Geometry on Canonical Domains in C^n

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As the generalizations of the Riemann's mapping theorem in a complex variable, three types of canonical domains of a bounded domain in C^n are studied by S. Bergman, M. Maschler and others.

The purpose of this paper is to investigate various properties, relations and distortion theorems on these canonical domains and the canonical mappings of the biholomorphic equivalent class of a bounded domain.

Unfortunately, the uniqueness of each one of these canonical domains of the biholomorphic equivalent class does not hold. In order to avoid this difficulty, lastly we shall define another canonical domain, i. e., the normal domain.

1. Introduction

By the Riemann's mapping theorem in a complex variable, we can get a disc as the canonical domain of the conformal equivalent class of a simply connected domain in C. But in several complex variables, it is known that a ball is not biholomorphically equivalent to a polycylinder. Therefore, even simply connected domains in C^n do not necessarily have the same canonical domain. This suggests complicated circumstances on the canonical domain of a bounded domain in C^n $(n\geqq 2)$.

Using extremal functions expressed in terms of the Bergman kernel function, three types of canonical domains of a bounded domain in C^n are defined and studied by S. Bergman[1], M. Maschler [7], [8], J. Mitchell [11] and others [9], [10], [12], [13].

The purpose of this paper is to investigate the various properties, relations and distortion theorems on the canonical domains and canonical mappings of the biholomorphic equivalent class of a bounded domain.

Unfortunately, the uniqueness of each one of these canonical domains of the biholomorphic equivalent class does not hold, since they depend on the initial conditions and the distinguished point. In order to avoid this difficulty, lastly we shall define another canonical domain, i. e., the normal domain.

2. Minimum problem and canonical domains

Let D be a bounded domain in C^n and $H_{p,A,t}{}^2(D)$ be the class of p-tuple vector functions $f(z)={}^t(f_1(z),\cdots,f_p(z))$, $z={}^t(z_1,\cdots,z_n)$ such that $f_j(z)$ $(j=1,\cdots,p)$ belong to the class $H^2(D)$ f Lebesgue square integrable holomorphic functions and $\mathcal{L}_t f=A$ (A: a given constant matrix of the type of $\mathcal{L}_t f$), where $\mathcal{L}_t$ denotes a bounded linear functional evaluated at t, which is called a distinguished point.

$\mathrm{vol}(D)$, $k_D(z,\bar{t})$, $T_D(z,\bar{t})$ and $M_{D,A}(z,t)$ denote the Euclidean volume of D, the Bergman kernel function of D, the Bergman metric tensor and the minimizing function $\varepsilon H_{p,A,t}{}^2(D)$ such that

$$\int_D ||M_{D,A}(z,t)||^2\omega_z \leqq \int_D ||f(z)||^2\omega_z,$$

$f\varepsilon H_{p,A,t}{}^2(D)$, respectively. ω_z denotes the Euclidean measure and

$$T_D(z,\bar{t})=D_z{}^*D_z\log k_D(z,\bar{t})$$
$$=(k(z,\bar{t})k_{11}(z,\bar{t})-k_{10}(z,\bar{t})k_{01}(z,\bar{t}))/k^2(z,\bar{t}),$$

where $D_z=\partial/\partial z=(\partial/\partial z_1,\cdots,\partial/\partial z_n)$, $D_z{}^*={}^t(\partial/\partial\bar{z})$ and

$$k_{ij}(z,\bar{t})=(D_z{}^*)^i(D_z)^j k_D(z,\bar{t}).$$

Lemma 2.1 *The minimizing function in $H_{p,A,t}{}^2(D)$ is given by*

(2.1) $M_{D,A}(z,t)=A(\Phi^*(t)\Phi(t))^{-1}\Phi^*(t)\phi(z)$, *where $\phi(z)$ denote an orthonormal base of the complex Hilbert space $H^2(D)$ and $\Phi(t)$ denotes $\mathcal{L}_t\ \phi$.*

The $\mathcal{L}^2$-minimum value of $M_{D,A}(z, t)$ is given by

$$(2.2)\quad \lambda_{D,A}(t) = Trace\int_D M_{D,A}(z,\ t)\ (M_{D,A}(z,\ t))^*\omega_z = Trace[A(\Phi^*(t)\Phi(t))^{-1}A^*].$$

Here and after A^* denotes the transposed conjugate matrix of A. It is clear that $\Phi^*(t)\Phi(t) = \mathcal{L}_t^*\mathcal{L}_t k_D(\cdot, \cdot)$ and $\Phi^*(t)\phi(z) = \mathcal{L}_t^* k_D(z, \cdot)$.

The proof of this lemma is given by the same manner as in [10], so we omitt this.

First, we enumerate some known resuls on minimum values (see [10]).

$$(2.3)\quad \lambda_{D,(1)}(t) = 1/k_D(t,\ \bar{t}),\quad \mathcal{L}_t f = (f(t)) = (1),$$

$$(2.4)\quad \lambda_{D,(0,1)}(t,\ u) = 1/[k_D(t,\ \bar{t})u^* T_D(t,\ \bar{t})u],$$

$\mathcal{L}_t f = (f(t), \partial_u f(t)) = (0,\ 1)$, where $\partial_u(\cdot)$ denotes $((\partial/\partial z)\cdot)u = (D_z\cdot)u$.

