BANACH ALGEBRAS AND THE WEIERSTRASS APPROXIMATION THEOREM

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Although the Weierstrass approximation theorem is interpreted by the terminology of Banach algebras, it was discovered and proved by K. Weicrstrass[15] with help of Riemann int·grals. Since then, there have been so many proofs with help of Riemann integrals. The shortest one, we believe, was given by E. Landau [9]. Besides these, two function-analysis-theoretical proofs were known; the one was given by N. Bourbaki [1] (cf. also S. Izumi [7]) by making use of absolute value functions and the other was given by K. Yosida [17] with help of the theory of one-parameter semi-groups originated by E. Hill [5] and K. Yosida [16].

Recently, the fundamental theorems in the theory of Banach algebras originated by I. Gelfand [3] (the spectrum radius theorem due to I. Gelfand and Banach field theorem due to S. Mazur and I. Gelfand) have been proved, with no help of the theory of function of a complex variable, by S. Kametani [8], C. E. Rickart [13],[14], and the second named author [10],[11]. (cf. also S. Ito[6] as to an elementary proof of the theorems with help of the theory of function of a complex variable.) The aim of this note is, as an application of the fundamental theorems in the theory of Banach algebras, to give an alternative proof of the Weierstrass approximation theorem in a genralized form given by M. H. Stone [12] and I. Gelfand [4].

In §1, we shall give an alternative proof of the fundamental theorems in the theory of Banach algebras with no help of Rieman-Radon integrals by modifying a method stated in the second named author [10],[11]. In §2, we shall deal with an alternative proof of the Weierstrass-Stone-Gelfand approximation theorem by making use of the fundamental theorems in the theory of Banach algebras combining with the local theory stated in the second named author [11].

§1. **BANACH ALGEBRAS.** By a *normed ring* A we mean an algebra over the field of comlex numbers C with a norm $\|\cdot\|$ enjoying the following conditions: $\|a\|\geqq O$; $\|a\|=O$ iff $a=O$, $\|\alpha a\|=|\alpha|\ \|a\|$, $\|a+b\|\leqq\|a\|+\|b\|$, and $\|ab\|\leqq\|a\|\ \|b\|$, where a, $b\in A$ and $\alpha\in C$. (No completeness condition is assumed on A.) We say that a normed ring is a *normed* field if it ts a field. By a *Banach algebra* we mean a normed ring, which is complete with respect to the norm. (I. Gelfand [1] originally used the terminology of normed rings to indicate Banach algebras defined here.) We say that a Banach algebra is a *Banach field* if it is a field. For the sake of simplicity, throughout this note, we are concerned with Banach algebras wiith a mutiplicative unit.

Let A be a Banach algebra. Denote by 1 the multiplicative unit of A. We may assume that $C\subset A$. We say that a complex number α is a *left* (*right*) *spectrum* of an element a in A if $\alpha-a$ has no left (right) inverse in A. A left (right) spectrum is called a *spectrum* if A is commutative. Write $\|a\|_o$ for sup ($|\alpha|$; α being a left or right spectrum of a). It will be shown later (in Lemma 1.5) that the set of left or right spectra is non-empty, but, anyway, we put $\|a\|_o=O$ if the set of left or right spectra of a would be empty. Write $\|a\|_\infty$ for $lim\ \|a^n\|^{1/n}$. The exstence of the limit will be shown later (in Lemma 1.2).

In this section, we give an alternative proof of the following

THEOREM A (*I. Gelfand*): $\| a \|_o = \| a \|_\infty$.

Let a be an arbitry (but fixed) element in A. Denote by $(a)'$ the set of those elements in A which commute with a, and by $(a)''$ the set of those elements which commute with all elements in $(a)'$. The set $(a)''$ then is a commutative Banach algebra containing a and C. Moreover, α is a spectrum of a in $(a)''$ if and only if it is a left or right spectrum of a in A. Hence, in order to prove Theorem A, we can assume without loss of generality that $A=(a)''$, or A is commutative.

