Optimization of Multimodal Transportation Network Flows

Hiroshi Matsui

Department of civil Engineering
(Received July 25, 1977)

In this paper we consider the optimization of the transportation network flows for multiclass travel modes which share the same roads. In building a model we divide travel units into several classes each of which has an individual cost function and an individual route selection criterion.

We show that the optimization problem of such multimodal network flows can be stated as a combined problem of the optimal modal split and traffic assignment in the network. In the last part of this paper we solve some problems for the 2-mode single OD network in order to illustrate the solution procedure and the application of the model.

1. INTRODUCTION

The minimum cost network flow problem can be stated as an optimal traffic assignmnt problem and has been dealt with many authors. The general form of the problem can be written as the problem that minimizes the total travel cost (e.g. travel time) in a given network for a fixed OD table. The previous works were, however, based on the assumption that all the tripmakers evaluated the network and selected their own route in the same manner. However, in the actual street network we can see different types of vehicles which share the same roads, for instance, trucks, passenger cars, buses, etc. The value of time and the operating cost of these vehicles are not same to all the tripmakers, and moreover, while some types of vehicles operated by a company may try to minimize their total travel cost, some may try to minimize each individual travel cost by using the cheapest route through the network.

The ordinary traffic assignment problems for multiclase user transportation network have been already dealt with by Dafermos[1], Netter[2], and Jeevananthan[3]. In this paper assuming that all the tripmakers are given a free choice of not only the routes but also the modes we consider the optimization of the transportation network flows for multiclass travel modes, each of which has an individual cost function on each link and/or an individual route selection criterion.

For the route selection criterion we will consider two types of assignment principle: the system-optimized and the user-optimized traffic pattern, which were first enunciated by Wardrop[4]. The first pattern is such that minimizes the total travel cost over the network and the second is such that minimizes each individual travel cost. It is known that both the assignment principles can be treated as mathematical programming problems.

The model developed here can be formulated as a multilevel optimization problem decomposed into a center problem that minimizes the total travel cost over the network by optimizing the modal split and some local problems that give the traffic assignment pattern to each mode. We will show that this multilevel optimization model is overcome by combining the traffic assignment and the modal split into one stage and describing them by one model. The combined model is then reformulated as an equivalent optimization problem which is solved.

2. PROBLEM FORMULATION

Let us consider the minimum cost network flow problem for two travel modes. It is easy to see that the problem could be extended to any number of travel modes.

To formulate the problem, the following notations are used:

N_i = the i-th OD flows $(i=1, 2, \cdots, n)$

x_i = the i-th OD flows by mode 1

y_i = the i-th OD flows by mode 2

x_i^k = the i-th OD flows by mode 1 along route k $(k=1, 2, \cdots, r_{xi})$

y_i^k = the i-th OD flows by mode 2 along route k $(k=1, 2, \cdots, r_{yi})$

t_{xj} = the flows by mode 1 on link j $(j=1, 2, \cdots, h)$

t_{yj} = the flows by mode 2 on link j

c_{xj} = the travel cost per unit flow by mode 1 on link j

c_{yj} = the travel cost per unit flow by mode 2 on link j

Among the above notations the following relations must be satisfied:

$$N_i = x_i + y_i \tag{1}$$

$$x_i = \sum_k x_i^k \tag{2}$$

$$y_i = \sum_k y_i^k \tag{3}$$

$$t_{xj} = \sum_i \sum_k \delta_{ij}{}^k x_i^k \tag{4}$$

$$t_{yj} = \sum_i \sum_k \delta_{ij}{}^k y_i^k \tag{5}$$

where

$$\delta_{ij}^k = \begin{cases} 1 & \text{if link j is on route k between the i-th OD pair} \\ 0 & \text{otherwise} \end{cases}$$

We assume that the link travel cost is a monotone increasing function of the link flows

$$c_{xj} = c_{xj}(t_{xj}, t_{yj}), \quad c_{yj} = c_{yj}(t_{xj}, t_{yj}) \tag{6}$$

The problem can be divided into several cases corresponding to the conbinations of different traffic modes.

Minimize Individual Cost to Mode 1 but Minimize Total Cost to Mode 2

For this case the flow pattern to mode 1 can be expressed as the following minimization problem:

$$\text{Minimize } F_x = \sum_j \int_0^{t_{xj}} c_{xj}(x)\,dx \tag{7}$$

$$\text{subject to } \sum_k x_i^k = x_i \quad i=1, 2, \cdots, n \tag{8}$$

$$\text{and} \quad x_i^k \geqq 0 \quad i=1, 2, \cdots, n, \ k=1, 2, \cdots, r_{xi} \tag{9}$$

It is well known that the solution of this problem, which we will call local problem 1, is interpreted as the user-optimized traffic pattern.