For an n-tuple vector function $f(z)\varepsilon H_{n,(0,E),t}{}^2(D)$ with $\mathcal{L}_t f = (1,\ D_z)_t f = (0,\ E)$, where E denotes the unit matrix E_n of order n, the minimizing function and the minimum value are given by

$$(2.5)\quad M_{D,(0,E)}(z,t) = T^{-1}[k\,k_{10}(z,\ \bar{t}) - k_{10}\,k(z,\ \bar{t})]/k^2$$

and

$$(2.6)\quad \lambda_{D,(0,E)}(t) = \mathrm{Trace}[kT]^{-1},$$

where $k_{ij}(z,\ \bar{t}) = (D_z^*)^i(D_z)^j k_D(z,\ \bar{t})$, $k_{ij} = k_{ij}(t,\ \bar{t})$ and $T = T_D(t,\ \bar{t})$. (2.6) is given by

$$(2.7)\quad \begin{bmatrix} k & k_{01} \\ k_{10} & k_{11} \end{bmatrix}^{-1} = \begin{bmatrix} 1/k + k_{01}(kT)^{-1}k_{10}/k, & -k_{01}(kT)^{-1}/k \\ -(kT)^{-1}k_{10}/k, & (kT)^{-1} \end{bmatrix}.$$

(i) Minimal domain in C^n

A bounded domain D is called the minimal domain with center at $\tau\ \varepsilon D$ (with respect to a *distinguished point t*) if $\mathrm{vol}(D) \leqq \mathrm{vol}(f(D))$ holds for any holomorphic map $f(z) = {}^t(f_1(z), \cdots, f_n(z))$, $z = {}^t(z_1, \cdots, z_n)$, which is locally one-to-one expect in a denumerable number of analytic segments of manifolds of complex dimensions $\leqq n-1$, with a single-valued Jacobian and $(f(t), \det(D_z f(t))) = (\tau,\ 1)$ [7], [8].

It is known that a domain D is a minimal domain with center at τ if and only if

$$(2.8)\quad M_{D,(1)}(z,\ \tau) = k_D(z,\ \bar{\tau})/k_D(\tau,\ \bar{\tau}) = 1,\quad z\ \varepsilon D,$$

or

$$(2.9)\quad 1/\mathrm{vol}(D) = k_D(\tau,\ \bar{\tau}) \leqq k_D(z, \bar{z}),\quad z\ \varepsilon D,$$

where the equality of (2.9) holds only for $z = \tau$ [6].

A holomorphic map $w(z)$, which maps a bounded schlicht domain D onto a minimal domain Δ with center at τ under the initial conditions $w(t) = \tau$, $\det(D_z w(t)) = 1$, satisfies

$$(2.10)\quad \det(D_z w(z)) = k_D(z,\ \bar{t})/k_D(t,\ \bar{t}),\quad z\ \varepsilon D.$$

This *minimal function* $w(z)$ may not be unique.

For $n=1$, $w(z) = \int_t^z k_D(z,\ \bar{t})/k_D(t,\ \bar{t})\,dz$ denotes the canonical mapping of the Riemann's mapping theorem in C.

(ii) Representative domain in C^n

For a bounded schlicht domain D, the image domain Δ of D under the mapping (*representative function*)

$$(2.11)\quad w(z) = M_{D,(0,E)}(z,\ t)/M_{D,(1)}(z,\ t) + \tau = T_D^{-1}(t,\ \bar{t})\int_t^z T_D(z,\ \bar{t})\,dz + \tau$$

is called the representative domain with center at τ (with respect to a distinguished point t).

A domain D is a representative domain with center at τ if and only if

$$(2.12)\quad M_{D,(0,E)}(z,\ \tau)/M_{D,(1)}(z,\ \tau) = T_D^{-1}(\tau,\ \bar{\tau})\int_\tau^z T_D(z,\ \bar{\tau})\,dz = z - \tau,\quad z\ \varepsilon D$$

or

$$(2.13)\quad T_D(z,\ \bar{\tau}) = T_D(\tau,\ \bar{\tau}),\quad z\ \varepsilon D.$$

Because of the biholomorphic relative invariance:

$$(2.14)\quad T_D(z,\ \bar{t}) = (D_z x(t))^* T_\Delta(x(z),\ \overline{x(t)})\,D_z x(z)$$

under any biholomorphic map $x(z)$ with $x(D) = \Delta$, the representative function (2.11) is biholomorphic invariant under $D_z x(t) = E$.

