On the Defect Relation of Holomorphic Curves for Moving Targets

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Let $f=[f_1,\cdots,f_{n+1}]$ be a transcendental holomorphic curve from $\boldsymbol{C}$ into the n dimensional complex projective space $P^n(\boldsymbol{C})$, $T(r,f)$ the characteristic function of f, X a subset of $\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$ in N-subgeneral position, where $N\geq n$ are positive integers, $X(0)=\{\boldsymbol{a}=(a_1,\cdots,a_{n+1})\in X\,|\,a_{n+1}=0\}$. Put

$$t(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\{\log\max_{1\leq j\leq n}|f_j(re^{i\theta})|-\log\max_{1\leq j\leq n}|f_j(e^{i\theta})|\}\,d\theta.$$

Then, we proved the following theorem in [9]:

Theorem A. For any $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q\in X$ $(2N-n+1<q<\infty)$,

$$\sum_{j=1}^{q}\omega(j)\delta(\boldsymbol{a}_j,f)\leq d+1+(n-d)\Omega,$$

where ω is a Nochka weight function for $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q$, $d=\Sigma_{\boldsymbol{a}_j\in X(0)}\omega(j)$ and $\Omega=\limsup_{r\to\infty}t(r,f)/T(r,f)$.

In this paper, a generalization of this theorem to moving targets is given, which is an improvement of a result by M. Ru and W. Stoll ([4]).

1 Introduction.

Let

$$f:\boldsymbol{C}\to P^n(\boldsymbol{C})$$

be a transcendental holomorphic curve from $\boldsymbol{C}$ into the n dimensional complex projective space $P^n(\boldsymbol{C})$, where n is a positive integer, and let

$$\hat{f}=(f_1,\cdots,f_{n+1}):\boldsymbol{C}\to\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$$

be a reduced representation of f. We then write

$$f=[f_1,\cdots,f_{n+1}].$$

Put

$$\|f(z)\|=\{\sum_{j=1}^{n+1}|f_j(z)|^2\}^{1/2}$$

and the characteristic function $T(r,f)$ of f is defined as follows (see [11]):

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta-\log\|f(0)\|.$$

Then,

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$$\lim_{r\to\infty}\frac{T(r,f)}{\log r}=+\infty$$

since f is transcendental.

We put

$$\mathcal{M}_o(f)=\{\alpha \mid meromorphic\ in\ |z|<\infty,\ T(r,\alpha)=S(r,f)\},$$

where $S(r,f)$ is any quantity satisfying

$$S(r,f)=o(T(r,f))$$

as $r\to\infty$, possibly outside a set of finite linear measure.

Let $\mathcal{F}$ be a subfield of $\mathcal{M}_o(f)$ containing $\boldsymbol{C}$ and

$$\mathcal{F}^{n+1}=\{(\alpha_1,\cdots,\alpha_{n+1})\mid \alpha_j\in\mathcal{F}\}.$$

We also use $S(z,f)$ which is any non-negative function defined on $\boldsymbol{C}$ satisfying

$$\int_0^{2\pi}\log^+ S(re^{i\theta},f)d\theta=S(r,f).$$

Throughout the paper we suppose that f is non-degenerate over $\mathcal{F}$.

For a holomorphic curve $b=[b_1,\cdots,b_{n+1}]$ from $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$ we put

$$\hat{b}=(b_1,\cdots,b_{n+1})\quad\text{and}\quad\tilde{b}=(\frac{b_1}{b_{j_o}},\cdots,\frac{b_{n+1}}{b_{j_o}}),$$

where b_{j_o} is the first element of $b_1,\cdots,b_{n+1}$ not identically equal to zero.

Let $\mathcal{F}(f)$ be the set of holomorphic curves $b=[b_1,\cdots,b_{n+1}]$ from $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$ satisfying $\tilde{b}\in\mathcal{F}^{n+1}$. For any $b=[b_1,\cdots,b_{n+1}]$ of $\mathcal{F}(f)$, we set

$$(b,f)=b_1f_1+\cdots+b_{n+1}f_{n+1}.$$

For b of $\mathcal{F}(f)$ we put

$$m(r,b,f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|b\|\|f\|}{|(b,f)|}d\theta,$$
$$N(r,b,f)=N(r,1/(b,f))$$

and

$$\delta(b,f)=\liminf_{r\to\infty}\frac{m(r,b,f)}{T(r,f)}.$$

These three quantities are independent of the choice of representations of the curves f and b..

Let $N(\geq n)$ be an integer and X be a subset of $\mathcal{F}(f)$ such that $\#X\geq N+1$. We say that X is in N-subgeneral position if and only if the set $\hat{X}=\{\hat{b}\mid b\in X\}$ is in N-subgeneral position; that is to say, any $N+1$ elements of $\hat{X}$ contain $n+1$ elements the determinant of which is not identically equal to zero. By definition, "n-subgeneral position" is "general position".

Ru and Stoll gave the following

Theorem A. For any $q(>2N-n+1)$ elements a_j $(j=1,\cdots,q)$ of $\mathcal{F}(f)$ in N-subgeneral position,

$$\sum_{j=1}^{q}\delta(a_j,f)\leq 2N-n+1$$

([4], Theorem II).

