A Probabilistic Method for Traffic Assignment

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It is the purpose of this paper to present a theory of traffic assifnment for the mathematical descriptions of the traffic assignment patterns on an urban and highway network. To do this, we discuss the probability of occurrence of a traffic assignment pattern and develop a new method, named Probability Maximization Method, which can derive the most probable assignment pattern under the given road and traffic conditions.

For a more realistic traffic assignment this method is furthermore extended to the one with capacity restraint. In this paper two extended traffic assignment models are proposed. One is a flow-dependent assignment model using a travel time function and the other with capacity constraints on links of the network. Though both the models are formulated as a nonlinear programming problem respectively, it is shown that the former can be solved by an iterative procedure and the latter by the SUMT method, one of the more effective methods of nonlinear programming.

1. Introduction

It is well known that drivers' choices among various alternative routes through a road network as they travel from some origin to destination are very various and uncertain. It seems to be caused by that drivers' behaviors at the time of route choices are normally different because the characteristics of a route such as route travel time or cost are judged differently by the drivers and they also don't necessarily have perfect knowledge on these route characteristics enough to judge correctly:

Most of the existing methods of traffic assignment are, however, deterministic and are based on the assumption that all the drivers evaluate the network in exactly the same manner. In studying more realistic traffic assignment problems it is important to take into account the different behaviors of drivers and also the effects of traffic congestion. In this paper through probabilistic approaches to these problems a new probabilistic method for traffic assignment is proposed.

Main features of this method are that it is formulated as a nonlinear programming problem and has strict logic as a theory of traffic assignment.

2. Formulation of the Problem

We consider a road network on which r OD flows are distributed. Between each OD pair we designate no more than s routes in the network. Now consider a trip distribution pattern composed of overall X trips on the network under the condition that the total travel time becomes E. If we distinguish the indiuidual trips one by one, we can recognize there are many micro-states, or combinations of trips, which make up this trip distribution pattern. By elements of combinatorial analysis the total number of such micro-states, $W(E)$, is

$$W_X(E)=(E+X-1)!/(X-1)!\,E! \tag{1}$$

where E must be integer.

Let $W_X(E+\varDelta E)$ be the total number of micro-states when E chages infinitestimally $\varDelta E$, then it is

$$W_X(E+\varDelta E)=\frac{(E+\varDelta E+X-1)!}{(X-1)!(E+\varDelta E)!} \tag{2}$$

Taking the ratio of Eq. (2) to Eq. (1), we have

$$W_X(E+\Delta E)/W_X(E)=(E+\Delta E+X-1)!(X-1)!E!/(E+X-1)!(X-1)!(E+\Delta E)!$$
$$=(E+X)(1+X/(E+1))(1+X/(E+2))\cdots\cdots$$
$$(1+X/(E+\Delta E-1)/(E+\Delta E) \quad (3)$$

By assuming $X>>1$ and $E>>\Delta E$ we omit ΔE in Eq. (3), then

$$\frac{W_X(E+\Delta E)}{W_X(E)}=(1+X/E)^{\Delta E} \quad (4)$$

Taking the logarithm of both sides of the above equation, we obtain

$$\log W_X(E+\Delta E)-\log W_X(E)=\Delta E\log(1+X/E)$$

or

$$\Delta\log W_X(E)=\gamma\Delta E \quad (5)$$

where we put

$$\gamma=\log(1+X/E) \quad (6)$$

On the other hand, the number of micro-states, $z(E)$, which give rise to the traffic assignment pattern $\{X_i^k\}$ under the condition that the total travel time becomes E is given by

$$z(E)=(X!/\prod_{i,k}X_i^k!) \quad (7)$$

also from elements of combinatorial analysis. Where X_i^k $(i=1, 2, \cdots, r,\ k=1, 2, \cdots, s)$ is the number of trips using the k-th route between the i-th OD pair. As a matter of course Eq. (1) is related to Eq. (7) by the following equation,

$$W_X(E)=\sum_{X_i^k}(X!/\prod_{i,k}X_i^k!) \quad (8)$$

where the summation is over X_i^k satisfying the total travel time constraint. Then if all micro-states are equally probable, the traffic assignment pattern $\{X_i^k\}$ that maximizes $z(E)$ is undoubtedly the most probable.

