On the Defect Relation for Exponential Curves

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For $n+1$ distinct complex numbers $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$, let $f=[e^{\lambda_1 z}, e^{\lambda_2 z}, \cdots, e^{\lambda_{n+1} z}]$ be an exponential curve, D be the convex polygon surrounding the points $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ and $X \subset \boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$ be in general position. We put $X^+=\{\boldsymbol{a}\in X \mid \delta(\boldsymbol{a}, f)>0\}$. It is well-known that $(*)\sum_{\boldsymbol{a}\in X^+}\delta(\boldsymbol{a}, f)\leq n+1$. In this paper we consider when the equality holds in $(*)$.

Theorem. Suppose that D is an $n+1-$gon. If the equality holds in $(*)$, then

$$X^+=\{a_1\boldsymbol{e}_1, a_2\boldsymbol{e}_2, \cdots, a_{n+1}\boldsymbol{e}_{n+1}\} \quad (a_1a_2\cdots a_{n+1}\neq 0),$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_{n+1}$ are the standard basis of $\boldsymbol{C}^{n+1}$.

When D is a segment, this theorem dose not hold.

1 Introduction.

Let $f=[f_1, \cdots, f_{n+1}]$ be a holomorphic curve from $\boldsymbol{C}$ into the n-dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation $(f_1, \cdots, f_{n+1}): \boldsymbol{C}\to\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$, where n is a positive integer. We use the following notations:

$$\|f(z)\|=(|f_1(z)|^2+\cdots+|f_{n+1}(z)|^2)^{1/2}$$

and for a vector $\boldsymbol{a}=(a_1, \cdots, a_{n+1})\in\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$

$$\|\boldsymbol{a}\|=(|a_1|^2+\cdots+|a_{n+1}|^2)^{1/2}, \quad (\boldsymbol{a}, f)=a_1f_1+\cdots+a_{n+1}f_{n+1}, \ (\boldsymbol{a}, f(z))=a_1f_1(z)+\cdots+a_{n+1}f_{n+1}(z).$$

The characteristic function $T(r, f)$ of f is defined as follows(see[4]):

$$T(r, f)=\frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta-\log\|f(0)\|.$$

On the other hand, put $U(z)=\max_{1\leq j\leq n+1}|f_j(z)|$, then it is known ([1]) that

$$T(r, f)=\frac{1}{2\pi}\int_0^{2\pi}\log U(re^{i\theta})d\theta+O(1). \tag{1}$$

We suppose throughout the paper that f is transcendental; that is to say,

$$\lim_{r\to\infty}\frac{T(r, f)}{\log r}=\infty$$

and that f is linearly non-degenerate over $\boldsymbol{C}$; namely, $f_1, \cdots, f_{n+1}$ are linearly independent over $\boldsymbol{C}$. It is well-known that f is linearly non-degenerate if and only if the Wronskian $W=W(f_1, \cdots, f_{n+1})$ of $f_1, \cdots, f_{n+1}$ is not identically equal to zero.

For meromorphic functions in the complex plane we use the standard notation of the Nevanlinna theory of meromorphic functions([2],[3]).

For $\boldsymbol{a}\in\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$, we write

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$$m(r, \boldsymbol{a}, f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|\boldsymbol{a}\|\|f(re^{i\theta})\|}{|(\boldsymbol{a}, f(re^{i\theta}))|}d\theta, \quad N(r, \boldsymbol{a}, f)=N(r, \frac{1}{(\boldsymbol{a}, f)}).$$

We then have the first fundamental theorem

$$T(r, f)=N(r, \boldsymbol{a}, f)+m(r, \boldsymbol{a}, f)+O(1)$$

([4], p.76). We call the quantity

$$\delta(\boldsymbol{a}, f)=1-\limsup_{r\to\infty}\frac{N(r, \boldsymbol{a}, f)}{T(r, f)}=\liminf_{r\to\infty}\frac{m(r, \boldsymbol{a}, f)}{T(r, f)}$$

the deficiency of $\boldsymbol{a}$ with respect to f. We have $0\leq\delta(\boldsymbol{a}, f)\leq 1$ from the first fundamental theorem since $N(r, \boldsymbol{a}, f)\geq 0$ for $r\geq 1$ and $m(r, \boldsymbol{a}, f)\geq 0$ for $r>0$.

Let X be a subset of $\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$ in general position; that is to say, $\sharp X\geq n+1$ and any $n+1$ elements of X generate $\boldsymbol{C}^{n+1}$. We denote by $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+1}$ the standard basis of $\boldsymbol{C}^{n+1}$.

Cartan([1]) gave the following

Theorem A (Defect relation). For any q elements $\boldsymbol{a}_j$ $(j=1, \cdots, q)$ of X $(n+1\leqq q<\infty)$,

$$\sum_{j=1}^{q}\delta(\boldsymbol{a}_j, f)\leq n+1.$$

For any $n+1$ distinct complex numbers $\lambda_1, \cdots, \lambda_{n+1}$, we define a curve f_e by

$$f_e=[e^{\lambda_1 z}, e^{\lambda_2 z}, e^{\lambda_{n+1} z}]. \tag{2}$$

We call it an exponential curve([4],p.94). It is easy to see that the curve f_e is non-degenerate and transcendental. It is an interesting problem to detemine

$$X^+=\{\boldsymbol{a}\in X\,|\,\delta(\boldsymbol{a}, f_e)>0\}$$

for which the equality

$$\sum_{\boldsymbol{a}\in X^+}\delta(\boldsymbol{a}, f_e)=n+1$$

holds when $n\geq 2$. When $n=1$, it is trivial that $X^+=\{a\boldsymbol{e}_1, b\boldsymbol{e}_2\}$ $(ab\neq 0)$.

The purpose of this paper is to give an answer to this problem for some cases.

The author owes to Professor Nakamura the solutions of several Diophantine equations at the beginning of this research.

2 Preliminaries

Let $f_e=[e^{\lambda_1 z}, e^{\lambda_2 z}, \cdots, e^{\lambda_{n+1} z}]$ be the exponential curve given in (2). Let ℓ be the length of the convex polygon D surrounding the points $\lambda_1, \cdots, \lambda_{n+1}$, where $\ell=2|\lambda_j-\lambda_k|$ if the convex polygon reduces to a segment with the endpoints λ_j and λ_k. Then, ℓ is equal to the circumference of the convex polygon spanned arround the $n+1$ points $\bar{\lambda}_1, \cdots, \bar{\lambda}_{n+1}$.

Lemma 1 ([4], pp.95-98). $T(r, f_e)=(\ell/2\pi)r+O(1)$.