(iii) Minimal domain of moment of inertia in C^n (shortly moment minimal domain)

Such a minimizing map $w(z)\varepsilon H_{n,(\tau,E),t}{}^2(D)$ that

$$\int_{\Delta(0)} ||w-\tau||^2\omega_w \leqq \int_\Delta ||f-\tau||^2\omega_f,\ f\ \varepsilon H_{n,(\tau,E),t}{}^2(D),$$

where $\Delta(0)$ and Δ are image domains under the mappings $w(z)$ and $f(z)$ of D, respectively, is called the *minimal function of moment of inertia* (shortly, *moment minimal function*) and the image domain $\Delta(0) = w(D)$ is called the moment minimal domain with center at $\tau = w(t)$ (with respect to a distinguished point t). The moment minimal function $w(z)$ satisfies

$$(2.15)\quad (w(z) - \tau)\det(D_z w(z)) = M_{D,(0,E)}(z,\ t).$$

A domain D is a moment minimal domain with center at $\tau\ \varepsilon D$ if and only if

(2.16) $M_{D,(0,E)}(z, \tau)=T_D^{-1}(\tau, \bar{\tau})\int_\tau^z \hat{T}_D(z, \bar{\tau})dz = z-\tau, \ z \in D,$

or

(2.17) $\hat{T}_D(z, \bar{\tau})=\hat{T}_D(\tau, \bar{\tau})=T_D(\tau, \bar{\tau}), \ z \in D,$

where

(2.18) $\hat{T}_D(z, \bar{\tau})=[k\, k_{11}(z, \bar{\tau})-k_{10}\, k_{01}(z, \bar{\tau})]/k^2,$ $k_{ij}=k_{ij}(\tau, \bar{\tau})$.

It is known that there exists a representative but nonminimal domain with the same center.

Definition 2.1 We call a point $\tilde{t} \in C^n$, which satisfies

(2.19) $\int_D (z-\tilde{t})\,\omega_z=0,$

to be the *center of gravity* of D.

$\tilde{t}$ is the center of gravity in the ordinary sense in E^{2n}. A bounded domain D has only one center of gravity:

(2.20) $\tilde{t}=\int_D z\omega_z/\mathrm{vol}(D)$

and it holds that for any $t \in C^n$

(2.21) $I(D, t)=I(D,\tilde{t})+|t-\tilde{t}|^2\mathrm{vol}(D)$,

where $I(D, t)$ denotes the moment of inertia of D with respect to $t: \int_D ||z-t||^2\omega_z$.

Notice that for a moment minimal domain D with center at $t_0 \in D$ $I(D, t_0)\geqq I(D, \tilde{t})$ holds, where the center of gravity $\tilde{t}$ of D may not belong to D. The equality holds if and only if $\tilde{t}=t_0 \in D$.

3. Relations among the canonical domains

Lemma 3.1 *Arbitrary two conditions among the following*

(i) *D is a minimal domain with center at τ,*

(ii) *D is a representative domain with center at τ and*

(iii) *D is a moment minimal domain with center at τ are sufficient to the remainder* [9].

We call a domain D with (i) and (ii) (consequently (iii)) a standard domain with center at τ. Necessary and sufficient conditions that D is a standard domain with center at τ are $k_D(z, \bar{\tau})=c$ and $k_{D,11}(z, \bar{\tau})=c'$ in D. Complete Carathéodory circular domains and in particular the classical Cartan domains with center at the origin are standard domains with center at the origin.

Hereafter, without loss of generality we shall treat canonical domains with center at the origin, since parallel transformations preserve the canonical properties.

Further, if confusions will not occur, without notice we shall sometimes use the abbreviated notations $k_{ij}(z, \bar{t})$ and k_{ij} instead of $(D_z{}^*)^i(D_z)^j k_D(z, \bar{t})$ and $(D_z{}^*)^i(D_z)^j k_D(t, \bar{t})$, respectively.

Lemma 3.2 *Let D be a bounded minimal domain with center at the origin in D, then we have*

(i) *for any function $f(z) \in H^2(D)$ with $f(0)=0$,*

$$\int_D f(z)\,\omega_z=0,$$

(ii) *the center of a minimal domain D is the center of gravity of D.*

(ii) *shows that the center of a minimal domain is unique (cf. [7]) and $I(D, 0)\leqq I(D, t)$, $t \in C^n$, holds, where the equality holds when and only when $t=0$.*

Proof. By (2.8) and the reproducing property of the Bergman kernel function we have

$$\int_D f(z)\,\omega_z=\int_D f(z)\,k_D(z, 0)/k_D(0, 0)\,\omega_z =f(0)/k_D(0, 0)=0,$$

which shows (i). We have, from (i), $\int_D z\omega_z=0$, which shows (ii) (see (2.21)).

Theorem 3.1 *Let D be a bounded minimal domain with center at the origin, then the following conditions are equivalent.*

(i) *D is a representative domain with center at the origin,*

(ii) *D is a moment minimal domain with center at the origin,*

(iii) *$k_{11}(z, 0)=k_{11}(0, 0)$ or $k_{10}(z, 0)=k_{11}(0, 0)z$ in D,*

(vi) *$I(D, 0)=vol(D)Trace[T_D^{-1}(0, 0)]$, which is equivalent to $\rho^2=Trace[T_D^{-1}(0, 0)]$. ρ denotes the radius of rotation of D.*

Proof. (i)$\Longleftrightarrow$(ii) is clear from Lemma 3.1.

A minimal domain D with center at the origin is also a representative domain with the same center if and only if $T_D(z, 0)=T_D(0, 0)$, i.e., $k_{11}(z, 0)/k(0, 0)=k_{11}(0, 0)/k(0,0)$, since $k_D(z, 0)=k(0, 0)$ holds in D. This shows (i)$\Longleftrightarrow$(iii).

