N-Adic Representations of the Partitions of n-Set and an Expression of Bell's Number B(n)

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(Received September 1, 1982)

A method for enumerating the whole of partitions of n-set $Nn=\{0,\cdots,n-1\}$ is obtained by using a certain set $\mathbf{D}_n$ of n-adic integers. And, also by using $\mathbf{D}_n$, the following rule is given for Bell's number $B(n)$, (the number of partitions of Nn) :

$$B(n)=\sum_{r=1}^{n}\sum_{k_1+\cdots+k_r=n-r}1^{k_1}2^{k_2}\cdots r^{k_r},$$

where $k_1,\cdots,k_r$ are non-negative integers.

1. n-adic representations of the partitions of n-set

For each n-adic integer $\sigma=(s_o,\cdots,s_{n-1})$ where $0\leqq s_i\leqq n-1(i=1,\cdots,n-1)$, let $D_n(\sigma)$ denote the following condition :

$s_o=0\wedge\forall i(0<i\leqq n-1\rightarrow\max_{j<i}s_j+1\geqq s_i)$,

and let $\mathbf{D}_n$ be $\{\sigma\mid D_n(\sigma),\sigma$ is an n-adic integer$\}$

Then we have the following.

PROPOSITION 1 Let $\mathbf{C}_n$ be the set of all partitions of the n-set $N_n=\{0,\cdots,n-1\}$. Then a 1-1 onto mapping g from $\mathbf{D}_n$ to $\mathbf{C}_n$ is given as follows : $g(\sigma)=\{\{i\mid s_i=0\},\{i\mid s_i=1\},\cdots,\{i\mid s_i=m(\sigma)\}\}$, where $\sigma=(s_o,\cdots,s_{n-1})$ and $m(\sigma)=\max_{0\leqq i\leqq n-1}s_i$.

Proof The proposition is clear in the case : n=1. For any σ and $\tau\varepsilon$ $\mathbf{D}_n$ where $\sigma=(s_o,\cdots,s_{n-1})\neq\tau=(t_o,\cdots,t_{n-1})$ and $n>1$, let I denote min $\{i\mid s_i\neq t_i, 0<i\leqq n-1\}$. Then, without loss of generality, $(s_o=t_o,\cdots,s_{I-1}=t_{I-1},)$ $s_I<t_I$. On the other hand, $\tau\in\mathbf{D}_n$ implies $\max_{j<I}t_j+1\geqq t_I>s_I$.

Then $\max_{j<I}s_j=\max_{i<I}t_j\geqq s_I$. So, $s_j\geqq s_I$ for some $j<I$. Let $J=\min\{j\mid s_j\geqq s_I$ and $j<I\}$, then $s_I=s_J=t_J$. [Because, $s_J\geqq s_I$ and, for every $j<J$, $s_j<s_I$. In the case : $J>0$, $\max_{j<J}s_j<s_I$. So, $s_J\leqq\max_{j<J}s_j+1\leqq s_I\leqq s_J$. In the case : $J=0$, $s_J=s_o=0\leqq s_I\leqq s_J$.] Thus, $g(\sigma)\ni\{i\mid s_i=s_I\}\ni J, I$, and $g(\tau)\ni\{i\mid t_i=s_I\}\ni J$ and $\{i\mid t_i=s_I\}\not\ni I$. So, $g(\sigma)\neq g(\tau)$. Therefore, the mapping g is 1-1.

For an arbitrary partition $C=\{A_1,\cdots,A_p\}$ of $N_n=\{0,\cdots,n-1\}$, let $d(k)=\min A_k(k=1,\cdots,p)$. Then, without loss of generality, we can assume that $d(1)=0$ and $d(1)<d(2)<\cdots<d(p)$ in the case : $p>1$. Let $\rho=(r_o,\cdots,r_{n-1})$ such that $r_i=j-1$ for all $i\in A_j(j=1,\cdots,p)$. Then we can easily see that $g(\rho)=\{A_1,\cdots,A_p\}$ and $\rho\in\mathbf{D}_n$. That is, g is a mapping from $\mathbf{D}_n$ onto $\mathbf{C}_n$.

