

Theory of Traffic Distribution through the Continuous-Time Absorbing Markov Process

Hiroshi MATSUI

Department of Civil Engineering

(Received September 8, 1969)

This work deals with the problem of traffic distribution on the streets of an urban traffic network. A stochastic model provides effective frameworks for analyzing the traffic flow patterns through the network. The conclusions derived from this stochastic model are compared with the data on actual traffic patterns on the midtown arterial streets in Kyoto and Nagoya.

1. Introduction

A convenient starting point for an analysis of the traffic behavior on the network is to notice that drivers face a series of decision points as they travel about the network. Viewing drivers' choices among various alternative routes at the intersections as a stochastic process has considerable descriptive appeal. Traffic distribution through Markov chain theory has been developed by Prof. Tsuna Sasaki,^{1,2)} however, the process used herein was discrete and only in a steady state.

In this work the problem is extended to the case in which the process may make a transition at random intervals, and make it possible to describe the traffic flow through the network both in a steady state and in a transient state. As the result the traffic flow on the streets can be described as a function of time. Another important aim of this work is to compare with data on the real traffic patterns on the actual street networks.

2. The Application of the Methods of Statistical Mechanics³⁾

First the model is presented in terms of analogues of physical laws of theory of molecular motions. Specially the distribution function used in this theory performs the important role.

In order to apply the distribution function to the traffic flow problem we consider a street network with s sources, or origins, from which $N_s(t)$ ($s=1,2,\dots,s$) trips begin

at time t , and r sinks, or destinations, in which $N_r(t)$ ($r=1,2,\dots,r$) trips end at time t . The network is divided up into a set of k equal street links (L in length) to predict link-to-link transition probabilities for traffic in the network.

We denote the number of cars on Link i at time t by $N_i(t)$ ($i=1,2,\dots,k$). $N_i(t)$ means the traffic density on Link i at time t because each link has equal length. We also denote the velocity distribution function on Link i at time t by $f_i(vt)$, then $N_i(t)$ is written as follows,

$$N_i(t) = \int_0^\infty f_i(vt) dv \dots \dots \dots (1)$$

Where, we take the integral range from 0 to ∞ for computational purposes.

To facilitate the formulation of the problem we consider a typical model of the street link between two adjacent intersections as shown in Fig. 1. In Fig. 1, vehicles

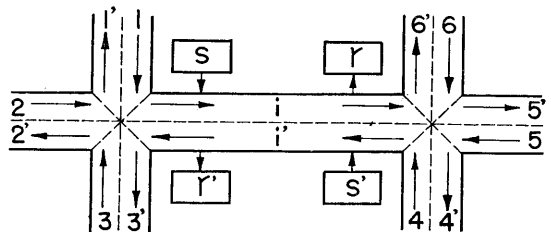


Fig. 1 Typical Model of the Street Link Between Two Adjacent Intersections

on Link i move out to the next Link 4, 5, and 6, or Sink r connected with Link i , with transition probabilities, P_{i4} (right-turn rate), P_{i5} (straight-going rate) and P_{i6} (left-turn rate), or an absorbing probability P_{ir} ,

which satisfy the following relation as

$$P_{i4} + P_{i5} + P_{i6} + P_{ir} = 1 \quad \dots\dots\dots (2)$$

on the other hand vehicles on the preceding Link 1, 2 and 3, or vehicles from

Source s connected with Link i enter Link i . In the results the movements of vehicles in and out of Link i in the infinitesimal range $(t, t + \Delta t)$ can be expressed as

$$\int f_i(v, t + \Delta t) dv - \int f_i(v, t) dv = \sum_{k=1,2,3} P_{ki} \int \frac{v \Delta t}{L} f_k(v, t) dv - \int \frac{v \Delta t}{L} f_i(v, t) dv + \int \frac{v \Delta t}{L} f_s(v, t) dv \dots\dots\dots (3)$$

Dividing both sides of this equation by Δt and taking the limit as $\Delta t \rightarrow 0$, then

$$\frac{\partial}{\partial t} \int f_i(v, t) dv = \sum_{k=1,2,3} \frac{P_{ki}}{L} \int v f_k(v, t) dv - \frac{1}{L} \int v f_i(v, t) dv + \frac{1}{L} \int v f_s(v, t) dv \dots\dots (4)$$