It has been proved mathematically in statistical mechanics that if X takes the large number, $W_X(E)$ is proportional to the maximum of $z(E)$. Then if we designate the maximum of $z(E)$ by $Z(E)$, the following relation should hold

$$W_X(E+\Delta E)/W_X(E)=Z(E+\Delta E)/Z(E) \quad (9)$$

It follows

$$\Delta\log W_X(E)=\Delta\log Z(E)=\gamma\Delta E \quad (10)$$

Subtracting $\Delta\log z(E)$ from both sides of the latter half relation in Eq. (10), we have

$$\Delta(\log Z(E)-\log z(E))=\Delta(\gamma E-\log z(E)) \quad (11)$$

In the above equation if $z(E)$ has its maximum, $Z(E)$, both sides become zero. Thus, we now conclude that the most probable traffic assignment pattern is given by minimizing

$$\gamma E-\log z(E) \quad (12)$$

under the given traffic and road conditions. It is noted that Eq. (12) is analogous to the Helmholtz free-energy function which has a minimum for a thermodynamic system in equilibrium.

Let t_i^k be the travel time via the k-th route between the i-th OD pair, then the total travel time in the network can be expressed by

$$E=\sum_i\sum_k X_i^k t_i^k \quad (13)$$

On the other hand, using Stirling's formula ($\log x!\fallingdotseq x\log x-x$) $\log z(E)$ can be expressed by

$$\log z(E)=X\log X-\sum_i\sum_k X_i^k\log X_i^k \quad (14)$$

where X is constant. Then we can rewrite the objective function as

$$\gamma\sum_i\sum_k X_i^k t_i^k-(-\sum_i\sum_k X_i^k\log X_i^k) \quad (15)$$

The second term of the above function we shall call trip assignment entropy with reference to the form of entropy in statistical mechanics. Now it is concluded that the most probable traffic assignment pattern is obtained by minimizing the objective function (15) subject to

$$\sum_k X_i^k=X_i \quad (16)$$

where X_i means the volume of the i-th OD flow. Eq. (16) represents the conservation law on the i-th OD flow.

Solutions to the problem can be obtained by Lagrangian method in general. Particularly if the link travel time is flow-independent, we obtain the solutions easily as

$$X_i^k=\frac{\exp(-\gamma t_i^k)}{\sum_k\exp(-\gamma t_i^k)}X_i \quad (17)$$

It is of considerable interest to note that the objective function (12) brings out the relation of the existing method to two extreme cases. If γ is so large that the second term of the objective function is negligible we have an optimal assignment pattern that minimzes the total travel time. In addition, if the link travel time is constant, it leads to an all-or-nothing assignment pattern. On the contrary, if $\gamma=0$, the problem is to maximize the trip assignment entropy function which means that all trips are assigned uniformly to each route. A real traffic assignment pattern is considered to lie in the region between these two extreme cases. The value of γ used for a real network assignment problem should be determined empirically. For instance such γ as puts an observed and computed assignment pattern in close agreement may be chosen.

This new method, which we shall call Probability Maximization Method, assigns each OD flow to some alternative routes and the assignment rate to each route is derived theoretically from within the model. That is, in this method there is no necessity for giving an empirical diversion curve.

In a similar way probabilistic consideration as mentioned above also can be applied to the traffic distribution problems, which has been already treated in the author's previous works.[1] The use of the entropy maximizing methodology in the analyses of traffic distribution has been explored by several authors. Sasaki showed[2], assuming a priori probability of traffic distribution, that the most probable traffic distribution pattern was found by maximizing the logarithm of a joint probability corresponding to a traffic distribution pattern. Wilson[3] set up an entropy function defined as the logarithm of the number of micro-states corresponding to a traffic distribution pattern by analogy with statistical mechanics and in his discussion the most probable distribution pattern was obtained by maximizing the entropy function subject to the fixed total travel cost constraint. Major differences of the author's model from Sasaki's and Wilson's models were that it was formulated without using a priori probability and also without the fixed total travel cost constraint.

3. Flow-Dependent Assignment

In studying more realistic traffic assignment problems it is necessary to take into account the effects of traffic congestion. The primary feature of the flow-dependent algorithm is its provision for link travel time functions. These functions may be any linear or nonlinear relation of link flow to travel time for that flow.

