Computer Program for Determinacy and Unfoldings in the Catastrophe Theory

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In order to apply the catastrophe theory to complicated problems in physics and engineering, we need to perform the essential computation of the strong determinacy, codimension, and test for transversality. This program (written in the BASIC language) will test for strong and local k-determinacy for a specified k, plus any unfolding offered in polynomial form for transversality, and then give the codimension.

§1. Introduction

The catastrophe theory was grown from the morphogenetic speculation by Thom[1], and developed by Zeeman[2]. It has found a broad range of successful applications in physics, in biology, and economics. The applications to the phase transition in physics are discussed by the text[3,4].

Computer programs, which will test the strong and local k-determinacy for specified k, and unfolding offered in polynomial form for transversality and give the codimension, have been published by Rockwood[5]. Since the program was written by ALGOL, we planned to rewrite it to the BASIC language, which is very popular and is easy to understand the algorithm. The program will be rewritten to PL/1, to PASCAL and C-language, because the run time should be shorten by using such computer languages.

In §2 a short survey for the catastrophe theory is given. In §3 and §4, we show the rules for the calculation of determinacy and unfoldings, briefly.

§2. Elementary Catastrophe Theory

2. 1 Structural Stability

We consider the phase portrait after any small perturbation of the system. The phase portrait is given by the integral curves, or trajectories defined by a flow on M (phase space). We would like to classify all possible dynamical systems up to some kind of topological equivalence. We need to find the simple enough order to classify and the complicated enough one to typical. This sometimes called "ying-yang" problem.

We say that a system is structurally stable if its phase portrait is not changed by sufficiently small perturbations. A structurally stable system preserves its basic form when its equations are perturbed. A family of germs $f(x, c)$ is structurally stable if any small perturbations of it is equivalent to it as unfolding, where x and c are the order parameters and the control variables, respectively.

There is a topology (the Whittney topology) on E in which the relevant germs form an open dense set. A property satisfied by an open dense set of objects is often said to be generic.

2. 2 Bifurcation

The phenomenon of bifurcation has been known at least since Poincare. Catastrophe theory (following a key idea of Smale[6] defines a bifurcation point to be a value of the parameter at which the topology of the phase portrait changes. This implies that a non-bifurcation point is one at which the topology dose not change, i. e., a point at which the system is structurally stable.

The essence of the catastrophe theory programme may be ; to obtaine a general understanding of what kinds of bifurcations can typically occur. The earliest results were applied to a more restricted area, known as elementary catastrophe.

Let $f : R^n \times R \rightarrow R$ be a parametrized family of smooth functions,

$$f(x, c)=f(x_1, \cdots, x_n ; c_1, \cdots, c_k). \tag{1}$$

Define the catastrophe set (or eqilibrium set)

Table I Elementary Catastrophes of Thom.

Symbol	k	Name	Germ	Perturbation	Corank	Codimension
A_2	1	fold	x^3	c_1x	1	1
$A_{\pm3}$	2	cusp	$\pm x^4$	$c_1x+c_2x^2$	1	2
A_4	3	swallowtail	x^5	$c_1x+c_2x^2+c_3x^3$	1	3
$A_{\pm5}$	4	butterfly	$\pm x^6$	$c_1x+c_2x^2+c_3x^3+c_4x^4$	1	4
A_6	5	wigwam	x^7	$c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5$	1	5
D_{-4}	3	elliptic umbilic	x^2y-y^3	$c_1x+c_2y+c_3y^2$	2	3
D_{+4}	3	hyperbolic umbilic	x^2y+y^3	$c_1x+c_2y+c_3y^2$	2	3
D_5	4	parabolic umbilic	x^2y+y^4	$c_1x+c_2y+c_3x^2+c_4y^2$	2	4
D_{-6}	5	2nd elliptic umbilic	x^2y-y^5	$c_1x+c_2y+c_3x^2+c_4y^2+c_5y^3$	2	5
D_{+6}	5	2nd hyperbolic umbilic	x^2y+y^5	$c_1x+c_2y+c_3x^2+c_4y^2+c_5y^3$	2	5
$E_{\pm6}$	5	symbolic umbilic	$x^3\pm y^4$	$c_1x+c_2y+c_3xy+c_4y^2+c_5xy^2$	2	5