Put $D=(\alpha;\ |\alpha| < \| a \|_o^{-1})$ and $\psi(\alpha)=(1-\alpha a)^{-1}$ for $\alpha \in D$, Then, $\psi(\alpha)$ is continuous in D. Put $g_n(\alpha)=n^{-1}\sum_{i=1}^{n} \psi\ (\alpha\zeta_i)$ for $\alpha \in D$, where $\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$ denotes the complete set of n-th roots of unity. We have from Lagrange's formula $g_n(\alpha)=(1-(\alpha a)^n)^{-1}$. Put $\Delta=(\alpha; lim\ (\alpha a)^n=O, \alpha$ being a positive real number). It is immediate that $\Delta=(O, \beta)$ for some $\beta > O$.

LEMMA 1.1: *$g_n(\alpha)$ is uuiformly equi-continuous in the wider sense in D.*

PROOF: Take any compact subset D' in D. We heve $\| \psi(\alpha) \| \leqq K'$ for $\alpha \in D'$, where K' is a positive constant depending only on D'. we then have

$$\| g_n(\alpha)-g_n(\beta) \| \leqq n^{-1}\sum_{i=1}^{n} \| (\alpha-\beta)a\psi(\zeta_i\alpha)\psi(\zeta_i\beta) \| = ((K')^2 \| a \|)\ |\alpha-\beta|.$$

This shows the assertion of Lemma 1.1.

LEMMA 1.2: $lim \| a^n \|^{1/n}=\beta^{-1}$.

PROOF: We first show that $(lim \| a^n \|^{1/n})^{-1} \leqq \beta$. Suppose the contrary. Select γ such that $\beta < \gamma < (lim \| a^n \|^{1/n})^{-1}$. There exists k such that $\| (\gamma a)^k \|^{1/k} < 1$, or $\| (\gamma a)^k \| < 1$. Hence $\| (\gamma a)^n \| \leqq \| (\gamma a)^k \|^s \| (\gamma a)^r \| \to O$ $(n \to \infty,\ n=sk+r,\ O \leqq r < k)$, or $\gamma \in \Delta$. This contradicts the construction of β. Hence we have the assertion.

We next show that $\beta \leqq (lim \| a^n \|^{1/n})^{-1}$. For any $O < \gamma < \beta$, we have $lim\ (\gamma a)^n=O$, and so $\| (\gamma a)^n \| \leqq L$ for all n. Hence $lim \| a^n \|^{1/n} \leqq \gamma^{-1}$, or $\gamma \leqq (lim \| a^n \|^{1/n})^{-1}$. This implie that $\beta \leqq (lim \| a^n \|^{1/n})^{-1}$.

From these results it follows that $lim \| a^n \|^{1/n}$ exists and equals to β^{-1}.

LEMMA 1.3: $\| a \|_o \leqq \beta^{-1}$.

PROOF: There is nothing to do if $\beta^{-1}=\infty$. Suppose $\beta^{-1} < \infty$. For $\beta^{-1} < \gamma^{-1} < \delta^{-1} < \infty$, we have $lim(\gamma a)^n=O$, or $\| (\gamma a)^n \| \leqq L$ for all n. Hence, $\| (\delta a)^n \| \leqq (\delta/\gamma)^n L$ for all n, and the series $\sum_{n=o}^{\infty} (\delta a)^n$ converges to the inverse of $1-\delta a$. This implies that $\| a \| \leqq \delta^{-1}$whenever $\beta^{-1} < \delta^{-1}$, or $\| a \|_o \leqq \beta^{-1}$.

LEMMA 1.4: $\beta^{-1} \leqq \| a \|_o$.