Let us define a travel time function on link $h(h=1, 2, \cdots, l)$ as

$$T_h\Big(\sum_i\sum_k {}_h\delta_i{}^k X_i{}^k\Big) \tag{18}$$

where

$$_h\delta_i{}^k=\begin{cases}1 & \text{if link } h\in\text{route } k \text{ between the } i\text{-th OD pair}\\ 0 & \text{if link } h\notin\text{route } k \text{ between the } i\text{-th OD pair}\end{cases}$$

The total travel time E is then given by

$$\sum_i\sum_k\sum_h {}_h\delta_i{}^k X_i{}^k T_k\Big(\sum_i\sum_k {}_h\delta_i{}^k X_i{}^k\Big) \tag{19}$$

Thus the problem is to find a set of $X_i{}^k$ to minimize

$$\sum_i\sum_k\sum_h {}_h\delta_i{}^k X_i{}^k T_h\Big(\sum_i\sum_k {}_h\delta_i{}^k X_i{}^k\Big) + \sum_i\sum_k X_i{}^k \log X_i{}^k \tag{20}$$

subject to Eq. (16). And the result is

$$X_i{}^k=\frac{\exp\{-\gamma\sum_{hk}\,{}_h\delta_i{}^k(T_h+\sum_i\sum_{kh}\,{}_h\delta_i{}^k X_i{}^k T_h')\}}{\sum_k \exp\{-\gamma\sum_{hh}\,{}_h\delta_i{}^k(T_h+\sum_i\sum_{kh}\,{}_h\delta_i{}^k X_i{}^k T_h')\}}X_i \tag{21}$$

where T_h' is the first derived function with respect to $X_i{}^k$. It is easily proved that if all T_h are convex and monotone increasing functions, the objective function (20) has only one minimum value.

The introduction of flow-dependent travel time considerably complicates the analysis of the assignment problem, but if we put

$$\sum_i\sum_k {}_h\delta_i{}^k X_i{}^k=Q_h \tag{22}$$

and we define F_h that satisfy

$$F_h(Q_h)=T_h+Q_hT_h'=\frac{d}{dQ_h}\{Q_hT_h\} \tag{23}$$

then the solutions can be rewritten as

$$X_i{}^k=\frac{\exp(-\gamma\sum_{hh}\,{}_h\delta_i{}^k F_h)}{\sum_k \exp(-\gamma\sum_{hh}\,{}_h\delta_i{}^k F_h)}X_i \tag{24}$$

Afterall the following iterative procedure is used for obtaining a set of $X_i{}^k$.

Step 1. Compute the initial $F_k{}^{(o)}$ based on the link travel time at free flow conditions.

Step 2. Assume γ and obtain a set of $X_i{}^k$ from Eq. (24).

Step 3. Compute again the new $\overset{*}{F}_k{}^{(o)}$ based on the new link travel time obtained by using the above $\{X_i{}^k\}$.

Step 4. $F_h{}^{(n)}\ (n=1, 2, \cdots)$ for the second and succeeding iterations are updated according to

$$F_h^{(n)} = \frac{mF_h^{(n-1)} + \overset{*}{F}_h^{(n-1)}}{m+1} \qquad (25)$$

where m is a constant given properly in order to stabilize the iterations.

Step 5. The previous $F_h^{(n-1)}$ is replaced by the new $F_k^{(n)}$.
Return to Step 2 and repeat the iterations until $F_h^{(n)}$ is sufficiently close to $F_h^{(n-1)}$.

Step 6. Based on the converged link travel times OD volumes are assigned and link volumes are accumulated.

An example problem

To demonstrate the use of this model an example problem is solved. In **Fig. 1** the problem

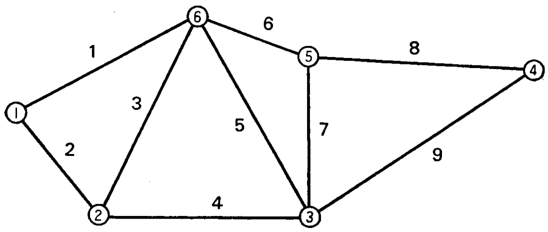

Fig. 1 Road network for the example problem

network is illustrated. OD flows are given in **Table 1**. Two routes are assumed between each OD pair on the network, which are expressed in the route matrix form.

