^2/|u|^2 \mid |u|=1\} \leqq (1-|f(z)|^2)^N/(1-||z||^2)^2,$$

where $||A||^2$ denotes the maximum characteristic value of A^*A, i. e., $||A|| = \sup\{|Au|/|u|\} \mid |u|=1\}$.

In particular we have the theorem of Ozaki-Matsuno:

(3.11) $||df(z)/dz||^2 \leqq (1-|f(z)|^2)/(1-|z|^2)^2$, $z \in B \equiv B_k(0, 1)$, for $f \in \mathrm{Hol}(B, B)$ $(k \geqq 2)$ and the usual Schwarz-Pick lemma:

(3.12) $|df(z)/dz|^2 \leqq \{(1-|f(z)|^2)/(1-|z|^2)\}^2$, $z \in \Delta$, for $f \in \mathrm{Hol}(\Delta, \Delta)$.

References

[1] S. Bergman, The kernel function and conformal mapping, 2nd ed., Math. Surveys, no. 5, A. M. S., Providence, R. I., 1970

[2] I. Burbea, Inequalities between intrinsic metrics, Proc. A. M. S. **67** (1977), 50-54

[3] I. Graham, Boundary behavior of the Carathéodory and Kobayashi metrics on strongly pseudoconvex domains in C^n with smooth boundary, Trans. A. M. S. **207** (1975), 219-240

[4] K. T. Hahn, On completeness of the Bergman metric and its subordinate metrics II, Pacific J. Math. **68** (1977), 437-446

[5] E. Hille and R. S. Phillips. Functional analysis and semigroups, A. M. S. Colloquium Publications, vol. 31, A. M. S. Providence, R. I., 1957

[6] S. Kato, On the Bergman metric on complex manifold, Mem. Fac. Liberal Arts and Edu. Yamanashi Univ. (1984), 7-9

[7] Y. Kubota, A note on holomorphic imbeddings of the classical Cartan domains into the unit ball, Proc. A. M. S. **85** (1982), 65-68

[8] K. H. Look, Schwarz lemma and analytic invariants, Scientia Sinica **VII** (1958), 453-504

[9] S. Ozaki and T. Matsuno, Note on bounded functions of several complex variables, Sci. Rep. T. K. D. Sect. A **5** (1955), 130-136

[10] S. Matsuura, Bergman kernel functions and the three types of canonical domains, Pacific J. Math. **33** (1970), 363-384

[11] ——, The generalized Martin's minimum problem and its applications in several complex variables, Trans. A. M. S. **208** (1975), 273-307