PROOF: suppose the contrary. We can select four real numbers $O < \alpha < \beta < \gamma < \delta < \varepsilon < \| a \|_o^{-1}$ such that $\delta-\alpha \leqq (2(K')^2 \| a \|)^{-1}$, where K' denotes the positive constant described in the proof of Lemma 1.1 ($D'=(\omega;\ |\omega| \leqq \varepsilon)$). Since $O < \alpha < \beta$, we have $lim\ (\alpha a)^n=O$, and so $lim\, g_n(\alpha)=1$, because of the continuity of $1/t$ $(|1-t|<1)$. Since $|\delta-\alpha|\ ((K')^2 \| a \|) \| o \|^{-1} \leqq 1/2$, we have from the proof of Lemma 1.1 $\| g_n(\delta)-g_n(\alpha) \| \leqq 1/2$. Hence there exists a natural numder N such that $\| g_n(\delta)-1 \| \leqq 3/4$ for all $n \geqq N$, or $\| 1-(\delta a)^n \| = \| g_n(\delta)^{-1} \| = \sum_{k=o} \| 1-g_n(\delta)^k \| \leqq 4$ for all $n \geqq N$. This implies that $\| (\delta a) \| \leqq 5$ forall $n \geqq N$. Hence we have $\| (\gamma a) \| = (\lambda/\delta)^n \| \leqq 5(\gamma/\delta)^n \| (\zeta a)$ for all $n \geqq N$, or $lim \| (\gamma a)^n \| =O$. We thus have $\gamma \in \Delta$. This is, however, impossible, because of the construction of β. Hence we must have $\beta^{-1} \leqq \| a \|_o$.

We did not assume so far that the set of spectra of a was non-empty, and we now have the following

LEMMA 1, 5: *a has at least one spectrum in A.*

PROOF: Suppose a has no spectrum in A. We have from Lemmas 1.3 and 1.4 $\beta = \|a\|_o^{-1} = \infty$, or $\lim (\alpha a)^n = O$ for all $\alpha > O$. Hence $2^n = \|(2aa^{-1})^n\| / \|1\| \leqq \|(2a)^n\| \ \|a^{-1}\|^n / \|1\| = \|((2\|a^{-1}\|a)n\| / \|1\| \to O$ $(n \to \infty)$. This is a contradiction. Thus a has at least one spect(um in A.

PROOF OF THEOREM A: From Lemma 1.5 it follows that the definition of $\|a\|_o$ is meaningful. And we have from Lemmas 1.2—1.4 $\|a\|_o = \|a\|_\infty$. This completes the proof.

The following Banach field theorem is almost a translation of Lemma 1.5:

THEOREM B (S. Mazur and I. Gelfand): Erery Banach field is isomorphic onto C.

PROOF: Let A be a Banach field and a be an element in A. The set $(a)''$ defined before is a field containing a and C. In view of Lemma 1.5, a has at least one spectrum in $(a)''$, say, α. The element $\alpha - a$ has no inverse in $(a)''$, and so $a = \alpha \in C$, This completes the proof.

REMARK (S. Kametani [8]): Theorem B holds for normed fields. This fact follows from the proof Lemmas 1.4 and 1.5.

§2. WEIERSTRASS APPROXIMATION THEOREM. In this section, we give an alternative proof of the weierstrass approximation theorem in a generalized form given by M. H. Stone [12] and I. Gelfand [4];

THEOREM C (Weierstrass-Stone-Gelfand); Let Ω be a compact Hausborff space and S be an algebra of real-valued continuous functions on Ω enjoying the following conditions: S contains 1, the function taking value 1 constantly, and if $\omega_1 \neq \omega_2$ $(\omega_1, \omega_2 \in \Omega)$, then there exists a function x in S such that $x(\omega_1) \neq x(\omega_2)$. Then, every real-valued continuous function on Ω can be uniformly approximated on Ω by functions in S.

Let Ω be a compact Hausdorff space. Denote by $C(\Omega)$ the algebra of complex-valued continuous functions on Ω. The algebra $C(\Omega)$ constitutes a commutative Banach algebra with a usual norm $\|\cdot\|$ defined by $\|f\| = sup(|f(\omega)|; \omega \in \Omega)$. Denote by $R(\Omega)$ the set of real-valued continuous functions on Ω. The set R (Ω) constitutes a semi-ordered set with a semi-ordering $f \leqq g$ defined by $f(\omega) \leqq g(\omega)$ for all $\omega \in \Omega$. We know that every algebraic homomorphism ρ of $C(\Omega)$ onto C satisfies the following conditions: (1) $\|\rho\| = 1$, (2) $\rho(R(\Omega)) \subset R$, the field of real numbers, and (3) $O \leqq \rho(f)$ whenever $f \in R(\Omega)$ and $O \leqq f$. We call such a mapping as ρ a pure state of $C(\Omega)$. For ω in Ω, the mapping: $f \to f(\omega)$ defines a pure state of $C(\Omega)$. Denote it by ρ_ω. Given ω in Ω, denote by $E(\omega)$ the set of functions e in $R(\Omega)$ such that ω is an inner point of $(\mu; e(\mu) = 1, \mu \in \Omega)$.