Table 1. OD table

(v. p. h)

①	②	③	④	⑤	⑥	
※	※	1,700	400	700	1,100	①
	※	※	800	1,500	1,300	②
		※	1,200	※	1,400	③
			※	※	900	④
				※	※	⑤
					※	⑥

These routes are fixed during the assignment process.

The following link travel time function is used for the problem.

$$T_h = a_h O_h + b_h \qquad h = 1, 2, \cdots, 9 \qquad (26)$$

where
T_h=link travel time in minutes;
Q_h=link volume in vehicles per hour;
a_h=empirically derived constant;
b_h=constant representing travel time at free flow conditions

The first route

link

route	1	2	3	4	5	6	7	8	9
K_1^1	0	1	0	1	0	0	0	0	0
K_2^1	1	0	0	0	0	1	0	1	0
K_3^1	1	0	0	0	0	1	0	0	0
K_4^1	1	0	0	0	0	0	0	0	0
K_5^1	0	0	0	1	0	0	0	0	1
K_6^1	0	0	1	0	0	1	0	0	0
K_7^1	0	0	1	0	0	0	0	0	0
K_8^1	0	0	0	0	0	0	0	0	1
K_9^1	0	0	0	0	1	0	0	0	0
K_{10}^1	0	0	0	0	0	1	0	1	0

The second route

link

route	1	2	3	4	5	6	7	8	9
K_1^2	1	0	0	0	1	0	0	0	0
K_2^2	0	1	0	1	0	0	0	0	1
K_3^2	0	1	1	0	0	1	0	0	0
K_4^2	0	1	1	0	0	0	0	0	0
K_5^2	0	0	1	0	0	1	0	1	0
K_6^2	0	0	0	1	0	0	1	0	0
K_7^2	0	0	0	1	1	0	0	0	0
K_8^2	0	0	0	0	0	0	1	1	0
K_9^2	0	0	0	0	0	1	1	0	0
K_{10}^2	0	0	0	0	1	0	0	0	1

Table 2. Coefficients of the link travel time function

link	a_h	b_h
$h=1$	0.0030	6.0
2	0.0015	3.0
3	0.0025	5.0
4	0.00225	4.5
5	0.0020	4.0
6	0.0009	1.8
7	0.00135	2.7
8	0.0025	5.0
9	0.0030	6.0

Eq. (26) is a linear travel function, which allows link travel time to increase linearly with flow. The coefficients of Eq. (26) are listed in **Table 2.**

The value of γ should be given empirically and in this case was assumed $\gamma=0.5$.

The solutions to the problem are shown in **Fig. 2** and **Table 3.**

The value of m used for the iterative procedure was 3 and required 21 iterations until obtained the converged solutions. The choice of m is considered to have considerable influence on the number of iterations. With respect to the determination of m the following computer experiences were gained. That is, the more the number of iterations, the less the computation time per iteration, on the contrary the fewer the number of iterations, the more the computation time per iteration. This result indicates that the overall effort required to solve a problem will be relatively insensitive to the choice of m. But we should also note that if too small a value of m is used, specially for assignment with a nonlinear travel time function, this iterative procedure may not converge.

As mentioned in the previous section, if $\gamma \to \infty$, the assignment pattern from the model leads to the optimal assignment pattern which minimizes the total travel time. **Fig. 3** and **Table 4** show the solutions to the problem at $\gamma=10$. It is remarkable that the assignment pattern shown in Fig. 3 and Taple 4 is very close not only to the assignment pattern according to total travel time minimization principle but also to the assignment pattern according to principle of equal times. As shown in Table 4 principle of equal times is approximately satisfied on 3 OD pairs of ②—④, ②—⑤ and ③—④. It is known that if a monotone increasing travel time function is used, the assignment patterns according to these two principles, which were both proposed by Wardrop[4], will become similar each other when the total flow increases. Particularly, if a travel time function is expressed by

