Analysis of Effects of Fluid Viscosity on Sound Propagation in Acoustic Wave Guide with Circular Section

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An exact analysis about the sound propagation in the acoustic wave guide with circular section is made and some numerical results concerning axisymmetric and non-axisymmetric acoustic modes are shown. The effects of the fluid viscosity, especially the damping of the sound, are discussed.

1. Introduction

The sound propagation in the acoustic wave guide with circular section is an important problem. However, in spite of many investigations about this topic, there appears to be no exact discussion about the effects of the viscosity of the fluid as the medium of sound. For a more rigorous consideration of the sound propagation, it is necessary to know the damping of the sound.

In this paper, an exact analysis in the presence of the fluid viscosity is presented, and some numerical results obtained using a computer are also shown.

2. Analysis

The present analysis is based on the linearized approximation of the Navier-Stokes equation

$$\rho\frac{\partial \boldsymbol{u}}{\partial t}=-\nabla p+\frac{4}{3}\nabla\nabla\cdot\boldsymbol{u}-\mu\nabla\times\nabla\times\boldsymbol{u}, \tag{1}$$

and the equation of continuity that is also linearized

$$\frac{\partial p}{\partial t}=-k\nabla\cdot\boldsymbol{u}, \tag{2}$$

where $\boldsymbol{u}$ is the particle velocity vector, p is the pressure, ρ is the density, k is the bulk modulus, μ is the coefficient of viscosity, t is time and ∇ is the three dimensional nabla operator. The coordinates system is shown in Fig. 1. Eq. (1) and Eq. (2) are valid, if, and only if, the amplitude of the wave is small. The velocity vector $\boldsymbol{u}$ can be divided into two terms, irrotational term $\boldsymbol{u}_1$ and rotational term $\boldsymbol{u}_2$.

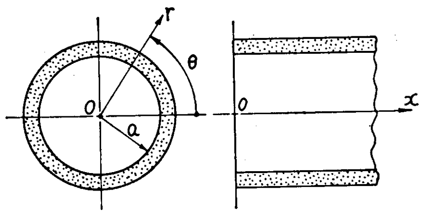

Fig. 1 Coordinates system.

$$\boldsymbol{u}=\boldsymbol{u}_1+\boldsymbol{u}_2,\quad \nabla\times\boldsymbol{u}_1=0,\quad \nabla\cdot\boldsymbol{u}_2=0. \tag{3}$$

The division of $\boldsymbol{u}$ by Eqs. (3) is convenient not only for obtaining solutions, but also for consideration of physical meanings of $\boldsymbol{u}$. Let us consider a plane wave or a spherical wave in infinite space. In these waves, the term $\boldsymbol{u}_1$ is always necessary, but the term $\boldsymbol{u}_2$ does not appear. On the other hand, if the wave is generated in a restricted space such as in a tube, both of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are always necessary for satisfaction of the condition that the velocity of the fluid in the vicinity of the wall must be eqaul to that of the wall. Therefore, we can conclude that the term $\boldsymbol{u}_1$ is always necessary for the wave caused by the compressibility of the fluid, but $\boldsymbol{u}_2$ appears only if the so-called boundary layer exists.

Using the relation given by Eqs. (3), Eq. (1) and Eq. (2) are rewritten as the followings.

$$\left.\begin{aligned}\rho\frac{\partial u_1}{\partial t}&=-\nabla p+\frac{4}{3}\mu\nabla\nabla\cdot u_1,\\ \frac{\partial p}{\partial t}&=-k\nabla\cdot u_1\end{aligned}\right\}\qquad(4)$$

$$\left.\begin{aligned}\rho\frac{\partial u_2}{\partial t}&=-\mu\nabla\times\nabla\times u_2,\\ \nabla\cdot u_2&=0\end{aligned}\right\}\qquad(5)$$

Elimination of U_1 from both of Eqs. (4) gives

$$\left(1+\frac{4}{3}\frac{\nu}{c^2}\frac{\partial}{\partial t}\right)\times\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial x^2}\right)p-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}p=0\qquad(6)$$

where c is the sound velocity $\sqrt{k/\rho}$ and ν is the kinematic viscosity μ/ρ. For a sinusoidal wave travelling along the x-axis, let

$$p=P(r)\cdot\cos n\theta\cdot\exp[j\beta(x-\alpha^*t)].\qquad(7)$$

In Eq. (7), α^* is the complex phase velocity, β is the longitudinal wave number and j is the imaginary unit. Substitution of Eq. (7) in to Eq. (6) gives an ordinary differential equation.

