ON A REPLACEMENT PROBLEM IN NETWORKS

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Summary. It is assumed that a given network has N nodes and $N(N-1)$ arcs which have the Probabilities to pass through another node. We have two choices in each arc whether we change the arc before and replace new arc with high probability, or not. Changing costs before and after are given. A method of obtaining the optimal path with optimal policy of replacement which makes the minimum expected cost from any node i to sink N in a directed network is desired. The problem is formulated by using the functional equations of dynamic progamming and which gives some informations in optimal policy. The uniqueness of functional equations and the convergence of successive approximations are shown. In the last a similiar problem with time-lag is considered.

1. **Introduction.** Bellman [2] has given a method of finding the shortest path for a network using the functional equations of dynamic programming. In this note we propose the method of obtaining the optimal path with optimal policy of replacement which makes the expected cost minimum. It is assumed that a given directed network has N nodes and $N(N-1)$ arcs which have the probabilities $\{P_{ij}\}$ not to pass through, moreover we are allowed in each arc (i, j) to replace it new arc with lower probability P_{ij}^* at beforehand with cost B_{ij}. In the second section the problem is formulated in the functional equations of dynamic programming, an analysis of replacement policy is given in the third section, and uniqueness and apporoximations in policy space are discussed in the following two sections. In the sixth section numerical examples are given. Contrary to the above problem of optimal type, a problem of adative type is formulated in the last section.

2. **Formulation.** In a directed network a set of N nodes is given and numbered in any fashion. In every ordered pair (i, j), $i, j=1, 2, \cdots\cdots N$, we call it arc (i, j) and which has the probability $\{P_{ij}\}$ not to pass through. We have two choices in each arc whether we change the arc with $\{P_{ij}\}$ before and replace the new arc with $\{P_{ij}^*\}$. Changing costs of arc (i, j) before and after are B_{ij} and A_{ij} respectively. The traverse cost is denoted dy C_{ij}. It is natural that we assume

$$A_{ij} < B_{ij} \tag{2.1}$$

$$P_{ij}^* < P_{ij} \tag{2.2}$$

Let

U_i = minimum expected cost traversing optimal path with optimal policy of replacement from node i to node N.

Employing the principle of optimality [1], we get

$$U_i = Min \begin{bmatrix} \underset{j \neq i}{Min} \left(P_{ij} \ (A_{ij}+U_j) + q_{ij} \ (C_{ij}+U_j)\right) \\ \underset{j \neq i}{Min} \left(B_{ij} + P_{ij}^* \ (A_{ij}+U_j) + q_{ij}^* \ (C_{ij}+U_j)\right) \end{bmatrix}, \quad (i = 1, 2, \cdots\cdots N-1) \tag{2.3}$$

$$U_N = 0. \tag{2.4}$$

where

$$q_{ij} = 1 - p_{ij} , \quad \text{and} \quad p_{ij}^* = 1 - q_{ij}^* \tag{2.5}$$

In the right hand of the upper equation means after-replacement and the lower before-replacement. If the arc (i, j) does not exsist, we consider the right hand of eq. (2.3) infinite. If it is no loop to return the node itself, we number the arc (i, j) as $i<j$ and can obtain the value of eq. (2.3) directly. But generally we cannot obtain it directly.

3. **Analysis in policy of replacement.** The equation (2.3) is reformulated in the following ;

$$U_i = Min \begin{bmatrix} \underset{j \neq i}{Min} \left[P_{ij}A_{ij} + q_{ij} \ C_{ij} + U_j\right] \\ \underset{j \neq i}{Min} \left[B_{ij} + P_{ij}^* \ A_{ij} + q_{ij}^* \ C_{ij} + U_j\right] \end{bmatrix} \tag{3.1}$$