The purpose of this paper is to give a result which contains this theorem and which is an extension of the theorem obtained in [9]. We use the standard notation of the Nevanlinna theory ([2],[3]).

2 Preliminaries

I. Let $f=[f_1,\cdots,f_{n-1}]$, $\mathscr{F}(f)$ etc. be as in Section 1.

Definition 1 ([8]). 1) We put

$$u(z)=\max_{1\le j\le n}|f_j(z)|$$

and

$$t(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log u(re^{i\theta})d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log u(e^{i\theta})d\theta.$$

2) $\limsup_{r\to\infty}\frac{t(r,f)}{T(r,f)}=\Omega$

This $t(r,f)$ is independent of the choice of reduced representations of f and it is easy to see that

(a) $u(z)\le\|f(z)\|$;

(b) $t(r,f)\le T(r,f)+O(1)$;

(c) $N(r,1/f_j)\le t(r,f)+O(1)$ $(j=1,\cdots,n)$;

(d) $0\le\Omega\le 1$.

We can easily give a holomorphic curve for which $\Omega<1$ (see [8]).

Lemma 1. For any $b=[b_1,\cdots,b_{n+1}]$ of $\mathscr{F}(f)$

(a) $b_i/b_j\in\mathscr{F}$ for any $1\le i\ne j\le n+1$ if $b_j\ne 0$; (b) $(b,f)\not\equiv 0$.

It is easy to see this lemma as $\mathscr{F}$ is a field and f is non-degenerate over $\mathscr{F}$ (see Prop.2 in [7]).

By Lemma 1 (b), we have the following

Proposition 1. For any b of $\mathscr{F}(f)$

(a) $m(r,b,f)+N(r,b,f)=T(r,f)+S(r,f)$;

(b) $0\le\delta(b,f)=1-\limsup_{r\to\infty}\frac{N(r,b,f)}{T(r,f)}\le 1$.

Lemma 2. For any $b=[b_1,\cdots,b_{n+1}]$ and $c=[c_1,\cdots,c_{n+1}]$ of $\mathscr{F}(f)$ such that $b_j\ne 0$, $c_k\ne 0$,

$$T\left(r,\frac{(b,f)/b_j}{(c,f)/c_k}\right)\le 2nT(r,f)+S(r,f)$$

We can prove this lemma as in Lemma 6([7]).

For any $b=[b_1,\cdots,b_{n+1}]$ of $\mathscr{F}(f)$ we set

$$\tilde{b}=(\frac{b_1}{b_{j_0}},\cdots,\frac{b_{n+1}}{b_{j_0}})=(g_1,\cdots,g_{n+1}),\qquad \|\tilde{b}\|=\frac{\|b\|}{|b_{j_0}|}$$

and for $F=(b,f)$

$$\tilde{F}=(\tilde{b},f)=\sum_{j=1}^{n+1}g_jf_j=\frac{(b,f)}{b_{j_0}}.$$

Further we set

$$m(r,\tilde{b},f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|\tilde{b}\|\|f\|}{|(\tilde{b},f)|}d\theta,$$

$$N(r,\tilde{b},f)=N(r,1/(\tilde{b},f))$$

and

$$\delta(\tilde{b}, f) = \liminf_{r\to\infty} \frac{m(r, \tilde{b}, f)}{T(r, f)}.$$

Then it is easy to see that

$$m(r, \tilde{b}, f) = m(r, b, f),$$
$$N(r, \tilde{b}, f) = N(r, b, f) + S(r, f)$$

and

$$\delta(\tilde{b}, f) = 1 - \limsup_{r\to\infty} \frac{N(r, \tilde{b}, f)}{T(r, f)} = \delta(b, f).$$

II. Let q be any integer satisfying $2N-n+1<q<\infty$ and put $Q=\{1, 2, \cdots, q\}$. Let

$$X = \{a_1, a_2, \cdots, a_q \mid a_j \in \mathscr{F}(f)\}$$

be in N-subgeneral position and put $\hat{X} = \{\hat{a}_j \mid a_j \in X\,;\, j=1, \cdots, q\}$.

Let $G(j_1, \cdots, j_k)(z)$ be the Gramian of $\hat{a}_{j_1}(z), \cdots, \hat{a}_{j_k}(z)$ where $1\le j_1<j_2<\cdots<j_k\le q$ and $2\le k\le n+1$. We put

$$I = \{(j_1, \cdots, j_k) \mid G(j_1, \cdots, j_k) \neq 0\}$$

and

$$\boldsymbol{S} = \{z \mid G(j_1, \cdots, j_k)(z) = 0,\ (j_1, \cdots, j_k) \in I\}.$$

Then, $\boldsymbol{S}$ is a countable subset of $\boldsymbol{C}$ clustering nowhere in $\boldsymbol{C}$. For $\phi \neq P \subset Q$ and $z \in \boldsymbol{C}$, let

$$H(z, P) = \textit{the linear subspace of } \boldsymbol{C}^{n+1} \textit{ spanned by } \{\hat{a}_j(z) \mid j \in P\}$$

and put

$$d(z, P) = \dim H(z, P).$$

Then, $d(z, P)$ is constant for $z \in \boldsymbol{C}-\boldsymbol{S}$ as in Lemma 3.2([4]), and so we put for $z \in \boldsymbol{C}-\boldsymbol{S}$

$$d(P) = d(z, P).$$

It is easy to see that if $P \subset Q$ and $N+1 \le \# P$, then $d(P) = n+1$.