Finally, we shall show that (ii)$\Longleftrightarrow$(iv). Let D be a minimal and also moment minimal domain with the same center at the origin, then we have from

(2.6), (2.9) and (2.16)

$$I(D,\ 0)=\int_D ||z||^2\omega_z=\int_D ||M_{D,(0,E)}(z,\ 0)||^2\omega_z$$
$$=\lambda_{D,(0,E)}(0)=\mathrm{Trace}[k_D(0,\ 0)\,T_D(0,0)]^{-1}$$
$$=\mathrm{vol}(D)\,\mathrm{Trace}[T_D^{-1}(0,\ 0)]=\mathrm{vol}(D)\,\rho^2,$$

since 0 is the center of gravity of D from Lemma 3.2. Therefore, we have (ii) ⇨ (iv).

The converse is true, since

$$I(D,0)=\mathrm{vol}(D)\,\mathrm{Trace}\,T_D^{-1}(0,\ 0)=\lambda_{D,(0,E)}(0)$$

and for any $f(z)$ in $(0,\ E)$-class

$$I(f(D),0)=\int_\Delta ||f||^2\omega_f=\int_D ||f(z)\det(D_z f(z))||^2\omega_z$$
$$\geqq\lambda_{D,(0,E)}(0)=I(D,\ 0)$$

hold, where $\Delta=f(D)$. This shows (iv) ⇨ (ii).

Example 3.1 In $B_n=\{z\ \varepsilon C^n\,|\,z^*z<1\}$, we have

$$k_D(z,\ 0)=(1-\zeta^*z)^{-(n+1)}n!/\pi^n|_{\zeta=0}=n!/\pi^n$$
$$=1/\mathrm{vol}(B_n),$$
$$T_D(z,\ 0)=(n+1)\,(E-z\zeta^*)^{-1}(1-\zeta^*z)^{-1}|_{\zeta=0}$$
$$=(n+1)E,$$
$$k_{D,11}(z,\ 0)=(n+1)!E/\pi^n,$$
$$I(B_n,\ 0)=\int_0^1 r^2S(r)\,dr=n\pi^n/(n+1)!=\rho^2\mathrm{vol}(B_n)$$

and

$$\mathrm{Trace}[T_D^{-1}(0,\ 0)]=\mathrm{Trace}[(n+1)E]=n/(n+1)=\rho^2,$$

where $\mathrm{vol}(B_n)=\pi^n/n!$ and $S(r)$ denotes the volume of $z^*z=r^2\ (0\leqq r\leqq 1)$.

Lemma 3.3 *Let D be a moment minimal domain with center at the origin, then the following conditions are equivalent.*

(i) *the origin is the center of gravity of D,*

(ii) $k_{D,01}(0,\ 0)=0$ *or* $k_{D,10}(0,\ 0)=0$.

Proof. From the Risez's theorem we have

$$\int_D z\omega_z=\int_D M_{D,(0,E)}(z,\ 0)\,\omega_z$$
$$=T_D^{-1}(0,\ 0)\int_D[k\ k_{10}(z,0)-k_{10}k(z,0)]/k^2\omega_z$$
$$=-(k^2T_D(0,\ 0))^{-1}k_{10},$$

which shows that (i) ⟺ (ii).

Theorem 3.2 *Let D be a moment minimal domain, whose center at the origin is also the center of gravity of itself, then*

$$k_{D,10}(z,\ 0)=k_{D,11}(0,0)\,z,\quad z\ \varepsilon D$$

holds. The converse is true.

Proof. For a domain D as above, from Lemma 3.3 and (2.16) we have

$$z=M_{D,(0,E)}(z,\ 0)=[k_{11}/k]^{-1}k_{10}(z,\ 0)/k.$$

Therefore, we have $k_{11}(z,\ 0)=k_{11}$ in D.

On the other hand, if D satisfies $k_{10}(z,\ 0)=k_{11}z$, then we have $k_{10}=0$ and $k_{11}(z,\ 0)=k_{11}$ in D. Therefore, we have

$$M_{D,(0,E)}(z,\ 0)=T_D^{-1}(0,\ 0)\,k_{10}(z,\ 0)/k$$
$$=k_{11}^{-1}k_{11}z=z,$$

which shows that D is a moment minimal domain with center at the origin. From Lemma 3.3 the origin is also the center of gravity of D.

Remark 3.1 The *Christoffel symbol* is expressed by the matrix $T_D^{-1}(z,\ \bar z)\,D_zT_D(z,\ \bar z)$ [5]. If D is a representative domain with center at t, then we easily have, from (2.13),

$$T_D^{-1}(t,\ \bar t)\,D_zT_D(t,\ \bar t)=0\ \text{(i. e., flat at } t\ \varepsilon D).$$

If D is a moment minimal domain, whose center at t is also the center of gravity of itself, then the Christoffel symbol equals to 0 at t. Indeed, differentiating

$$z-t=M_{D,(0,E)}(z,\ t)$$
$$=(kT_D)^{-1}k(z,\ \bar t)\int_t^z T_D(z,\ \bar t)\,dz$$

two times with respect to z, we have the result.