The left-hand side of the above equation is equal to $\frac{dN_i(t)}{dt}$ from Eq. (1). And

$\int v f_k(v, t) dv$ ($k=1, 2, 3$), $\int v f_i(v, t) dv$ or $\int v f_s(v, t) dv$ on the right-hand side of Eq. (4)

represents the traffic volume of Link k ($k=1, 2, 3$), Link i , or Source s respectively. Then denoting these volumes by $Q_k(t)$ ($k=1, 2, 3$), $Q_i(t)$ and $Q_s(t)$, the equation is written as

$$\frac{dN_i(t)}{dt} = \sum_{k=1,2,3} \frac{P_{ki}}{L} Q_k(t) - \frac{1}{L} Q_i(t) + \frac{1}{L} Q_s(t) \dots\dots\dots (5)$$

If each $Q_i(t)$ be explained in the form of the products of $N_i(t)$ by $V_i(t)$, the mean speed on Link k , Eq. (5) is also rewritten as

$$\frac{dN_i(t)}{dt} = \sum_{k=1,2,3} \frac{P_{ki}}{L} V_k(t) N_k(t) - \frac{1}{L} V_i(t) N_i(t) + \frac{1}{L} V_s(t) N_s(t) \dots\dots\dots (6)$$

Herein, it is noticed that $N_s(t)$ and $V_s(t)$ on Source s are imaginary measures.

As we arrange Eq. (6) by putting $\frac{P_{ki}}{L} V_k(t) = A_{ki}(t)$, $-\frac{1}{L} V_i(t) = A_{ii}(t)$ and $\frac{1}{L} V_s(t) = A_{si}(t)$, we have the following fundamental equations on traffic distribution as

$$\frac{dN_i(t)}{dt} = \sum_{k=1,2,3,i,s} A_{ki}(t) N_k(t) \quad (i=1, 2, \dots, k) \dots\dots\dots (7)$$

In the Case that Traffic Density is Low

Each vehicle runs in free speed with no restriction due to the presence of other vehicles. Therefore, we may presume that the mean speed on each link be constant, having no change with time t , then $A_{ki}(t) = A_{ki}$ ($k=1, 2, 3, i, s$), and Eqs. (7) are rewritten as follows,

$$\frac{dN_i(t)}{dt} = \sum_{k=1,2,3,i,s} A_{ki} N_k(t) \dots\dots\dots (8)$$

or, using the generating traffic volume $Q_s(t)$ on Source s ,

$$\frac{dN_i(t)}{dt} = \sum_{k=1,2,3,i} A_{ki} N_k(t) + \frac{1}{L} Q_s(t) \dots\dots\dots (8)'$$

Eq. (8) or (8)' is a set of k constant-coefficient differential equations of first order on $N_i(t)$ ($i=1, 2, \dots, k$).

In matrix form we may write Eq. (8)' as

$$\frac{dN(t)}{dt} = AN(t) + Q^*(t) \dots\dots\dots (9)$$

where,

$N(t)$ is a k -component column vector with entries $N_i(t)$ ($i=1, 2, \dots, k$).

A is a k -by- k matrix whose entry is A_{ii} ($i, j=1, 2, \dots, k$).

$Q^*(t)$ is a k -component column vector with entries $\frac{1}{L} Q_s(t)$ ($s=1, 2, \dots, s$).

If we take the Laplace transform of Eq. (9), we have

$$sn(s) - n(0) = An(s) + Q^*(s)$$

then

$$n(s) = n(0)(sI - A)^{-1} + Q^*(s)(sI - A)^{-1} \dots\dots\dots(10)$$

Where, I is a k -by- k identity matrix, and $n(0)$ is the initial condition vector. Hence the traffic density vector $N(t)$ may be found by inverse transformation of Eq. (10).

In a steady state we have

$$\lim_{t \rightarrow \infty} N(t) = \lim_{s \rightarrow 0} sn(s) = \lim_{s \rightarrow 0} n(0)(sI - A)^{-1} + \lim_{s \rightarrow 0} Q^*(s)(sI - A)^{-1} = (-A)^{-1} \lim_{s \rightarrow 0} sQ^*(s) \dots\dots\dots(11)$$

In this case it is clear that the traffic density on each link is independent of the initial conditions.