Lemma 2. 1) For any $\boldsymbol{a}=(a_1, \cdots, a_{n+1})\in\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$ such that $a_j\neq 0$ $(j=1, \cdots, n+1)$,

$$\delta(\boldsymbol{a}, f_e)=0, \quad \text{or} \quad \limsup_{r\to\infty}N(r, \boldsymbol{a}, f_e)/T(r, f_e)=1.$$

2) For a vector $\boldsymbol{b}=(b_1, \cdots, b_{n+1})\in\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$ such that $b_{j_k}\neq 0$ $(1\leq j_1<\cdots<j_m\leq n+1)$ and $b_j=0$ $(j\neq j_1, \cdots, j_m\,;\, m\geq 2)$

$$\delta(\boldsymbol{b}, f_e)=1-\ell'/\ell,$$

where ℓ' is the length of the convex polygon spanned arround the points $\lambda_{j_1},\cdots,\lambda_{j_m}$.

Proof. 1) $\boldsymbol{e}_1,\cdots,\boldsymbol{e}_{n-1},\boldsymbol{a}$ are in general position and by Theorem A,

$$\sum_{j=1}^{n-1}\delta(\boldsymbol{e}_j, f_e)+\delta(\boldsymbol{a}, f_e)\leq n+1. \tag{3}$$

On the other hand, $\delta(\boldsymbol{e}_j, f_e)=1$ $(j=1,\cdots,n+1)$. This and (3) imply that $\delta(\boldsymbol{a}, f_e)=0$.

2) Let $g=[e^{\lambda_{j_1}z},\cdots,e^{\lambda_{j_m}z}]$. Then, by Lemma 1,

$$T(r, g)=(\ell'/2\pi)r+O(1). \tag{4}$$

By 1) of this lemma and (4) we obtain

$$\delta(\boldsymbol{b}, f_e)=1-\limsup_{r\to\infty}\frac{N(r,\boldsymbol{b},f_e)}{T(r,f_e)}=1-\limsup_{r\to\infty}\frac{N(r,\boldsymbol{b},g)}{T(r,g)}\cdot\frac{T(r,g)}{T(r,f_e)}=1-\frac{\ell'}{\ell}.$$

3 The case when D is a convex $n+1-$gon

In this section we consider the exponential curve $f_e=[e^{\lambda_1 z}, e^{\lambda_2 z},\cdots,e^{\lambda_{n+1}z}]$, where $\lambda_1,\lambda_2,\cdots,\lambda_{n+1}$ are distinct and vertices of a convex $n+1-$gon D. We number without loss of generality the vertices λ_j $(j=1,\cdots,n+1)$ in ascending sequence as one goes arround D in the positive direction. For convenience we put $\lambda_q=\lambda_i$ when $(n+1)\mid(q-i)$ for any integer q.

We use the following notations for $i=1,2,\cdots,n+1$:

$$s_i^{p-1}=|\lambda_i-\lambda_{i-p}|\quad(1\leq p\leq n+1).$$

Particularly we put $s_i^0=\ell_i$ and note that $s_i^n=0$ $(i=1,2,\cdots,n+1)$.

Further we note that for any integer q and $i=1,2,\cdots,n+1$

$$l_q=l_i,\quad s_q^{p-1}=s_i^{p-1}$$

when $(n+1)\mid(q-i)$, and we have the following relation.

$$s_i^{p-1}=s_{i-p}^{n-p}\quad(1\leq p\leq n). \tag{5}$$

We have

$$\ell=\sum_{i=1}^{n+1}\ell_i.$$

In this case, by Lemma 1

$$T(r, f_e)=(\ell/2\pi)r+O(1). \tag{6}$$

For a vector $\boldsymbol{a}=(a_1,a_2,\cdots,a_{n-1})\in\boldsymbol{C}^{n-1}-\{\boldsymbol{0}\}$ and for $i=1,2,\cdots,n+1$; $j=1,2,\cdots,n$ we say that $\boldsymbol{a}$ is of

(a) 0-type if $a_m\neq0$ $(m=1,\cdots,n+1)$;

(b) $(i,j)-$type if $a_i=a_{i-1}=\cdots=a_{i+j-1}=0$, $a_m\neq0$ $(m\neq i, i+1,\cdots,i+j-1)$,

where $a_q=a_i$ when $(n+1)\mid(q-i)$ for any integer q; and we write

$\boldsymbol{a}_0$ for any 0-type vector and $\boldsymbol{a}_i^j$ for any $(i,j)-$type vector.

From now on throughout this section we suppose without loss of generality that $\ell=1$. From Lemma 2 and the relation (6) we have the following

Proposition 1. (a) $\delta(\boldsymbol{a}_0, f_e)=0$;

(b) $\delta(\boldsymbol{a}_i^j, f_e)=\sum_{k=1}^{j}\ell_{i-k-1}-s_{i-1}^j$ $(i=1,\cdots,n+1;\ j=1,\cdots,n)$;

(c) In particular $\delta(\boldsymbol{a}_i^n, f_e)=1$ $(i=1, \cdots, n+1)$.

For $\boldsymbol{a}=(a_1, \cdots, a_{n-1})\in \boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$, we put

$$\boldsymbol{a}(0)=\{i \mid a_i=0 \ \ (1\leq i\leq n+1)\}.$$

Lemma 3. Let $\boldsymbol{a}=(a_1, a_2, \cdots, a_{n+1})$ be a vector in $\boldsymbol{C}^{n-1}-\{\boldsymbol{0}\}$ satisfyinfg $\delta(\boldsymbol{a}, f_e)>0$. Then there exist vectors $\boldsymbol{a}_{i_1}^{j_1}, \cdots, \boldsymbol{a}_{i_k}^{j_k}$ $(k\geq 1)$ such that

1) $\delta(\boldsymbol{a}, f_e)=\sum_{\nu=1}^{k}\delta(\boldsymbol{a}_{i_\nu}^{j_\nu}, f_e)$;

2) the sets $\boldsymbol{a}_{i_1}^{j_1}(0), \cdots, \boldsymbol{a}_{i_k}^{j_k}(0)$ are mutually disjoint and $\boldsymbol{a}(0)=\cup_{\nu=1}^{k}\boldsymbol{a}_{i_\nu}^{j_\nu}(0)$.

Proof. By lemma 2-1), the set $\boldsymbol{a}(0)$ is not empty. Put

$$I=\{i \mid a_i=0,\ a_{i-1}\neq 0;\ 1\leq i\leq n+1\} \quad \text{and} \quad M=\{m \mid a_{m-1}=0,\ a_m\neq 0;\ 1\leq m\leq n+1\}.$$

It is easy to see that $\sharp I=\sharp M$. Put $\sharp I=k\geq 1$. Let

$$I=\{i_1<i_2<\cdots<i_k\} \quad \text{and} \quad M=\{m_1<m_2<\cdots<m_k\}.$$

(1) When $i_1<m_1$, it is easy to see that

$$1\leq i_1<m_1<i_2<m_2<\cdots<i_k<m_k\leq n+1$$

and

$$\{i \mid a_i=0\}=\cup_{\nu=1}^{k}\{i_\nu, i_\nu+1, \cdots, i_\nu+j_\nu-1\}, \tag{7}$$

where $j_\nu=m_\nu-i_\nu$ $(\nu=1, \cdots, k)$.