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}+\zeta^2\right)P(r)=0,\qquad(8)$$

where

$$\zeta^2=\beta^2\left(\frac{\alpha^{*2}}{c^2-\frac{4}{3}j\nu\beta\alpha^*}-1\right)\qquad(9)$$

Since the pressure p should have a certain finite value, we must choose the solution of Eq. (8) as the following.

$$P(r)=AJ_n(\zeta r)\qquad(10)$$

where A is an undetermined coefficient and $J_n(\zeta r)$ is nth order Bessel's function of the first kind. Now, let

$$u_1=[U_{1x}\cos n\theta,\ U_{1r}\cos n\theta,\ U_{1\theta}\sin n\theta]\times\exp[j\beta(x-\alpha^*t)]\qquad(11)$$

Substitution of Eq. (11) with Eq. (7) and Eq. (10) into the first of Eq. (4) yields

$$\left.\begin{aligned}U_{1x}&=A'\frac{1}{\rho\alpha^*}J_n(\zeta r),\\ U_{1r}&=-A'\frac{\zeta}{2\rho\beta\alpha^*}[J_{n-1}(\zeta r)-J_{n+1}(\zeta r)],\\ U_{1\theta}&=A'\frac{\zeta}{2\rho\beta\alpha^*}[J_{n-1}(\zeta r)+J_{n+1}(\zeta r)],\end{aligned}\right\}\qquad(12)$$

where

$$A'=A\left(1-\frac{4}{3}j\frac{\nu}{c^2}\beta\alpha^*\right).\qquad(13)$$

In the next place, let us consider about u_2. Let

$$u_2=[U_{2x}\cos n\theta,\ U_{2r}\cos n\theta,\ U_{2\theta}\sin n\theta]\times\exp[j\beta(y-\alpha^*t)].\qquad(14)$$

Eq. (14) is corresponding to u_1 given by Eq. (11). Substituting Eq. (14) into Eqs. (5) and rearranging expressions, we have the following ordinary differential equations.

$$\left.\begin{aligned}&\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}+\eta^2\right)U_{2x}=0,\\ &\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}+\eta^2\right)U_{2r}-\frac{2n}{r_2}U_{2\theta}=0,\\ &\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}+\eta^2\right)U_{2\theta}-\frac{2n}{r^2}U_{2r}=0,\end{aligned}\right\}\qquad(15)$$

where

$$\eta^2=\frac{j\beta\alpha^*}{\nu}-\beta^2,\qquad(16)$$

The solution of the first of Eq. (15) is obtained as

$$U_{2x}=B_1J_n(\eta r),\qquad(17)$$

where B_1 is an undetermined coefficient. The second and the third of Eqs. (15) are evidently in a dual relation. Therefore, solutions of these simultaneous equations must be in some dual relations. If we assume that $U_{2\theta}=U_2r$, we obtain

$$U_2r=U_{2\theta}=B_2\cdot J_{n+1}(\eta r)\qquad(18)$$

and if we assume that $U_{2\theta}=-U_2r$, we obtain

$$U_2r=-U_{2\theta}=B_3\cdot J_{n-1}(\eta r)\qquad(19)$$

B_2 and B_3 are also undetermined coefficients. Thus, we have

$$\left.\begin{aligned}U_{2r}&=B_2J_{n+1}(\eta r)+B_3J_{n-1}(\eta r)\\ U_{2\theta}&=B_2J_{n+1}(\eta r)-B_3J_{n-1}(\eta r)\end{aligned}\right\}\qquad(20)$$

The three coefficients, B_1, B_2 and B_3 can not be independent each other, because u_2 must satisfy the second of Eq. (5). From this fact,

$$B_1=\frac{j\eta}{\beta}(B_2-B_3)\qquad(21)$$

Substitution of Eq. (21) into Fq. (18) yields

$$U_{2x}=\frac{j\eta}{\beta}(B_2-B_3)J_n(\eta r)\qquad(22)$$

Now, we must consider about the boundary condition. In this paper, any movement of the wall of the wave guide is not permitted. Thus,

$$\left.\begin{aligned}[U_{1x}+U_{2x}]_{r=a}&=0,\\ [U_{1r}+U_{2r}]_{r=a}&=0,\\ [U_{1\theta}+U_{2\theta}]_{r=a}&=0.\end{aligned}\right\}\qquad(23)$$

where a is the radius of the wave guide. Substituting the expressions about U_{1x}, U_{2x}, etc. derived so far into the boundary conditions given by Eq. (23), we have simultaneous homogeneous equations for A', B_2

and B_3.