2. A nonrecursive expression of the n-th Bell's number B(n).

By **PROPOSITION 1,** the number $B(n)$ of partitions of n-set $N_n=\{0,\cdots,n-1\}$, is equal to the number of the elements of $\mathbf{D}_n$. Thereby, the following proposition holds.

PROPTOSITION 2 $B(n)=\sum_{r-1}^{n}\sum_{k_1+\cdots+k_r=n-r}1^{k_1}\cdot 2^{k_2}\cdots r^{k_r}$, where $k_1,\cdots,k_r$ are nonnegative integers.

Proof For r, $o\leqq r\leqq n$, let $d_1+\cdots+d_r=n$ and $d_i\geqq 1(i=1,\cdots,r)$. Moreover, let $Q(d_1,\cdots,d_r)$ denote

$\{(s_o,\cdots,s_{n-1})\varepsilon\ D_n\mid\forall j(1\leqq j\leqq r\rightarrow(s_{d_1+\cdots+d_{j-1}}=j-I\wedge\forall i(d_1+\cdots+d_{j-1}\leqq i<d_l+\cdots+d_j\rightarrow 0\leqq s_i\leqq j-1))))\}$

where $d_1+\cdots+d_{j-1}$ denotes 0 when $j=1$.

We represent n-adic integer $\sigma=(s_o,\cdots,s_{n-1})$ by a line-graph which connects the points $(0,s_o),(l,s_1),\cdots,(n-1,s_{n-1})$ one by one.

