MONOTONICITY OF AN INITIAL BUSY PERIOD IN A PRIORITY QUEUING SYSTEM

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This paper deals with a single-server priority queuing system consisting of several terminals with single buffers. The service rule is a policy on a priority basis, that is, the terminal with the highest input rate having the highest priority, ···, and the terminal with the lowest input rate having the lowest priority. It is shown that the initial busy period has a monotonicity property under the service rule.

1. Introduction

This paper deals with a priority queuing system. The system consists of M terminals with single buffers and the server attends to one of busy terminals. The service time of each job at all terminals is assumed to be a unit time. The system is observed at time $t=0, 1, 2, \cdots$. The i-th terminal $(i=1, 2, \cdots, M)$ generates one new job in each period with probability a_i, independently of the other terminals and independently from period to period. It can be assumed without loss of generality that $1 \geqq a_1 \geqq a_2 \geqq \cdots\cdots \geqq a_M >0$. A new job arriving at a terminal in service is not allowed to enter into its buffer. The service rule is a policy on a priority basis, the terminal with the highest input rate having the highest priority, ···, and the terminal with the lowest input rate having the lowest priority. The purpose of this paper is to show monotonicity of an initial busy period under the service rule. It should be noted that the monotonicity leads to the optimality of the policy on the priority basis mentioned above[1].

In the real world the above system is found in data networks using packet switching techniques[2,3,4], and is closely related to communication systems with polling[5,6]. Any optimal service rule, however, has never been discussed for a single-server queuing system consisting of terminals with finite buffer spaces. An optimal service rule in a multiclass queuing system with infinite buffer spaces has been studied by Harrison[7]. Furthermore, Wan Tcha and Pliska[8] have dealt with this system with feedback.

This paper is organized as follows. In Section 2, the preliminaries are given. In Section 3, it is proved that the initial busy period has the monotonicity under the service rule.

2. Preliminaries

The system is observed at time $t=0, 1, 2, \cdots$ The state of the system is described by the vector $i(t)=(i_1(t), \cdots, i_M(t))$, where $i_k(t)$, $k=1, \cdots, M$, $t=0, 1, 2, \cdots$ represents the number of jobs at the k-th terminal and takes the value 0 or 1. The state space consisting of all possible states is denoted by S. Denote by $A(i)$ the set $\{k \mid i_k=1, k=1, \cdots, M\}$ and define $l(i)$ by $l(i)=\min\{k \mid k \varepsilon A(i)\}$. The system in state i next moves to state $j=(j_1, \cdots, j_M)$ with the transition probability $P(i, j)$ given by

$$P(i,j)=\prod_{k \notin A(i)} \{(1-a_k)+j_k(2a_k-1)\} \quad \text{for } j_{l(i)}=0,\ j_m=1\ (m \neq l(i) \varepsilon A(i))$$
$$=0, \quad \text{otherwise.} \tag{1}$$

Let P be the $2^M \times 2^M$-dimensional matrix whose (i, j) component is $P(i, j)$.

Let $t(i)$ be the initial busy period, i. e., the first epoch at which the system starting from the initial state i is cleared of jobs. Define $F(i)$ as the expectation of $d^{t(i)}$, where d is an arbitrary constant between 0 and 1. Here, we assume that $F(0)=1$. Let S' be the set $S-\{0\}$. It is clear by (1) that $F(i)$ satisfies the following relation for any $i \varepsilon S'$:

$$F(i)=dP(i, 0)+d \sum_{j \varepsilon S'} P(i, j)F(j). \tag{2}$$

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3. Monotonicity of Initial Busy Period

In order to prove the monotonicity of the initial busy peoriod, it suffices to show that of $F(i)$. For this purpose, let a partial order $i<j$ be defined for $i=(i_1, \cdots, i_M)$ and $j=(j_1, \cdots, j_M)$ as the relation that $i_k=0$, $i_l=1$, $j_k=1$, $j_l=0$ for $k<l$ and $i_m=j_m$ for $m \neq k, l$. If $F(i) \leqq (<) F(j)$ for all i and j such that $i<j$, then the column vector F is said to be (strictly) monotone.

Let $t_m(i)$ be the first epoch at which the system consisting of the first terminal to the m-th terminal $(2 \leqq m \leqq M-1)$ is cleared of jobs, given that the initial state is $i=(i_1, i_2, \cdots, i_M)$. Since $t_m(i)$ is not affected by i_k, $k=m+1, \cdots, M$, define $F_m(i_m)$ as the expectation of $d^{t_m(i)}$ given that the initial state is $i_m=(i_1, \cdots, i_m)$. Denote by S_m the state space consisting of all possible i_m, and by S_m' the set $S_m-\{0_m\}$, where 0_m stands for the state with $i_k=0$, $k=1, \cdots, m$.