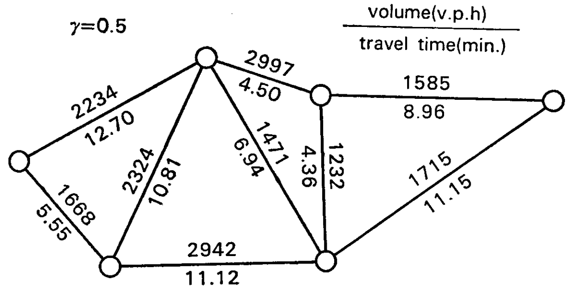

Fig. 2 Assignment resulting from the use of linear travel time function-link volume and travel time-

Table 3. Assignment resulting from the use of linear travel time function -route volume and travel time- $\gamma=0.5$

OD	first route		second route	
	assigned volume (v. p. h)	route travel time (min.)	assigned volume (v. p. h)	route travel time (min.)
① — ③	1,454	16.62	246	19.64
① — ④	311	26.16	89	27.77
① — ⑤	652	17.20	48	20.81
① — ⑥	1,024	12.70	76	16.31
② — ④	636	22.27	164	24.27
② — ⑤	741	15.31	759	15.48
② — ⑥	1,295	10.31	5	18.06
③ — ④	949	11.15	251	13.33
③ — ⑥	1,178	6.94	222	8.86
④ — ⑥	858	13.46	42	13.09

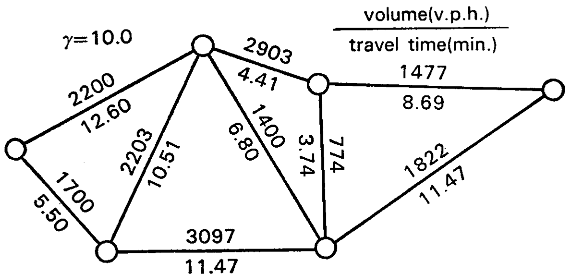

Fig. 3 Assignment for $\gamma=10$ -link volume and travel time-

Table 4. Assignment for $\gamma=10$
-route volume and travel time-

$\gamma=10.0$

OD	first route		second route	
	assigned volume (v. p. h)	route travel time (min.)	assigned volume (v. p. h)	route travel time (min.)
① — ③	1,700	17.02	0	19.40
① — ④	400	25.71	0	28.48
① — ⑤	700	17.01	0	20.47
① — ⑥	1,100	12.60	0	16.06
② — ④	721	22.93	79	23.62
② — ⑤	824	14.92	676	15.21
② — ⑥	1,300	10.51	0	18.27
③ — ④	1,101	11.47	99	12.44
③ — ⑥	1,400	6.80	0	8.16
④ — ⑥	900	13.11	0	18.27

$$T_h(Q_h)=aQ_h^b \qquad a\geq 0 \quad b\geq 0 \tag{27}$$

these two assignment patterns can be proved to become equivalent perfectly.

4. Assignment with Capacity Constraints

Another means of expression of capacity restraint is the use of the capacity constraint tn each link of the network. Capacity constraint can prevent link flow in excess of link capacity from being assigned to the link.

Probability Maximization Method with capacity constraints is formulated as follows:
minimize

$$\gamma\sum_i\sum_k X_i^k\, t_i^k+\sum_i\sum_k X_i^k \log X_i^k \tag{28}$$

subject to

$$\sum_k X_i^k=X_i \tag{29}$$

and

$$\sum_i\sum_k{}_h\, \delta_i^k\, X_i^k\leq C_h \tag{30}$$

where Eq. (30) means the capacity constraints and C_h is capacity on link h. Then this assignment problem must be solved by some constrained nonlinear programming method.

SUMT (sequential unconstrained minimization technique[5]), which is one of the more effective methods of constrained nonlinear programming, is one of penalty function methods which solve by transforming a constrained nonlinear programming problem into an unconstrained problem. For example, suppose we want to solve the above assignment problem expressed by Eq. (28) through Eq. (30), we form a new unconstrained objective function

$$f(X,\ r_M)=r\sum_i\sum_k X_i^k\, t_i^k+\sum_i\sum_k X_i^k \log X_i^k + (r_M)^{-\frac{1}{2}}\sum_i\Big(\sum_k X_i^k-X_i\Big)^2 + r_M\sum_h \lambda_h/\Big(C_h-\sum_i\sum_k{}_h\, \delta_i^k X_i^k\Big) \tag{31}$$

where r_M, a penalty factors, are positive and form a monotonically decreasing sequence of values $(r_0>r_1>r_2 \cdots r_M>r_{M+1}\cdots>0)$. The procedure of the SUMT method is based on the minimization of a new function $f(X,\ r_M)$ over a strictly monotone decreasing sequence of r-values $\{r_M\}$. Under certain conditions, it has been proved that the sequence of unconstrained minima of $f(X,\ r_M)$ will approach a solution of the original problem as r_M goes to zero. The essential requirement is the convexity of the f function. In this case Eq. (31) can be easily shown to be convex so that this asignment problem will be well treated by the SUMT method.