Similarly the flow pattern to mode 2 can be defined by the following minimization problem:

$$\text{Minimize } F_y = \sum_i \sum_j \sum_k \delta_{ij}^k y_i^k c_{yj} \tag{10}$$

$$\text{subject to } \sum_k y_i^k = y_i \quad i=1, 2, \cdots, n \tag{11}$$

$$\text{and} \quad y_i^k \geqq 0 \quad i=1, 2, \cdots, n, \ k=1, 2, \cdots, r_{yi} \tag{12}$$

The solution of this problem, which we will call local problem 2, gives the system-optimized traffic pattern.

On the other hand the center problem that minimizes the total travel cost to all the modes over the network can be formulated as follows:

$$\text{Minimize } F = \sum_i \sum_j \sum_k \delta_{ij}^k (x_i^k c_{xj} + y_i^k c_{yj}) \tag{13}$$

$$\text{subject to } \sum_k x_i^k + \sum_k y_i^k = N_i \tag{14}$$

$$\text{and} \quad x_i^k \geqq 0, \ y_i^k \geqq 0 \tag{15}$$

The solution of the center problem also must satisfy the necessary conditions for the optimal solutions of the previous two local problems. That is to say, the minimum cost problem for multimodal network flows can be treated as a two-level optimization problem decomposed into a center problem and some local problems corresponding to the number of modes.

It should be noted that the local problem to mode 1 can be recast as that of minimizing the following Largrange function:

$$\phi = \sum_j \int_0^{t_{xj}} c_{xj}(x)\,dx - \sum_i \lambda_i \left(\sum_k x_i^k - x_i\right) \tag{16}$$

where λ_i $(i=1, 2, \cdots, n)$ are the Lagrange multipliers.

From the Kuhn-Tucker theorem we can write the necessary and sufficient conditions for the minimum solution of the above problem as

$$\left.\begin{array}{l} \text{if } x_i^k > 0, \text{ then} \\ \dfrac{\partial \phi}{\partial x_i^k} = \sum_j \delta_{ij}^k c_{xj} - \lambda_i = 0 \\ \text{if } x_i^k = 0, \text{ then} \\ \dfrac{\partial \phi}{\partial x_i^k} = \sum_j \delta_{ij}^k c_{xj} - \lambda_i \geqq 0 \\ \text{and} \\ \dfrac{\partial \phi}{\partial \lambda_i} = -\sum_k x_i^k + x_i = 0 \end{array}\right\} \text{for all i and k} \tag{17}$$

where it is noted that $\sum_j \delta_{ij}^k c_{xj}$ gives the average travel cost on route k between the i-th OD pair.

These conditions imply that the average travel costs on all the routes actually used are equal, and any unused routes have average costs greater than or equal to routes with positive flow, which just satisfies the definition of the user-optimized traffic pattern.

Similarly, for the local problem to mode 2 we can have the following Lagrange function:

$$\phi=\sum_i\sum_j\sum_k \delta_{ij}^k\, y_i^k c_{yj}-\sum_i \mu_i\,(\sum_k y_i^k-y_i) \tag{18}$$

For this case we can see that the minimization of the function (18) is equivalent to the following equilibrium conditions:

$$\left.\begin{array}{l}\text{if } y_i^k>0,\ \text{then}\\ \dfrac{\partial\phi}{\partial y_i^k}=\sum_j \delta_{ij}^k\,(c_{yj}+y_i^k c'_{yj})-\mu_i=0\\ \text{if } y_i^k=0,\ \text{then}\\ \dfrac{\partial\phi}{\partial y_i^k}=\sum_j \delta_{ij}^k\,(c_{yj}+y_i^k c'_{yi})-\mu_i\geqq 0\\ \text{and}\\ \dfrac{\partial\phi}{\partial \mu_i}=-\sum_k y_i^k+y_i=0\end{array}\right\}\text{for all i and k} \tag{19}$$

where c'_{yj} is the first partial derivative with with respect to y_i^k.