$$M=\{(x,\ c)\mid df(x,\ c)=0\},\qquad (2)$$

the singularity set

$$\Sigma=\{(x,\ c)\mid df(x,\ c)=0,\ \det d^2f(x,\ c)=0\},\qquad (3)$$

the catastrophe map

$$\chi\ :\ M\rightarrow R,\qquad (4)$$

$$\chi(x,\ c)=c,\qquad (5)$$

and the bifurcation set

$$B=\chi(\Sigma)=\{c\mid (x,\ c)\in\Sigma \text{ for some } x\}.\qquad (6)$$

Smallness of a perturbation is dealt with by definning a topology, the Whitney C^∞ topology. We can now define a critical point of a function to be structurally stable if all nearby critical points of functions in the Whitney C^∞ topology are equivalent to it.

Morse Lemma. Let $f\ :\ R^n\rightarrow R$ be smooth, with a critical point at the origin. The following are equivalent.

(a) f is structurally stable.

(b) f is non-degenerate, i.e., the Hessian matrix $[\partial^2 f/\partial x_i\partial x_j]$ has non-zero determinant at $x=0$.

(c) f is equivalent to a Morse function $\pm x_1^2\pm x_2^2 \cdots \pm x_n^2$.

Note that we can express that a generic critical point is structurally stable.

We seek generic families defined by a mathematical condition of transversalitiy which implies structural stability. It turns out that with up to five parameters, this condition characterizes almost all families. Thom's classification theorem are listed in Table I. The names are standard 'pet' names ; the symbol is part of a systematic notation due to Arnol'd[7]. These families of functions and their generalizations to larger number of parameters, are the elementary catastrophes. The reason for insisting that $k<5$ is that, for larger k, the classification becomes infinite.

2. 3 Determinacy and Co-dimension

Suppose $f\ :\ R^n\rightarrow R$ is smooth, and defined near 0. Choose a coordinate system $(x_1,\ \cdots,\ x_n)$ on R. The jet of f is defined as ;

$$jf=f(0)+\Sigma\frac{\partial f}{\partial x_i}x_i+\frac{1}{2}\Sigma\frac{\partial^2 f}{\partial x_i\partial x_j}x_ix_j+\cdots.\qquad (7)$$

The k-jet $j^k f$ is the Taylor series up to and including terms of order k. We say that f is k-determined if, whenever $g\in E_n$ has $j^kg=j^kf$, it follows that g is right equivalent to f. Where E_n is the set of all functions $R^n\rightarrow R$. If f is k-determined then f is equivalent j^kf, though of as a polynomial functions ; but the converse need not be true. A germ is 1-determined if its linear part is nonzero, that is, its derivative does not vanish. The 1-determined germ is not a singularity. It can be shown that f is 2-detremined if and only if $\det(H)\neq 0$; and in this case f is right equivalent to

$$\pm x_1^2\pm x_2^2\pm\cdots\pm x_n^2.\qquad (8)$$

A germ equivalent to (8) is said to be Morse. Here we define the co-dimension of f

$$\mathrm{cod}(f)=\dim_R \mathrm{m}_n/\Delta(f)\qquad (9)$$

where $\Delta(f)$ is the Jacobian ideal ;

$$g_1\frac{\partial f}{\partial x_1}+\cdots+g_n\frac{\partial f}{\partial x_n}\qquad (10)$$

for arbitrary germ g_i. The m_n is the set of $f\in E_n$ such that $f(0)=0$. The analogous concept in F_n is

$$\mathrm{cod}\ (jf) = \dim_R \mathrm{M}_n / j\Delta(f), \tag{11}$$

where F_n is the set of formal power series and $\mathrm{M}_n = j(\mathrm{m}_n)$. Morse germs are precisely those of codimension 0. A small perturbation of a function of co-dimension c can have at most $c+1$ critical points. The number l of negative signs in (8) is the index of f, and f is an l-saddle.

It is safe to truncate a k-determined germ at degree k of its Taylor series. Suppose that f is not 2-determinate, so that $\det(\mathrm{H}) = 0$. Let the rank of the matrix H be r, and call $n-r$ its corank. A useful result, called the Spilitting Lemma, says that f is right equivalent to a germ of the form

$$g\ (x_1, \cdots, x_{n-r}) \pm x_{n-r+1}^2 \pm \cdots \pm x_n^2 \tag{12}$$

We give an example of the computation of codimension.

