数値積分に於ける函数解析的方法

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A Method of Functional Analysis in Numerical Integrals

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In the first place we prove that the Banach space $C^{(n)}/P_{n-1}$, where $C^{(n)}$ is the Banach space of all functions who have the n-th continuous derivatives and P_{n-1} is the closed subspace in $C^{(n)}$ spaned by all the $n-1$ dimensional polynomials, is isomorphic to the space of all the continuous functions. Applying F. Riesz's theorem to the above fact, we show that the linear functional on the $C^{(n)}/P_{n-1}$ can be written in the form of the Stieltjes-integral. Many error functionals in numerical integrals are the functionals on the space $C^{(n)}/P_{n-1}$ for some integers n, and we have the systematic method of the estimation of the following type:

$$|E(f)| \leqq \|E\| \max|f^{(n)}(x)|,$$

where $\|E\|$ is the norm of the error functional $E(f)$ which may be calculeted as the total variation of the function of bounded variation assined to the error functional $E(f)$. As an example, the case of Simpson's rule is to be showed at the end of this paper.

I. We shall consider the formula

$$\int_a^b W(x) f(x)\, dx = \sum_{k=0}^{m} W_k f(x_k) + E(f), \qquad \text{-(1)}$$

where $W(x)$ is a positive weight function, and $x_0, x_1, \cdots\cdots, x_m$ are $m+1$ abscissa. When this formula is considered on the space of all the continuous functions on the closed interval (a, b), the error $E(f)$ is regarded as a continuous linear functional on the space, and by the theorem of F. Riesz, we can express $E(f)$ in the integral form.
This implies the existence of the estimatoin of $E(f)$ in the following type,

$$|E(f)| \leqq \|E\| \max_{x \in (a,b)} |f(x)|, \qquad \text{-(2)}$$

where $\|E\|$ is the functional norm which may be calculated as the total variation of the function of bounded variation assined to the functional $E(f)$.

The fact that the estimation (2) holds for any continuous function, on the other hand, weakens the accuracy of the approximation.

But the error functional $E(f)$ in (1) is often assumed to have the degree of precision γ which is defined as the largest of integers satisfing the equation $E(x^n)=0$ for any integer $n \leqq \gamma$, and we shall consider the method of restriction of the error functional to certain classes of continuous functions.

It seems that the recent developments in high-speed autmatic computers may aid the global treatments in numerical analysis.

See, for an example of the estimation of the error of this type, Davis 2). And we shall give the example of our method in the case of Simpson's rule in the last paragraph.

II. Let $C^{(n)}$ be the space of all the continuous functions which are defined on the closed interval (a,b) and have the n-th continuous derivatives, and $\|f\|_n$ be the norm of $f\epsilon C^{(n)}$ defined by

$$\|f\|_n = \max_{i=0,1,\cdots,n}(\max_{x\epsilon(a,b)}|f^{(i)}(x)|),$$

where $f^{(i)}$ is the i-th derivative of f.

Then $C^{(n)}$ becomes the Banach space with this norm. We shall denote the closed subspace in $C^{(n)}$ spaned by all the $n-1$ dimentional polynomials by P_{n-1} and we construct the quatient space $K_n=C^{(n)}/P_{n-1}$ in the ordinary manner. Let $[f]$ be an element in K_n, which is the coset represented by f, and $\|[f]\|$ be the norm in K_n defined by

$$\|[f]\| = \inf\|f+p\|_n,$$

where inf is to be taken over $p\epsilon P_{n-1}$.

Lemma *The Banach space K_n is eqquivalent to the space $C^{(0)}$ in the sence of the terminology of Banach*

Proof. We shall define the operator D_n which trasform K_n into C_0:

$$D_n[f]=f^{(n)}.$$

It is clear that the operator D_n is one-one mapping, and moreover

$$\|D_n[f]\|_0 = \|f^{(n)}\|_0 \leqq \max_{i=0,1,\cdots,n}|(f^{(i)}+p)(x)| \quad \text{for any } p\epsilon P_{n-1},$$

that is, $\quad \|D_n[f]\|_0 \leqq \|[f]\|_0$

Then there is a inverse operator of D_n and for some positive number $M>0$, we have

$$M\|[f]\| \leqq \|D_n[f]\|_0 \leqq \|[f]\|.$$

Any continuous function can be considered as the n-th continuos derivative of some continuous function, and so D_n is the onto mapping. This complete the proof.