Ru and Stoll gave the following

Lemma 3([4], p.486). Let $X=\{a_j \mid j \in Q\}$ be a subset of $\mathscr{F}(f)$ in N-subgeneral position. Then for every $z \in \boldsymbol{C}-\boldsymbol{S}$, there exist a Nochka weight function

$$\omega : Q \to (0, 1]$$

and a Nochka constant $\theta \ge 1$ such that

(a) $0<\omega(j)\theta \le 1$ for all $j \in Q$;
(b) $q-2N+n-1 = \theta\left(\sum_{j=1}^{q} \omega(j) - n - 1\right)$;
(c) If $\phi \neq P \subset Q$ with $\# P \le N+1$, then $\sum_{j\in P} \omega(j) \le d(P)$;
(d) $(N+1)/(n+1) \le \theta \le (2N-n+1)/(n+1)$;

Remark 1. If $\# A = N+1$, then $H(z, A) = \boldsymbol{C}^{n+1}$ and $\{\hat{a}_j(z) \mid j \in B(z)\}$ generates $\boldsymbol{C}^{n+1}$ for $z \in \boldsymbol{C}-\boldsymbol{S}$.

Lemma 4 ([4], Theorem 3.3). Let $\omega : Q \to (0, 1]$ be a Nochka weight function given in Lemma 3 and let $\{E_j \mid j \in Q\}$ be a family of functions $E_j : \boldsymbol{C}-\boldsymbol{S} \to [1, \infty)$. Take $A \subset Q$ with $0<\# A \le N+1$ and $z \in \boldsymbol{C}-\boldsymbol{S}$. Then, there is a subset $B=B(z)$ of A such that $\# B(z) = d(A)$ and $\{\hat{a}_j(z) \mid j \in B(z)\}$ is a basis of $H(z, A)$ and such that

$$\prod_{j\in A} E_j(z)^{\omega(j)} \leq \prod_{j\in B} E_j(z).$$

Put

$$X(0) = \{a_j = [a_{j1}, \cdots, a_{jn+1}] \in X \mid a_{jn-1} = 0\} \quad \text{and} \quad \widetilde{X}(0) = \{\widetilde{a}_j \mid a_j \in X(0)\}.$$

Then, $0 \leq l = \sharp X(0) \leq N$. Without loss of generality we put

$$X(0) = \{a_{p+1}, \cdots, a_{p+l}\},$$

where $q - l = p$. Further we put

$$G_k = (a_{p+k}, f), \qquad \widetilde{G}_k = (\widetilde{a}_{p+k}, f) \ (k = 1, \cdots, l)$$

and

$$d = \sum_{k=1}^{l} \omega(p+k),$$

where $\omega : Q \to (0, 1]$ is a Nochka weight function for X. When $l > 0$ we have the following

Lemma 5. For any $z \in \boldsymbol{C} - \boldsymbol{S}$ such that $\widetilde{G}_k(z) \neq 0, \infty$ for $k = 1, \cdots, l$,

(I) When d is an integer, there are linearly independent vectors $\widehat{a}_{p+i_1}(z), \cdots, \widehat{a}_{p+i_d}(z)$ such that

$$|\widetilde{G}_{i_1}(z) \cdots \widetilde{G}_{i_d}(z)| \leq \prod_{k=1}^{l} |\widetilde{G}_k(z)|^{\omega(p+k)}.$$

(II) When d is not an integer, there are linearly independent vectors $\widehat{a}_{p+i_1}(z), \cdots, \widehat{a}_{p+i_{[d]+1}}(z)$ such that

$$|\widetilde{G}_{i_1}(z) \cdots \widetilde{G}_{i_{[d]+1}}(z)| \leq S(z, f) u(z)^{[d]+1-d} \prod_{k=1}^{l} |\widetilde{G}_k(z)|^{\omega(p+k)}.$$

Proof. For a point z satisfying the condition given above, we suppose for brevity that

$$|\widetilde{G}_1(z)| \leq |\widetilde{G}_2(z)| \leq \cdots \leq |\widetilde{G}_l(z)|.$$

(A) (resp. (B)). We choose $i_1, \cdots, i_d$ (*resp.* $i_1, \cdots, i_{[d]+1}$) as follows:

(i) $i_1 = 1$.

(ii) Suppose that $i_1, \cdots, i_{\mu-1}$ are chosen for $\mu \geq 2$. Then we choose i_μ as follows ($\mu \leq d$ (resp. $\mu \leq [d]+1$)): "i_μ is the least number in $\{i_{\mu-1}+1, \cdots, l\}$ such that $\widehat{a}_{p+i_1}(z), \cdots, \widehat{a}_{p+i_\mu}(z)$ are linearly independent." Then, $\widetilde{G}_{i_1}(z), \cdots, \widetilde{G}_{i_d}(z)$ (resp. $\widetilde{G}_{i_1}(z), \cdots, \widetilde{G}_{i_{[d]+1}}(z)$) satisfy the inequality in (I) (resp. (II)).