As easily seen, the inequalities

$$P_{ij}A_{ij}+q_{ij}C_{ij}+U_j \gtreqless B_{ij}+P_{ij}^*A_{ij}+q_{ij}^*C_{ij}+U_j$$

implies

$$P_{ij}-P_{ij}^* \gtreqless \frac{B_{ij}}{A_{ij}-C_{ij}} \tag{3.2}$$

This means that if we pass from node i to node j, it is better to replace the arc (i,j) beforehand by new (afterwards), when the first (second) inequality occurs. Let $J^{(i)}$ be the set of indices, $1, 2, \cdots\cdots, i-1, i+1, \cdots\cdots N$, then decompose $J^{(i)}$ into two disjoint parts $J_1{}^{(i)}$ and $J_2{}^{(i)}$;

$$j^{(i)} = J_1{}^{(i)} \cap J_2{}^{(i)}$$

where

$$J_1{}^{(i)} = \left\{ j \in J^{(i)} ; P_{ij} ; P_{ij}-P_{ij}^* \geqq \frac{B_{ij}}{A_{ij}-C_{ij}} \right\},$$

and

$$J_2{}^{(i)} = \text{the complement of } J_1{}^{(i)} \text{ in } J^{(i)}.$$

Then (3.1) reduces to the following equation

$$U_i = Min \begin{bmatrix} \underset{j \in J_2{}^{(i)}}{Min} \left[P_{ij}A_{ij}+q_{ij}C_{ij}+U_j\right] \\ \underset{j \in J_1{}^{(i)}}{Min} \left[B_{ij}+P_{ij}^*A_{ij}+q_{ij}^*C_{ij}+U_j\right] \end{bmatrix} \tag{3.3}$$

The equality (3.3)enables the calculation simpler than of (3.1).

4. **Uniqueness of solution.** The uniqueness of solution of (2.3) is shown by slight modifications of Bellman [2]. Let $\{U_i\}$ and $\{V_i\}$ are two solutions and k be an integer such that

$U_k - V_k = \underset{i=1,\cdots,N}{Max} \{U_i - V_i\}.$

Suppose that the minimum for U_k is attained at r and the policy of replacement in the arc (k, r) is the after-replacement, and the minimum for V_k is attained at s and the policy in the arc (k, s) is the before-replacement. Then we get

$$U_k = P_{kr}(A_{kr} - C_{kr}) + C_{kr} + U_r \leqq B_{ks} + P_{ks}^*(A_{ks} - C_{ks}) + C_{ks} + U_s$$

$$V_k = B_{ks} + P_{ks}^*(A_{ks} - C_{ks}) + C_{ks} + V_s.$$

This implies that

$$U_k - V_k \leqq U_s - V_s,$$

that is,

$$U_k - V_k = U_s - V_s.$$

Clearly we see $k \neq s$. Repeating this Procedure for U_s, we get for some $n \neq s$ and $n \neq k$,

$$U_k - V_k = U_s - V_s = U_n - V_n.$$

At the mean time, the member of nodes is finite and we exhaust the set of all nodes, and we have

$$U_1 - V_1 = U_2 - V_2 = \cdots\cdots = U_N - V_N = 0.$$

The similar method is applicable for the remaining three cases with the results that $U_i = V_i$ $(j = 1, 2, \cdots\cdots, N)$.

5. **An algorithm.** The method of successive approximations in policy space yield us numerical solutions of (2.3). Let $\{U_i{}^{(0)}\}$ be an initial approximation and then define $\{U_i{}^{(k)}\}$ inductively by the followings,

$$U_i{}^{(k+1)} = Min \begin{bmatrix} \underset{j \neq i}{Min} \left[P_{ij}\ (A_{ij} + U_j{}^{(k)}) + q_{ij}(C_{ij} + U_i{}^{(k)})\right] \\ \underset{j \neq i}{Min} \left[B_{ij} + P_{ij}^*(A_{ij} + U_j{}^{(k)}) + q_{ij}^*(C_{ij} + U_j{}^{(k)})\right] \end{bmatrix} \tag{5.1}$$

$$(i = 1, 2, \cdots\cdots, N-1)$$

$$U_N{}^{(k+1)} = 0. \tag{5.2}$$

$$(k = 0, 1, 2, \cdots\cdots)$$

It is known in [3] that the iterations in (5.1) converges in a finite number of steps for any choice of $\{U_i{}^{(0)}\}$. Here we consider only the case where

$$U_i{}^{(0)} = Min \begin{bmatrix} P_{iN}A_{iN} + q_{iN}C_{iN} \\ B_{iN} + P_{iN}^*A_{iN} + q_{iN}^*C_{iN} \end{bmatrix} \tag{5.3}$$

$$(i = 1, 2, \cdots\cdots, N-1)$$

, and

$$U_N{}^{(0)} = 0. \tag{5.4}$$

We shall show that $\{U_i{}^{(k)}\}$ is monotone, non-increasing with respect to k for any i.