First, we consider the case where $m=2$. From (2), $F_2(10)$, $F_2(01)$ and $F_2(11)$ satisfy the following relations :

$$F_2(10)=d(1-a_2)+da_2F_2(01)$$
$$F_2(01)=d(1-a_1)+da_1F_2(10)$$
$$F_2(11)=dF_2(01).$$

That is, the 3-dimensional column vector $F_2(S_2')=[F_2(10), F_2(01), F_2(11)]^T$ satisfies

$$F_2(S'_2)=dB_2+dT_2F_2(S'_2), \tag{3}$$

where the 3-dimensional column vector B_2 and 3×3 -dimensional matrix T_2 are given for constants $c_{10}=1-a_1$ and $c_{11}=a_1$, by

$$B_2=\begin{pmatrix} 1-a_2 \\ c_{10} \\ 0 \end{pmatrix} \text{ and } T_2=\begin{pmatrix} 0 & a_2 & 0 \\ c_{11} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4}$$

Let B_m be the transition probability vector from state $i_m \varepsilon S'_m$ to state $j_m=0_m$ and T_m be the transition probability matrix from state $i_m \varepsilon S'_m$ to state $j_m \varepsilon S'_m$. For $m=2, \cdots, M-1$, define for vector or matrix 0 whose components are all zero,

$$c_{m,0}=(1-a_m)c_{m-1,0}, \tag{5}$$
$$\mathbf{c}_{m,1}=[(1-a_m)\mathbf{c}_{m-1,1}, a_mc_{m-1,0}, a_m\mathbf{c}_{m-1,1}], \tag{6}$$
$$\mathbf{c}_m=[c_{m,0}, \mathbf{c}_{m,1}], \tag{7}$$
$$B_{m+1}=\begin{pmatrix} (1-a_{m+1})B_m \\ c_{m,0} \\ 0 \end{pmatrix} \tag{8}$$

and

$$T_{m+1}=\begin{pmatrix} (1-a_{m+1})T_m & a_{m+1}B_m & a_{m+1}T_m \\ \mathbf{c}_{m,1} & 0 & 0 \\ 0 & B_m & T_m \end{pmatrix}, \tag{9}$$

where $c_{1,0}$, $c_{1,1}$, B_2 and T_2 are given by (4). The following lemma holds :

Lemma 1. For $m=2, \cdots, M-1$, the $(2^{m+1}-1)$-dimensional column vector B_{m+1} and $(2^{m+1}-1) \times (2^{m+1}-1)$-dimensional matrix T_{m+1} satisfy (5) through (9) and the $(2^{m+1}-1)$-dimensional column vector $F_{m+1}(S'_{m+1})=[F_{m+1}(S'_m,0)^T, F_{m+1}(0_m,1), F_{m+1}(S'_m,1)^T]^T$ satisfies

$$F_{m+1}(S'_{m+1})=dB_{m+1}+dT_{m+1}F_{m+1}(S'_{m+1}). \tag{10}$$

Proof. It follows from (1) that the transition probability matrix is given by $[B_m, T_m]$. Using B_m and T_m, and noting that $\mathbf{c}_m$ is the transition probability vector from state $i_m=0_m$ to state $j_m \varepsilon S_m$, we can obtain

$$F_{m+1}(S'_m, 0)=d(1-a_{m+1})B_m+d(1-a_{m+1})T_m \times F_{m+1}(S'_m, 0)+da_{m+1}B_mF_{m+1}(0_m,1) + da_{m+1}T_mF_{m+1}(S'_m,1), \tag{11}$$
$$F_{m+1}(0_m,1)=dc_{m,0}+d\mathbf{c}_{m,1}F_{m+1}(S'_m,0) \tag{12}$$

and

$$F_{m+1}(S'_m,1)=dB_mF_{m+1}(0_m,1)+dT_mF_{m+1}(S'_m,1). \tag{13}$$

Summarizing (11) through (13) leads to (5) through (10).

□

Since $d^n[T_{m+1}]^n$ converges to 0 as n tends to infinity, the inverse matrix $(I-dT_{m+1})^{-1}$ exists for identity matrix I. Therefore, (10) with m replaced by m−1 leads to

$$F_m(S'_m)=(I-dT_m)^{-1}dB_m. \tag{14}$$

Define $\tilde{F}_m(S'_m)$ as $F_m(S'_m)$ with d replaced by $(1-a_{m+1})$d. From (12), the next lemma is obtained, because $0 \leqq (1-a_{m+1})d<1$, B_m and T_m are independent of d, and $\tilde{F}_m(S'_m)=0$ for $(1-a_{m+1})d=0$.

Lemma 2. If $F_m(S'_m)$ is (strictly) monotone for any $d(0<d<1)$, then $\tilde{F}_m(S'_m)$ is also (strictly) monotone for any $d(0<d<1)$.

Lemma 3.

(a) $$F_{m+1}(S'_m,1)=F_{m+1}(0_m,1)F_m(S'_m). \tag{15}$$

(b) $$F_{m+1}(S'_m,0)=[1-F_{m+1}(0_m,1)]\tilde{F}_m(S'_m) + F_{m+1}(0_m,1)F_m(S'_m). \tag{16}$$

Proof.

(a) From (13), $F_{m+1}(S'_m,1)$ is given by

$$F_{m+1}(S'_m,1)=F_{m+1}(0_m,1)(I-dT_m)^{-1}dB_m.$$

Consequently, (14) yields (15).