The above conditions are very similar to the previous conditions (17). The only substantive difference is that we have substituted $\sum_j \delta_{ij}^k\,(c_{yj}+y_i^k c'_{yj})$ for $\sum_j \delta_{ij}^k c_x^j$. $\sum_j \delta_{ij}^k(c_{yj}+y_i^k c'_{yj})$ is called the marginal cost of flow on route k between the i-th OD pair.

We can now reformulate the center problem by using the equivalent conditions (17) and (19) as additional constraints.

$$\text{Minimize } F=\sum_i\sum_j\sum_k \delta_{ij}^k\,(x_i^k c_{xj}+y_i^k c_{yj}) \tag{13}$$

$$\text{subject to}\quad \sum_k x_i^k+\sum_k y_i^k=N_i \tag{14}$$

$$\left.\begin{array}{ll}\sum_j \delta_{ij}^k c_{xj}-\lambda_i=0 & (\text{if } x_i^k>0)\\ \sum_j \delta_{ij}^k c_{xj}-\lambda_i\geqq 0 & (\text{if } x_i^k=0)\end{array}\right\} \tag{20}$$

$$\left.\begin{array}{ll}\sum_j \delta_{ij}^k\,(c_{yj}+y_i^k c'_{yj})-\mu_i=0 & (\text{if } y_i^k>0)\\ \sum_j \delta_{ij}^k\,(c_{yj}+y_i^{k\prime} c_{yj})-\mu_i\geqq 0 & (\text{if } y_i^k=0)\end{array}\right\} \tag{21}$$

$$\text{and}\quad x_i^k\geqq 0,\ y_i^k\geqq 0 \tag{15}$$

If the routes used by mode 1 and by mode 2 are given in advance, we can obtain the solution by solution by solving the above minimization problem.

Minimize Individual Cost to Each Mode

Next we consider the case where the users of each mode try to minimize their individuel costs. For this case the local problem to mode 1 is the same as local problem 1 described by (7), (8) and (9), and the local problem to mode 2 can be defined similarly by replacing t_{xj}, c_{xj}, x_i^k and x_i by t_{yj}, c_{yj}, y_i^k and y_i in local problem 1.

In consequence we can have the center problem of minimizing (13) subject to the constraints (14), (15), (20) and

$$\left.\begin{array}{ll}\sum_j \delta_{ij}^k c_{yj}-\mu_i=0 & (\text{if } y_i^k>0)\\ \sum_j \delta_{ij}^k c_{yj}-\mu_i\geqq 0 & (\text{if } y_i^k=0)\end{array}\right\} \tag{22}$$

Minimize Total Cost to to Each Mode

For this case the local problem to mode 2 becomes the same as local problem 2 described by (10), (11) and (12), and the local problem to mode 1 can be given by replacing t_{yj}, c_{yj}, y_i^k and y_i by t_{xj}, c_{xj}, x_i^k and x_i respectively in local problem 2 reversely to the preceding case. We now have the center problem of minimizing (13) subject to the constraints (14), (15), (21) and

$$\left.\begin{array}{ll}\sum_j \delta_j^{ik}\,(c_{xj}+x_i^k c'_{xj})-\lambda_i=0 & (\text{if } x_i^k>0)\\ \sum_j \delta_{ij}^k\,(c_{xj}+x_i^k c'_{xj})-\lambda_i\geqq 0 & (\text{if } x_i^k=0)\end{array}\right\} \tag{23}$$

Minimize Total Cost to All Modes

In the above case the total cost to each mode has been minimized as the local problems before minimizing the overall travel cost in the network as the center problem. Here we minimize the total cost to all modes more directly. It is obvious that the solutions of this problem and the previous problem will not (always) be the same. Moreover the objective function for this problem will never have a larger value than any of the objective functions for the other problems mentioned above.

This problem can be formulated as that of minimizing (13) subject to the constraints (14) and (I5) only.

Mathematically all the multimodal network flow problems stated above can be dealt with as a nonlinear programming and can be solved by recent nonlinear programming techniques. If we use linear increasing cost functions, then the problem can be recast as that of minimizing the quadratic objective function subject to the linear constraints.

3. NUMERCAL EXAMPLES

In order to illustrate the solution procedure and the application of the models proposed above, we will solve two example problems pertaining to the network flows of passenger cars and buses. In the first example we consider the problem that minimizes the overall travel time in the network, based on the assumption that buses try to minimize their total travel time while passenger cars prefer to choose the shortest route. In the second example we consider the problem that simply minimizes the total travel time to both the modes without constraints related to route choice.

We will consider a single OD network shown in Fig. 1.