(2) When $m_1<i_1$, we have that

$$1\leq m_1<i_1<m_2<i_2<\cdots<m_k<i_k\leq n+1$$

and

$$\{i \mid a_i=0\}=\cup_{\nu=1}^{k}\{i_\nu, i_\nu+1, \cdots, i_\nu+j_\nu-1\}, \tag{8}$$

where $j_\nu=m_{\nu+1}-i_\nu$ $(\nu=1, \cdots, k-1)$ and $j_k=n+1+m_1-i_k$.

From (7) or (8) it is easy to see that

1) $\delta(\boldsymbol{a}, f_e)=\sum_{\nu=1}^{k}(\sum_{\mu=0}^{j_\nu}\ell_{i_\nu-1-\mu}-s_{i_\nu-1}^{j_\nu})=\sum_{\nu=1}^{k}\delta(\boldsymbol{a}_{i_\nu}^{j_\nu}, f_e)$;

2) $\boldsymbol{a}_{i_1}^{j_1}(0), \cdots, \boldsymbol{a}_{i_k}^{j_k}(0)$ are mutually disjoint and $\boldsymbol{a}(0)=\cup_{\nu=1}^{k}\boldsymbol{a}_{i_\nu}^{j_\nu}(0)$.

For any $X\subset \boldsymbol{C}^{n-1}-\{\boldsymbol{0}\}$ in general position, we put as in Section 1,

$$X^{+}=\{\boldsymbol{a}\in X \mid \delta(\boldsymbol{a}, f_e)>0\}.$$

Proposition 2. $\sharp X^{+}\leq n(n+1)$.

Proof. As X^{+} is in general position, we have the inequality

$$\sharp\{\boldsymbol{a}=(a_1, \cdots, a_i, \cdots, a_{n+1})\in X^{+} \mid a_i=0\}\leq n$$

for any $i=1, 2, \cdots, n+1$, so that we have our proposition.

Theorem 1. Let d_i be the number of vectors of $(i+1, n)$-type in X^{+} $(i=1, \cdots, n+1)$. Then, $0\leq d_i\leq 1$ and we have the inequality

$$\sum_{\boldsymbol{a}\in X^{+}}\delta(\boldsymbol{a}, f_e)\leq n+1-\sum_{i=1}^{n-1}(1-d_i)(\ell_{i-1}+\ell_i-s_{i-1}^{1}).$$

Proof. We have the following equalities for some non-negative integers x_i^j $(i=1, \cdots, n+1;\ j=1, \cdots, n)$

$$\Delta \equiv \sum_{\boldsymbol{a}\in X^+} \delta(\boldsymbol{a}, f_e) = \sum_{i=1}^{n+1}\sum_{j=1}^{n} x_i^j \delta(\boldsymbol{a}_i^j, f_e) \tag{9}$$

by Lemma 3-1), and for $i=1, \cdots, n+1$ and $p=1, \cdots, n$

$$\sharp\{\boldsymbol{a}=(a_1, \cdots, a_{n+1})\in X^+ \mid a_i=a_{i+1}=\cdots=a_{i+p-1}=0\} = \sum_{j=p}^{n}\sum_{k=0}^{j-p} x_{i-k}^j \tag{10}$$

by Lemma 3-2) since for each i $(1\leq i\leq n+1)$ and p $(1\leq p\leq n)$

$$\{i, i+1, \cdots, i+p-1\}\subset \boldsymbol{a}_{i-k}^j(0) \quad (j=p, \cdots, n;\ k=0, \cdots, j-p).$$

As X^+ is in general position, for $i=1, \cdots, n+1$ and $p=1, \cdots, n$ we have the inequalities

$$\sharp\{\boldsymbol{a}=(a_1, \cdots, a_{n+1})\in X^+ \mid a_i=a_{i+1}=\cdots=a_{i+p-1}=0\} \leq n+1-p,$$

which imply with (10) that x_i^j $(i=1, \cdots, n+1;\ j=1, \cdots, n)$ must satisfy the following inequalities for $i=1, \cdots, n+1$:

$$\sum_{j=p}^{n}\sum_{k=0}^{j-p} x_{i-k}^j \leq n+1-p \quad (p=1, \cdots, n). \tag{11}$$

As $\delta(\boldsymbol{a}_i^1, f_e)>0$, from (9) and (11) for $p=1$, we have the inequality

$$\begin{aligned}
\Delta &\leq \sum_{i=1}^{n+1}\{n-\sum_{j=2}^{n}\sum_{k=0}^{j-1} x_{i-k}^j\}\delta(\boldsymbol{a}_i^1, f_e) + \sum_{i=1}^{n+1}\sum_{j=2}^{n} x_i^j\delta(\boldsymbol{a}_i^j, f_e)\\
&= \sum_{i=1}^{n+1}\{n-\sum_{j=2}^{n}\sum_{k=0}^{j-1} x_{i-k}^j\}(\ell_{i-1}+\ell_i-s_{i-1}^1) + \sum_{i=1}^{n+1}\sum_{j=2}^{n} x_i^j\delta(\boldsymbol{a}_i^j, f_e)\\
&= 2n-n\sum_{i=1}^{n+1} s_i^1 - \sum_{i=1}^{n+1}\sum_{j=2}^{n} x_i^j\{\sum_{k=0}^{j-1}(\ell_{i+k-1}+\ell_{i-k}-s_{i+k-1}^1)\} + \sum_{i=1}^{n+1}\sum_{j=2}^{n} x_i^j\delta(\boldsymbol{a}_i^j, f_e)
\end{aligned}$$

since $\Sigma_{i=1}^{n+1}\ \ell_i=1$, which is equal to

$$2n-n\sum_{i=1}^{n+1} s_i^1 - \sum_{i=1}^{n+1}\sum_{j=2}^{n} x_i^j\{\delta(\boldsymbol{a}_i^j, f_e) - \sum_{k=0}^{j-1}(\ell_{i-k-1}+\ell_{i+k}-s_{i+k-1}^1)\} \equiv E_1.$$