The computational procedure is as follows:

Step 1. As a starting point for the process, select a point X_0 such as
$C_h-\sum_i\sum_k{}_h\, \delta_i^k X_i^k\geq 0$ for all h. Where, equality constraints have no need for taking into account, then such an initial interior point will be easily found.

Step 2. Select r_0, the initial value of r_M, determine the minimum of $f(X,\ r_0)$. The technique used to minimize f will be describe later. The initial value r_0 must be given a numerical value in actual computations. Choice of r_0 will relate to the total number of computation required to obtain the solution. It seems difficult to determine the best r_0 strictly because it will relate to scaling of the problem. Fiacco and McCormick recommened[6] some theoretical methods for selecting the initial value r_0, which will be helpful for our croblem.

Step 3. Select r_1 such as $0<r_1<r_0$ and determine the minimum of $f(X, r_1)$.

Step 4. Continuing in this manner, a sequence of points $(X(r_M))$, are generated that respectively minimize $f(X, r_M)$. The sequence of f-minimization converges to the optimum of the primal objective function.

The assignment problem transformed in the form of Eq. (31) can be solved by some of unconstrained nonlinear programming methods, which are much easier to solve than constrained nonlinear programming methods. In this paper we will use Flecher-powell method to minimize $f(X, r_M)$, which is one of the gradient search methods using the conjugate directions.

An example problem

To illustrate the computational techniques described above and to investigate the effectiveness of the assignment model by the use of the SUMT method an example problem is solved. The road network and OD pattern used here

Table 5. Assigned volumes at each reduction in r_M

N		0	1	2	3	4	5	6
M			37	12	16	14	10	9
r_k			10^4	10^3	10^2	10	1	10^{-1}
OD ①—③	X_1^1	100	1012	1111	1213	1241	1247	1247
	X_1^2	100	668	583	485	459	452	453
①—④	X_2^1	100	157	185	198	204	203	203
	X_2^2	100	221	209	200	195	197	197
①—⑤	X_3^1	100	493	508	510	511	511	512
	X_3^2	100	187	186	188	188	188	188
①—⑥	X_4^1	100	796	800	803	804	804	804
	X_4^2	100	287	294	295	296	296	296
②—④	X_5^1	100	617	582	582	576	579	576
	X_5^2	100	162	212	216	223	221	221
②—⑤	X_6^1	100	537	648	692	702	701	701
	X_6^2	100	944	846	806	798	799	799
②—⑥	X_7^1	100	1188	1180	1156	1132	1129	1129
	X_7^2	100	95	115	142	168	171	171
③—④	X_8^1	100	830	837	839	840	840	840
	X_8^2	100	353	358	359	359	359	360
③—⑥	X_9^1	100	1190	1060	983	955	952	953
	X_9^2	100	194	335	416	445	448	447
④—⑥	X_{10}^1	100	460	599	655	682	683	682
	X_{10}^2	100	421	295	244	218	217	218

N : step number M : number of function evaluations per step are same as used in the previous section. In this paper, however, each link travel time is assumed a constant and its value is taken to be the same as b_h of Eq. (26). Let the capacities of links be all 3,000 v. p. h and let the value of γ be 0.5.

We start from the initial values of X_i^k all 100. Let the initial valhe of r_M be 10^4 and let the reduction of r_M be determined by the simple relation $r_M = r_{M-1}/c$, where $c>0$ is a constant and assumed 10 in this example.