In fact, put for $1 \leq m \leq d-1$ (resp. $1 \leq m \leq [d]$)

$$\sigma(m) = i_{m+1} - 1 \quad \text{and} \quad \varphi(m) = \sum_{k=1}^{\sigma(m)} \omega(p+k).$$

We first note that

$$\prod_{k=1}^{\sigma(d-1)} |\widetilde{G}_k(z)|^{\omega(p+k)} \cdot |\widetilde{G}_{i_d}(z)|^{d-\varphi(d-1)} \leq \prod_{k=1}^{l} |\widetilde{G}_k(z)|^{\omega(p+k)} \tag{1}$$

(resp.

$$\prod_{k=1}^{\sigma([d])} |\widetilde{G}_k(z)|^{\omega(p+k)} \cdot |\widetilde{G}_{i_{[d]+1}}(z)|^{[d]+1-\varphi([d])} \leq \prod_{k=1}^{l} |\widetilde{G}_k(z)|^{\omega(p+k)} \cdot (S(z, f) u(z)^{[d]+1-d}) \tag{2}$$

since

$$|\widetilde{G}_{i_d}(z)| \leq |\widetilde{G}_k(z)| \ (i_d < k) \ (\textit{resp.}\ |\widetilde{G}_{i_{[d]+1}}(z)| \leq |\widetilde{G}_k(z)| \ (i_{[d]+1} < k) \text{ and } |\widetilde{G}_l(z)| \leq S(z, f) u(z)).$$

Then, by using Lemma 3(c), we have

$$|\widetilde{G}_1(z) \cdots \widetilde{G}_{i_d}(z)| \leq \prod_{k=1}^{\sigma(d-1)} |\widetilde{G}_k(z)|^{\omega(p+k)} \cdot |\widetilde{G}_{i_d}(z)|^{d-\varphi(d-1)} \tag{3}$$

which is equivalent to

$$\sum_{\nu=1}^{d}\log|\tilde{G}_{i_\nu}(z)|\leq\sum_{k=1}^{\sigma(d-1)}\omega(p+k)\log|\tilde{G}_k(z)|+(d-\varphi(d-1))\log|\tilde{G}_{i_d}(z)|. \tag{4}$$

We prove (4) as follows. We first note that by Lemma 3(c)

$$\varphi(m)\leq m \quad (m=1,\cdots,d-1 \text{ (resp. } [d])). \tag{5}$$

By the choice of $\{i_1,\cdots,i_d\}$, we have the following inequalities.

$$\log|\tilde{G}_{i_1}(z)|\leq\sum_{k=1}^{\sigma(1)}\omega(p+k)\log|\tilde{G}_k(z)|+(1-\varphi(1))\log|\tilde{G}_{i_2}(z)|;$$

$$\log|\tilde{G}_{i_m}(z)|\leq\sum_{k=i_m}^{\sigma(m)}\omega(p+k)\log|\tilde{G}_k(z)|+(m-\varphi(m))\log|\tilde{G}_{i_{m-1}}(z)|-(m-1-\varphi(m-1))\log|\tilde{G}_{i_m}(z)|$$

$(m=2,\cdots,d-1)$;

$$\log|\tilde{G}_{i_d}(z)|=\log|\tilde{G}_{i_d}(z)|.$$

Adding all these d inequalities side by side, we have

$$\sum_{\nu=1}^{d}\log|\tilde{G}_{i_\nu}(z)|\leq\sum_{k=1}^{\sigma(d-1)}\omega(p+k)\log|\tilde{G}_k(z)|+(d-\varphi(d-1))\log|\tilde{G}_{i_d}(z)|,$$

which is the desired inequality. From (1) and (3) we have (I).

(resp. We can also prove

$$|\tilde{G}_{i_1}(z)\cdots\tilde{G}_{i_{[d]+1}}(z)|\leq\sum_{k=1}^{\sigma([d])}|\tilde{G}_k(z)|^{\omega(p+k)}\cdot|\tilde{G}_{i_{[d]+1}}(z)|^{[d]+1-\varphi([d])} \tag{6}$$

as in (3). From (2) and (6), we have (II).)

3 Defect relation

Let $f=[f_1,\cdots,f_{n+1}]$, $\mathscr{F}(f)$, X, $X(0)$, $\tilde{X}$ and $\tilde{X}(0)$ etc. be as in Section 2. Then, we have the following theorem.

Theorem. Put $d=\Sigma_{k=1}^{l}\omega(p+k)$. Then, the following inequality holds:

$$\sum_{j=1}^{q}\omega(j)\delta(a_j,f)\leq d+1+(n-d)\Omega,$$

where $q=p+l$ and ω is a Nochka weight function from $Q=\{1,\cdots,q\}$ into $(0,1]$ given in Lemma 3.