Here, by Proposition 1(b) for $j\leq n$ the equality

$$\begin{aligned}
&\delta(\boldsymbol{a}_i^j, f_e) - \sum_{k=0}^{j-1}(\ell_{i+k-1}+\ell_{i-k}-s_{i-k-1}^1)\\
&\qquad = \sum_{k=0}^{j} \ell_{i+k-1} - s_{i-1}^j - \sum_{k=0}^{j-1}(\ell_{i-k-1}+\ell_{i+k}-s_{i+k-1}^1)\\
&\qquad = \sum_{k=0}^{j-1} s_{i-k-1}^1 - \sum_{k=0}^{j-2} \ell_{i+k} - s_{i-1}^j
\end{aligned}$$

holds, so that we have

$$\begin{aligned}
E_1 &\equiv 2n-n\sum_{i=1}^{n+1} s_i^1 + \sum_{i=1}^{n+1} x_i^2(s_{i-1}^1+s_i^1-\ell_i-s_{i-1}^2) + \sum_{i=1}^{n+1}\sum_{j=3}^{n} x_i^j\{\sum_{k=0}^{j-1} s_{i+k-1}^1 - \sum_{k=0}^{j-2}\ell_{i+k} - s_{i-1}^j\}\\
&= 2n+(n-2)\sum_{i=1}^{n+1} s_i^1 - \sum_{i=1}^{n+1}\{n-1-x_i^2-\sum_{j=3}^{n}\sum_{k=0}^{j-2} x_{i-k}^j\}(s_{i-1}^1+s_i^1) - \sum_{i=1}^{n+1} x_i^2(\ell_i+s_{i-1}^2) - \sum_{i=1}^{n+1}\sum_{j=3}^{n}\sum_{k=0}^{j-2} x_{i-k}^j(s_{i-1}^1+s_i^1)\\
&\quad + \sum_{i=1}^{n+1}\sum_{j=3}^{n} x_i^j\{\sum_{k=0}^{j-1} s_{i-k-1}^1 - \sum_{k=0}^{j-2}\ell_{i+k} - s_{i-1}^j\} \equiv E_2.
\end{aligned}$$

As $s_{i-1}^1+s_i^1>\ell_i+s_{i-1}^2$ $(i=1, \cdots, n+1)$, by (11) for $p=2$, we have the inequality

$$
\begin{aligned}
E_2 \le\, & 2n+(n-2)\sum_{i=1}^{n+1} s_i^1-\sum_{i=1}^{n+1}(n-1-x_i^2-\sum_{j=3}^{n}\sum_{k=0}^{j-2} x_{i-k}^j)(s_{i-1}^2+\ell_i)-\sum_{i=1}^{n+1} x_i^2(\ell_i+s_{i-1}^2)-\sum_{i=1}^{n+1}\sum_{j=3}^{n} x_i^j\sum_{k=0}^{j-2}(s_{i+k-1}^1+s_{i+k}^1)\\
&+\sum_{i=1}^{n+1}\sum_{j=3}^{n} x_i^j\{\sum_{k=0}^{j-1} s_{i+k-1}^1-\sum_{k=0}^{j-2}\ell_{i-k}-s_{i-1}^j\}\\
=\, & 2n+(n-2)\sum_{i=1}^{n+1} s_i^1-(n-1)-(n-1)\sum_{i=1}^{n+1} s_i^2\\
&+\sum_{i=1}^{n-1}\sum_{j=3}^{n} x_i^j\{\sum_{k=0}^{j-1} s_{i+k-1}^1-\sum_{k=0}^{j-2}\ell_{i+k}-s_{i-1}^j-\sum_{k=0}^{j-2}(s_{i+k-1}^1+s_{i-k}^1)+\sum_{k=0}^{j-1}(\ell_{i-k}+s_{i+k-1}^2)\}\\
=\, & n+1+(n-2)\sum_{i=1}^{n-1} s_i^1-(n-1)\sum_{i=1}^{n-1} s_i^2+\sum_{i=1}^{n+1}\sum_{j=3}^{n} x_i^j\{\sum_{k=0}^{j-2} s_{i-k-1}^2-\sum_{k=0}^{j-2} s_{i-k}+s_{i-1}^j\}\\
=\, & n+1+(n-2)\sum_{i=1}^{n+1} s_i^1-(n-1)\sum_{i=1}^{n+1} s_i^2+\sum_{i=1}^{n-1} x_i^3(s_{i-1}^2+s_i^2-s_i^1-s_{i-1}^3)+\sum_{i=1}^{n+1}\sum_{j=4}^{n} x_i^j\{\sum_{k=0}^{j-2} s_{i-k-1}^2-\sum_{k=0}^{j-2} s_{i+k}-s_{i-1}^j\}\\
=\, & n+1+(n-2)\sum_{i=1}^{n-1} s_i^1+(n-3)\sum_{i=1}^{n-1} s_i^2-\sum_{i=1}^{n-1}\{n-2-x_i^3-\sum_{j=4}^{n}\sum_{k=0}^{j-3} x_{i-k}^j\}(s_{i-1}^2+s_i^2)-\sum_{i=1}^{n-1} x_i^3(s_i^1+s_{i-1}^3)\\
&-\sum_{i=1}^{n-1}\sum_{j=4}^{n} x_i^j\sum_{k=0}^{j-3}(s_{i+k-1}^2+s_{i+k}^2)+\sum_{i=1}^{n-1}\sum_{j=4}^{n} x_i^j\{\sum_{k=0}^{j-2} s_{i-k-1}^2-\sum_{k=0}^{j-3}(s_{i-k}^1-s_{i-1}^j)\}\equiv E_3.
\end{aligned}
$$

As $s_{i-1}^2+s_i^2>s_i^1+s_{i-1}^3$ $(i=1,\cdots,n+1)$, by (11) for $p=3$, we have the inequality

$$
\begin{aligned}
E_3\le\, & n+1+(n-2)\sum_{i=1}^{n-1} s_i^1+(n-3)\sum_{i=1}^{n-1} s_i^2-\sum_{i=1}^{n-1}\{n-2-x_i^3-\sum_{j=4}^{n}\sum_{k=0}^{j-3} x_{i-k}^j\}(s_i^1+s_{i-1}^3)\\
&-\sum_{i=1}^{n-1} x_i^3(s_i^1+s_{i-1}^3)-\sum_{i=1}^{n-1}\sum_{j=4}^{n} x_i^j\sum_{k=0}^{j-3}(s_{i-k-1}^2+s_{i-k}^2)+\sum_{i=1}^{n-1}\sum_{j=4}^{n} x_i^j\{\sum_{k=0}^{j-2} s_{i-k-1}^2-\sum_{k=0}^{j-3} s_{i-k}^1-s_{i-1}^j\}\\
=\, & n+1+(n-3)\sum_{i=1}^{n-1} s_i^2-(n-2)\sum_{i=1}^{n-1} s_i^3\\
&+\sum_{i=1}^{n-1}\sum_{j=4}^{n} x_i^j\{\sum_{k=0}^{j-2} s_{i-k-1}^2-\sum_{k=0}^{j-3} s_{i-k}^1-s_{i-1}^j-\sum_{k=0}^{j-3}(s_{i-k-1}^2+s_{i-k}^2)+\sum_{k=0}^{j-3}(s_{i-k}^1+s_{i-k-1}^3)\}\\
=\, & n+1+(n-3)\sum_{i=1}^{n-1} s_i^2-(n-2)\sum_{i=1}^{n-1} s_i^3+\sum_{i=1}^{n-1}\sum_{j=4}^{n} x_i^j\{\sum_{k=0}^{j-3} s_{i-k-1}^3-\sum_{k=0}^{j-4} s_{i-k}^2-s_{i-1}^j)\}\equiv E_4.
\end{aligned}
$$