After 6 calculations of f-minima we obtained the solutions to the problem. The value of $r_0 = 10^4$ was reduced to $r_6 = 10^{-1}$. The resulting assignment pattern is given in **Fig. 4**. Assigned volumes to the routes, X_i^k, at each reduction in γ_M are given in **Table 5**. It is noted that all

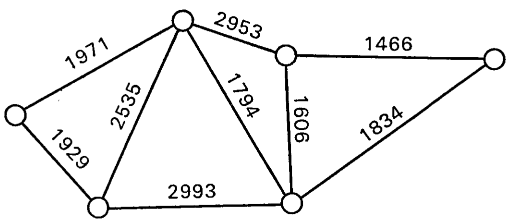

Fig. 4 Assignment resulting from the use of capacity constraints -link volume-

equality constraints are more and more closely satisfied as $r_M \to 0$. It is also noted that each link volume is restricted within that link capacity.

The choice of the initial r_0 and the factor c by which r_M is reduced have considerable influence on the effectiveness of the SUMT method. Consequently the question arises as to whether certain values of these parameters are to be prefered to others with respect to reducing the total number of calculations required to compute the solution. Then we tested the sensitivity of the method to various choices of r_0 and c.

The results are shown in **Table 6** and **7**. An obvious relation between the initial value of r_M and the amount of computational effort couldn't be found. However, it should be noted there is some danger that the resulting minimum will exceed some of the constraints if r_0 is decreased below a certain value, on the other hand, when r_0 is increased beyond a certain value, the resulting minimum is forced so far into the interior of the feasible region, then inordinate computation time is required to solve the problem.

Table 6. Relation between r_0 and the amount of computation

r_0	10^4	10^5	10^6
N	6	7	7
M	98	118	100
CPU time (sec)	48.74	54.33	49.24

Table 7. Relation between c and the amount of computation

c	2	5	10	15
N	11	6	6	5
M	104	80	88	81
M/N	9.5	13.3	14.7	16.2
CPU time (sec)	45.74	37.41	40.25	39.54

N : number of iterations of f-minima
M : total number of function evaluations

With respect to the choice of c, the computer experience shows that the overall effort required to solve a problem seems to be relatively insensitive to the choice of c.

For the assignment problem discussed above a linear programming also can be applied by the linear approximation of the objective function. But this procedure suffers from the handicap that the number of variables increases inordinately.

5. An Alternative Formulation

The assignment models considered in the previous sections is formulated in path-flow form which treats flows assigned to routes as variables. However the path-flow formulation has the disadvantage of requiring the determination of the first, second, etc, shortest route through the network before the assignment process, which will be cumbersome computationally for a large scale assignment problem. In this section we state Probability Maximization Method in link-flow form, which treats flows assigned to links as variables and show the technique to solve that problem.

In link-flow form let X_{ij}^k be the k-th OD flow on link (i, j). Let t_{ij} be the link travel time and C_{ij} the capacity. Let X^k be the k-th OD flow. Then the problem is to minimize

$$\gamma\sum_i\sum_j\sum_k t_{ij}X_{ij}^k+\sum_i\sum_j\sum_k X_{ij}^k \log X_{ij}^k \qquad (32)$$

subject to

$$\sum_j(X_{ij}^k-X_{ji}^k)=\begin{cases} X^k & \text{for } i=\text{origin node} \\ -X^k & \text{for } i=\text{destination node} \\ 0 & \text{otherwise} \end{cases} \qquad (33)$$

$$\sum_k X_{ij}^k \leq C \qquad (34)$$

and

$$X_{ij}^k \geq 0 \qquad (35)$$

where Eq. (33) means the node conservation law for each OD flow. In the above formulation the link travel time t_{ij} is allowed to be either flow-independent or flow-dependent. For this assignment problem the SUMT method will be also effective. The assignment model in link-flow form, however, has handicap that the numbef of variables increases inordinately for a large scale assignment problem.

6. Conclusion

In this paper for the purpose of the mathematical descriptions of the real traffic assignment pattern on a road network two assignment models with capacity restraint were proposed. These models derive the most probable assignment pattern under the given road and traffic conditions and their major features are that they are formulated as constrained nonlinear programming problems and have strict logic as a theory of traffic assignment. For further problems we must investigate the applicability of the assignment pattern by the models to the real pattern through many practical assignment problems.

With respect to capacity restraint the cases of using travel time functions and using capacity constraints were considered in this paper. Though both have advantages and disadvantages respectively, the assignment model with travel time functions seems superior to the model with capacity constraints in practicality and computational simplicity.

7. References

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