(b) It follows from (11) and (13) that

$$F_{m+1}(S'_m,0)=(1-a_{m+1})dB_m+(1-a_{m+1})dT_m \times F_{m+1}(S'_m,0)+a_{m+1}F_{m+1}(S'_m,1).$$

Thus,

$$F_{m+1}(S'_m,0)=[I-(1-a_{m+1})dT_m]^{-1}[(1-a_{m+1})dB_m + a_{m+1}F_{m+1}(S'_m,1)]. \tag{17}$$

On the other hand, (14) and (15) lead to the following relation :

$$\begin{aligned}
&a_{m+1}[I-(1-a_{m+1})dT_m]^{-1}F_{m+1}(S'_m,1)\\
&=a_{m+1}\sum_{n=0}^{\infty}(1-a_{m+1})^n d^n[T_m]^n F_{m+1}(O_m,\ 1)F_m(S'_m)\\
&=a_{m+1}F_{m+1}(O_m,1)\sum_{n=0}^{\infty}(1-a_{m+1})^n d^n[T_m]^n\\
&\quad\times(I-dT_m)^{-1}dB_m\\
&=a_{m+1}F_{m+1}(O_m,1)\sum_{n=0}^{\infty}(1-a_{m+1})^n d^n[T_m]^n\\
&\quad\times\sum_{k=0}^{\infty}d^{k+1}[T_m]^k B_m\\
&=a_{m+1}F_{m+1}(O_m,1)\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}(1-a_{m+1})^n d^{k+1}\\
&\quad\times[T_m]^k B_m\\
&=a_{m+1}F_{m+1}(O_m,1)\sum_{k=0}^{\infty}\sum_{n=0}^{k}(1-a_{m+1})^n d^{k+1}\\
&\quad\times[T_m]^k B_m\\
&=F_{m+1}(O_m,1)\sum_{k=0}^{\infty}[1-(1-a_{m+1})^{k+1}]d^{k+1}[T_m]^k B_m\\
&=dF_{m+1}(O_m,1)(I-dT_m)^{-1}B_m\\
&\quad-(1-a_{m+1})dF_{m+1}(O_m,1)[I-(1-a_{m+1})dT_m]^{-1}B_m.
\end{aligned} \tag{18}$$

Combining (17) with (18) yields

$$F_{m+1}(S'_m,0)=[1-F_{m+1}(O_m,1)][I-(1-a_{m+1})dT_m]^{-1}\times(1-a_{m+1})dB_m+F_{m+1}(O_m,1)(I-dT_m)^{-1}dB_m. \tag{19}$$

Using (14), (19) is rewritten as (16). The proof is completed. □

Noting that from the definition of $F_{m+1}(O_m,1)$, $0<F_{m+1}(O_m,1)<1$, we have the following lemma from Lemmas 2 and 3.

Lemma 4. If $F_m(S'_m)$ is (strictly) monotone for any d $(0<d<1)$, then $F_{m+1}(S'_m,0)$ and $F_{m+1}(S'_m,1)$ are also (strictly) monotone for any $d\,(0<d<1)$.

Lemma 5.

(a) $F_{m+1}(O_{m-1},1,0)\geqq F_{m+1}(O_{m-1},0,1)$.

(b) $F_{m+1}(i_{m-1},1,0)\geqq F_{m+1}(i_{m-1},0,1)$ for all $i_{m-1}\varepsilon S'_{m-1}$.

In particular, when $1\geqq a_1>a_2>\cdots>a_M>0$, the strict inequalities in (a) and (b) hold.

Proof.

(a) From Lemma 3(b), it follows that

$$F_{m+1}(S'_m,0)=\tilde{F}_m(S'_m)+F_{m+1}(O_m,1)[F_m(S'_m)-\tilde{F}_m(S'_m)]. \tag{20}$$

Combining (12) with (20), we have

$$F_{m+1}(O_m,1)=dc_{m,0}+dc_{m,1}[\tilde{F}_m(S'_m)+F_{m+1}(O_m,1)[F_m(S'_m)-\tilde{F}_m(S'_m)]]. \tag{21}$$

Since $\tilde{F}_m(S'_m)=[\tilde{F}_m(S'_{m-1},0)^{T},\tilde{F}_m(O_{m-1},1),\tilde{F}_m(S'_{m-1},1)^{T}]^{T}$, it follows from (5) and (6) that

$$c_{m,1}\tilde{F}_m(S'_m)=(1-a_m)c_{m-1,1}\tilde{F}_m(S'_{m-1},0)+a_m c_{m-1,0}\tilde{F}_m(O_{m-1},1)+a_m c_{m-1,1}\tilde{F}_m(S'_{m-1},1).$$

Hence, using (6) again, (21) is rewritten as

$$\begin{aligned}
F_{m+1}(O_m,1)&=d(1-a_m)c_{m-1,0}+d(1-a_m)c_{m-1,1}\\
&\quad\times\tilde{F}_m(S'_{m-1},0)+da_m c_{m-1,0}\tilde{F}_m(O_{m-1},1)\\
&\quad+da_m c_{m-1,1}\tilde{F}_m(S'_{m-1},1)+dF_{m+1}(O_m,1)c_{m,1}[F_m(S'_m)\\
&\quad-\tilde{F}_m(S'_m)]\\
&=dc_{m-1,0}+dc_{m-1,1}\tilde{F}_m(S'_{m-1},0)-da_m c_{m-1,0}[1\\
&\quad-\tilde{F}_m(O_{m-1},1)]-da_m c_{m-1,1}[\tilde{F}_m(S'_{m-1},0)\\
&\quad-\tilde{F}_m(S'_{m-1},1)]+dF_{m+1}(O_m,1)c_{m,1}[F_m(S'_m)\\
&\quad-\tilde{F}_m(S'_m)].
\end{aligned} \tag{22}$$