Theorem 3.3 *Let D be a bounded homogeneous Lu Qi-Keng domain ($k_D(z,\ \zeta)\neq 0$ in $D\times D^*$) and also a representative domain with center at the origin, then D is a minimal domain with the same center if and only if the biholomorphic invariant*

(3.1) $J_D(z,\ \zeta)\equiv\det\ T_D(z,\ \zeta)/k_D(z,\ \zeta)=constant$ *holds in $D\times D^*$.*

In particular, homogeneous standard domains, say classical Cartan domains, have the property (3.1) (cf. [7] Corollary 1).

Proof. If D is a refresentative and also minimal domain with center at the origin, we have

$$J_D(z,\ 0)=\det\ T_D(z,\ 0)/k_D(z,\ 0)$$
$$=\det\ T_D(0,\ 0)/k_D(0,\ 0)=J_D(0,\ 0),\quad z\ \varepsilon D.$$

For any transitive map $h_\zeta(z)$ with $h_\zeta(0)=\zeta,\ \ \zeta\ \varepsilon D$, we have, from the biholomorphic relative invariances of $k_D(z,\ \bar t)$ and $\det\ T_D(z,\ \bar t)$,

$$J_D(z,\ 0)=\det\ T_D(z,\ 0)/k_D(z,\ 0)$$
$$=\det\ T_D(h_\zeta(z),\ \zeta)/k_D(h_\zeta(z),\ \zeta)$$
$$=J_D(h_\zeta(z),\ \zeta)=J_D(w,\ \zeta),\quad (w,\ \zeta)\,\varepsilon D\times D,$$

which shows (3.1).

Converse is true from (3.1), (2.13) and (2.8).

4. Distortions in canonical domains

We define the sets of points

$c(D)\equiv\{t\ \varepsilon D\,|\,k_D(t,\ \bar{t})=1/\mathrm{vol}(D)\}$

and

$m(D)\equiv\{t\ \varepsilon D\,|\,k_D(t,\ \bar{t})\leqq\min_{z\varepsilon D} k_D(z,\ \bar{z})\}$ [3].

If $k_D(z,\ \bar{z})$ becomes infinite everywhere on ∂D, say D is a pseudoconvex domain of holomorphy or a homogeneous domain [2], [6], then

$$m(D)\neq\phi \text{ and } m(D)\supset c(D).$$

The set $c(D)$ consists of at most one point of D, and is nonempty if and only if $c(D)=m(D)$.

D is a minimal domain with center at t if and only if $t=c(D)\neq\phi$ as is stated before.

Theorem 4.1 *Let D be a bounded homogeneous domain and $F(z)$ be a biholomorphic map of D onto $\Delta\equiv F(D)$. If $f(z)$ is a holomorphic map of D into Δ, then we have the generalized Schwarz lemma*

(4.1) $|det(D_zf(z))|^2\leqq k_D(z,\ \bar{z})/k_\Delta(f(z),\ \overline{f(z)})$
$=det\ T_D(z,\ \bar{z})/det\ T_\Delta(f(z),\ \overline{f(z)}),\ z\ \varepsilon D.$

Proof. Let $G(z)$ be a holomorphic map of D into itself with a fixed point $t\ \varepsilon D$, then we have

(4.2) $|\det(D_zG(t))|\leqq 1,$

since $((D_zG(t))^*)^k(D_zG(t))^k\leqq R^2T_D(t,\ \bar{t}),\ k=1,2,\cdots,$ hold from the fundamental theorem of K.H. Look (see [10]). Set $G(z)=(F^{-1}\circ h_\alpha\circ f)(z)$ for a transitive map $h_\alpha(w)$ of Δ(Δ: homogeneous) with $h_\alpha(\alpha)=\beta=F(t)$ and $\alpha=f(t)$, then $G(z)$ maps D holomorphically into D with $G(t)=t$. From (4.2) we have

$$|\det(D_zG(t))|=|\det(D_z(F^{-1}\circ h_\alpha\circ f)(t))|\leqq 1.$$

Noting that $dF^{-1}/dF=(D_zF(z))^{-1}$, we have

$$|\det(D_zf(t))|\leqq|\det(D_zF(t))|/|\det(D_wh_\alpha(\alpha))|.$$

The biholomorphic relative invariances of the Bergman kernel function and the Bergman metric tensor give us

$$k_D(t,\ \bar{t})=k_\Delta(\beta,\ \bar{\beta})|\det(D_zF(t))|^2,$$
$$k_\Delta(\alpha,\ \bar{\alpha})=k_\Delta(\beta,\ \bar{\beta})|\det(D_wh_\alpha(\alpha))|^2$$

and

$$\det\ T_D(t,\ \bar{t})=\det\ T_\Delta(\beta,\ \bar{\beta})|\det(D_zF(t))|^2$$
$$\det\ T_\Delta(\alpha,\ \bar{\alpha})=\det\ T_\Delta(\beta,\ \bar{\beta})|\det(D_wh_\alpha(\alpha))|^2.$$

Therefore, we obtain the result, since we may take t to be an arbitrary point in D.

Corollary 4.1 *If $f(z)$ maps a bounded homogeneous domain D into itself, then we have*

(4.3) $|det(D_zf(z))|^2\leqq k_D(z,\ \bar{z})/k_D(f(z),\ \overline{f(z)}),$
$z\ \varepsilon D$ [2], [6].

In particular, for $\tau_0\ \varepsilon m(D)$, which is nonempty, we have

(4.4) $|det(D_zf(\tau_0))|\leqq 1$ (*cf.* (4.2)).