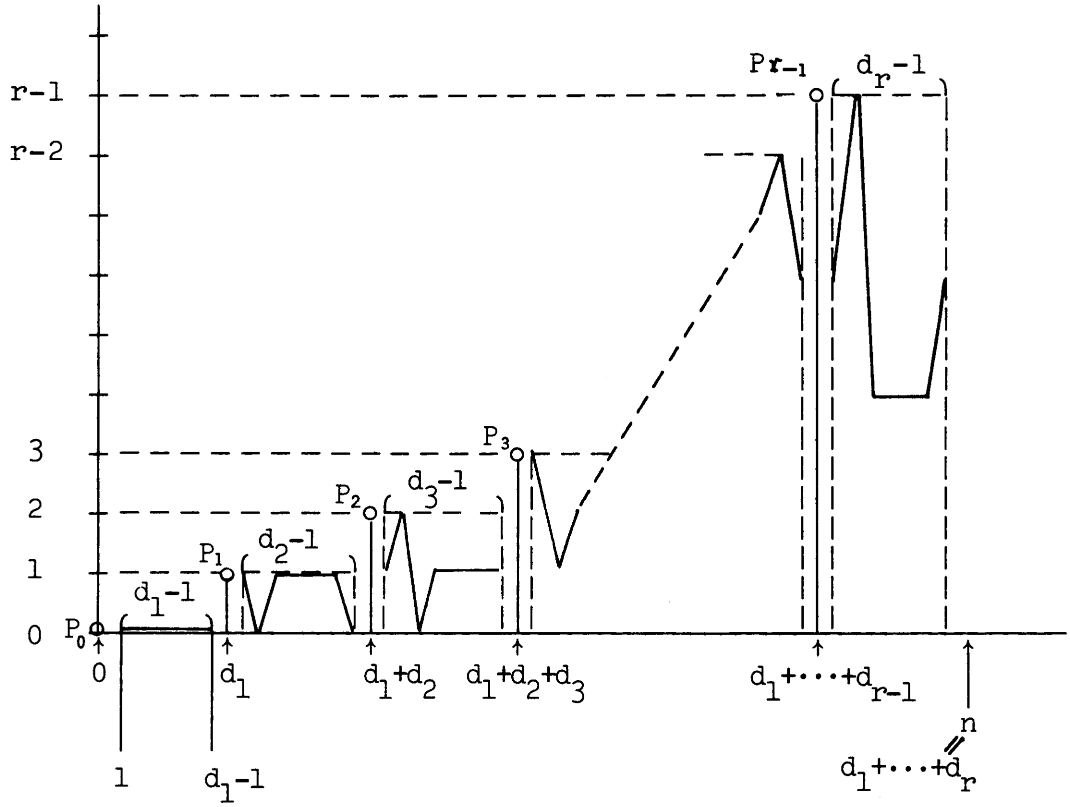

Fig. 1 A line graph which represents an element of $Q(d_1, \cdots, d_r)$, where $d_1+\cdots+d_r=n$ and $d_1 \geqq 1, \cdots, d_r \geqq 1$.

Then $Q(d_1, \cdots, d_r)$ is represented by the set of all the line-graphs, each of which intersects in $P_o=(0, 0)$ $P_1=(d_1, 1)$, $P_2=(d_1+d_2, 2)$, $\cdots$, $P_{r-1}=(d_1+\cdots+d_{r-1}, r-1)$ and has the segment for every interval $[d_1+\cdots+d_{j-1}+1 \leqq i \leqq d_1+\cdots+d_j-1]$ in a limited region $0 \leqq s_i \leqq j-1$.

Therefore, we can calculate the number of the elements of $Q(d_1, \cdots, d_r)$ as follows :

$$\#(Q(d_1, \cdots, d_r)) = 1^{d_1-1} 2^{d_2-1} \cdots\cdots r^{d_r-1}. \tag{1}$$

On the other hand, it holds clearly that

$$\mathbf{D}_n = \bigcup_{1 \leqq r \leqq n} \bigcup_{\substack{d_1+\cdots+d_r=n \\ d_1 \geqq 1, \cdots, d_r \geqq 1}} Q(d_1, \cdots, d_r)$$

Moreover, let $1 \leqq r$, $r \leqq n$, $d_1+\cdots+d_r=n$, $d_i \geqq 1 (i=1, \cdots, r)$, $d'_1+\cdots+d'_{r'}=n$ and $d'_i \geqq 1$ $(i=1, \cdots, r')$. Then $(d_1, \cdots, d_r) \neq (d'_1, \cdots, d'_{r'})$ implies $Q(d_1, \cdots, d_r) \cap Q(d'_1, \cdots, d'_{r'}) = \phi$.

Therefore, $B(n) = \#(\mathbf{D}_n)$ (the number of $\mathbf{D}_n$)

$$= \sum_{1 \leqq r \leqq n} \sum_{\substack{d_1+\cdots+d_r=n \\ d_1 \geqq 1, \cdots, d_r \geqq 1}} \#(Q(d_1, \cdots, d_r)). \tag{2}$$

Thus, by (1) and (2), PROPOSITION 2 holds.

3. An enumeration of partitions of n-set.

For every two n-adic integers $\sigma=(s_o, \cdots, s_{n-1})$ and $\tau=(t_o, \cdots, t_{n-1})$, let $\sigma > \tau$ mean that $\exists_k (0 \leqq k \leqq n-1 \wedge (s_o=t_o \wedge \cdots \wedge s_{k-1}=t_{k-1} \wedge s_k > t_k))$.

Then, we define inductively the k-th element $\rho(k)$ of $\mathbf{D}_n (k=1, \cdots, B(n))$ as follows :

(i) $\rho(1)=(\overbrace{0, \cdots, 0}^{n})$

(ii) $\rho(k+1)=\min\{\sigma \mid \rho(k) < \sigma$ and $\sigma \in \mathbf{D}_n\}$, for each $k < B(n)$.

Then, a procedure for enumerating all the partitions of n-set is also obtained by using $\mathbf{D}_n$.