On the other hand, Lemma 3 implies that

$$F_{m+1}(S'_m,1)=F_{m+1}(S'_m,0)-[1-F_{m+1}(O_m,1)]\tilde{F}_m(S'_m). \tag{23}$$

Combining (22) and (12) with (23) gives us

$$\begin{aligned}
F_{m+1}(O_m,1)&=dc_{m-1,0}+dc_{m-1,1}\tilde{F}_m(S'_{m-1},0)\\
&\quad-da_m c_{m-1,0}[1-\tilde{F}_m(O_{m-1},1)]-da_m[1-\tilde{F}_m(O_{m-1},\\
&\quad1)]\times c_{m-1,1}\tilde{\tilde{F}}_{m-1}(S'_{m-1})\\
&\quad+dF_{m+1}(O_m,1)c_{m,1}[F_m(S'_m)-\tilde{F}_m(S'_m)]\\
&=\tilde{F}_m(O_{m-1},1)-da_m[1-\tilde{F}_m(O_{m-1},1)][c_{m-1,0}\\
&\quad+c_{m-1,1}\tilde{\tilde{F}}_{m-1}(S'_{m-1})]+dF_{m+1}(O_m,1)c_{m,1}[F_m(S'_m)\\
&\quad-\tilde{F}_m(S'_m)]+a_{m+1}d[c_{m-1,0}+c_{m-1,1}\tilde{F}_m(S'_{m-1},0)],
\end{aligned}$$

where $\tilde{\tilde{F}}_{m-1}(S'_{m-1})$ denotes $F_{m-1}(S'_{m-1})$ with d replaced by $(1-a_{m+1})(1-a_m)d$. Thus we obtain

$$\begin{aligned}
&\tilde{F}_m(O_{m-1},1)-F_{m+1}(O_m,1)\\
&=da_m[1-\tilde{F}_m(O_{m-1},1)][c_{m-1,0}+c_{m-1,1}\\
&\quad\times\tilde{\tilde{F}}_{m-1}(S'_{m-1})]-dF_{m+1}(O_m,1)c_{m,1}[F_m(S'_m)\\
&\quad-\tilde{F}_m(S'_m)]-a_{m+1}d[c_{m-1,0}+c_{m-1,1}\tilde{F}_m(S'_{m-1},0)].
\end{aligned}$$

Therefore, Lemma 3 leads to :

$$\begin{aligned}
&F_{m+1}(O_{m-1},1,0)-F_{m+1}(O_{m-1},0,1)\\
&=[1-F_{m+1}(O_m,1)]\tilde{F}_m(O_{m-1},1)-F_{m+1}(O_m,1)[1\\
&\quad-F_m(O_{m-1},1)]\\
&=da_m[1-\tilde{F}_m(O_{m-1},1)][c_{m-1,0}+c_{m-1,1}\\
&\quad\times\tilde{\tilde{F}}_{m-1}(S'_{m-1})]+F_{m+1}(O_m,1)[\{F_m(O_{m-1},1)\\
&\quad-\tilde{F}_m(O_{m-1},1)\}-dc_{m,1}\{F_m(S'_m)-\tilde{F}_m(S'_m)\}]\\
&\quad-a_{m+1}d\{c_{m-1,0}+c_{m-1,1}\tilde{F}_m(S'_{m-1},0)\}\ .
\end{aligned} \tag{24}$$

Using (12), (24) is rewritten as

$$\begin{aligned}
&F_{m+1}(O_{m-1},1,0)-F_{m+1}(O_{m-1},0,1)\\
&=da_m[1-\tilde{F}_m(O_{m-1},1)][c_{m-1,0}+c_{m-1,1}\\
&\quad\times\tilde{\tilde{F}}_{m-1}(S'_{m-1})]+dF_{m+1}(O_m,1)[\{c_{m-1,1}F_m(S'_{m-1},0)\\
&\quad-c_{m,1}F_m(S'_m)\}-\{c_{m-1,1}\tilde{F}_m(S'_{m-1},0)-c_{m,1}
\end{aligned}$$

$$\times \tilde{F}_m(S'_m)\}] - a_{m+1} d\{c_{m-1,0} + \mathbf{c}_{m-1,1}\tilde{F}_m(S'_{m-1},0)\}\ \{1 - F_{m+1}(O_m,1)\}. \tag{25}$$

On the other hand, from (6) and (7), it follows that

$$\mathbf{c}_{m,1}F_m(S'_m)$$
$$= (1-a_m)\mathbf{c}_{m-1,1}F_m(S'_{m-1},0) + a_m c_{m-1,0}F_m(O_{m-1},1) + a_m \mathbf{c}_{m-1,1}F_m(S'_{m-1},1)$$
$$= (1-a_m)\mathbf{c}_{m-1,1}F_m(S'_{m-1},0) + a_m\mathbf{c}_{m-1,1}F_m(S'_{m-1},0) + a_m c_{m-1,0}F_m(O_{m-1},1) + a_m\mathbf{c}_{m-1,1}[F_m(S'_{m-1},1) - F_m(S'_{m-1},0)]. \tag{26}$$