Corollary 4.2 *In Theorem 4.1, since $\tau_0\ \varepsilon m(\Delta)$ exists, we have*

(4.5) $|det(D_zf(z))|^2\leqq k_D(z,\ \bar{z})/k_\Delta(\tau_0,\ \bar{\tau}_0),\ z\ \varepsilon D.$

In particular, if τ_0 belongs to $c(\Delta)$, we have

(4.6) $|det(D_zf(z))|^2\leqq vol(\Delta)k_D(z,\ \bar{z}),\ z\ \varepsilon D.$

Theorem 4.2 *Let D be a bounded domain with $t_0\ \varepsilon m(D)$ and $F(z)$ be a biholomorphic map of D onto $F(D)$ with $\tau_0=F(t)\ \varepsilon m(F(D))$ $(t\neq t_0)$, then we have*

(4.7) $|det(D_zF(t))|^2\geqq k_D(t_0,\ \bar{t}_0)/k_{F(D)}(\tau_0,\ \bar{\tau}_0)$
$\geqq|det(D_zF(t_0))|^2.$

In particular, if D is a bounded homogeneous domain and $w=f(z)$ maps D holomorphically into $F(D)$, then we have (4.7) *and further*

(4.8) $|det(D_zF(t))|\geqq max\{|det(D_zf(t))|,$
$|det(D_zf(t_0))|\}.$

Proof. We easily have, for $\tau=F(t_0)$,

$$|\det(D_zF(t))|^2=k_D(t,\ \bar{t})/k_{F(D)}(\tau_0,\ \bar{\tau}_0)$$
$$\geqq k_D(t_0,\ \bar{t}_0)/k_{F(D)}(\tau_0,\ \bar{\tau}_0)\geqq k_D(t_0,\ \bar{t}_0)/k_{F(D)}(\tau,\ \bar{\tau})$$
$$=|\det(D_zF(t_0))|^2.$$

If D is a bounded homogeneous domain, $m(D)$ is nonempty and $F(D)$ is homogeneous with $m(F(D))\neq\phi$. Therefore, we have, for $F(t)=\tau_0$,

$$|\det(D_zF(t))|^2=k_D(t,\ \bar{t})/k_{F(D)}(F(t),\ \overline{F(t)})$$
$$\geqq k_D(t,\ \bar{t})/k_{F(D)}(f(z),\ \overline{f(z)})$$
$$=(k_D(t,\ \bar{t})/k_D(z,\bar{z}))(k_D(z,\ \bar{z})/k_{F(D)}(f(z),\ \overline{f(z)})$$
$$\geqq(k_D(t,\ \bar{t})/k_D(z,\ \bar{z}))|\det(D_zf(z))|^2,\ z\ \varepsilon D,$$

since from Theorem 4.1 for $\Delta=F(D)$

$$|\det(D_zf(z))|^2\leqq k_D(z,\ \bar{z})/k_{F(D)}(f(z),\ \overline{f(z)})$$

holds. Hence from $k_D(z,\ \bar{z})\geqq k_D(t_0,\ \bar{t}_0)$ we easily have (4.8).

Theorem 4.3 *Let D and $F(D)$ be bounded minimal domains with center at $t_0\ \varepsilon c(D)$ and $\tau_0\ \varepsilon c(F(D))$, respectively, where $F(z)$ maps D biholomorphically onto $F(D)$ with $F(t)=\tau_0$, then we have*

(4.9) $|det(D_zF(t))|^2\geqq vol(F(D))/vol(D)$
$\geqq|det(D_zF(t_0))|^2,$

where the equality signs hold if and only if $t=t_0$.

In particular, if $F(z)$ is a volume preserving biholomorphic map, then we have

(4.10) $|det(D_zF(t))|\geqq 1\geqq|det(D_zF(t_0))|$ [7].

Proof. From (4.7) and (2.9) we have (4.9) and (4.10).

Remark 4.2 If $\det(D_zF(t))=1$, $F(z)$ is a minimal map of D with center at τ_0. Therefore, $\mathrm{vol}(F(D))\leqq\mathrm{vol}(D)$. Then we have $|\det(D_zF(t_0))|\leqq1$, where equality holds if and only if $\mathrm{vol}(F(D))=\mathrm{vol}(D)$.

Corollary 4.2 *Let D and Δ be bounded minimal domains with center at $t_0\ \varepsilon c(D)$ and $\tau_0\ \varepsilon c(\Delta)$, respectively. If there exists a biholomorphic map of D onto Δ with $F(t)=\tau_0$, then we have*

$$(4.11)\quad \det(D_zF(z))=\det(D_zF(t))k_D(z,\bar{t})/k_D(t,\bar{t}) =\det(D_zF(t))M_{D,(1)}(z,t)$$

and

$$(4.12)\quad ck_D^{-1}(t,\bar{t})|k_D(z,\bar{t})|\leqq|\det(D_zF(z))| \leqq c|k_D(z,\bar{t})|\ (cf.\ (4.5)\ and\ (4.6)),$$

where $c=[vol(D)\,vol(\Delta)]^{1/2}$ and the equality signs in (4.12) hold when and only when $t=t_0$.