$$U_i{}^{(1)} - U_i{}^{(0)} = Min \{\underset{j \neq i}{Min} [P_{ij}(A_{ij} - C_{ij}) + C_{ij} + U_j{}^{(0)}], \underset{j \neq i}{Min} [B^{ij} + P^{ij*}(A_{ij} - C_{ij}) + C_{ij} + U_j{}^{(0)}]\}$$

$$- Min \{P_{iN}A_{iN} + q_{iN}C_{iN},\ B_{in} + P_{in}^*\ A_{iN} + q_{in}^*C_{iN}\}$$

$$\leqq 0. \qquad (j \neq N)$$

On the one hand, for any k, we easily have

$$U_i{}^{(k+1)} - U_i{}^{(k)} \leqq U_j{}^{(k)} - U_j{}^{(k-1)} \tag{5.5}$$

for any j $(\neq i)$.

This implies that

$$U_i^{(k+1)} \leqq U_i^{(k)} \tag{5.6}$$

for any i.

It follows that there exsists a finite U_i^* such that

$$\lim_{k\to\infty} U_i^{(k)} = U_i^* \tag{5.7}$$

and clearly $\{U_i^*\}$ is the unique solution of (2.3).

As Bellman noticed in [2] that $U_i^{(k)}$ represents the minimal expected cost for a optimal path with at most k steps. This fact implies the foregoing inequalities $U_i^{(k+1)} \leqq U_i^{(k)}$ and that at most $(N-1)$ iterations, $U_i^{(k)}$ converges to U_i^*.

6. **Numerical examples.** We consider the network of Fig. 1. In this network we can number every arc (i, j) as $i<j$ and can obtain the solution directly.

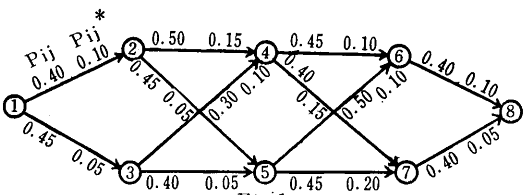

Fig. 1

In each arc the right value means $\{P_{ij}\}$ and the left $\{P_{ij}^*\}$, moreover

$$\left.\begin{aligned} A_{ij} &= A = 18 \\ B_{ij} &= B = 3 \\ C_{ij} &= C = 2 \end{aligned}\right\} \quad \text{for any } i,\ j.$$

$U_1 = 29.60$	$(B, 3)$
$U_2 = 23.60$	$(B, 5)$
$U_3 = 22.00$	$(A, 4)$
$U_4 = 15.40$	$(A, 7)$
$U_5 = 16.00$	$(B, 6)$
$U_6 = 7.80$	$(A, 8)$
$U_7 = 7.60$	$(B, 8)$
$U_8 = 0.00$	

Fig. 2 gives the optinal path with optimal policy of replacement. A and B denote after-replacement and before-replacement respectively.

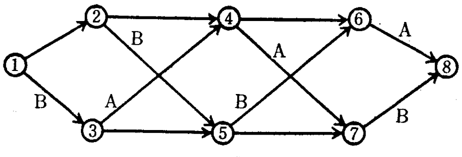

Fig. 2

In the second example we consider the Fig. 3. In this network we cannot number every arc (i, j) as $i<, j$ so we apply the successive approximations.

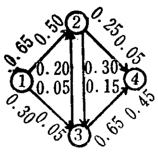

Fig. 3

$$\left.\begin{array}{l} A_{ij}=A=18 \\ B_{ij}=B=3 \\ C_{ij}=C=2 \end{array}\right\} \text{ for any } i,\ j.$$

$$\begin{cases} U_1{}^{(0)}=\infty & \\ U_2{}^{(0)}=5.8 & (B,4) \\ U_3{}^{(0)}=12.2 & (B,4) \\ U_4{}^{(0)}=0.0 & \end{cases}$$

$$\begin{cases} U_1{}^{(1)}=18.0 & (A,3) \\ U_2{}^{(1)}=5.8 & (B,4) \\ U_3{}^{(1)}=11.0 & (A,2) \\ U_4{}^{(1)}=0.0 & \end{cases}$$

$$\begin{cases} U_1{}^{(2)}=16.8 & (B,3) \\ U_2{}^{(2)}=5.8 & (B,4) \\ U_3{}^{(2)}=11.0 & (A,2) \\ U_4{}^{(2)}=0.0 & \end{cases}$$

