AN ALGORITHM IN A HILBERT SPACE

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1. **Introduction.** Many algorithms for integro-differential equations in a Hilbert space have been studied extensively with the progresses of the high speed automatic computers, and these methods have many extensions and variations.

We shall consider the minimal problem on a subspace V

$$\inf_{x \varepsilon V} \| Ax-f \| = \lim \| Ax-x \| = \| y-f \|, \quad y \ \varepsilon \ H_2 \quad \cdots\cdots (1)$$

for A, a bounded linear operator defined on a separable Hilbert space H_1 and has the range in another Hilbert space H_2.

If the given element x belongs to the range S of A, the above problem reduces to the operational equation

$$Ax=f, \quad \cdots\cdots (1)$$

and the equation of this type have been studied under the some conditions. For a finite dimentional vector space, M. R. Hestenes proposed the so called conjugate gradient method in [1], and W. V. Petryshyn studied for a K-positive operators on a Hilbert space and its some variations. The familier equation of this type is the case where A is a self-adjoint and positive definite operator.

Our method is, in this sense, an extension for these problems.

Another method for the problem (1) is found in V. V. Ivanov [2].

2. **Notations and the principal theorem.** Let H_1 and H_2 be Hilbert spaces and we shall suppose that H_1 is separable. A is a bounded linear operator defined on H_1 and its range S is in H_2. the solution of the problem (2) is denoted by u, and the residual of x, by $r(x)=f-Ax$.

The family of the expanding subspaces $\{V_n\}$ is the family of the finite dimentional subspaces in H_1 which covers the whole space H_1, that is, $\bigcup_{n=1}^{\infty} V_n$ is dense in H_1. For a convenience, we shall agree with denoting the norms in H_1 and H_2 by the same notations $\| \cdot \|$, and the inner products by the same notations (,).

The metric function $P(x)$ is defined by the following:

$$P(x) = \| Ax-f \| .$$

THEOREM 1.

Let V be a linear subspace of H_1 and x_0 be in V, then the following two conditions are equivalent each other:

1. $\| f-Ax_0 \| \leqq \| f-A(x_0+z) \|$ *for any* $z \varepsilon V$.
2. $(f-Ax_0, \ Az)=0$ *for any* $z \varepsilon V$.

PROOF. The condition 2. implies that

$$\| f-A(x_0+z) \|^2 - \| f-Ax_0 \|^2 = -2 \ Real(Az, f-Ax_0) + \| Az \|^2 = \| Az \|^2 \geqq 0$$

for any $z \ \varepsilon \ V$.

Conversely if we assume $(f-Ax_0, Az) \neq 0$, the non-zero scalar α exists such that

$$\frac{\partial p(x_0+\lambda z)}{\partial \lambda}\Big|_{\lambda=\alpha}=1$$

and the metric function P(x) has the minimum M on the line $x=x_0+\lambda z$,

where $\alpha=(Az, f-Ax_0)/\|Az\|^2\neq 0$

and $M=-|(f-Ax_0.Az)|^2/\|Az\|^2+\|f-Ax_0\|^2$.

And we get $\|f-Ax_0\|^2-\|f-A(x_0+\alpha z)\|^2=|(f-Ax_0, Az)|^2\|Az\|^2\geqq 0$.

This completes the proof.

3. **An algorithm.** The theorem 1 enables us to use the projective method for the construction of the iterative sequence whcih converges to the solution.

LEMMA 1. *Let V_n be a memder of the expanding subspaces spaned by a linearly independent seguence* $\{e_1, e_2, \dots, e_n\}$, *and x_n be the minimal point on V_n such that*

$$min\ \{\|f-Ax\| \ :\ x\ \varepsilon V_n\}=\|f-Ax_n\|,$$

then x_n takes the following form :

$$x_n=\sum_{k=1}^{n} c_k e_k,$$

where

$$(Ae_j, Ae_k)=0,\ \ j=1,\ 2,\dots,\ k-1$$

$$c_k=(f-Ax_{k-1}, Ae_k)/\|Ae_k\|^2,$$

or

$$(f-Ax_{k-1}, Ae_k)=0$$

$$c_k=0.$$

The last case happens, when $Ae_k=0$.

PROOF. By the theorem 1 and the relation

$$(f-Ax_k, Ae_j)-(f-Ax_k-1, Ae_j)=-c_k(Ae_k, Ae_j),$$

it is easy to prove the above lemma.

LEMMA 2. *Let G be a closed subspace in H_1 and for the family of the expanding subspaces $\{V_n\}$ put*

$$G_n=V_n\cap G,\ \ n=1,\ 2,\dots,$$

then for a starting point x_0, the iterative sequence $\{Ax_n\}$, where

$$x_{n+1}=x_n+(f-Ax_n, Ae_n)e_n/\|Ae_n\|^2, \qquad \text{(2)}$$

converges to the soluion y of (1) *and we get*

$$lim\|Ax_n-f\|=inf\ \{\|Ax-f\| \ :\ x\varepsilon G_n\}=\|y-f\|$$

PROOF. Put $P(x_n)=\xi_n$, then ξ_n is a non-increasing and we denote the limit of the sequence $\{\xi_n\}$ by ξ.