We have the traffic volume $Q_i(t)$ on each link for the solutions, by multiplying the traffic density $N_i(t)$ by the mean speed V_i .
In the Case that Traffic Density is High

The speed of each vehicle will be reduced

and congestion will occur. Under the conditions of this, we can't assume any longer a constant travel speed function for each link. In this case, for example, if we assume

$$V_i(t) = \alpha_i - \frac{\beta_i}{B_i} N_i(t) \dots\dots\dots(12)$$

the form of Eq. (6) is then

$$\frac{dN_i(t)}{dt} = \sum_{k=1,2,3} \frac{P_{ki}}{L} N_k(t) \left\{ \alpha_k - \frac{\beta_k}{B_k} N_k(t) \right\} - \frac{1}{L} N_i(t) \left\{ \alpha_i - \frac{\beta_i}{B_i} N_i(t) \right\} + \frac{1}{L} Q_i(t) \dots\dots\dots(13)$$

where,

α_i is a constant representing travel speed at free flow conditions on Link i .

β_i is an empirically derived constant on Link i .

B_i is the number of lanes of Link i .

Eq. (13) is a set of k nonlinear differential equations of second order on $N_i(t)$ ($i=1, 2, \dots, k$). From Eq. (13) we have the traffic volume on each street link in flow of all conditions theoretically.

3. The Application of the Methods of the Continuous-Time Absorbing Markov Process⁴⁾

The traffic distribution model presented in the previous chapter can be also led by using a continuous-time absorbing Markov process technique.

As preparation for this chapter, first let's review the basic concepts of the continuous-time Markov process. We now denote by $P_{ij}(t + \Delta t/t)$ the transition probability with which a process that is now in state at time t will make a transition to state j at time $t + \Delta t$. For the continuous-time process we may prefer the following transition rate

$A_{ij}(t)$ of a process to the transition probability.

Let's define a transition rate $A_{ij}(t)$ by

$$A_{ij}(t) = \lim_{\Delta t \rightarrow 0} \frac{P_{ij}(t + \Delta t/t) - P_{ij}(t/t)}{\Delta t} \dots\dots\dots(14)$$

Because of $\sum P_{ij}(t/t') = 1$,

$$A(t) \geq 0 \quad (i \neq j) \quad \sum_j A_{ij}(t) = 0 \dots\dots(15)$$

From Eq. (15) each row of the transition rate matrix $A = (A_{ij})$ amounts to zero. A matrix whose row amounts to zero is called a differential matrix.

Next we consider a kind of Markov processes quite different from ergodic processes. A state in a Markov process is called an absorbing state if it is impossible to leave it. A Markov process is absorbing if it has at least one absorbing state, and from the other states it is possible to go to the absorbing state sooner or later.

In this chapter we also denote the traffic density on Link i at time t by $N_i(t)$, then we have the variation of the traffic density on each link in the short time interval $(t, t + \Delta t)$, using the transition probabilities, in the form

$$N_i(t + \Delta t) - N_i(t) = \sum_{k=1,2,3} P_{ki}(t + \Delta t/t) N_k(t) - \sum_{k=4,5,6,r} P_{ik}(t + \Delta t/t) N_i(t) + P_{si}(t + \Delta t/t) N_s(t) \dots\dots\dots(16)$$

Where, the first term of the left-hand side shows the increased vehicles on Link i from the preceding Link 1, 2, and 3 in a short time interval Δt . $P_{ij}(t+\Delta t/t)$ is a transition probability with which a vehicle on Link k ($k=1, 2, 3$) at time t will make a transition to Link i at time $t+\Delta t$. Similarly the second term shows the decreased vehicles

leaving Link i for the next Link 4, 5 and 6, or Sink r in the same time interval. The third term shows the increased vehicles from Source s . $N_s(t)$ is also the imaginary measure.