$$\left.\begin{array}{l} J_n(\zeta a)\cdot A'+\frac{j\eta}{\beta}J_n(\eta a)\cdot B_2 \\ \quad -\frac{j\eta}{\beta}J_n(\eta a)\cdot B_3=0 \\ -\frac{j\zeta}{2\beta}[J_{n-1}(\zeta a)-J_{n+1}(\zeta a)]\cdot A' \\ \quad +J_{n+1}(\eta a)\cdot B_2+J_{n-1}(\eta a)\cdot B_3=0 \\ \frac{j\zeta}{2\beta}[J_{n-1}(\zeta a)+J_{n+1}(\zeta a)]\cdot A' \\ \quad +J_{n+1}(\eta a)\cdot B_2-J_{n-1}(\eta a)\cdot B_3=0 \end{array}\right\} \tag{24}$$

According to the theorem of the linear algebra, there are non-trivial solutions of Eqs. (24), if, and only if, the determinant of coefficient in Eqs. (24) is zero. Thus, we have

$$\begin{vmatrix} 1 & \frac{j\eta}{\beta} & -\frac{j\eta}{\beta} \\ -\frac{j\zeta}{2\beta}[1/\varphi_{(n-1),\xi}-\varphi_{n,\xi}] & \varphi_{n,\eta} & 1/\varphi_{(n-1),\eta} \\ \frac{\zeta 2}{2\beta}[1/\varphi_{(n-1),\xi}+\varphi_{n,\xi}] & \varphi_{n,\eta} & 1/\phi_{(n-1),\eta} \end{vmatrix}=0 \tag{25}$$

In Eq. (25), the following expression is used for simplicity.

$$\varphi_{n,z}\equiv\frac{J_{n+1}(za)}{J_n(za)} \tag{26}$$

Expanding the determinant of Eq. (25) and rearranging the expression, we obtain

$$2\beta^2\frac{\varphi_{n,\eta}}{\varphi_{(n-1),\eta}}+\zeta\eta\left[\frac{\varphi_{n,\xi}}{\varphi_{(n-1),\eta}}+\frac{\varphi_{n,\eta}}{\varphi_{(n-1),\xi}}\right]=0 \tag{27}$$

If $n=0$, namely, the wave is an axi-symmetric one, Eq. (27) can be simplified as

$$\beta^2\varphi_{0,\eta}+\zeta\eta\varphi_{0,\xi}=0 \tag{28}$$

Eq. (27) and Eq. (28) are the equations for obtaining the complex phase velocity α^*, when the value of the wave number β, consequently, of the wave length λ is given. These equations are the so-called complex transcendental ones. Therefore, these can be solved only by numerical methods. In this paper, the steepest descent method is utilized. The phase velocity α and the logarithmic decrement δ can be calculated from the value of α^*, using the following relations.

$$\alpha=\mathrm{Re}\alpha^* \tag{29}$$

$$\delta=-2\pi\mathrm{Im}\alpha^*/\mathrm{Re}\alpha^* \tag{30}$$

3. Numerical Results

On the sound wave in the presence of the fluid viscocity, both of the phase velocity and the damping modulus are matters of concern. The phase velocity is slightly smaller than that in the absence of the fluid viscosity. However, the decline of the phase velocity caused by the viscosity is negligiblly small, moreover the phase velocity in the absence of viscosity can be calculated using the following simple equation.

$$\alpha=c\sqrt{\frac{\varepsilon^2}{\beta^2}+1} \tag{31}$$

where ε is obtained from the roots of the following equation

$$J_{n-1}(\varepsilon a)-J_{n+1}(\varepsilon a)=0 \tag{32}$$

For reference, the earlier ten roots of Eq. (32) are given in the Table 1. In this table, n of (n, m) in the Mode Shape column is the number of the nodal diameters of the pressure distribution and m is the number of nodal circles. For example, Fig. 2 shows

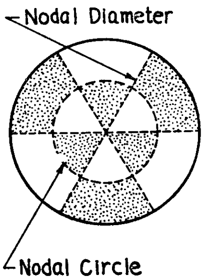

Fig. 2 Schematic illustration of mode shape. This figure shows the mode (3, 1).

the mode of (3, 1).

Hence, attention will be given only to the damping modulus. Let $\bar{\lambda}=\lambda/a$ and $\bar{\nu}=\nu/ac$ be dimensionless parameters corresponding to λ and ν respectively. Fig.