We put for $3\le q\le n-1$

$$E_{q-1}=n+1+(n-q)\sum_{i=1}^{n-1} s_i^{q-1}-(n-q+1)\sum_{i=1}^{n-1} s_i^q+\sum_{i=1}^{n-1}\sum_{j=q-1}^{n} x_i^j\{\sum_{k=0}^{j-q} s_{i-k-1}^q-\sum_{k=0}^{j-(q+1)} s_{i-k}^{q-1}-s_{i-1}^j\}.$$

Then, by the inequalities

$$s_{i-1}^p+s_i^p>s_i^{p-1}+s_{i-1}^{p-1}\quad(p=1,\cdots,n-1;\ i=1,\cdots,n+1)$$

and by (11), we apply the mathematical induction to obtain the following inequalities

$$\Delta\le E_1\le E_2\le\cdots\le E_{n-1}\le E_n=n+1+\sum_{i=1}^{n+1} s_i^{n-2}-2\sum_{i=1}^{n+1} s_i^{n-1}+\sum_{i=1}^{n+1} x_i^n(s_{i-1}^{n-1}+s_i^{n-1}-s_i^{n-2}-s_{i-1}^n),$$

which is equal to

$$n-1+\sum_{i=1}^{n-1} s_i^1+\sum_{i=1}^{n-1} d_{i-1}(\ell_{i-2}+\ell_{i-1}-s_{i-2}^1)=n-1+\sum_{i=1}^{n-1} s_{i-1}^1+\sum_{i-1}^{n-1} d_i(\ell_{i-1}+\ell_i-s_{i-1}^1)$$

since $\Sigma_{i=1}^{n-1}\, s_i^{n-1}=\Sigma_{i=1}^{n+1}\,\ell_i=1$ and $x_i^n=d_{i-1}$, so that $0\le d_i\le 1$ by (11) for $p=n$.

As $\Sigma_{i=1}^{n-1}\,(\ell_{i-1}+\ell_i)=2$,

$$n-1+\sum_{i=1}^{n-1}s_{i-1}^1+\sum_{i=1}^{n-1}d_i(\ell_{i-1}+\ell_i-s_{i-1}^1)=n+1+\sum_{i=1}^{n-1}(1-d_i)s_{i-1}^1-\sum_{i=1}^{n-1}(1-d_i)(\ell_{i-1}+\ell_i)$$
$$=n+1-\sum_{i=1}^{n-1}(1-d_i)(\ell_{i-1}+\ell_i-s_{i-1}^1).$$

We complete the proof of our theorem.

Corollary 1. Suppose that D is a convex $n+1$-gon and $\sum_{\boldsymbol{a}\in X^+}\delta(\boldsymbol{a},f_e)=n+1$. Then

$$X^-=\{a_1\boldsymbol{e}_1,\cdots,a_{n+1}\boldsymbol{e}_{n+1}\},\quad (a_1\cdots a_{n-1}\neq 0).$$

4 The case when D is a segment

In this section we consider the exponential curve $f_e=[e^{\lambda_1 z},e^{\lambda_2 z},\cdots,e^{\lambda_{n+1}z}]$, where $\lambda_1,\cdots,\lambda_{n-1}$ are distinct and on a segment L. We number without loss of generality the points λ_j $(j=1,\cdots,n+1)$ as follows:

(i) The points λ_1 and λ_{n-1} are the endpoints of L.

(ii) The points λ_j $(j=1,\cdots,n+1)$ are in ascending sequence as one goes from λ_1 to λ_{n+1} on L.

For convenience for $q\in\boldsymbol{Z}$ we put $\lambda_q=\lambda_i$ when $(n+1)|(q-i)$ for $i=1,\cdots,n+1$.

We use the following notation for $q\in\boldsymbol{Z}$: $|\lambda_q-\lambda_{q-1}|=\ell_q$. Recall that $\ell=2|\lambda_1-\lambda_{n+1}|$.

We note that for $q\in\boldsymbol{Z}$, $\ell_q=\ell_i$ when $(n+1)|(q-i)$ for $i=1,\cdots,n+1$ and that $\ell=\Sigma_{i=1}^{n-1}\ell_i$. Then we have $\Sigma_{i=1}^{n}\ell_i=\ell_{n-1}=\ell/2$. In this case, by Lemma 1

$$T(r,f_e)=(\ell/2\pi)r+O(1). \tag{12}$$

Lemma 4. For $\boldsymbol{a}=(a_1,a_2,\cdots,a_n,a_{n+1})\in\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$, if $\delta(\boldsymbol{a},f_e)>0$, then $a_1=0$ or $a_{n+1}=0$.

Proof. Suppose to the contrary that $a_1\neq 0$ and $a_{n-1}\neq 0$. Then, by Lemma 2-2), $\delta(\boldsymbol{a},f_e)=0$, which is a contradiction. We have our lemma.

For a vector $\boldsymbol{a}=(a_1,a_2,\cdots,a_n,a_{n+1})\in\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$, we say that $\boldsymbol{a}$ is of

(i) $(1,j)$-type if $a_1=a_2=\cdots=a_j=0$ and $a_m\neq 0$ $(j+1\leq m\leq n+1)$;

(ii) $(n+1,j)$-type if $a_{n+1}=\cdots=a_{n-2-j}=0$ and $a_m\neq 0$ $(1\leq m\leq n+1-j)$, where $1\leq j\leq n$ and we write $\boldsymbol{a}_1^j$ for any $(1,j)$-type vector and $\boldsymbol{a}_{n+1}^j$ for any $(n+1,j)$-type vector.

From now on throughout this section we suppose without loss of generality that $\ell=1$. Then by Lemma 2 and (12) we have the following

Proposition 3. For $j=1,\cdots,n$

(a) $\delta(\boldsymbol{a}_1^j,f_e)=2\sum_{k=1}^{j}\ell_k$, (b) $\delta(\boldsymbol{a}_{n+1}^j,f_e)=2\sum_{k=1}^{j}\ell_{n+1-k}$.

Note that $\delta(\boldsymbol{a}_1^n,f_e)=\delta(\boldsymbol{a}_{n+1}^n,f_e)=1$ as $\Sigma_{i=1}^n\ell_i=1/2$.