From the relation of $P_{ij}(t/t)=0$ ($i \neq j$), we have

$$\begin{aligned} N_i(t+\Delta t) - N_i(t) = & \sum_{k=1,2,3} \{P_{ki}(t+\Delta t/t) - P_{ki}(t/t)\} N_k(t) \\ & - \sum_{k=4,5,6,r} \{P_{ik}(t+\Delta t/t) - P_{ik}(t/t)\} N_i(t) \\ & + \{P_{si}(t+\Delta t/t) - P_{si}(t/t)\} N_s(t) \dots \dots \dots (17) \end{aligned}$$

As we divide both sides of this equation by Δt and take the limit as $\Delta t \rightarrow 0$, we obtain

$$\frac{dN_i(t)}{dt} = \sum_{k=1,2,3} A_{ki}(t) N_k(t) - \sum_{k=4,5,6,r} A_{ik}(t) N_i(t) + A_{si}(t) N_s(t) \dots \dots \dots (18)$$

Where, $A_{ki}(t)$, $A_{ik}(t)$ and $A_{si}(t)$ are the transition rates as defined by Eq. (14). From Eq. (15) the equation may be rewritten as

$$\frac{dN_i(t)}{dt} = \sum_{k=1,2,3,i,s} A_{ki}(t) N_k(t) \dots \dots \dots (19)$$

Eq. (19) and Eq. (7) in the preceding chapter are entirely identical, and we find that the transition rate $A_{ij}(t)$ can be represented by $\frac{P_{ij}}{L} V_i(t)$.

If a Markov process is stationary, and a transition rate is independent of time t , namely if $A_{ij}(t) = A_{ij}$, then we have

$$\frac{dN_i(t)}{dt} = \sum_{k=1,2,3,i,s} A_{ki}(t) N_k(t) \dots \dots \dots (20)$$

The above equation is identical with Eq.(8).

From the properties of the continuous-time absorbing Markov process we can know the probability that the process will end up in a given absorbing state and how long it will take for the process to be absorbed on the average, which will make it possible to discuss on the transition probability from origin to destination and travel time of each trip.

Consider n absorbing states, or sinks, n generating states, or sources and $l-n$ transient states, or street links. Where, generating states and transient states we will call

non-absorbing states. We now define the following canonical differential matrix whose entries are transition rates A_{ij} between $l+n$ states.

$$A = \begin{array}{cc|cc} \mathbf{O} & \mathbf{O} & & \\ \hline & & \mathbf{R} & \mathbf{Q} \\ \hline \end{array} \begin{array}{l} n \text{ absorbing states} \\ n \text{ generating states} \\ l-n \text{ transient states} \end{array}$$

$\underbrace{\quad}_n \quad \underbrace{\quad}_n \quad \underbrace{\quad}_{l-n}$
ab. sts. ge. sts. tra. sts.

Where, \mathbf{R} is a l -by- n matrix, \mathbf{Q} is a l -by- l matrix and \mathbf{O} is a matrix whose entries are all zero.

From the properties of the continuous-time absorbing Markov process, it is recognized that, ij -entry of the inverse matrix $(-\mathbf{Q})^{-1}$ gives the mean time spent in a certain non-absorbing state j before absorption on a certain starting state i , and ij -entry of \mathbf{R} gives the transition rate from a certain non-absorbing state to a certain absorbing state j . Then by computing the product $(-\mathbf{Q})^{-1}\mathbf{R}$, we can obtain the probability that an absorbing process will be absorbed in state j on depending on the starting state i . And let $U_i (i=1, 2, \dots, n)$ be the traffic volume generated on generating state s , then the traffic volume $V_i (i=1, 2, \dots, n)$ absorbed in absorbing state r in a steady state can be given by

$$\underbrace{(V_1, V_2, \dots, V_n)}_n = \underbrace{(U_1, U_2, \dots, U_n)}_n \underbrace{(0, 0, \dots, 0)}_{l-n} (-\mathbf{Q})^{-1}\mathbf{R} \dots \dots \dots (21)$$

4. Relation to OD Traffic Flow

The problem of traffic distribution on the network requires the following basic character-

ristics in general. Namely, in a steady state,

(1) Each generating traffic volume U_i will coincide with each absorbing one V_i , respectively.

(2) The OD traffic volume calculated by the product of matrices will coincide with the actual ones.

These two characteristics through an absorbing Markov chain have been already discussed by Prof. Tsuna Sasaki²⁾. The extension to the case of a continuous-time absorbing Markov process can be made easily.