LEMMA 2.1: *Given any pure state ρ of $C(\Omega)$, there exists a point ω in Ω such that $\rho(e) \neq O$ whenever $e \in E(\omega)$.*

PROOF: Suppose the coutrary. Given any ω in Ω, there exists e_ω in $E(\omega)$ such that $\rho(e_\omega) = O$. Put $U_\omega = (\mu; e_\omega(\mu) > 1/2, \mu \in \Omega)$. The set $U\omega$ is open. Since Ω is compact, there exists a finite subset $\{\omega_1, \omega_2, \ldots, \omega_n\}$ of Ω such that $\Omega = \bigcup_{\iota=1}^{n} U\omega_\iota$, or $1/2 \leqq \sum_{\iota=1}^{n} e_{\omega\iota}$. In view of (3), we get $1/2 \leqq \rho(\sum_{\iota=1}^{n} e_{\omega\iota}) = \sum_{\iota=1}^{n} \rho(e_{\omega\iota}) = O$. This is a contradiction. Thus the proof is completed.

LEMMA 2.2: *Given any pure state ρ of $C(\Omega)$, there exists a point ω in Ω such that $\rho = \rho_\omega$.*

PROOF: We fix a point ω described in Lemma 2.1. It must be shown that $\rho = \rho_\omega$. By

Urysohn's lemma, given any e' in $E(\omega)$ such that $ee'=e'$. Hence $\rho(e)\rho(e')=\rho(e')$, or $\rho(e)=1$. Suppose now $\rho\neq\rho_\omega$. There exists f in $C(\Omega)$ such that $\rho(f)\neq f(\omega)$. put $g=(\rho(f)-f(\omega))^{-1}(f-f(\omega))$. Since $g(\omega)=O$, given any $\varepsilon>O$, there exists e in $E(\omega)$ such that $\|eg\|\leqq\varepsilon$, or $|\rho(g)|=|\rho(eg)|\leqq\varepsilon$. This implies that $\rho(g)=O$, which contradicts the construction of g. Hence we must have $\rho=\rho_\omega$.

Let S be a uniformly closed algebra of real-valued continuous functions on Ω with the properties stated in Theorem C. In order to prove Theorem C, we need only to show that $S=R(\Omega)$.

LEMMA 2.3: *If* $1\leqq x\in S$, *then* $x^{-1}\in S$.

PROOF: Suppose $1\leqq x\leqq\alpha$ $(1<\alpha)$. Since $\|1-\alpha^{-1}x\|\leqq 1-\alpha^{-1}<1$, $\alpha^{-1}\sum^{\infty}_{n=0}(1-\alpha^{-1}x)^n$ uniformly converges to x^{-1}. Hence, $x^{-1}\in S$.