Combining (26) with (23), we have

$$\mathbf{c}_{m,1}F_m(S'_m)$$
$$= \mathbf{c}_{m-1,1}F_m(S'_{m-1},0) + a_m c_{m-1,0}F_m(O_{m-1},1) - a_m\mathbf{c}_{m-1,1}[1 - F_m(O_{m-1,1})]\tilde{F}_{m-1}(S'_{m-1}). \tag{27}$$

From (27), it is immediate that

$$\mathbf{c}_{m-1,1}F_m(S'_{m-1},0) - \mathbf{c}_{m,1}F_m(S'_m)$$
$$= a_m[1 - F_m(O_{m-1},1)]\mathbf{c}_{m-1,1}\tilde{F}_{m-1}(S'_{m-1}) - a_m c_{m-1,0}F_m(O_{m-1},1). \tag{28}$$

and

$$\mathbf{c}_{m-1,1}\tilde{F}_m(S'_{m-1},0) - \mathbf{c}_{m,1}\tilde{F}_m(S'_m)$$
$$= a_m[1 - \tilde{F}_m(O_{m-1},1)]\mathbf{c}_{m-1,1}\tilde{\tilde{F}}_{m-1}(S'_{m-1}) - a_m c_{m-1,0}\tilde{F}_m(O_{m-1},1). \tag{29}$$

Combining (28) and (29) with (25) leads to

$$F_{m+1}(O_{m-1},1,0) - F_{m+1}(O_{m-1},0,1)$$
$$= dD_m + d[1 - F_{m+1}(O_m,1)]a_m[1 - \tilde{F}_m(O_{m-1},1)] \times \mathbf{c}_{m-1,1}\tilde{\tilde{F}}_{m-1}(S'_{m-1}) + da_m F_{m+1}(O_m,1)[1 - F_m(O_{m-1},1)]\ \mathbf{c}_{m-1,1}\tilde{F}_{m-1}(S'_{m-1}) - da_{m+1}[1 - F_{m+1}(O_m,1)]\ [c_{m-1,0} + \mathbf{c}_{m-1,1}\tilde{F}_m(S'_{m-1},0)],$$

where for notational convenience D_m is defined as

$$D_m = a_m[1 - \tilde{F}_m(O_{m-1},1)]c_{m-1,0} - a_m F_{m+1}(O_m,1)c_{m-1,0} \times [F_m(O_{m-1},1) - \tilde{F}_m(O_{m-1},1]. \tag{30}$$

It follows from (12) and (30) that

$$D_m = a_m[1 - \tilde{F}_m(O_{m-1},1)]c_{m-1,0} - a_m F_{m+1}(O_m,1)c_{m-1,0} \times [\mathbf{c}_{m-1,1}d\{F_m(S'_{m-1},0) - \tilde{F}_m(S'_{m-1},0)\} + da_{m+1}\{c_{m-1,0} + \mathbf{c}_{m-1,1}\tilde{F}_m(S'_{m-1},0)\}]$$
$$= a_m[1 - \tilde{F}_m(O_{m-1},1)]c_{m-1,0} - a_m F_{m+1}(O_m,1)c_{m-1,0} \times [\mathbf{c}_{m-1,1}d\{\mathbf{1}_m - \tilde{F}_m(S'_{m-1},0)\} + da_{m+1}\{c_{m-1,0} + \mathbf{c}_{m-1,1}\tilde{F}_m(S'_{m-1},0)\}] + a_m F_{m+1}(O_m,1)c_{m-1,0}d\mathbf{c}_{m-1,1}[\mathbf{1}_m - F_m(S'_{m-1},0)]$$
$$= a_{m+1}c_{m-1,0}[1 - F_{m+1}(O_m,1)] + (a_m - a_{m+1})c_{m-1,0}[1 - F_{m+1}(O_m,1)] - a_m c_{m-1,0}[F_{m+1}(O_{m-1},1,0) - F_{m+1}(O_{m-1},0,1)],$$

where $\mathbf{1}_m$ denotes the m-dimensional column vector $\mathbf{1}_m = (1,1,\cdots,1)^T$.

By using Lemma 3, we obtain

$$F_{m+1}(O_{m-1},1,0) - F_{m+1}(O_{m-1},0,1)$$
$$= [1 - F_{m+1}(O_m,1)]\tilde{F}_m(O_{m-1},1)[(a_m - a_{m+1})/(1 - a_{m+1}) - da_m\mathbf{c}_{m-1,1}\{\hat{F}_{m-1}(S'_{m-1}) - \tilde{F}_{m-1}(S'_{m-1})\}]$$
$$- da_m[c_{m-1,0} + \mathbf{c}_{m-1,1}\tilde{F}_{m-1}(S'_{m-1})] \times [F_{m+1}(O_{m-1},1,0) - F_{m+1}(O_{m-1},0,1)], \tag{31}$$

where $\hat{F}_{m-1}(S'_{m-1})$ denotes $F_{m-1}(S'_{m-1})$ with d replaced by $(1-a_{m+1})d$. From (31), it holds that

$$F_{m+1}(O_{m-1},1,0) - F_{m+1}(O_{m-1},0,1)$$
$$= [1 - F_{m+1}(O_m,1)]\tilde{F}_m(O_{m-1},1)I_m(a_{m+1})/[1 + da_m\{c_{m-1,0} + \mathbf{c}_{m-1,1}\tilde{F}_{m-1}(S'_{m-1})\}], \tag{32}$$

where

$$I_m(a_{m+1}) = (a_m - a_{m+1})/(1 - a_{m+1}) - a_m d\mathbf{c}_{m-1,1} \times [\hat{F}_{m-1}(S'_{m-1}) - \tilde{F}_{m-1}(S'_{m-1})] \quad \text{for } 0 \leqq a_{m+1} \leqq a_m. \tag{33}$$