Now we divide the submatrix Q in the preceding chapter into four parts as follows,

$$Q = \left\{ \begin{array}{cc} Q_1 & Q_2 \\ O & Q_3 \end{array} \right\} \begin{array}{l} n \\ l-n \end{array} \quad \dots\dots\dots(22)$$

Similarly R is defined as

$$R = \left\{ \begin{array}{c} O \\ R_1 \end{array} \right\} \begin{array}{l} n \\ l-n \end{array} \quad \dots\dots\dots(23)$$

First in order to equalize the absorbing volume to the generating volume in a steady state as,

$$(U_1, U_2, \dots, U_n) = (V_1, V_2, \dots, V_n) \dots\dots(24)$$

let's compute $(-Q)^{-1}R$ by the product of the matrices, (22) by (23), then we obtain

$$\left\{ \begin{array}{c} -Q_1^{-1}Q_2Q_3^{-1}R_1 \\ Q_3^{-1}R_1 \end{array} \right\} \begin{array}{l} n \\ l-n \end{array} \quad \dots\dots\dots(25)$$

Here, putting $-Q_1^{-1}Q_2Q_3^{-1}R_1 = P_0$ and $Q_3^{-1}R_1 = P_0'$, it should be noted that P_0 implies the transition matrix from origin to destination.

In vector form Eq. (21) can be written as

$$(U^*O)(-Q)^{-1}R = (U^*O) \begin{pmatrix} P_0 \\ P_0' \end{pmatrix} = (U^*P_0) \quad \dots\dots\dots(26)$$

Where, $U^* = (U_1, U_2, \dots, U_n)$. Hence Eq.(24) is expressed as

$$U^* = U^*P_0 \quad \dots\dots\dots(27)$$

From the relation of Eq. (27), we can not specify U^* and P_0 independently. Then we

will give the sum of the generating traffic volume U ,

$$U = U_1 + U_2 + \dots + U_n \quad \dots\dots\dots(28)$$

Therefore we can have the solutions from Eq. (27) and Eq. (28) simultaneously, which satisfy two properties mentioned above.

To facilitate the formulation of the problem consider a typical network with one source and one sink, as shown in Fig. 2.

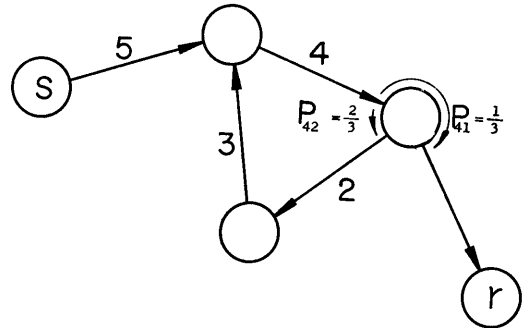


Fig. 2 Street Network for the Example Problem

Further, consider the constant traffic volume Q per unit of time generated on the source, moving forward along a link with a transition rates between each link as follows,

$$\left. \begin{array}{l} A_{s5} = a \quad A_{54} = a \quad A_{41} = \frac{1}{3}a \quad A_{42} = \frac{2}{3}a \\ A_{23} = a \quad A_{34} = a \quad A_{1r} = a \end{array} \right\} \quad \dots\dots\dots(29)$$

Then the fundamental equations associated with links are as follows,

$$\left. \begin{array}{l} \frac{dQ_1(t)}{dt} = \frac{1}{3}aQ_4(t) - aQ_1(t) \\ \frac{dQ_2(t)}{dt} = \frac{2}{3}aQ_4(t) - aQ_2(t) \\ \frac{dQ_3(t)}{dt} = aQ_2(t) - aQ_3(t) \\ \frac{dQ_4(t)}{dt} = aQ_5(t) + aQ_3(t) - aQ_4(t) \\ \frac{dQ_5(t)}{dt} = aQ - aQ_5(t) \end{array} \right\} \quad \dots\dots\dots(30)$$