For $\boldsymbol{a}=(a_1,\cdots,a_{n+1})\in\boldsymbol{C}^{n-1}-\{\boldsymbol{0}\}$, we put

$$\boldsymbol{a}(0)=\{i|a_i=0\ (1\leq i\leq n+1)\}.$$

Lemma 5. Let $\boldsymbol{a}=(a_1,a_2,\cdots,a_n,a_{n-1})\in\boldsymbol{C}^{n-1}-\{\boldsymbol{0}\}$ satisfying $\delta(\boldsymbol{a},f_e)>0$.

(a) When $a_1=0$, $a_{n-1}\neq 0$, there exists an integer j_1 $(1\leq j_1\leq n)$ satisfying

1) $\delta(\boldsymbol{a},f_e)=\delta(\boldsymbol{a}_1^{j_1},f_e)$ and 2) $\boldsymbol{a}(0)\supset\boldsymbol{a}_1^{j_1}(0)$.

(b) When $a_1\neq 0$, $a_{n+1}=0$, there exists an integer j_2 $(1\leq j_2\leq n)$ satisfying

2) $\delta(\boldsymbol{a}, f_e)=\delta(\boldsymbol{a}^{j_2}_{n+1}, f_e)$ and 2) $\boldsymbol{a}(0)\supset\boldsymbol{a}^{j_2}_{n+1}(0)$

(c) When $a_1=0$, $a_{n+1}=0$, there exists two integers i and j $(1\le i,\ j; i+j\le n)$ satisfying

1) $\delta(\boldsymbol{a}, f_e)=\delta(\boldsymbol{a}^i_1, f_e)+\delta(\boldsymbol{a}^j_{n+1}, f_e)$ and 2) $\boldsymbol{a}(0)\supset\boldsymbol{a}^i_n(0)\cup\boldsymbol{a}^j_{n+1}(0)$, $\boldsymbol{a}^i_1(0)\cap\boldsymbol{a}^j_{n+1}(0)=\phi$.

Proof. Note that $a_1=0$ or $a_{n+1}=0$ if $\delta(\boldsymbol{a}, f_e)>0$ by Lemma 4.

(a) Let j_1 be the number satisfying $a_1=\cdots=a_{j_1}=0$ and $a_{j_1+1}\ne 0$ $(1\le j_1\le n)$. Then, by Lemma 2 and Proposition 3(a)

$$\delta(\boldsymbol{a}, f_e)=1-2(\ell_{j_1+1}+\cdots+\ell_n)=2\sum_{k=1}^{j_1}\ell_k=\delta(\boldsymbol{a}^{j_1}_1, f_e)$$

and we have 2) by Lemma 2-2).

(b) Let j_2 be the number satisfying $a_{n+1}=\cdots=a_{n+2-j_2}=0$ and $a_{n+1-j_2}\ne 0$ $(1\le j_2\le n)$. Then, by Lemma 2 and Proposition 3(b)

$$\delta(\boldsymbol{a}, f_e)=1-2\sum_{k=1}^{n-j_2}\ell_k=\sum_{k=1}^{j_2}\ell_{n+1-k}=\delta(\boldsymbol{a}^{j_2}_{n+1}, f_e)$$

and we have 2) by Lemma 2-2).

(c) Let i be the number satisfying $a_1=\cdots=a_i=0$ and $a_{i+1}\ne 0$ $(1\le i\le n-1)$ and j the number satisfying $a_{n+1}=\cdots=a_{n+2-j}=0$ and $a_{n+1-j}\ne 0$ $(1\le j\le n-1)$. As $i+1\le n+1-j$, $i+j\le n$. Then, by Lemma 2 and Proposition 3

$$\delta(\boldsymbol{a}, f_e)=1-2(\ell_{i+1}+\cdots+\ell_{n-j})=2\left(\sum_{k=1}^{i}\ell_k+\sum_{k=1}^{j}\ell_{n+1-k}\right)=\delta(\boldsymbol{a}^i_1, f_e)+\delta(\boldsymbol{a}^j_{n+1}, f_e)$$

and we have 2) by Lemma 2-2).

Theorem 2. Let d_i be the number of vectors of type $a\boldsymbol{e}_i$ $(a\ne 0)$ in X^- $(i=1,\cdots,n+1)$. Then, $0\le d_i\le 1$ $(i=1,\cdots,n+1)$ and we have the inequality

$$\sum_{\boldsymbol{a}\in X^-}\delta(\boldsymbol{a}, f_e)\le n+1-2\{(1-d_1)\ell_1+(1-d_{n+1})\ell_n\}.$$

Proof. By Lemma 5, for some non-negative integers x_j, y_j $(j=1,\cdots,n)$ we have the equalities

$$\Delta\equiv\sum_{\boldsymbol{a}\in X^-}\delta(\boldsymbol{a}, f_e)=\sum_{j=1}^{n}\{x_j\delta(\boldsymbol{a}^j_1, f_e)+y_j\delta(\boldsymbol{a}^j_{n+1}, f_e)\}\tag{13}$$

and for $p=1,\cdots,n$

$$\#\{\boldsymbol{a}=(a_1,\cdots,a_{n+1})\in X^-\mid a_1=\cdots=a_p=0\}=\sum_{j=p}^{n}x_j,\tag{14}$$

$$\#\{\boldsymbol{a}=(a_1,\cdots,a_{n+1})\in X^-\mid a_{n+2-p}=\cdots=a_{n+1}=0\}=\sum_{j=p}^{n}y_j,\tag{15}$$

since for each p $(1\le p\le n)$

$$\{1,\cdots,p\}\subset\boldsymbol{a}^j_1(0)\ (j=p,\cdots,n)\ \text{and}\ \{n+2-p,\cdots,n+1\}\subset\boldsymbol{a}^j_{n+1}(0)\ (j=p,\cdots,n).$$

As X^- is in general position, we have the inequalities

$$\#\{\boldsymbol{a}=(a_1,\cdots,a_{n+1})\in X^-\mid a_1=\cdots=a_p=0\}\le n+1-p$$

and

$$\#\{\boldsymbol{a}=(a_1,\cdots,a_{n+1})\in X^-\mid a_{n+2-p}=\cdots=a_{n+1}=0\}\le n+1-p$$

for $p=1,\cdots,n$, which imply with (13) and (14) that x_j and y_j $(j=1,\cdots,n)$ must satisfy the following inequalities for $p=1,\cdots,n$:

$$\sum_{j=p}^{n}x_j\le n+1-p\tag{16}$$

and

$$\sum_{j=p}^{n} y_j \leq n+1-p. \tag{17}$$

By Proposition 3 and (13) we obtain

$$\frac{\Delta}{2} = \sum_{j=1}^{n} \{x_j \sum_{k=1}^{j} \ell_k + y_j \sum_{k=1}^{j} \ell_{n+1-k}\}.$$