Define $h(t)$ as $h(t) = t(1-a)^{t-1}d^t$. Then,

$$\frac{d}{dt}h(t) = (1-a)^{t-1}d^t + t(1-a)^{t-1}d^t ln[(1-a)d]$$
$$= [1 + t ln\{(1-a)d\}](1-a)^{t-1}d^t.$$

Hence, defining t^* as $t^* = -1/ln[(1-a)d]$, we have

$$\max_t\ [h(t)] = h(t^*)$$
$$= (1-a)^{-1}t^*[(1-a)d]^{t^*}$$
$$= (1-a)^{-1}e^{-1}t^*. \tag{34}$$

On the other hand, the following holds : for $a \geqq 0$,

$$\frac{\partial}{\partial a}[(1-a)de^{ad}] = [-1 + (1-a)d]de^{ad} < 0$$

and

$$\lim_{a\to 0}[(1-a)de^{ad}] = d \leqq 1.$$

Therefore,

$$[(1-a)de^{ad}] \leqq d \leqq 1,$$

and

$$-ln[(1-a)d] \geqq ln[e^{ad}].$$

Hence,

$$-ad/ln[(1-a)d] \leqq 1. \tag{35}$$

Since $a^t - b^t = (a-b)(a^{t-1} + a^{t-2}b + \cdots + b^{t-1})$,

$$I_m(a_{m+1})$$
$$= [(a_m - a_{m+1}) - (1 - a_{m+1})a_m d\mathbf{c}_{m-1,1}\{\hat{F}_{m-1}(S'_{m-1}) - \tilde{F}_{m-1}(S'_{m-1})\}]/(1 - a_{m+1})$$
$$= [(a_m - a_{m+1}) - (1 - a_{m+1})a_m d\mathbf{c}_{m-1,1}\{E\{(1 - a_{m+1})^t d^t \mid S'_{m-1}\} - E\{(1 - a_m)^t d^t \mid S'_{m-1}\}\}]/(1 - a_{m+1}) \tag{36}$$
$$= [(a_m - a_{m+1}) - (1 - a_{m+1})a_m d\mathbf{c}_{m-1,1}(a_m - a_{m+1}) \times E\{d^t \sum_{l=0}^{t-1}(1 - a_{m+1})^{t-l-1}(1 - a_m)^l | S'_{m-1}\}]/(1 - a_{m+1}) \tag{37}$$

Using $1 - a_{m+1} \geqq 1 - a_m$, it follows from (34) and (37) that

$$I_m(a_{m+1}) \geqq [1 - (1 - a_{m+1})a_m d\mathbf{c}_{m-1,1}\mathrm{E}\{t(1 - a_{m+1})^{t-1}d^t | S'_{m-1}\}](a_m - a_{m+1})/(1 - a_{m+1})$$
$$\geqq [1 - (1 - a_{m+1})a_m d\mathbf{c}_{m-1,1}\{-\mathbf{1}_m(1 - a_{m+1})^{-1}e^{-1}/ln\{(1 - a_{m+1})d\}\}](a_m - a_{m+1})/(1 - a_{m+1})$$
$$= [1 - \{e^{-1}a_m/a_{m+1}\}\mathbf{c}_{m-1,1}\ \mathbf{1}_m\{-a_{m+1}d/ln\{(1 - a_{m+1})d\}\}](a_m - a_{m+1})/(1 - a_{m+1}). \tag{38}$$

Since $c_{m-1,0}+c_{m-1,1}\,\mathbf{1}_m=1$ and $c_{m-1,1}\,\mathbf{1}_m<1$, it holds from (38) and (39) that

$$I_m(a_{m+1})\geqq 0 \text{ for } a_m e^{-1}\leqq a_{m+1}\leqq a_m. \tag{39}$$

On the other hand, by (33),

$$\lim_{a_{m+1}\to 0} I_m(a_{m+1})=a_m[1-dc_{m-1,1}\{\hat{F}_{m-1}(S'_{m-1}) -\tilde{F}_{m-1}(S'_{m-1})\}]\geqq 0. \tag{40}$$

Equation (36) yields

$$\frac{\partial}{\partial a_{m+1}}[(1-a_{m+1})I_m(a_{m+1})]=-1+a_m dc_{m-1,1} \times[E\{(1-a_{m+1})^t d^t \mid S'_{m-1}\}-E\{(1-a_m)^t d^t \mid S'_{m-1}\}]+(1-a_{m+1})a_m dc_{m-1,1}E\{t(1-a_{m+1})^{t-1}d^t \mid S'_{m-1}\}$$

and

$$\frac{\partial^2}{\partial^2 a_{m+1}}[(1-a_{m+1})I_m(a_{m+1})]=-a_m dc_{m-1,1} \times \mathrm{E}\,[t(1-a_{m+1})^{t-1}d^t \mid S'_{m-1}]-a_m dc_{m-1,1}\mathrm{E}[t(1-a_{m+1})^{t-1}d^t \mid S'_{m-1}]-(1-a_{m+1})a_m dc_{m-1,1} \times E[t(t-1)(1-a_{m+1})^{t-2}d^t|S'_{m-1}]\leqq 0.$$