Where, we put $VN_i(t) = Q_i(t)$ ($i=1, 2, \dots, 5$), supposing the mean speed on each link be equally constant. Giving the initial conditions $Q_i(0) = 0$ ($i=1, 2, \dots, 5$), we have the solutions as

$$\left. \begin{aligned}
 Q_1(t) &= \left\{ 1 - \frac{1-r}{6r} e^{-art} + \frac{1-7r}{6r} e^{\frac{a(r-3)}{2}t} \cos \frac{\sqrt{3}}{2} a(1-r)t \right. \\
 &\quad \left. + \frac{\sqrt{3}}{2} r(1-r) e^{\frac{a(r-3)}{2}t} \sin \frac{\sqrt{3}}{2} a(1-r)t \right\} Q \\
 Q_2(t) &= \left\{ 2 - \frac{1-r}{3r} e^{-art} + \sqrt{3} r(1-r) e^{\frac{a(r-3)}{2}t} \sin \frac{\sqrt{3}}{2} a(1-r)t \right. \\
 &\quad \left. + \frac{1-3r}{3r} e^{\frac{a(r-3)}{2}t} \cos \frac{\sqrt{3}}{2} a(1-r)t \right\} Q \\
 Q_3(t) &= \left\{ 2 + e^{-at} - \frac{1}{3r} e^{art} + \sqrt{3} r(1-r) e^{\frac{a(r-3)}{2}t} \sin \frac{\sqrt{3}}{2} a(1-r)t \right. \\
 &\quad \left. + r(r-3) e^{\frac{a(r-3)}{2}t} \cos \frac{\sqrt{3}}{2} a(1-r)t \right\} Q \\
 Q_4(t) &= \left\{ 3 - \frac{(1-r)^2}{2r} e^{-art} - \frac{\sqrt{3}}{2} (1-r)(1-3r) e^{\frac{a(r-3)}{2}t} \sin \frac{\sqrt{3}}{2} a(1-r)t \right. \\
 &\quad \left. - r t + \frac{3}{2} r^2 (1-r)^2 e^{\frac{a(r-3)}{2}t} \cos \frac{\sqrt{3}}{2} a(1-r)t \right\} Q \\
 Q_5(t) &= (1 - e^{-at}) Q
 \end{aligned} \right\} \dots\dots\dots (31)$$

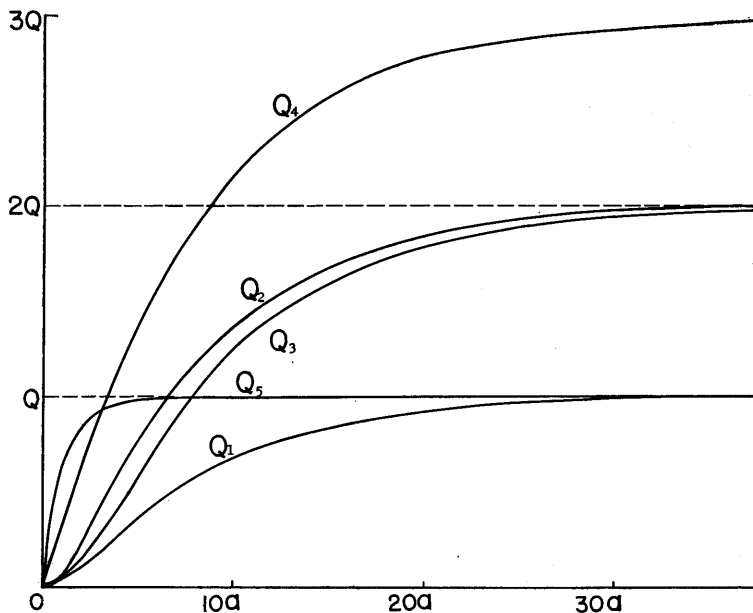


Fig. 3 Traffic Volume Variation by Time

Where, we put $r = a \left(1 - \sqrt{\frac{2}{3}} \right)$. Fig. 3 illustrates the curves of these solutions.

As we take the limit as $t \rightarrow \infty$ in Eqs. (31), we have the solutions in a steady state as

$$\left. \begin{aligned}
 \lim_{t \rightarrow \infty} Q_1(t) &= Q & \lim_{t \rightarrow \infty} Q_2(t) &= 2Q & \lim_{t \rightarrow \infty} Q_3(t) &= 2Q \\
 \lim_{t \rightarrow \infty} Q_4(t) &= 3Q & \lim_{t \rightarrow \infty} Q_5(t) &= Q
 \end{aligned} \right\} \dots\dots\dots (32)$$