Example ; $F(x, y) = x^3 + y^3$.
Here the Jacobian ideal j consists of all power series of the form $g_1 3x^2 + g_2 3y^2$, or equivalently $g_1 x^2 + g_2 y^2$. If we let Δ_1 be the set of power series $g_1 x^2$ and Δ_2 the set of $g_2 y^2$, (Fig. 1) Exactly three monomials, x, y, and xy, are in M_2 but missing from the region formed by both Δ_1 and Δ_2, which represents $j\Delta$. It follows that $\mathrm{cod}(jf) = 3$. Hence also $\mathrm{cod}(f) = 3$.

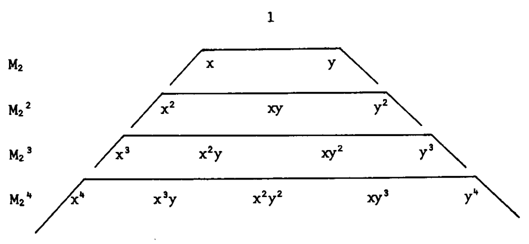

Fig. 1 Monomials in F_2 and the chain of subspaces of $M_2{}^k$ of F_2.

2.4 Unfolding

An unfolding of a singularity is a parameterized family of perturbations.

Let $f \in E_n$. Then an l-parameter unfolding of f is a germ $F \in E_n + l$, that is, a real valued germ of a function

$$F(x_1, \cdots, x_n, \varepsilon_1, \cdots, \varepsilon_l) = F(x, \varepsilon), \text{ such that } F(x, 0) = f(x). \tag{13}$$

Two unfoldings are equivalent if each can be induced from the other. An l-parameter unfolding is versal if all other unfoldings can be induced from it ; universal if in addition l is minimal.

A function f has a universal unfolding if and only if it has finite co-dimension. In this case a universal unfolding is given by

$$f(x, u) = f(x) + u_1 c_1(x) + \cdots + u_l c_l(x) \tag{14}$$

where $c_1, \cdots, c_l$ form a basis for M_n modulo $j\ \Delta(f)$, that is, $\mathrm{M}_n / j\Delta(f)$.

For the case of $f(x, y) = x^3 + y^3$, a basis for M_2 modulo $j\Delta(f)$ is given by x, y and xy. So a universal unfolding is

$$f(x, y, u_1, u_2, u_3) = x^3 + y^3 + u_1 x + u_2 y + u_3 xy. \tag{15}$$

Let R^n denote n-dimensional real Euclidean space. A smooth (that is, infinitely differentiable) function $f : R^n \to R$ has a singularity at $x \in \mathrm{R}^n$ if its derivative $df(x)$ vanishes. In coordinate form, $f(x_1, \cdots, x_n)$ is a real-valued function of n real variables, and

$$df(x) = \left(\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_n}(x)\right) = (0, \cdots, 0). \tag{16}$$

By translation of coordinates we may assume $x = 0$ and this is usualy done.

§3. Rules for Determinacy

A simple algorithm has been developed by Mather for deciding if a function $f(x)$ is determinate and, if so, how much of its Taylor series must be retained to capture its qualitative properties[8].

The algorithm proceeds in a number of simple steps : [9]

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1. Assume that $f(x)$ is k-determinate.
2. Let $\mathrm{m}(x)$ be the sequence of monomials in $x_1, x_2, \cdots, x_l$ of degree 1, 2, ⋯ :

$$\mathrm{m}(x) = x_1, \cdots, x_l : x_1^2, x_1 x_2, \cdots, x_l^2, x_1^3, \cdots. \tag{17}$$

3. Compute the set of polynomials $R_{ij}(x)$ defined by

$$R_{ij}(x) = j^{k+1}\left\{\frac{\partial f}{\partial x_i} m_j(x)\right\}. \tag{18}$$

4. Can all monomials of degree $k+1$ be written as linear superpositions of $R_{ij}(x)$ with constant coefficients ?

If $f(x)$ is k-determinate, the answer to this queation is "yes". Unfortunately the theorem underlying this algorithm is not an "if and only if" theorem, so the answer may be yes if $f(x)$ is not $k-$determinate.