By (16) and (17) for $p=1$ we have the inequality

$$\begin{aligned}\frac{\Delta}{2} &\leq (n-\sum_{j=2}^{n} x_j)\ell_1 + (n-\sum_{j=2}^{n} y_j)\ell_n + \sum_{j=2}^{n} \{x_j \sum_{k=1}^{j} \ell_k + y_j \sum_{k=1}^{j} \ell_{n+1-k}\} \\ &= n(\ell_1+\ell_n) + \sum_{j=2}^{n} \{x_j \sum_{k=2}^{j} \ell_k + y_j \sum_{k=2}^{j} \ell_{n+1-k}\}\end{aligned}$$

by (16) and (17) for $p=2$

$$\begin{aligned}&\leq n(\ell_1+\ell_n) + (n-1-\sum_{j=3}^{n} x_j)\ell_2 + (n-1-\sum_{j=3}^{n} y_j)\ell_{n-1} + \sum_{j=3}^{n} \{x_j \sum_{k=2}^{j} \ell_k + y_j \sum_{k=2}^{j} \ell_{n+1-k}\} \\ &= n(\ell_1+\ell_n) + (n-1)(\ell_2+\ell_{n-1}) + \sum_{j=3}^{n} \{x_j \sum_{k=3}^{j} \ell_k + y_j \sum_{k=3}^{j} \ell_{n+1-k}\}\end{aligned}$$

by using (16) and (17) for $p=3, \cdots, n$ successively

$$\begin{aligned}&\leq \sum_{j=1}^{p-1} (n+1-j)(\ell_j+\ell_{n+1-j}) + \sum_{j=p}^{n} \{x_j \sum_{k=p}^{j} \ell_k + y_j \sum_{k=p}^{j} \ell_{n+1-k}\} \\ &\leq \sum_{j=1}^{p-1} (n+1-j)(\ell_j+\ell_{n+1-j}) + (n+1-p-\sum_{j=p+1}^{n} x_j)\ell_p + (n+1-p-\sum_{j=p+1}^{n} y_j)\ell_{n+1-p} \\ &\qquad + \sum_{j=p+1}^{n} \{x_j \sum_{k=p}^{j} \ell_k + y_j \sum_{k=p}^{j} \ell_{n+1-k}\} \\ &= \sum_{j=1}^{p} (n+1-j)(\ell_j+\ell_{n+1-j}) + \sum_{j=p+1}^{n} \{x_j \sum_{k=p+1}^{j} \ell_k + y_j \sum_{k=p+1}^{j} \ell_{n+1-k}\} \quad (p=3, \cdots, n-1) \\ &\leq \sum_{j=1}^{n-1} (n+1-j)(\ell_j+\ell_{n+1-j}) + x_n\ell_n + y_n\ell_1 \\ &= \frac{n+1}{2} - \ell_1(1-d_1) - \ell_n(1-d_{n+1})\end{aligned}$$

since $x_n = d_{n+1}$, $y_n = d_1$ and

$$\begin{aligned}\sum_{j=1}^{n-1} (n+1-j)(\ell_j+\ell_{n+1-j}) &= \sum_{j=1}^{n} (n+1-j)(\ell_j+\ell_{n+1-j}) - \ell_1 - \ell_n \\ &= \sum_{j=1}^{n} (n+1-j)\ell_j + \sum_{j=1}^{n} (n+1-j)\ell_{n+1-j} - \ell_1 - \ell_n \\ &= \sum_{j=1}^{n} j\ell_{n+1-j} + \sum_{j=1}^{n} (n+1-j)\ell_{n+1-j} - \ell_1 - \ell_n \\ &= \sum_{j=1}^{n} (n+1)\ell_{n+1-j} - \ell_1 - \ell_n = \frac{n+1}{2} - \ell_1 - \ell_n.\end{aligned}$$

We have our theorem.

We write $\boldsymbol{e}_i^j$ for any vector of type

$$a_i\boldsymbol{e}_i + \cdots + a_j\boldsymbol{e}_j,$$

where $1\leq i\leq j\leq n+1$ and $a_i\cdots a_j\neq 0$.

Proposition 4. (a) $\boldsymbol{e}_i^{n+1}=\boldsymbol{a}_1^{i-1}$ $(2\leq i\leq n+1)$.
(b) $\boldsymbol{e}_1^j=\boldsymbol{a}_{n+1}^{n+1-j}$ $(1\leq j\leq n)$.
(c) $\delta(\boldsymbol{e}_i^j, f_e)=\delta(\boldsymbol{a}_1^{i-1}, f_e)+\delta(\boldsymbol{a}_{n+1}^{n+1-j}, f_e)$ $(2\leq i<j\leq n)$.
(d) $\delta(\boldsymbol{e}_i^i, f_e)=1$ $(1\leq i\leq n+1)$.
Proof. It is trivial that (a) and (b) hold by the definitions of $\boldsymbol{e}_i^j$, $\boldsymbol{a}_1^j$ and $\boldsymbol{a}_{n+1}^j$.
(c) By Lemma 2 and Proposition 3

$$\begin{aligned}\delta(\boldsymbol{e}_i^j, f_e)&=1-2(\ell_i+\cdots+\ell_{j-1})\\&=2\,(\sum_{k=1}^{i-1}\ell_k+\sum_{k=j}^{n}\ell_k)=\delta(\boldsymbol{a}_1^{i-1}, f_e)+\delta(\boldsymbol{a}_{n+1}^{n+1-j}, f_e).\end{aligned}$$

(d) As $\boldsymbol{e}_i^i=a\boldsymbol{e}_i$ $(a\neq 0)$, it is trivial that $\delta(\boldsymbol{e}_i^i, f_e)=1$.

Lemma 6. Suppose that,

$$\sum_{\boldsymbol{a}\in X^+}\delta(\boldsymbol{a}, f_e)=n+1. \tag{18}$$

Then, for each $\boldsymbol{a}\in X^+$ one of the following cases holds:
(a) When $a_1=0$ and $a_{n+1}\neq 0$, there exists an integer i $(1\leq i\leq n)$ satisfying

$$\boldsymbol{a}=\boldsymbol{e}_{i+1}^{n+1}.$$

(b) When $a_1\neq 0$ and $a_{n+1}=0$, there exists an integer j $(1\leq j\leq n)$ satisfying

$$\boldsymbol{a}=\boldsymbol{e}_1^{n+1-j}.$$

(c) When $a_1=0$ and $a_{n+1}=0$, there exist two integers i and j $(1\leq i, j; i+j\leq n)$ satisfying

$$\boldsymbol{a}=\boldsymbol{e}_{i+1}^{n+1-j}.$$

Proof. We use the same notation as in the proof of Theorem 2. From (18), Theorem 2 and its proof we obtain that $d_1=d_{n+1}=1$ and for $p=1, \cdots, n$

$$\left.\begin{aligned}\sum_{j=p}^{n}x_j=n+1-p\\ \sum_{j=p}^{n}y_j=n+1-p\end{aligned}\right\} \tag{19}$$