By the above lemma we can appy the theorem of F. Riesz to represent a linear functional of K_n in the integral form. His theorem states that a linear functional $F(f)$ on the $C^{(0)}$ can be written in the following integral form:

$$F(f) = \int_a^b f(x)d\lambda(x),$$

and the norm of F is equal to the total variation of $\lambda(x)$, where $\lambda(x)$ is defined as follows;

$$\lambda(x)=F(\varphi_{x,0}),$$

$$\varphi_{x,0}(t) = \begin{cases} 1 & (a\leqq t\leqq x), \\ 0 & (x<t\leqq b). \end{cases}$$

Let $\varphi_{x,n}(t)$ be the function defined as follows;

$$\varphi_{x,n}(t)=\begin{cases}(t-x)/n! & (a\leqq t\leqq x),\\ 0 & (x<t\leqq b).\end{cases}$$

Then we have

Theorem. *A linear functional E on the space K_n can be represented in the following integral form:*

$$E([f])=\int_a^b f^{(n)}(x)d\lambda_n(x), \qquad -(3)$$

where $\lambda_n(x) = E(\varphi_{x,n})$,

and the norm of E is equal to the total variation of $\lambda_n(x)$.

Proof. For a linear functional $E(f)$ on K_n, put

$$\overline{E}\,(D_n[f])=E([f]).$$

Then the following inequality shows that $\overline{E}$ is the continuous linear functional on the space $C^{(0)}$;

$$|\overline{E}\,(D_n[f])|=|E([f])|\leqq\|E\|\,\|[f]\|\leqq M^{-1}\,\|E\|\,\|D_n[f]\|_0.$$

Applying F. Riesz's theorem to $\overline{E}$, we have the above theorem.

If the error functional in (1) has the degree of precision γ, it is easy to see that the restriction of the error functional to $C^{(n)}$, $(n\leqq\gamma+1)$, may be regarded as the linear continuous functional on the space K_n. We shall agree to use the same notation for the given error functional and the restricted one.

Corollary. *Let $E(f)$ be the error functional in* (1) *whose degree of precision is γ, then $E(f)$ can be represented in the following form;*

$$E(f)=\int_a^b f^{(n)}(x)d\alpha_n(x),\ f(x)\epsilon C^{(n)},$$

where an integer $n\leqq\lambda+1$ and $\alpha_n(x)$ is defined by

$$\alpha_n(x)=(1/n!)\int_a^b(t-x)^n w(t)dt-\sum_{k\geq x}^{x}W_k(x_k-x)\ . \qquad -(4)$$

III. To illustrate our method, we shall employ the simple case where $w(x)=1$ in (4), and particulary the rule is Simpson's 3-point rule.

The function of bounded variation $\alpha_n(x)$ takes the following form;

$$\alpha_n(x)=(1/n!)\quad\left\{(x-x_0)^{n+1}/n+1-W_0(x_0-x)^n-W_1(x_1-x)^n-\cdots\cdots-W_k(x_k-x)^n\right\},$$

in each segment $(x_k, x_{k+1}[$, and it is easy to compute the total variation of $\alpha_n(x)$.

Let $E(f)$ be the linear functional generated by Simpsons 3-points rule. By regarding $E(f)$ as the functional on the space K_n for each $n\leqq4$,

we have the following estimations:

a) When $f(x)$ is a continuous function, the estimation,

$$|E(f)|\leqq 2h\max_{x\ni(a,b)}|f(x)|,$$

holds.

b) When $f(x)$ has the 1-th continuous derivative, the estimation,

$$|E(f)| \leqq 3h^2 \max_{x \in (a,b)} |f^{(1)}(x)|,$$

holds.

c) When $f(x)$ has the 2-th continuous derivative, the estimation,

$$|E(f)| \leqq (16/81)h^3 \max_{x \in (a,b)} |f^{(2)}(x)|,$$

holds.

d) When $f(x)$ has the 3-th continuous derivative, the estimation,

$$|E(f)| \leqq 8h^4 \max_{x \in (a,b)} |f^{(3)}(x)|,$$

holds.

e) When $f(x)$ has the 4-th continuous derivative, the estimation,

$$|E(f)| \leqq (1/90)h^5 \max_{x \in (a,b)} |f^{(4)}(x)|,$$

holds.

In each case, $h=(b-a)/2$.

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