Hence, $(1-a_{m+1})I_m(a_{m+1})$ is concave with respect to a_{m+1}. This property of $(1-a_{m+1})I_m(a_{m+1})$ being concave and relations (39) and (40) lead to

$$(1-a_{m+1})I_m(a_{m+1})\geqq 0 \text{ for } 0\leqq a_{m+1}\leqq a_m.$$

Since $1-a_{m+1}\geqq 0$, it holds that

$$I_m(a_{m+1})\geqq 0 \text{ for } 0\leqq a_{m+1}\leqq a_m.$$

Thus, from (32),

$$F_{m+1}(O_{m-1},1,0)-F_{m+1}(O_{m-1},0,1)\geqq 0. \tag{41}$$

In particular, when $1\geqq a_1>a_2>\cdots>a_M>0$, the strict inequality holds. The proof of Lemma 5(a) is completed.

(b) From Lemma 3, we note that

$$\begin{pmatrix} F_{m+1}(S'_{m-1},0,0)\\ F_{m+1}(O_{m-1},1,0)\\ F_{m+1}(S'_{m-1},1,0)\end{pmatrix}=F_{m+1}(S'_m,0) =[1-F_{m+1}(O_m,1)]\begin{pmatrix}\tilde{F}_m(S'_{m-1},0)\\ \tilde{F}_m(O_{m-1},1)\\ \tilde{F}_m(S'_{m-1},1)\end{pmatrix} +F_{m+1}(O_m,1)\begin{pmatrix}F_m(S'_{m-1},0)\\ F_m(O_{m-1},1)\\ F_m(S'_{m-1},1)\end{pmatrix}, \tag{42}$$

$$\begin{pmatrix} F_{m+1}(S'_{m-1},0,1)\\ F_{m+1}(O_{m-1},1,1)\\ F_{m+1}(S'_{m-1},1,1)\end{pmatrix}=F_{m+1}(S'_m,1) =F_{m+1}(O_m,1)\begin{pmatrix}F_m(S'_{m-1},0)\\ F_m(O_{m-1},1)\\ F_m(S'_{m-1},1)\end{pmatrix}, \tag{43}$$

$$F_{m+1}(S'_{m-1},0,1)=F_{m+1}(O_m,1)[\{1-F_m(O_{m-1},1)\}\tilde{F}_{m-1}(S'_{m-1}) +F_m(O_{m-1},1)F_{m-1}(S'_{m-1})] \tag{44}$$

and

$$\tilde{F}_m(S'_{m-1},1)=\tilde{F}_m(O_{m-1},1)\hat{F}_{m-1}(S'_{m-1}). \tag{45}$$

Combining (42) through (45) gives us

$$F_{m+1}(S'_{m-1},1,0)-F_{m+1}(S'_{m-1},0,1)=[\{1-F_{m+1}(O_m,1)\}\tilde{F}_m(O_{m-1},1)-F_{m+1}(O_m,1)\{1-F_m(O_{m-1},1)\}]\hat{F}_{m-1}(S'_{m-1})+F_{m+1}(O_m,1)\{1-F_m(O_{m-1},1)\}\{\hat{F}_{m-1}(S'_{m-1})-\tilde{F}_{m-1}(S'_{m-1})\}. \tag{46}$$

By (42), (46) is rewritten as

$$F_{m+1}(S'_{m-1},1,0)-F_{m+1}(S'_{m-1},0,1) =[F_{m+1}(O_{m-1},1,0)-F_{m+1}(O_m,1)]\hat{F}_{m-1}(S'_{m-1}) +F_{m+1}(O_m,1)[1-F_m(O_{m-1},1)][\hat{F}_{m-1}(S'_{m-1}) -\tilde{F}_{m-1}(S'_{m-1})]. \tag{47}$$

Noting that $\hat{F}_{m-1}(i_{m-1})-\tilde{F}_{m-1}(i_{m-1})\geqq 0$ for all $i_{m-1}\varepsilon S'_{m-1}$ because $(1-a_{m+1})d\geqq(1-a_m)d$, we see from (41) and (47) that

$$F_{m+1}(i_{m-1},1,0)\geqq F_{m+1}(i_{m-1},0,1) \quad \text{for all } i_{m-1}\varepsilon S'_{m-1}.$$

In particular, when $1\geqq a_1>a_2>\cdots>a_M>0$, the strict inequality holds. Thus the proof of Lemma 5 is completed. □

Theorem 1. For any $d(0<d<1)$, F is monotone. In particular, if $1\geqq a_1>a_2>\cdots>a_M>0$, then F is strictly monotone for any $d(0<d<1)$.