We can obtain the solution by two iterations and show the policy of replacement in Fig. 4.

$$\begin{array}{ll} U_1=16.8 & (B,3) \\ U_2=5.8 & (B,4) \\ U_3=11.0 & (A,2) \\ U_4=0.0 & \end{array}$$

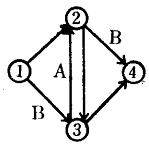

Fig. 4

7. **An adaptive problem.** In the foregoing problem, we consider the expected cost independent of time. Here we shall give an another problem which includes time as a parameter. The two problems differ essentially from each other in the point that the last problem is that of adative type and the former is of optimal one. Let $U_i(t)$ be the expected cost of traversing from node i to node N with time t using optimal policy Let $P_{ij}(t)$ be the density function of failure in arc (i,j) at time t, and A_{ij}, B_{ij} and C_{ij} are the same as in the third section. Δ and Δ' denote the time-lags of after-replacement of $P_{ij}(t)$ and $P_{ij}{}^*(t)$ respectively. Then we have

$$U_i(t)=\operatorname*{Min}\left[\begin{array}{l} \operatorname*{Min}_{j\neq i}\Big[\int_0^t P_{ij}(t-s)[A_{ij}+U_j(s-\Delta)]ds+\int_0^t q_{ij}(t-s)\,[C_{ij}+U_j(s)]\,ds\Big] \\ \operatorname*{Min}_{j\neq i}\Big[[B_{ij}+\int_0^{t-\Delta'} P_{ij}{}^*(t-\Delta'-s)[A_{ij}+U_j(s-\Delta)]ds+\int_0^{t-\Delta'} q_{ij}(t-\Delta'-s)\,[C_{ij}+U_j(s)]ds\Big] \end{array}\right] \quad (7.1)$$

$$(i=1,2,\cdots\cdots N-1)$$

$$U_N(t)=0 \qquad (7.2)$$

We choose an initial sequence

$$U_i^{(0)}(t)=\underset{}{Min}\begin{bmatrix}\int_0^t p_{in}(t-s)A_{iN}ds+\int_0^t q_{iN}(t-s)C_{iN}ds \\ B_{iN}+\int_0^{t-\Delta'}p_{iN}^*(t-\Delta'-s)A_{iN}ds+\int_0^{t-\Delta'}q_{iN}(t-\Delta'-s)C_{ij}ds\end{bmatrix} \tag{7.3}$$

$$(i=1,2,\cdots\cdots,N-1)$$

$$U_N^{(0)}(t)=0. \tag{7.4}$$

Employing the successive approximations, we again get

$U_i^{(k+1)}(t)=$

$$Min\begin{bmatrix}\underset{j\neq i}{Min}\left[\int_0^t p_{ij}(t-s)[A_{ij}+U_j^{(k)}(s-\Delta)]ds+\int_0^t q_{ij}(t-s)[C_{ij}+U_j^{(k)}(s)]ds\right] \\ \underset{j\neq i}{Min}\left[B_{ij}+\int_0^{t-\Delta'}p_{ij}^*(t-\Delta'-s)[A_{ij}+U_j^{(k)}(s-\Delta)]ds+\int_0^{t-\Delta'}q_{ij}(t-\Delta'-s)[C_{ij}+U_j^{(k)}(s)]ds\right]\end{bmatrix} \tag{7.5}$$

$$(i=1,2,\cdots\cdots,N-1)$$

$$U_N^{(k+1)}(t)=0$$

$$(k=0,1,\cdots\cdots) \tag{7.6}$$

As before discussion, we see that $U_i^{(k)}(t)$ represents a minimum cost for path with time t, and this implies the inequality

$$U_i^{(k+1)}(t)\leqq U_i^{(k)}(t) \tag{7.7}$$

and we see that $\{U_i^{(k)}(t)\}^\infty_{k=0}$ converges, *i. e.*

$$\lim_{k\to\infty} U_i^{(k)}(t)=U_i^*(t) \tag{7.8}$$

Clearly $U_i^*(t)$ is a solution of (7.1).

References

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