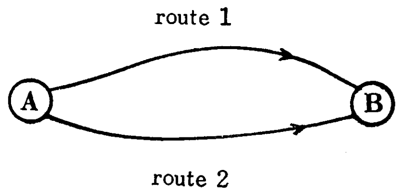

Fig. 1.

The following notations are used:

N = the number of person trips from A to B. [trips/hour]

x_i = the flows of passenger cars along route i, i=1, 2. [vehicles/hour]

y_i = the flows of buses along route i. [vehicles/hour]

C_{xi}= the travel travel time by passenger car on route i. [minutes]

C_{yi}= the travel time by bus on route i. [minutes]

S_x = the average occupancy rate for passenger car. [persons/vehicle]

S_y = the average average occupancy rate for bus. [persons/vehicle]

E = the passenger car equivalent of for bus.

Let us denote by t_i the total flows along route i. Then the following relation exists:

$$t_i=x_i+Ey_i \qquad i=1,\ 2 \tag{24}$$

Moreover, we assume that the travel time on route i is given by the linear function of the route flows t_i.

$$C_{xi}=a_{xi}\frac{t_i}{c_i}+b_{xi},\ C_{yi}=a_{yi}\frac{t_i}{c_i}+b_{yi}\ \ i=1,\ 2 \tag{25}$$

where a_{xi} and a_{yi} are empirically derived constants, b_{xi} and b_{yi} are constants representing travel time at free flow conditions and c_i is capacity of route i.

We also assume that

$$S_x = 1.2,\ S_y=50,\ E=1.75,\ c_1=c_2=2000,$$

and

$$a_{x1} = 45,\ a_{x2}=49.5,\ a_{y1}=30,\ a_{y2}=33,$$
$$b_{x1} = 15,\ b_{x2}=16.5,\ b_{y1}=30,\ b_{y2}=33,$$

then we have

$$C_{x1} = 0.0225x_1+0.03938y_1+15$$
$$C_{x2} = 0.02475x_2+0.04331y_2+16.5$$
$$C_{y1} = 0.015x_1+0.02625y_1+30$$
$$C_{y2} = 0.0165x_2+0.02888y_2+33$$

Example 1

The objective function to be minimized is given by

$$F = S_x(x_1C_{x1}+x_2C_{x2})+S_y(y_1C_{y1}+y_2C_{y2}) \tag{26}$$

This may be worked out

$$F = 0.027x_1^2+0.0297x_2^2+1.3125y_1^2+1.4438y_2^2 +7.5473x_1y_1+0.877x_2y_2+18x_1+19.8x_2 +1,500y_1+1,515y_2 \tag{27}$$

x_i and y_i (i=1, 2) must satisfy the conservation laws

$$S_x(x_1+x_2)+S_y(y_1+y_2)=N \tag{28}$$

This is then

$$1.2\,x_1+1.2\,x_2+50\,y_1+50\,y_2-N=0 \tag{29}$$

If the passenger cars are present on both routes, the travel time by passenger car on the two routes are equal. That is

$$C_{x1}=C_{x2} \qquad \text{for } x_1,\ x_2>0 \tag{30}$$

This may be rewritten as

$$0.0225\,x_1-0.02475\,x_2+0.03938\,y_1 -0.04331\,y_2-1.5=0 \tag{31}$$

Similarly if the buses use both routes, the marginal travel time by bus on the two routes are equal, then

$$C_{y1}+y_1\,C'_{y1}=C_{y2}+y_2\,C'_{y2} \quad \text{for } y_1,\ y_2>0 \tag{32}$$

This may be rewritten as

$$0.015\,x_1-0.0165\,x_2+0.0525\,y_1 -0.05775\,y_2-3=0 \tag{33}$$

Hence the first example problem can be recast as that of minimizing the quadratic objective function (27) subject to the linear constraints (29), (31) and (33) and easily solved with Lagrange multipliers.

It should be noted that both routes are not always occupied by the passenger cars and buses as the number of the total users N varies. If $x_1=0$ and/or $x_2=0$, we may neglect the constraint (31), and also if $y_1=0$ and/or $y_2=0$, we may neglect the constraint (33).