Proof. Put for $j=1,\cdots,q$

$$a_j=[a_{j1},\cdots,a_{jn+1}],\quad \tilde{a}_j=(g_{j1},\cdots,g_{jn+1}),\quad F_j=(a_j,f),\quad \tilde{F}_j=(\tilde{a}_j,f)$$

and

$$E_j=\frac{\|\tilde{a}_j\|\|f\|}{|\tilde{F}_j|}=\frac{\|a_j\|\|f\|}{|F_j|}. \tag{7}$$

For any integer s, let $V(s)$ be the vector space generated by

$$\{\prod_{k=1}^{n+1}\prod_{j=1}^{q}g_{jk}^{s(j,k)}\mid\sum_{k=1}^{n+1}\sum_{j=1}^{q}s(j,k)\leq s,\ s(j,k)\geq 0 \textit{ and integer}\}$$

over $\boldsymbol{C}$ and put

$$d(s)=\dim V(s).$$

Then, $V(s)$ is a subspace of $V(s+1)$ and

$$\liminf_{s\to\infty}\frac{d(s+1)}{d(s)}=1 \tag{8}$$

by the deduction to absurdity since $d(s)\le\binom{q(n+1)+s}{s}$ (see [5],[6]).

Let

$$b_1,\cdots,b_{d(s)},b_{d(s)+1},\cdots,b_{d(s+1)}$$

be a basis of $V(s+1)$ such that

$$b_1,\cdots,b_{d(s)}$$

form a basis of $V(s)$. Then, the functions

$$\{b_tf_k\,|\,t=1,\cdots,d(s+1),k=1,\cdots,n+1\}$$

are linearly independent over $\boldsymbol{C}$. We put

$$W=W(b_1f_1,b_2f_1,\cdots,b_{d(s+1)}f_{n+1}),$$

where $W(g,\cdots,h)$ is the Wronskian of the functions $g,\cdots,h$. Note that

$$N(r,W)=S(r,f).$$

Let $z(\neq 0)$ be a point of $\boldsymbol{C}-\boldsymbol{S}$ and at which none of $\{\tilde{F}_j\}_{j=1}^q$ has pole or zero and none of $\{g_{jn+1}\}_{j=1}^p$ has zero. Note that we have only to consider the case $p\geqq N+1$. We rearrange $\{\tilde{F}_j(z)\}_{j=1}^p$ as follows:

$$|\tilde{F}_{j_1}(z)|\le|\tilde{F}_{j_2}(z)|\le\cdots\le|\tilde{F}_{j_{N+1}}(z)|\le\cdots\le|\tilde{F}_{j_p}(z)|,$$

where $1\le j_1,\cdots,j_p\le p$. Then, we have

$$\|f(z)\|\le S(z,f)|\tilde{F}_{j_k}(z)|\ (k=N+1,\cdots,p), \tag{9}$$

$$|\tilde{F}_{j_k}(z)|\le S(z,f)\|f(z)\|\ (k=1,\cdots,p) \tag{10}$$

and for any $j_k(\le p)$

$$\|f(z)\|\le S(z,f)(|f_1(z)|^2+\cdots+|f_n(z)|^2+|\tilde{F}_{j_k}(z)|^2)^{1/2} \tag{11}$$

$$\le\begin{cases}S(z,f)u(z) & \text{if } |\tilde{F}_{j_k}(z)|\le u(z),\\ S(z,f)|\tilde{F}_{j_k}(z)| & \text{otherwise}\end{cases} \tag{12}$$

since the $n+1-$th element of $\tilde{a}_{j_k}$ is different from zero at z for any $j_k(\le p)$.

By (9) we have at the point z

$$\left(\prod_{j=1}^{q}E_j^{\omega(j)}\right)^{d(s)}\le S(z,f)\left(\prod_{\nu=1}^{N+1}E_{j_\nu}^{\omega(j_\nu)}\cdot\prod_{k=1}^{l}E_{p+k}^{\omega(p+k)}\right)^{d(s)}\equiv J_1. \tag{13}$$

We want to estimate this J_1. By Lemma 4 we have

$$J_1\le S(z,f)\left(\prod_{i=1}^{n+1}\frac{\|f\|}{|\tilde{H}_i|}\right)^{d(s)}, \tag{14}$$

where $\tilde{H}_1,\cdots,\tilde{H}_{n+1}$ are chosen from $\{\tilde{F}_{j_1},\cdots,\tilde{F}_{j_{N+1}},\tilde{G}_1,\cdots,\tilde{G}_l\}$ and are linearly independent over $\mathscr{F}$. We put

$$\tilde{H}_\mu=(\tilde{a}_{i_\mu},f)\ (\mu=1,\cdots,n+1),\ \boldsymbol{H}=\{\tilde{a}_{i_\mu}\}_{\mu=1}^{n+1}$$

and note that

$$\boldsymbol{H}-\tilde{X}(0)\neq\phi.$$

We put

$$J_2=\left(\frac{\|f\|^{n+1}}{|\tilde{H}_1\cdots\tilde{H}_{n+1}|}\right)^{d(s)}$$

(I) The case when for any μ such that $\tilde{a}_{i_\mu} \in \boldsymbol{H} - \tilde{X}(0)$

$$u(z) < |\tilde{H}_\mu(z)|$$

and for some j_ν $(1 \leq \nu \leq N+1)$

$$|\tilde{F}_{j_\nu}(z)| \leq u(z),$$

or when for some μ such that $\tilde{a}_{i_\mu} \in \boldsymbol{H} - \tilde{X}(0)$

$$|\tilde{H}_\mu(z)| \leq u(z).$$

In this case, we have by (11) and (12)

$$\|f(z)\| \leq S(z, f) u(z)$$

and

$$J_2 \leq S(z, f)\left(\frac{u(z)^{n+1}}{|\tilde{H}_1 \cdots \tilde{H}_{n+1}|}\right)^{d(s)}. \tag{15}$$