This algorithm can be carried out in a systematic diagramatic way for functions $f(x, y)$ of two variables when $\partial f/\partial x$ and $\partial f/\partial y$ are monomials. The monomials 1 ; x, y : x, xy, $\cdots$. and so on, are arranged in a triangular array â la Pascal. (Fig. 3)

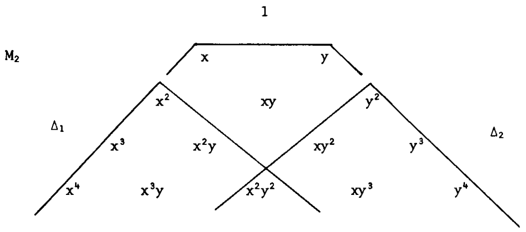

Fig. 2 The Jacobian ideal of $f(x, y) = x^3 + y^3$. The three monomials x, y and xy show that the co-dimension is 3.

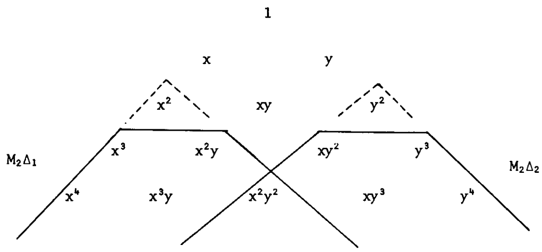

Fig. 3 The determinacy of $f(x, y) = x^3 + y^3$.

§4. Rules for Unfolding

Mather has also developed an algorithm to determine the universal unfolding of a function $f(x)$[9].

1. Find k, the determinacy of $f(x)$. It is sufficient to work with the polynomials $\bar{f}(x) = j^k f(x)$.
2. Let $nj(x)$ be the sequence of monomials in x_1, x_2, $\cdots$, x_l of degree 0, 1, 2, $\cdots$:

$$nj(x) : 1 ; x_1, x_2, \cdots, x_1 ; x_1^2, \cdots \tag{19}$$

3. Assume that $F(x ; a)$ is an r-dimensional unfolding of $\bar{f}(x)$.
   Define

$$T_j(x) = \frac{\partial}{\partial a_j} j^{k+1} F(x ; Q)|_{a=0} \tag{20}$$

4. List all polynomials

$$S_{ij}(x) = j^k \left\{ \frac{\partial \hat{f}}{\partial x_i} n_j(x) \right\}. \tag{21}$$

5. Can all monomials of degree $< k$ be expressed in the form

$$\text{any monomial of degree} < k = \Sigma s_{ij} S_{ij}(x) + \Sigma t_j T_j(x), \tag{22}$$

   Where s_{ij}, t_j are real number ? If the answer is "yes", the $F(x ; a)$ is a versal unfolding of $f(x)$.
6. Is $T_j(x)$ a minimal set ? If the answer is "yes", the $F(x ; a)$ is a universal unfolding of $\bar{f}(x)$.

§5. Computer Program

Computer program which tests the determinacy and the unfolding for specified k, have been published by Rockwood et. al. in ALGOL[5]. We translated it into BASIC. We also planned to translate it into PASCAL, because it is easy to perform the rewrite the program from ALGOL. Since this program need the very large memory size for calculation, we translate it into BASIC at first. We are now planning to rewrite it to PASCAL and C-language, because it takes a long machine time for calculation of the

Table II Sample output of calculation.

```
Welcome to the world of CATASTROPHE theory
86/07/22
14 : 52 : 51
Number of variables = 2
Number of terms     = 2
 It is now the time to input the polynomials
 coef  x  y  z
  1    3  0              x3
  1    0  3                 y3
What k do you want to try ? (0 to stop) 3
MONOMIAL INDEX = 2
  * * * strongly determined * * *
HOW MANY UNFOLDING TERMS DO YOU HAVE ? 0
CODIMENSION = 4
ADDITIONAL UNFOLDING TERMS ARE
  1    1          xy
  0    1            y
  1    0              x
  0    0
ENTER 0 FOR ANOTHER UNFOLDING
UNFOLDING TERMS -1
What k do you want to try ?
0 to stop 0
THANK YOU !
```

unfolding of the function f. Our program is able to test the determinacy and the unfolding offered only in polynomial form. An example is listed in Table II.

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