The solutions for this example are given as follows:

for $0 \leqq N < 80$

$x_1 = 0.8333\,N$

$x_2 = y_1 = y_2 = 0$

$F = 0.01875\,N^2 + 15\,N$

for $80 \leqq N < 1,164$

$x_1 = 0.4365\,N + 31.7$

$x_2 = 0.3968\,N - 31.7$

$y_1 = y_2 = 0$

$F = 0.009822\,N^2 + 15.7143\,N$

for $1164 \leqq N < 1375$

$x_1 = -0.4798\,N + 1098.3$

$x_2 = -0.3718\,N + 862.9$

$y_1 = 0.04044\,N - 47.1$

$y_2 = 0$

$F = -0.003\,N^2 + 45.5628\,N - 17370.8$

for $1375 \leqq N < 7502$

$x_1 = y_2 = 0$

$x_2 = 0.02069\,N + 266.0$

$y_1 = 0.0195\,N - 6.4$

$F = 0.000512\,N^2 + 29.6656\,N - 2154.3$

for $7502 \leqq N$

$x_1 = x_2 = 0$

$y_1 = 0.01048\,N + 27.2$

$y_2 = 0.009524\,N - 27.2$

$F = 0.000275\,N^2 + 31.4283\,N - 2040.8$

Figure 2 shows the variation in percentage of the users by modes and by routes as the number of the total users varies. We see that in this case x_i and y_i $(i=1,\ 2)$ are not positive at the same time. However, this does not imply that such a solution does not exist in gereral.

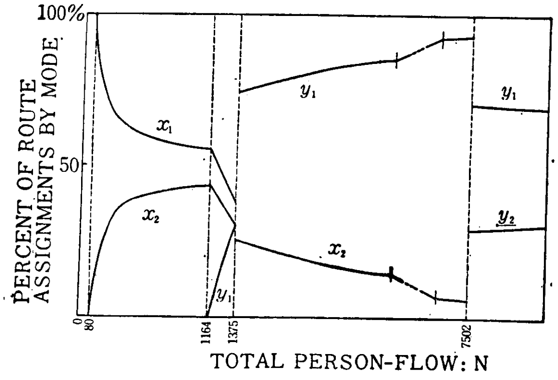

Fig. 2. Relationship between route assignments by mode and total person-flow

Example 2

Here we consider the problem of minimizing the previous objective function (27) subject to the constraint (29) and the non-negativity restriction

$$x_i \geqq 0, \quad y_i \geqq 0 \quad i=1,\ 2 \tag{34}$$

For this problem we have the following solutions:

for $0 \leqq N < 40$

The solutions are the same as those for $0 \leqq N < 80$ in the previous example.

for $40 \leqq N < 1147$

$x_1 = 0.4365\,N + 15.9$

$x_2 = 0.3968\,N - 15.9$

$y_1 = y_2 = 0$

$F = 0.009821\,N^2 + 15.7142\,N - 14.3$

for $1147 \leqq N < 1584$

$x_1 = -1.179\,N + 1867.8$

$x_2 = -0.3204\,N + 806.3$

$y_1 = 0.05599\,N - 64.2$

$y_2 = 0$

$F = -0.007928\,N^2 + 56.4091\,N - 23341.3$

for $1584 \leqq N < 7502$

The solutions are the same as those for $1375 \leqq N < 7502$ in the previous example.

for $7502 \leqq N$

The solutions are the same as those for $7502 \leqq N$ in the previous example.

Figure 3 shows the variation in percentage of the users by modes and by routes as N varies for the second example.

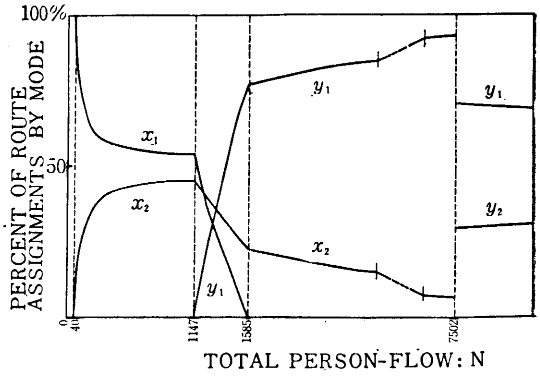

Fig. 3. Relationship between route assignments by mode and total person-flow

4. CONCLUSION

By the model proposed in this paper we can obtain not only the optimal traffic assignment but also the optimal modal split simultaneously for the multi-modal network flows. However, the major aim of the model may prefer to determine the optimal split of travel modes which share the same transportation network.

The typical application may be the optimization of the street network flows of passenger cars and buses in peak-hour, as shown in the example problems. By the model we can evaluate the optimal bus use of urban roads and, moreover, the planning of bus priority treatments such as bus lanes

5. REFERENCES

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