Now, as $\tilde{H}_1, \cdots, \tilde{H}_{n+1}$ are linearly independent over $\mathscr{F}$, it holds that

$$\{b_1\tilde{H}_1,\ b_2\tilde{H}_1,\ \cdots,\ b_{d(s)}\tilde{H}_{n+1}\}$$

are linearly independent over $\boldsymbol{C}$. Since $\tilde{F}_j = (\tilde{a}_j, f)$, these $(n+1)d(s)$ functions can be represented as linear combinations of

$$\{b_t f_k \mid 1 \leq t \leq d(s+1),\ 1 \leq k \leq n+1\}$$

with constant coefficients:

$$(b_1\tilde{H}_1,\ b_2\tilde{H}_1,\ \cdots,\ b_{d(s)}\tilde{H}_{n+1}) = (b_1 f_1,\ b_2 f_1,\ \cdots,\ b_{d(s+1)} f_{n+1}) D_1,$$

where D_1 is an $(n+1)d(s+1) \times (n+1)d(s)$ matrix the elements of which are constants and the rank of which is equal to $(n+1)d(s)$. Let D_2 be an $(n+1)d(s+1) \times (n+1)\{d(s+1) - d(s)\}$ matrix consisting of constant elements such that the matrix

$$D = (D_1\ D_2)$$

is regular. Put for $L = (n+1)\{d(s+1) - d(s)\}$

$$(K_1,\ \cdots,\ K_L) = (b_1 f_1,\ b_2 f_1,\ \cdots,\ b_{d(s+1)} f_{n+1}) D_2.$$

then

$$(b_1\tilde{H}_1,\ b_2\tilde{H}_1,\ \cdots,\ b_{d(s)}\tilde{H}_{n+1},\ K_1,\ \cdots,\ K_L) = (b_1 f_1,\ b_2 f_1,\ \cdots,\ b_{d(s+1)} f_{n+1}) D \tag{16}$$

from which we obtain

$$W(b_1\tilde{H}_1, b_2\tilde{H}_1,\ \cdots,\ K_L) = (\det D) W, \qquad \det D \neq 0 \tag{17}$$

where $W = W(b_1 f_1,\ b_2 f_1,\ \cdots,\ b_{d(s+1)} f_{n+1})$. We then have from (11)

$$\begin{aligned}
\frac{1}{\left(\Pi_{k=1}^{n+1} |\tilde{H}_k|\right)^{d(s)}} &= \frac{|W(b_1\tilde{H}_1,\ \cdots,\ K_L)|}{|W|\,|\det D|} \cdot \frac{1}{\left(\Pi_{k=1}^{n+1} |\tilde{H}_k|\right)^{d(s)}} \\
&= \frac{1}{|\det D|\,|W|} \cdot \frac{|W(b_1\tilde{H}_1,\ \cdots,\ K_L)|}{\left(\Pi_{k=1}^{n+1} |\tilde{H}_k|\right)^{d(s)}} \\
&\leq S(z, f) \frac{u(z)^L}{|W|} \cdot \frac{|W(b_1\tilde{H}_1,\ b_2\tilde{H}_1,\ \cdots,\ K_L)|}{|b_1\tilde{H}_1 \cdot b_2\tilde{H}_1 \cdots K_L|}
\end{aligned} \tag{18}$$

since $|\widetilde{H}_k(z)|\le S(z,f)\|f(z)\|$, $|K_j(z)|\le S(z,f)\|f(z)\|$ and $\|f(z)\|\le S(z,f)u(z)$ in this case.

From (15) and (18) we have

$$J_2\le S(z,f)\frac{u(z)^{(n+1)d(s+1)}}{|W|}\cdot\frac{|W(b_1\widetilde{H}_1,b_2\widetilde{H}_1,\cdots,K_L)|}{|b_1\widetilde{H}_1\cdot b_2\widetilde{H}_1\cdots K_L|} \tag{19}$$

(II) The case when for any μ such that $\widetilde{a}_{i_\mu}\in \boldsymbol{H}-\widetilde{X}(0)$

$$u(z)<|\widetilde{H}_\mu(z)|$$

and for any j_ν $(1\le\nu\le N+1)$

$$u(z)<|\widetilde{F}_{j_\nu}(z)|.$$

In this cae, by (11) and (12) we have for any j_ν $(\nu=1,\cdots,N+1)$

$$\|f(z)\|\le S(z,f)|\widetilde{F}_{j_\nu}|$$

and from (13) we have

$$J_1\le S(z,f) \tag{20}$$

when $l=0$, and when $l>0$

$$\begin{aligned}J_1&\le S(z,f)\left(\prod_{k=1}^{l}E_{p+k}^{\omega(p+k)}\right)^{d(s)}\\&\le S(z,f)\frac{\|f(z)\|^{dd(s)}}{\left(\prod_{k=1}^{l}|\widetilde{G}_k(z)|^{\omega(p+k)}\right)^{d(s)}}\end{aligned} \tag{21}$$