Next, take the canonical differential matrix for this example as

$$A = \begin{matrix} & \begin{matrix} r & s & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} r \\ s \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & 0 & a \\ a & 0 & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & a & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & a & 0 \\ 0 & 0 & \frac{1}{3}a & \frac{2}{3}a & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 0 & a & -a \end{pmatrix} \end{matrix} \quad \dots\dots\dots (33)$$

Then the matrix Q can be expressed as

$$Q = \begin{pmatrix} -a & 0 & 0 & 0 & 0 & a \\ 0 & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & -a & a & 0 & 0 \\ 0 & 0 & 0 & -a & a & 0 \\ 0 & \frac{1}{3}a & \frac{2}{3}a & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & a & -a \end{pmatrix} \quad \dots\dots\dots (34)$$

Computing the inverse of Q , we have

$$(-Q)^{-1} = \begin{matrix} & \begin{matrix} r & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} r \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} \frac{1}{a} & \frac{1}{a} & \frac{2}{a} & \frac{2}{a} & \frac{3}{a} & \frac{1}{a} \\ 0 & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a} & \frac{3}{a} & \frac{3}{a} & \frac{3}{a} & 0 \\ 0 & \frac{1}{a} & \frac{2}{a} & \frac{3}{a} & \frac{3}{a} & 0 \\ 0 & \frac{1}{a} & \frac{2}{a} & \frac{2}{a} & \frac{3}{a} & 0 \\ 0 & \frac{1}{a} & \frac{2}{a} & \frac{2}{a} & \frac{3}{a} & \frac{1}{a} \end{pmatrix} \end{matrix} \quad \dots\dots\dots (35)$$

In case of this example, we may remark only the first row, concerned with Source s , of the above matrix. Each entry of the first row shows that vehicles generated on Source s are to spend travel time $1/a$ on Link 1 and 5, $2/a$ on Link 2 and 3, and $3/a$ on Link 4 on the average before absorption.

Furthermore, dividing the ij -entry of $(-Q)^{-1}$ by the travel time for passing along Link j (in this case this value is $\frac{L}{V_j} = \frac{1}{a}$ in common for $j=1, 2, \dots, 5$), we can obtain the components which give the mean number of times to be in each non-absorbing state j as

$$\begin{matrix} & \begin{matrix} r & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} r \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 & 3 & 0 \\ 0 & 1 & 2 & 3 & 3 & 0 \\ 0 & 1 & 2 & 2 & 3 & 0 \\ 0 & 1 & 2 & 2 & 3 & 1 \end{pmatrix} \end{matrix} \quad \dots\dots\dots (36)$$

Multiplying this matrix by the generating traffic volume vector, we have the traffic volume on each link in a steady state as

$$(Q, 0, 0, 0, 0, 0) \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 & 3 & 0 \\ 0 & 1 & 2 & 3 & 3 & 0 \\ 0 & 1 & 2 & 2 & 3 & 0 \\ 0 & 1 & 2 & 2 & 3 & 1 \end{pmatrix} = (Q, Q, 2Q, 2Q, 3Q, Q) \dots\dots\dots(37)$$

It should be noted that these results coincide with the previous ones obtained in Eq. (32).

5. The Theoretical Analysis of the Traffic Distribution on the Actual Midtown Street Network

To demonstrate how this model will be

applied to simulate traffic flow behavior on the urban street network, we apply this model to the analysis of the traffic distribution on the midtown arterial street networks in Kyoto and Nagoya.

We will consider the simplified arterial street networks shown as in Fig. 4 for Kyoto, and in Fig. 5 for Nagoya. The net-