Given ω in Ω, denote by $F(\omega)$ the set of functions e in S such that $e(\omega)=1$ and such that $O\leqq e\leqq 1$. ($F(\omega)$ is non-empty. In fact, $1-\|x^2(1+x^2)^{-1}\|^{-1}x^2(1+x^2)^{-1}\in F(\omega)$ whenever $x(\omega)=O$ and $O\neq x\in R(\Omega)$.) Put $\|f\|_{\omega,n}=inf(\|e^{2n}f\|;e\in F(\omega))$ for all $f\in C(\Omega)$ $(1\leqq n<\infty)$. It is immediate that $\|f\|_{\omega,n}\leqq\|f\|_{\omega,n+1}$ and $\|f\|^2_{\omega,n}\leqq\|f^2\|_{\omega,n+1}$ $(1\leqq n<\infty)$. Put $\|f\|_\omega=sup(\|f\|_{\omega,n};1\leqq n<\infty)$ for $f\in C(\Omega)$. Then $\|\cdot\|$ satisfies the following conditions: $\|f\|\geqq\|f\|_\omega\geqq O$, $\|\alpha f\|_\omega=|\alpha|\ \|f\|_\omega$, $\|f+g\|_\omega\leqq\|f\|_\omega+\|g\|_\omega$, $\|fg\|_\omega\leqq\|f\|_\omega\|g\|_\omega$, and $\|f^2\|_\omega=\|f\|^2_\omega$, where f, $g\in C(\Omega)$ and $\alpha\in C$. Put $J_\omega=(f;\|f\|_\omega=O,\ f\in\Omega(\Omega))$. Note that $(1-e)f\in J_\omega$ whenever $e\in F(\omega)$ and $f\in C(\Omega)$. In fact, $\|e^n(1-e)\|\leqq(n+1)^{-1}/(1+n^{-1})^n\to O$ $(n\to\infty)$. The quotient algebra $C(\Omega)/J_\omega$ constitutes a Banach algebra with the norm $\|\cdot\|$ defined by $\|\iota_\omega(f)\|=\|f\|_\omega$, where ι_ω denotes the natural homooorphism of $C(\Omega)$ onto $C(\Omega)/J_\omega$. (This norm is independent of a choice of a representative f of $\iota_\omega(f)$. The completeness condition of $C(\Omega)/J_\omega$ follows from $\|x\|\geqq\|x\|_1$ $(x\in C(\Omega)/J_\omega)$, where $\|\cdot\|_1$ denotes the induced norm of $C(\Omega)/J_\omega$ due to J_ω, andso $\|x\|\leqq c\|x\|_1$ for some c iin R (See S. Banach[1)].) Moreover, we have $\|x^2\|=\|x\|^2$ for any x in $C(\Omega)/J_\omega$.

LEMMA 2.4: $C(\Omega)/J_\omega\cong C$.

PROOF: We first show that any pure state ρ of $C(\Omega)$, which vanishes on J_ω, coincides with ρ_ω. We have from Lemma 2.2 $\rho=\rho_{\omega'}$ for some ω' in Ω. If $\omega\neq\omega'$, there exists x in S such that $x(\omega)\neq x(\omega')$. Put $y=(x(\omega')-x(\omega))^{-1}(x-x(\omega))$ and $e=1-\|y^2(1+y^2)^{-1}\|^{-1}y^2(1+y^2)^{-1}$. we then have $y(\omega)=O, y(\omega')=1$, $e\in F(\omega)$, and $e(\omega')\neq 1$. This contradicts $1-e\in J\omega$ and $1-e(\omega')=O$. Hence, $\rho=\rho_\omega$. Obviously, $\rho\omega$ vanishes on $J\omega$.

Denote by $M\omega$ the set of functions f in $C(\Omega)$ such that $f(\omega)=O$. Since $\rho\omega$ vanishes on $J\omega$, we next show that $J_\omega=M_\omega$. Suppose the contrary. There exists a non-zero element x in M_ω/J_ω. Since $\|x\|_\infty=lim\|x^{2n}\|^{1/2n}=\|x\|\neq O$, by Theorem A, there exists a non-zero spectrum α of x. Then the ideal J of $C(\Omega)/J_\omega$ generated by $\alpha\iota_\omega(1)-x$ does not contain the multiplicative unit $\iota_\omega(1)$ of $C(\Omega)/J_\omega$. Hence, by Zorn's lemma, there exists a maximal ideal M of $C(\Omega)/J_\omega$ containing J. The quotient algebra of $C(\Omega)/J_\omega$ by M is a Banach field, and so, by theorem B, there exists a pure state ρ of $C(\Omega)$, which vanishes on $\iota_\omega^{-1}(M)$ $(\supset J_\omega)$. Since ρ vanishes on J_ω, we have $\rho=\rho_\omega$. This contraeicts $\rho(\iota_\omega^{-1}(x))\neq O$. Hence $J_\omega=M_\omega$, and so $C(\Omega)/J_\omega=\mathcal{O}(\Omega)/M_\omega\cong C$. Thus the proof is completed.

LEMMA 2.5: *Let* $(e_i;1\leqq i\leqq n)$ *be a set of functions in* $R(\Omega)$ *which satisfies the following conditions*: $O\leqq e_i\leqq 1$ $(1\leqq i\leqq n)$ *and* $1/2\leqq\sum_{i=1}^{n}e_i$. *Then, there exists a natural number* N *such that* $\|(\sum_{i=1}^{n}e_i^N)/(\sum_{i=1}^{n}e_i^{N+1})\|\leqq 3$.