When $l>0$ we put

$$J_3=1/\left(\prod_{k=1}^{l}|\widetilde{G}_k(z)|^{\omega(p+k)}\right)^{d(s)}.$$

When d is a positive integer, by Lemma 5(I) there are d functions $\widetilde{G}_{i_1},\cdots,\widetilde{G}_{i_d}$ linearly independent over $\mathscr{F}$ such that

$$J_3\leqq 1/|\widetilde{G}_{i_1}(z)\cdots\widetilde{G}_{i_d}(z)|^{d(s)}. \tag{22}$$

When d is not an integer, by Lemma 5(II) there are $[d]+1$ functions $\widetilde{G}_{i_1},\cdots,\widetilde{G}_{i_{[d]+1}}$ linearly independent over $\mathscr{F}$ such that

$$J_3\leqq S(z,f)u(z)^{([d]+1-d)d(s)}/|\widetilde{G}_{i_1}(z)\cdots\widetilde{G}_{i_{[d]+1}}(z)|^{d(s)}. \tag{23}$$

We put

$$<d>=\begin{cases}d & \text{if } d \text{ is an integer,}\\ [d]+1 & \text{otherwise.}\end{cases}$$

Now we can find $\boldsymbol{e}_{i_{<d>+1}},\cdots,\boldsymbol{e}_{i_n}$ such that

$$\widetilde{a}_{i_1},\cdots,\widetilde{a}_{i_{<d>}},\boldsymbol{e}_{i_{<d>+1}},\cdots,\boldsymbol{e}_{i_n},\boldsymbol{e}_{n+1}$$

are linearly independent over $\mathscr{F}$, where

$$\boldsymbol{e}_1,\cdots,\boldsymbol{e}_{n+1}$$

are the standard basis of $\boldsymbol{C}^{n+1}$. Then,

$$\widetilde{G}_{i_1},\cdots,\widetilde{G}_{i_{<d>}},f_{i_{<d>+1}},\cdots,f_{i_n},f_{n+1}$$

are linearly independent over $\mathscr{F}$. Put

$$\tilde{H}_j = \begin{cases} \tilde{G}_{i_j} & (j=1, \cdots, <d>), \\ f_{i_j} & (j=<d>+1, \cdots, n), \\ f_{n+1} & (j=n+1) \end{cases}$$

(We use the same notation as in the case (I) for simplicity.) Then, as in the case of (I), there are $K_1, \cdots, K_L$ satisfying (16), (17) and we have the following inequality at z as in (18)

$$\begin{aligned} \frac{1}{\Pi_{k=1}^{<d>} |\tilde{G}_{i_k}(z)|^{d(s)}} &\leq \frac{(\|f(z)\| u(z)^{n-<d>})^{d(s)}}{\Pi_{k=1}^{n+1} |\tilde{H}_k|^{d(s)}} \\ &= \frac{(\|f(z)\| u(z)^{n-<d>})^{d(s)}}{\Pi_{k=1}^{n+1} |\tilde{H}_k|^{d(s)}} \cdot \frac{|W(b_1\tilde{H}_1, \cdots, K_L)|}{|\det D| \, |W|} \\ &\leq S(z,f) \|f(z)\|^L \frac{(\|f(z)\| u(z)^{n-<d>})^{d(s)}}{|W|} \cdot \frac{|W(b_1\tilde{H}_1, \cdots, K_L)|}{|b_1\tilde{H}_1 \cdots K_L|} \end{aligned} \tag{24}$$

since $|f_{i_j}(z)| \leq u(z)$ if $i_j \leq n$ by Definition 1 and for any j, $|K_j(z)| \leq S(z,f)\|f(z)\|$ as in (10). Putting

$$n(s) = (n+1)d(s+1) - (n-d)d(s),$$

from (21),(22),(23) and (24) we have

$$J_1 \leq S(z,f) \frac{\|f(z)\|^{n(s)} u(z)^{(n-d)d(s)}}{|W|} \cdot \frac{|W(b_1\tilde{H}_1, \cdots, K_L)|}{|b_1\tilde{H}_1 \cdots K_L|} \tag{25}$$

Since

$$u(z)^{(n+1)d(s+1)} \leq \|f(z)\|^{n(s)} u(z)^{(n-d)d(s)},$$

from (13),(14),(19),(20) and (25) we have the inequality

$$d(s) \sum_{j=1}^{q} \omega(j) \log \frac{\|a_j(z)\| \|f(z)\|}{|F_j|} \leq \log^+ \frac{\|f(z)\|^{n(s)} u(z)^{(n-d)d(s)}}{|W|} + \sum_{\{H_1, \cdots, H_{n-1}\}} \log^+ \frac{|W(b_1\tilde{H}_1, \cdots, K_L)|}{|b_1\tilde{H}_1 \cdots K_L|} + \log^+ S(z,f),$$

where the sum $\Sigma_{\{H_1, \cdots, H_{n-1}\}}$ is taken over all $\{H_1, \cdots, H_{n-1}\}$ which are linearly independent over $\mathscr{F}$ chosen from $\{F_1, \cdots, F_q, f_1, \cdots, f_{n+1}\}$. This inequality is independent of $z \in \boldsymbol{C} - \boldsymbol{S}$ and at which none of $\{\tilde{F}_j\}_{j=1}^q$ has pole of zero and none of $\{g_{jn+1}\}_{j=1}^p$ has zero.