PROPOSITION 3 For each k, $1 \leqq k < B(n)$, by using $\rho(k)=(s_o, \cdots, s_{n-1})$, we can calculate $\rho(k+1)=$

$(s'_o, \cdots, s'_{n-1})$ as follows :

Let $I = \max\{i \mid 0 < i \leqq n-1 \wedge \max_{j<i} s_j + 1 > s_i\}$. (3)

$$\left.\begin{aligned} &\text{Then } s'_o = s_o, \cdots, s'_{I-1} = s_{I-1},\\ &s'_I = s_I + 1,\\ &s'_{I+1} = \cdots = s'_{n-1} = 0. \end{aligned}\right\} \quad (4)$$

Proof First, we notice that the final element of $\mathbf{D}_n$ is $\rho(B(n)) = (0, 1, \cdots, n-1)$.

So, if $\rho(k) = (s_o, \cdots, s_{n-1})$ is not final in $\mathbf{D}_n$(that is, $k < B(n)$), then $\exists_i (0 < i \leqq n-1 \wedge \max_{j<i} s_j + 1 > s_i)$holds. Therefore, $I = \max\{i \mid 0 < i \leqq n-1 \wedge \max_{j<i} s_j + 1 > s_i\}$ is definable. Let τ denote $(s'_0, \cdots, s'_{n-1})$which is obtained by the above calulation (4) from $\rho(k) = (s_o, \cdots, s_{n-1})$, where $k < B(n)$. Then, to prove that $\rho(k+1) = \tau$, it suffices to verify the following :

(i) $\rho(k) < \tau$ and $\tau \varepsilon\ \mathbf{D}_n$

(ii) $\rho(k) < \mu < \tau$ implies $\mu \notin \mathbf{D}_n$.

The former part of (i) is clear, since $\rho(k) = (s_0, \cdots, s_{n-1})$ and $\tau = (s_o, \cdots, s_{I-1}, s_I + 1, 0, \cdots, 0)$. $\rho(k) = (s_o, \cdots, s_{n-1}) \in \mathbf{D}_n$ and (4) imply that $\max_{j<i} s'_j + 1 \geqq s'_i$ holds clearly, for each $i = 1, \cdots, I-1, I+1, \cdots, n-1$, and it also holds for $i = I$, since $\max_{j<I} s'_j + 1 = \max_{j<I} s_j + 1 > s_I = s'_I - 1$. And $s'_o = s_o = 0$. So, $\tau = (s'_o, \cdots, s'_{n-1}) \in D_n$.

The hypothesis of (ii) is written thus $\rho(k) = (s_o, \cdots, s_{I-1}, s_I, s_{I+1}, \cdots, s_{n-1}) < \mu = (m_o, \cdots, m_{I-1}, m_I, m_{I+1}, \cdots, m_{n-1}) < \tau = (s_o, \cdots, s_{I-1}, s_I + 1, 0, \cdots\cdots, 0)$.

Thus, it implies that $I < n-1$ and $(m_o, \cdots, m_{I-1}) = (s_o, \cdots, s_{I-1})$, and then $(m_I, \cdots, m_{n-1}) < (s_I + 1, 0, \cdots, 0)$. So, $m_I < s_I + 1$. Moreover, $m_I \geqq s_I$. Thus, $m_I = s_I$.

So, $(s_{I+1}, \cdots, s_{n-1}) < (m_{I+1}, \cdots, m_{n-1})$.

Therefore, for some $J > I$, $(s_o, \cdots, s_{J-1}) = (m_o, \cdots, m_{J-1})$ and $s_J < m_J$. So, by (3) and $J > I$, $m_J > s_J = \max_{j<J} s_j + 1 = \max_{j<J} m_j + 1$.

Thus, $\mu \notin \mathbf{D}_n$.

Acknowledgements

The author would like to express his thanks to Professor A. Nozaki and Professor H. Narushima for kindly advices and valuable suggestions.

References

1) A. Nijenhuis and H. S. Wilf, Combinatorial Algorithm (Academic Press, New York, 1975)