PROOF: We first show that, given ω in Ω, there exist a natural number N_ω and a neighbourhood V_ω at ω such that $(\sum_{i=1}^{n} e_k(\omega')^m)/(\sum_{i=1}^{n} e_i(\omega')^{m+1} \leqq 3$ for all $m \geqq N_\omega$ and for all $\omega' \in V_\omega$. We can assume without loss of generalty that $e_1(\omega) = \cdots = e_k(\omega)$ and $e_j(\omega) < e_1(\omega)$ $(k < j \leqq n)$. ($(j; k < j \leqq n)$ might be empty.) Given $O < \eta < 1$, there exists a neighbourhood V_ω at ω such that $e_i(\omega')/p(\omega') \leqq 1-\eta$ $(1 \leqq i \leqq k)$ and $1-\eta \leqq e_j(\omega')/p(\omega')$ $(k < j \leqq n)$ for all $\omega' \in V_\omega$, where $p(\omega') = Max(e_i(\omega'); 1 \leqq i \leqq n)$.

Put $q_i = e_i(\omega')/p(\omega')$ $(1 \leqq i \leqq n)$. Then, we have

$$| 1/p(\omega') - (\sum_{i=1}^{n} e_i(\omega')^m)/(\sum_{i=1}^{n} e_i(\omega')^{m+1}) 1) |$$
$$= (1/p(\omega')) | 1 - (\sum_{i=1}^{n} q_i^m)/(\sum_{i=1}^{n} q_i^{m+1}) |$$
$$\leqq 2(\sum_{i=1}^{n} q_i^m (1-q_i)) \leqq 2(k\eta + 2(n-k)(1-\eta)^m)$$

for all $\omega' \in V_\omega$. We can select η and N_ω such that $2(k\eta + 2(n-k)(1-\eta)^m) \leqq 1$ for all $m \geqq N_\omega$. We then have $(\sum_{i=1}^{n} e_i(\omega')^m)/(\sum_{i=1}^{n} e_i(\omega')^{m+1}) \leqq 3$ for all $m \geqq N_\omega$. and for all $\omega' \in V_\omega$.

We associate N_ω and V_ω with each ω in Ω. Since Ω is compact, there exists a finite subset $\{\omega_1, \omega_2, \ldots, \omega_n\}$ of Ω such that $\Omega = \bigcup_{i=1}^{n} V_{\omega i}$. Put $N = Max(N_{\omega k}; 1 \leqq i \leqq n)$. It is now immediate that N is the natural number in question. Thus the proof is completed.

PROOF OF THEOREM C: Let S be a uniformly closed algebra of real-valued continuous functions on Ω with the properties stated in Theorem C. It must be shown that $S = R(\Omega)$. Let f be a function in $R(\Omega)$ and $\varepsilon > O$. By the proof of Lemma 2.4, given ω in Ω, we have $\| f - f(\omega) \|_\omega = O$. Hence there exsits e_ω in $F(\omega)$ such that $\| e_\omega (f - f(\omega)) \| < \varepsilon/3$. Put $U_\omega = (\mu; e_\omega(\mu) > 1/2, \mu \in \Omega)$. The set U_ω is open. Since Ω is compact, there exists a finite subset $\{\omega_1, \omega_2, \ldots, \omega_n\}$ of Ω suoh that $\Omega = \bigcup_{i=1}^{n} U_{\omega i}$, or $1/2 \leqq \sum_{i=1}^{n} e_{\omega i}$. Put $e_i = e_{\omega i}$ $(1 \leqq i \leqq n)$. The set $(e_i; 1 \leqq i \leqq n)$ satisfies the conditions stated in Lemma 2.5. Hence, by Lemma 2.5, we have $\| f - (\sum_{i=1}^{n} f(\omega_i) e_i^{N+1}) / (\sum_{i=1}^{n} e_i^{N+1}) \| < (\varepsilon/3) \| (\sum_{i=1}^{n} e_i^{N+1}) \leqq \varepsilon$. We thus have $f \in S$, or $R(\Omega) = S$. This completes the proof.

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