In particular, when $t=t_0$ and $F(t_0)=\tau_0$ hold, we have

$$(4.13)\quad \det(D_zF(z))=\det(D_zF(t_0)),\ z\ \varepsilon D,$$

and

$$(4.14)\quad |\det(D_zF(t_0))|=[vol(F(D))/vol(D)]^{1/2}.$$

Further, if $vol(D)=vol(\Delta)$ holds, we have

$$(4.15)\quad \det(D_zF(z))=e^{i\theta},\ z\varepsilon D,$$

where $i=\sqrt{-1}$ and θ denotes a real constant.

Theorem 4.4 *Let D and Δ be representative domains with center at t_0 and τ_0, respectively. If there exists a biholomorphic map $F(z)$ of D onto Δ with $F(t)=\tau_0$, then we have, from (2.11) and (2.14),*

$$(4.16)\quad F(z)=(D_zF(t))T_D^{-1}(t,\bar{t})\int_t^z T_D(z,\bar{t})dz+F(t).$$

In particular, for $t=t_0$ and $F(t_0)=\tau_0$ we have

$$(4.17)\quad F(z)=(D_zF(t_0))(z-t_0)+\tau_0.$$

Corollary 4.3 *Let D be a bounded homogeneous standard domain with center at t_0, then D can not have more than one center as a representative domain.*

Proof. Suppose that D is a representative domain with two centers t_0 and t_1 in D, and $h(z)$ is a transitive map of D onto itself with $h(t_0)=t_1$, then we have from Theorem 4.4

$$D_zh(z)=D_zh(t_1),\ z\ \varepsilon D.$$

On the other hand, since D is a minimal domain with center at t_0, we have, from (4.9) for $t_1\neq t_0$,

$$|\det(D_zh(t_1))|>1>|\det(D_zh(t_0))|.$$

This is a contrdiction.

Remark 4.3 For a minimal and also moment minimal domain the similar result holds, since the differential equation $w(z)\det(D_zw(z))=z$ with $w(0)=0$ and $D_zw(0)=E$ has a unique solution $w=z$.

Theorem 4.5 *Let D be a bounded homogeneous domain and the biholomorphic image domains Δ_w and Δ_ζ of D be the representative domains of D with center at the origin with respect to $t\ \varepsilon D$ $(w(t)=0)$ and $\tau\ \varepsilon D$ $(\zeta(\tau)=0)$, respectively. Then the map $\zeta(w)$ of Δ_w onto Δ_ζ with $\zeta(0)=0$ is given by the linear map*

$$(4.18)\quad \zeta=(D_zh_t(t))w,\ w\ \varepsilon\Delta_w,$$

where $h_t(z)$ denotes the transitive map of D onto itself with $h_t(t)=\tau$.

Proof. The representative functions are given by

$$w=T_D^{-1}(t,\bar{t})\int_t^z T_D(z,\bar{t})dz \text{ and}$$

$$\zeta=T_D^{-1}(\tau,\bar{\tau})\int_\tau^x T_D(x,\bar{\tau})dx.$$

For a transitive map $x=h_t(z)$ of D with $h_t(t)=\tau$, we have, from (2.14),

$$T_D(z,\bar{t})=(D_zh_t(t))^*T_D(x,\bar{\tau})(D_zh_t(x)).$$

So, we obtain

$$\zeta=T_D^{-1}(\tau,\bar{\tau})\int_\tau^x T_D(x,\bar{\tau})dx =(D_zh_t(t))T_D^{-1}(t,\bar{t})\int_t^z T_D(z,\bar{t})dz=(D_zh_t(t))w.$$

Remark 4.4 Under the same situation in Theorem 4.5, for two minimal domains Δ_w and Δ_ζ the map $\zeta=\zeta(w)$ of Δ_w onto Δ_ζ with $\zeta(0)=0$ satisfies

$$\det(D_w\zeta(w))=\det(D_zh_t(t)),\ w\ \varepsilon\Delta_w.$$

Example 4.1 Let B_n be the unit ball in C^n, then B_n is a representative domain with center at the origin. For the representative functions $w(z)$ with $w(t)=0$ and $\zeta(z)$ with $\zeta(\tau)=0$ we have $\zeta=(D_zh_t(t))w$, where $h_t(z)$ denotes the transitive map of B_n onto itself and

$$D_zh_t(t)=[(1-||\tau||^2)(E-\tau\tau^*)]^{1/2}U[(1-||t||^2)\cdot(E-tt^*)]^{-1/2},$$

where U is a constant unitary matrix.

This shows that the representative domains Δ_w with $w(t)=0$ and Δ_ζ with $\zeta(\tau)=0$ have the slight distortion between them. On the other hand, for the unit disc we have

$$\zeta=e^{i\theta}(1-|\tau|^2)(1-|t|^2)^{-1}w,$$

which gives a similar transformation of the unit disc.

5. Normal domain

If D is a minimal domain with center at the origin, the image domain of D under any map $w=Az$ with $\det(A)=1$ is also a minimal domain with center at the origin because of the biholomorphic relative invariance of the Bergman kernel function and (2.8). Thus, even minimal domains belonging to a biholomorphic equivalent class of a ball under the conditions $f(0)=0$ and $\det(D_z f(0))=1$ are not unique.