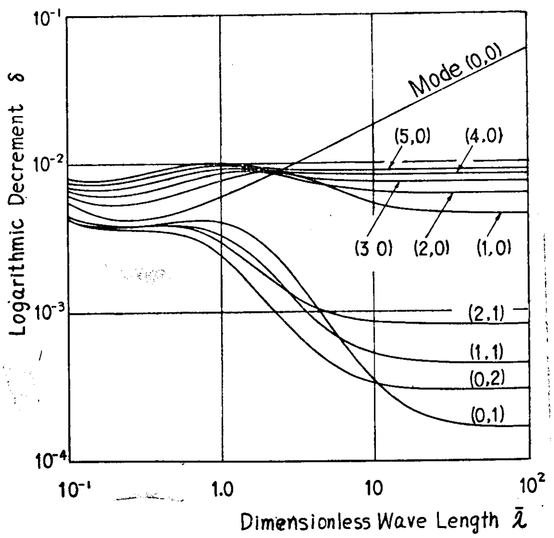

Fig. 3 Relation between δ and $\bar{\lambda}$. ($\nu=10^{-5}$)

3 shows the relation between the logarithmic decrement δ and the dimensionless wave length $\bar{\lambda}$ about the earlier ten modes, when $\bar{\nu}=10^{-5}$. In this figure, it appears that the modes can be classified into three groups according to magnitude of δ in the long wave length region. The first group includes only the (0, 0) mode, the second includes the modes of (1, 0), (2, 0) and so on, and the third consists of the modes (0, 1), (1, 1) and so on. Main reason of possibility of this classification can be found in the profiles of the axial component of the particle velocity $\boldsymbol{u}$. Fig. 4

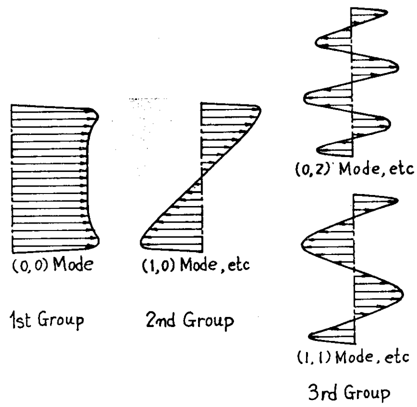

Fig. 4 Rough schematic illustrations of distribution of U_x.

shows rough schematic illustrations of the profiles of u_x. If the profile of u_x of one mode is similar to that of another mode, the values of δ of these modes are evidently close each other.

In Fig. 5, δ of the earlier three modes are shown, when the value of $\bar{\nu}$ varies. The value of δ of any mode is not proportional to $\bar{\nu}$, but it appears that δ is roughly proportional to $\sqrt{\bar{\nu}}$ in long wave length region.

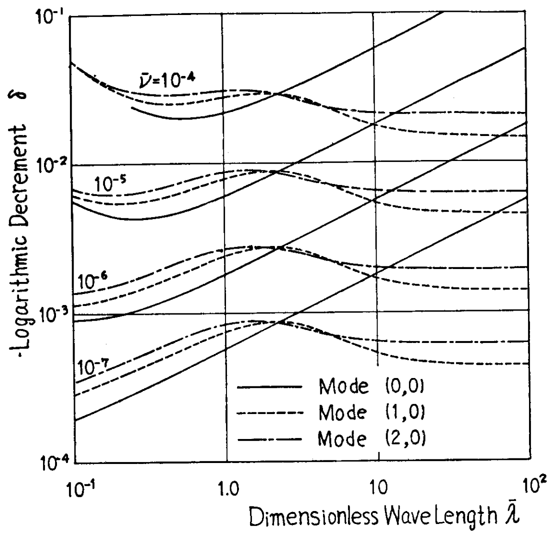

Fig. 5 Relation between δ and $\bar{\lambda}$, when $\bar{\nu}$ varies.

4. Conclusion

The modes of the sound travelling in the circular wave guide are classified into some groups according to the value of δ in long wavelength region, and δ of every mode is roughly proportional to $\sqrt{\bar{\nu}}$.

Table 1. Roots of Eq. (32)

No.	Mode Shape	εa
1	(0, 0)	0
2	(1, 0)	1.841183
3	(2, 0)	3.054236
4	(0, 1)	3.831706
5	(3, 0)	4.201189
6	(4, 0)	5.317553
7	(1, 1)	5.331443
8	(5, 0)	6.415616
9	(2, 1)	6.706133
10	(0, 2)	7.015587