Proof. Suppose that $F_m(S'_m)$ is monotone for any d $(0<d<1)$. We note that

$$F_{m+1}(S'_{m+1})=\begin{pmatrix}F_{m+1}(S'_m,0)\\ F_{m+1}(O_m,1)\\ F_{m+1}(S'_m,1)\end{pmatrix}=\begin{pmatrix}F_{m+1}(S'_{m-1},0,0)\\ F_{m+1}(O_{m-1},1,0)\\ F_{m+1}(S'_{m-1},1,0)\\ F_{m+1}(O_{m-1},0,1)\\ F_{m+1}(S'_{m-1},0,1)\\ F_{m+1}(O_{m-1},1,1)\\ F_{m+1}(S'_{m-1},1,1)\end{pmatrix}.$$

For i_{m-1} such that $i_l=1$ and $i_k=0$ for $k\neq l$, the states $(O_{m-1},1,0)$ and $(i_{m-1},0,0,)$ $\varepsilon(S'_m,0)$ satisfy the relation

$$(O_{m-1},0,1)<(O_{m-1},1,0,)<(i_{m-1},0,0). \tag{48}$$

By (48) and Lemmas 4 and 5(a), we obtain for such i_{m-1},

$$F_{m+1}(O_{m-1},0,1)\leqq F_{m+1}(O_{m-1},1,0) \leqq F_{m+1}(i_{m-1},0,0). \tag{49}$$

For i_{m-1} and $i'_{m-1}\varepsilon S'_{m-1}$ such that $i_l=1$, $i_n=0$, $i'_l=1$, $i'_n=1$ and $i_k=i'_k=0$ for $k\neq l$, n, the states $(i_{m-1},1,0)$and $(i'_{m-1},0,0)\varepsilon(S'_m,0)$ satisfy the following relation :

$$(O_{m-1},1,1)<(i_{m-1},0,1)<(i_{m-1},1,0) =(i_1,\cdots,i_{n-1},0,i_{n+1},\cdots,i_{m-1},1,0)$$

$< (i_1, \cdots, i_{n-1}, 1, i_{n+1}, \cdots, i_{m-1}, 0, 0)$
$= (i'_{m-1}, 0, 0)$ (50)

Lemmas 4 and 5 and (50) give us for such i_{m-1} and i'_{m-1},

$$F_{m+1}(0_{m-1}, 1, 1) \leqq F_{m+1}(i_{m-1}, 1, 0) \leqq F_{m+1}(i'_{m-1}, 0, 0). \quad (51)$$

For i_{m-1} and $i'_{m-1} \varepsilon S'_{m-1}$ such that $i_l=0$, $i'_l=1$ and $i_k=i'_k$ for $k \neq l$, the states $(i_{m-1}, 1, 0)$ and $(i'_{m-1}, 0, 0)$ satisfy the following partial order :

$$(i_{m-1}, 0, 1) < (i_{m-1}, 1, 0) = (i_1, \cdots, i_{l-1}, 0, i_{l+1}, \cdots, i_{m-1}, 1, 0) < (i_1, \cdots, i_{l-1}, 1, i_{l+1}, \cdots, i_{m-1}, 0, 0) = (i'_{m-1}, 0, 0). \quad (52)$$

Combining Lemmas 4 and 5 with (52), we have for such i_{m-1} and $i'_{m-1} \varepsilon S'_{m-1}$,

$$F_{m+1}(i_{m-1}, 0, 1) \leqq F_{m+1}(i_{m-1}, 1, 0) \leqq F_{m+1}(i'_{m-1}, 0, 0). \quad (53)$$

For i_{m-1} and $i'_{m-1} \varepsilon S'_{m-1}$ such that $i_l=0, i'_l=1$ and $i_k=i'_k$ for $k \neq l$, the state $(i'_{m-1}, 1, 0) \varepsilon (S'_m, 0)$ satisfies the relation

$$(i_{m-1}, 1, 1) = (i_1, \cdots, i_{l-1}, 0, i_{l+1}, \cdots, i_{m-1}, 1, 1) < (i_1, \cdots, i_{l-1}, 1, i_{l+1}, \cdots, i_{m-1}, 0, 1) = (i'_{m-1}, 0, 1) < (i'_{m-1}, 1, 0).$$

In a way similar to (53), we have for such i_{m-1} and $i'_{m-1} \varepsilon S'_{m-1}$,

$$F_{m+1}(i_{m-1}, 1, 1) \leqq F_{m+1}(i'_{m-1}, 0, 1) \leqq F_{m+1}(i'_{m-1}, 1, 0). \quad (54)$$

From (49), (51), (53), (54) and Lemma 4, we see that if $F_m(S'_m)$ is monotone for any $d (0<d<1)$, then $F_{m+1}(S'_{m+1})$ is also monotone for any $d (0<d<1)$. Therefore, since by (3), $F_2(S'_2)$ is monotone, it follows by induction on m that $F_M(S'_M)$ is monotone for any d $(0<d<1)$. In particular, when $1 \geqq a_1 > \cdots > a_M > 0$, $F_M(S'_M)$ is strictly monotone for any $d (0<d<1)$ because from Lemmas 4 and 5 strict inequalities in (49), (51), (53) and (54) hold. Thus the proof is completed.

□

4. Conclusion

In this paper, we deal with a single-server priority queuing system consisting of M terminals with single buffers. It is proved that the initial busy period has the monotonicity under the service rule on the priority basis, the first terminal with a_1 having the highest priority, ⋯, and the M-th terminal with a_M having the lowest priority, where $a_1 \geqq a_2 \geqq \cdots \geqq a_M$. The monotonicity leads to the optimality of the service rule mentioned above. It should be noted that this service rule is optimal among dynamic probabilistic policies depending on the entire history[9].

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