The inequality

$$\xi_n^2-\xi^2\geqq-\{|t|^2\|Az\|^2+2Real(tAz, Ax_n-f)\}$$

folds for any complex number, and we have

$$[\text{Real }(Ax_n-f, Az)]^2\leqq \|Az\|^2\ (\xi_n^2-\xi^2).$$

Putting $z=x_n-x_m$, we get

$$\|A(x_n-x_m)\|^2\leqq \|A(x_n-x_m)\|\ \sqrt{(\xi^2_n-\xi^2)+(\xi^2_m-\xi^2)}.$$

This implies that $\{Ax_n\}$ is a Cauchy sequence in H_2 and there exists an element y in H_2 such that

$$\lim Ax_n = y,$$

For any $\varepsilon > 0$, and $x \varepsilon G$ there is an element x^* in H_1 and an integer N such that

$$\| Ax^* - f \| \leqq \| Ax - f \| + \varepsilon \| A \| \text{ and } x^* \varepsilon G_N.$$

Since

$$\| y - f \| \leqq \| Ax_n - f \| \quad \text{for any } n$$

and x_n is the minimal point on G_n, we get

$$\| y - f \| \leqq \| Ax - f \| + \varepsilon \| A \|.$$

This shows that

$$\| y - f \| = \inf \{ \| Ax - f \| : x \varepsilon G \}$$

The last part of the assertion in the lemma is obvious.

Corollary. *If A is an invertible bounded linear operator, the iterative sequence* (3) *converges to the solution u of the equation*

$$Ax = f.$$

PROOF. The operator A being invertible, $\} x_n \}$ is the Cauchy sequence in H_1, and there exists an element $\hat{u}$ in H_1 to which the sequence $\{ x_n \}$ converges in norm. We shall show that $\hat{u}$ is the solution u.

For any $\varepsilon > 0$ there exists a $x\varepsilon \; \varepsilon \bigcap_{n=1}^{\infty} V_n$ such that

$$\| u - \hat{u} - z_\varepsilon \| < \varepsilon / \| A \|,$$

and there is the least integer N such that $z_\varepsilon \; \varepsilon \; V_n$ for any $n \geqq N$.

Then by the theorem 1 we get

$$(A(u - x_n), Az_\varepsilon) - 0 \text{ for } n \geqq N.$$

This implies

$$(A(u - \hat{u}), Az_\varepsilon) = 0,$$

and we have

$$\| A(u - \hat{u}) \|^2 = (A(u - \hat{u}), A(u - \hat{u} - z_\varepsilon)) \leqq \| A \| \; \| A(u - \hat{u}) \| \; \| u - \hat{u} - z_\varepsilon \| < \varepsilon.$$

Then

$$\| A(u - \hat{u}) \| \leqq \| A \| \; \| u - \hat{u} - z_\varepsilon \| < \varepsilon,$$

and the invertibility of A implies that $u = \hat{u}$.

we shall remark that a linearly indenpent sequence $\{ e_1, e_2, \ldots \}$ which is A-orthogonal, that is, which satisfies the condition $(Ae_j, Ae_k) = 0$ for $j \neq k$, can be constructed from any linearly independent sequence $\{ q_1, q_2, \ldots \}$ by the iteration

$$e_n = \gamma_n q_n - \sum_{k=1}^{n-1} (Aq_n, Ae_k) e_k / \| Ae_k \|^2,$$

where γ_n is the normalizing factor of e_n.

Combining these facts we have the following theorem.

THEOREM 2.

Let $\{ V_n \}$ be the expanding subspaces of H_1 where V_n is spaned by the linearly independent A-orthogonal sequence constructed for a bounded linear operator A, then for any starting point x_0 the sequence $\{ Ax_n \}$ defined by

$$x_{n+1} = x_n + c_n e_n, \quad \text{where } c_n = (f - Ax_n, Ae_n) e_n / \| Ae_n \|^2,$$

converges to the solution y of (1).

Moreover if A is invertible and f is in S, the sequence $\{x_n\}$ defined above converges to the solution u of the operational equation (2), *and*

$$\| Ax_n - f \| \leqq \| Ax - f \| \quad \text{for any } x \in V_n, \; n=1, 2, \ldots .$$

4. **An error estimation.** Noticing $x_n \in V_{n+1}$ in the last assertion in the theorem 2, we see that the residual $r(x_n)=f-Ax_n$, and when A is invertible and f is in S, the error $E(x)=u-x$ too, decrease in its norm at the each step of the iteration.

we shall give an error estimation for the last case.

THEOREM 3.

Let x_n be the same as in the theorem 2 and let $E_n=u-x_n$. Then we have

$$\| E_{n+1} \|^2 \leqq \| A^{-1} \|^2 (1-k_n) \| E_n \|^2$$

where

$$k_n = (Ae_n, f-Ax_n) / \| Ae_n \|^2 \| f-Ax_n \|^2 \leqq 1.$$

PROOF.

$$\| E_{n+1} \|^2 \leqq \| A^{-1} \|^2 \| A(u-x_n-c_n e_n) \|^2$$

$$= P(x_n)^2 \{1-(Ae_n, A(u-x_n)) / \| Ae_n \|^2 P(x_n)^2\} \| A^{-1} \|^2$$

$$\leqq \| A^{-1} \|^2 (1-k_n) \| E_n \|^2 .$$

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