The image domain of a representative domain D with center at the origin under any map $w=Az$ is also a representative domain with center at the origin. Therefore, representative domains belonging to the same biholomorphic equivalent class are not unique. But the representative domain belonging to the equivalent class under the conditions $f(0)=0$ and $D_z f(0)=E$ is uniquely determined (see Theorem 4.4).

Further, any one of the three types of canonical domains of a domain D depends on a distinguished point t in D and the initial conditions with respect to t (see Theorem 4.5).

Now, we wish to define the normal domain (a sort of a representative domain) as a natural canonical domain.

Definition 5.1 The image domain Δ_w of a bounded domain D under the map (*normal map*)

(5.1) $w=T_D^{-1/2}(t, \bar{t})\int_t^z T_D(z, \bar{t})\,dz$

is called the ***normal domain*** of D with center at the origin (with respeat to a distinguished point t), where $T_D^{1/2}(t, \bar{t})$ denotes a regular matrix P such that $P^*P=T_D(t, \bar{t})$ (positive definite Hermitian Matrix) holds.

Lemma 5.1 *For a biholomorphic map $\zeta(z)$ of a bounded domain D onto Δ with $\zeta(t)=\tau$ we have*

$$(5.2)\quad T_D^{-1/2}(t, \bar{t})\int_t^z T_D(z, \bar{t})\,dz = UT_\Delta^{-1/2}(\tau, \bar{\tau})\int_\tau^\zeta T_\Delta(\zeta, \bar{\tau})\,d\zeta,$$

where U denotes a constant unitary matrix.

Proof. As $dz^*T_D^*(z, \bar{t})\,T_D(t, \bar{t})\,T_D(z, \bar{t})\,dz$ is biholomorphically invariant from (2.14), then we have (5.2) with a constant unitary matrix U. Indeed, U is a unitary matrix and must be holomorphic with respect to z, and thus $U^*=U^{-1}$ must be holomorphic with respect to z. Therefore, U must be a constant unitary matrix.

Lemma 5.2 *A necessary and sufficient condition that a domain Δ is a normal domain with center at the origin is*

(5.3) $T_\Delta(\zeta, 0)=T_\Delta^{1/2}(0, 0)\,U^*$, $\zeta \in \Delta$,

i. e.,

(5.4) $T_\Delta(\zeta, 0)=T_\Delta(0, 0)$, $\zeta \in \Delta$, *and* $T_\Delta(0, 0)=E$, *which shows that Δ is a sort of a representative domain.*

Proof. For a normal map $\zeta=T_D^{-1/2}\int_0^z T_D(z, 0)\,dz$ we have

$$\zeta=T_D^{-1/2}\int_0^z T_D(z, 0)\,dz=UT_\Delta^{-1/2}\int_0^\zeta T_\Delta(\zeta, 0)\,d\zeta.$$

Differentiating both sides of this, we have

$$E=UT_\Delta^{-1/2}(0, 0)\,T_\Delta(\zeta, 0).$$

Hence we have the result. Converse is true.

Theorem 5.1 *Normal domains of the biholomorphic equivalent class of a bounded domain with respect to the corresponding distinguished points are uniquely determined up to unitary matrices.*

Theorem 5.2 *Normal domains of the biholomorphic equivalent class of a bounded homogeneous domain with r spect to arbitary distinguished points are uniquely determined up to unitary matrices.*

Proof. Let Δ_w and Δ_ζ be normal domains with center at the origin with respect to distinguished points t and τ, respectively, then we have the normal maps

$$w=T_D^{-1/2}(t, \bar{t})\int_t^z T_D(z, \bar{t})\,dz,$$

$$\zeta=T_D^{-1/2}(\tau, \bar{\tau})\int_\tau^x T_D(x, \bar{\tau})\,dx$$

and

$$T_D^{-1/2}(t, \bar{t})\int_t^z T_D(z, \bar{t})\,dz = UT_D^{-1/2}(\tau, \bar{\tau})\int_\tau^x T_D(x, \bar{\tau})\,dx$$

for a transitive map $x=h_t(z)$ of D onto itself with $h_t(t)=\tau$. Therefore, we obtain $w=U\zeta$.

Example 5.1 (i) If D is a bounded homogeneous domain, then a transitive map $\zeta=\zeta(z)$ with $\zeta(t)=\tau$ is given by

$$T_D^{-1/2}(t, \bar{t})\int_t^z T_D(z, \bar{t})\,dz = UT_D^{-1/2}(\tau, \bar{\tau})\int_\tau^\zeta T_D(\zeta, \bar{\tau})\,d\zeta.$$

(ii) For the unit ball $B_n=B$ in C^n, we have a normal map $w(z)$ of B with $w(t)=0$ as

$$w=T_B^{-1/2}(t,\ \bar{t})\int_t^z T_B(z,\ \bar{t})\,dz$$

$$=\sqrt{n+1}\,(E-tt^*)^{-1/2}(z-t)\,(1-t^*z)^{-1}(1-t^*t)^{1/2}.$$

$w(B)$ is a ball of radius $\sqrt{n+1}$ with center at 0.

(iii) The normal domain $w(B)$ of the unit ball is also a ball with $T_{w(B)}(0,\ 0)=E$.

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