The solutions of (19) are as follows:

$$x_1=x_2=\cdots=x_n=1 \quad\text{and}\quad y_1=y_2=\cdots=y_n=1.$$

This implies that

$$\sum_{\boldsymbol{a}\in X^+}\delta(\boldsymbol{a}, f_e)=\sum_{j=1}^{n}(\delta(\boldsymbol{a}_1^j, f_e)+\delta(\boldsymbol{a}_{n+1}^j, f_e))=n+1. \tag{20}$$

Let $\boldsymbol{a}=(a_1, \cdots, a_{n+1})$ be in X^+. Then by Lemma 5 we have the followings.
(a) When $a_1=0$ and $a_{n+1}\neq 0$, there exists an integer i $(1\leq i\leq n)$ satisfying

$$\delta(\boldsymbol{a}, f_e)=\delta(\boldsymbol{a}_1^i, f_e) \quad\text{and}\quad \boldsymbol{a}(0)\supset\boldsymbol{a}_1^i(0)$$

from which we obtain the inequality

$$\sharp\boldsymbol{a}(0)\geq\sharp\boldsymbol{a}_1^i(0)=i. \tag{21}$$

(b) When $a_1 \neq 0$ and $a_{n-1}=0$, there exists an integer j $(1 \leq j \leq n)$ satisfying

$$\delta(\boldsymbol{a}, f_e)=\delta(\boldsymbol{a}^j_{n-1}, f_e) \quad \text{and} \quad \boldsymbol{a}(0) \supset \boldsymbol{a}^j_{n-1}(0)$$

from which we obtain the inequality

$$\sharp \boldsymbol{a}(0) \geq \sharp \boldsymbol{a}^j_{n-1}(0)=j. \tag{22}$$

(c) When $a_1=a_{n-1}=0$, there exists two integers i and j $(1 \leq i, j; i+j \leq n)$ satisfying

$$\delta(\boldsymbol{a}, f_e)=\delta(\boldsymbol{a}^i_1, f_e)+\delta(\boldsymbol{a}^j_{n-1}, f_e) \quad \text{and} \quad \boldsymbol{a}(0) \supset \boldsymbol{a}^i_1(0) \cup \boldsymbol{a}^j_{n-1}(0),\ \boldsymbol{a}^i_1(0) \cap \boldsymbol{a}^j_{n-1}(0)=\phi$$

from which we obtain the inequality

$$\sharp \boldsymbol{a}(0) \geq \sharp \boldsymbol{a}^i_1(0)+\sharp \boldsymbol{a}^j_{n-1}(0)=i+j. \tag{23}$$

On the other hand as X^- is in general position we obtain the inequality

$$\sum_{\boldsymbol{a} \in X^+} \sharp \boldsymbol{a}(0) \leq n(n+1) \tag{24}$$

as in the proof of Proposition 2. From (21) through (24) we obtain our lemma.

Corollary 2. Suppose that D is a segment and $\Sigma_{\boldsymbol{a} \in X^+}\delta(\boldsymbol{a}, f_e)=n+1$. Then, X^+ must coinside with one of the following sets X_0 and X_k $(k=1, \cdots, n-1)$ when they are in general position.

(I) $X_0=\{\boldsymbol{e}^j_1 \mid j=1, \cdots, n\} \cup \{\boldsymbol{e}^{n-1}_i \mid i=2, \cdots, n+1\}$.

(II) For any integer k $(1 \leq k \leq n-1)$ and any integers $i_1, \cdots, i_k; j_1, \cdots, j_k$ satisfying the conditions (i) $2 \leq i_1 < i_2 < \cdots < i_k \leq n$, (ii) $j_1, \cdots, j_k$ are distinct and (iii) $2 \leq i_\nu \leq j_\nu \leq n$ $(\nu=1, \cdots, k)$,

$$X_k=\{\boldsymbol{e}^j_1 \mid 1 \leq j \leq n;\ j \neq j_1, \cdots, j_k\} \cup \{\boldsymbol{e}^{n-1}_i \mid 2 \leq i \leq n+1;\ i \neq i_1, \cdots, i_k\} \cup \{\boldsymbol{e}^{j_\nu}_{i_\nu} \mid \nu=1, \cdots, k\}.$$

Proof. We have only to see that for $k=0, 1, \cdots, n-1$

$$\sum_{\boldsymbol{a} \in X_k} \delta(\boldsymbol{a}, f_e)=n+1.$$

(I)

$$\sum_{\boldsymbol{a} \in X_0} \delta(\boldsymbol{a}, f_e)=\sum_{j=1}^{n} \delta(\boldsymbol{e}^j_1, f_e)+\sum_{i=2}^{n-1} \delta(\boldsymbol{e}^{n-1}_i, f_e)=2\sum_{j=1}^{n}(\ell_j+\cdots+\ell_n)+2\sum_{i=2}^{n-1}(\ell_1+\cdots+\ell_{i-1})$$
$$=2+2\sum_{j=2}^{n}(\ell_1+\cdots+\ell_n)=n+1.$$

(II)

$$\sum_{\boldsymbol{a} \in X_k} \delta(\boldsymbol{a}, f_e)=2\sum_{j=1}^{n}(\ell_j+\cdots+\ell_n)-2\sum_{\nu=1}^{k}(\ell_{j_\nu}+\cdots+\ell_n)+2\sum_{i=2}^{n-1}(\ell_i+\cdots+\ell_{i-1})-2\sum_{\nu=1}^{k}(\ell_1+\cdots+\ell_{i_\nu-1})$$
$$+2\sum_{\nu=1}^{k}(\ell_{j_\nu}+\cdots+\ell_n)+2\sum_{\nu=1}^{k}(\ell_1+\cdots+\ell_{i_\nu-1})$$
$$=\sum_{\boldsymbol{a} \in X_0} \delta(\boldsymbol{a}, f_e)=n+1.$$

Example. 1) The set $X_{n-1}=\{\boldsymbol{e}^i_i \mid i=1, \cdots, n+1\}$ is in general position.

2) One of X_{n-2}: In (II) of Corollary 2, for $k=n-2$, put $i_\nu=\nu+1$, $j_\nu=\nu+2$ $(\nu=1, \cdots, n-2)$. Then, we obtain the set

$$\{\boldsymbol{e}^1_1, \boldsymbol{e}^2_1, \boldsymbol{e}^3_2, \boldsymbol{e}^4_3, \cdots, \boldsymbol{e}^n_{n-1}, \boldsymbol{e}^{n-1}_n, \boldsymbol{e}^{n-1}_{n-1}\}$$

which is in general position.

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