Integrating this inequality with respect to θ from 0 to 2π, where $z = re^{i\theta}$, we obtain

$$d(s) \sum_{j=1}^{q} \omega(j) m(r, a_j, f) \leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{\|f(z)\|^{n(s)} u(z)^{(n-d)d(s)}}{|W|} d\theta + S(r,f) \tag{26}$$

since

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{|W(b_1\tilde{H}_1, \cdots, K_L)|}{|b_1\tilde{H}_1 \cdots K_L|} d\theta = S(r,f)$$

as in [1] by Lemma 2 and by the inequality

$$T(r, K_j/b_1H_1) \leq 2nT(r,f) + S(r,f) \quad (j=1, \cdots, L)$$

which we can prove as in Lemma 2 since $b_t \in \mathscr{F}$ $(t=1, \cdots, d(s+1))$ and since

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ S(re^{i\theta}, f) d\theta = S(r,f).$$

Now,

$$\log^+ \frac{\|f(z)\|^{n(s)} u(z)^{(n-d)d(s)}}{|W|} = \log \max\{\|f(z)\|^{n(s)} u(z)^{(n-d)d(s)}, |W|\} - \log|W|$$

and

$$\begin{aligned}|W| &= |b_1f_1\cdots b_{d(s+1)}f_{n+1}|\frac{|W|}{|b_1f_1\cdots b_{d(s+1)}f_{n+1}|}\\ &\le S(z,f)\|f(z)\|^{d(s+1)}u(z)^{nd(s+1)}\frac{|W|}{|b_1f_1\cdots b_{d(s+1)}f_{n+1}|}\\ &\le S(z,f)\|f(z)\|^{n(s)}u(z)^{(n-d)d(s)}\frac{|W|}{|b_1f_1\cdots b_{d(s+1)}f_{n+1}|}\end{aligned}$$

since $u(z)\le\|f(z)\|$. Using these relations we have from (26)

$$d(s)\sum_{j=1}^{q}\omega(j)m(r,a_j,f)\le n(s)T(r,f)+(n-d)d(s)t(r,f)-N(r,1/W)+S(r,f), \tag{27}$$

which reduces to

$$d(s)\sum_{j=1}^{q}\omega(j)\delta(a_j,f)\le(n+1)d(s+1)-(n-d)d(s)-(n-d)d(s)\Omega$$

since $n(s)=(n+1)d(s+1)-(n-d)d(s)$.

Dividing both sides of this inequality by $d(s)$ and letting $s\to\infty$ so that $\frac{d(s+1)}{d(s)}$ tends to 1 according to (8), we obtain

$$\sum_{j=1}^{q}\omega(j)\delta(a_j,f)\le d+1+(n-d)\Omega.$$

Remark 2 (Second fundamental inequality). For any positive ϵ,

$$\sum_{j=1}^{q}\omega(j)m(r,a_j,f)\le(d+1+\epsilon)T(r,f)+(n-d)t(r,f)+S(r,f).$$

In fact, let s be so large that $d(s+1)/d(s)<1+\epsilon$ by (8), we have this inequality from (27) immediately.

Corollary 1. Under the same assumption as in Theorem,

$$\sum_{j=1}^{q}\delta(a_j,f)\le 2N-n+1-\frac{(N+1)(n-d)(1-\Omega)}{n+1}.$$

Proof. We can easily prove this corollary by applying Lemma 3(a),(b) and (d) to Theorem as usual.

As in Definition 3 in [10], we can definie X to be maximal or ν-maximal in the sense of subgeneral position. By using this notion, we have the following

Corollary 2. Let X be ν-maximal in the sense of subgeneral position. Then we have the inequality

$$\sum_{j=1}^{q}\delta(a_j,f)\le 2N-n+1-\frac{(N+1)(n-\nu)(1-\Omega)}{n+1}.$$

In fact, the inequality $d\le\nu$ holds in this case and we have this corollary from Corollary 1 immediately.

Corollary 3 ([9], Theorem 3). For any $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q\in\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$ $(2N-n+1<q<\infty)$ in N-subgeneral position, we have the following inequalities:

(A) $\sum_{j=1}^{q}\omega(j)\delta(\boldsymbol{a}_j,f)\le d+1+(n-d)\Omega$;

(B) $\sum_{j=1}^{q}\delta(\boldsymbol{a}_j,f)\le 2N-n+1-\frac{(N+1)(n-d)(1-\Omega)}{n+1}$,

where ω is a Nochka weight function for $X=\{\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q\}$ and $d=\Sigma_{\boldsymbol{a}_j\in X(0)}\omega(j)$, $X(0)=\{\boldsymbol{a}_j=(a_{j1},\cdots,a_{jn+1})\in X\mid a_{jn+1}=0\}$.

By taking $\mathscr{F}=\boldsymbol{C}$ in Theorem and Corollary 1 we have this corollary immediately.

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