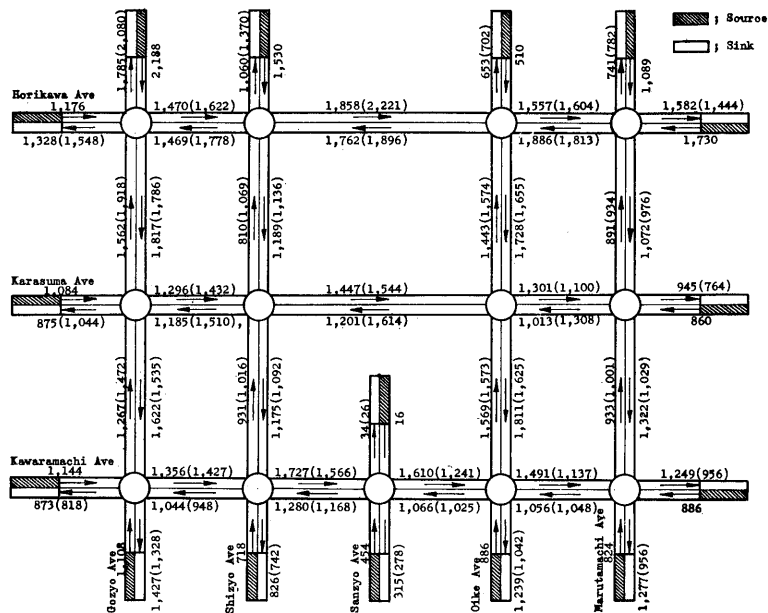


Fig. 4 Street Network and Traffic Distribution in Kyoto
(Parenthesized Numbers are Data Surveyed in 1965)

work for Kyoto consists of 16 sources, 16 sinks and 36 street links, the one for Nagoya consists of 14 sources, 14 sinks and 34 street links. Right-turn rate, straight-going rate and left-turn rate at each intersection, and the traffic volume on each street are given by survey data for both networks. It is assumed that there will be no generation and extinction of vehicles on each link

joining a pair of signalized intersections for the simplification of the problem.

The problem is to determine the traffic volumes on the streets in a steady state. The results computed by using the stochastic model through the Markov process are then shown in Fig. 4 and 5 and compared with survey data.

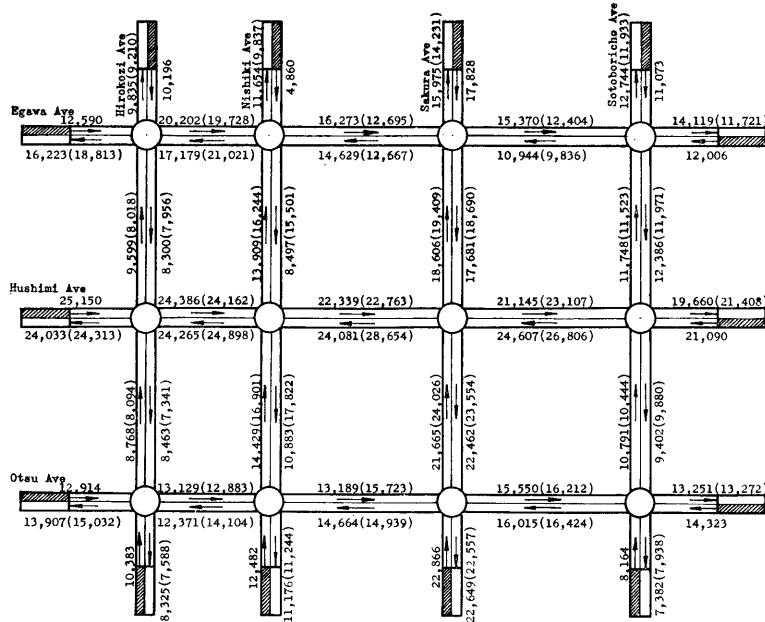


Fig. 5 Street Network and Traffic Distribution in Nagoya
(Parenthesized Numbers are Data Surveyed in 1964)

6. Conclusion

The traffic distribution model led by using a continuous-time absorbing Markov process technique is available for the accurate simulation of traffic on the street network not only in the present but also in the near future.

One of the most important applications is to measure of road network efficiency under various operational controls and to determine the road network improvements.

7. Acknowledgement

The author would like to express his

sincere appreciation to Professor T. Sasaki for his helpful suggestions.

References

- (1) Tsuna Sasaki: "Theory of Traffic Flow", Series of Traffic Engineering No. 3 (1965)
- (2) Tsuna Sasaki: "Theory of Traffic Assignment Through Absorbing Markov Process", Transactions of the Japan Society of Civil Engineers No. 121 (1965)
- (3) D. Ter. Haar: "Elements of Statistical Mechanics", Translated by Tomoyasu Tanaka and Kazuyoshi Ikeda (1964)
- (4) J. G. Kemeny, H. Mirkil, J. L. Snell and G. L. Thompson: "Finite Mathematical Structures" (1961)