On Asymptotic Points of Holomorphic Curves

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Let $f=[f_1,\cdots,f_{n-1}]$ be a transcendental holomorphic curve from $\boldsymbol{C}$ into the n dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1,\cdots,f_{n+1}) : \boldsymbol{C}\to\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\},$$

where n is a positive integer.

Definition (asymptotic point). Let $\boldsymbol{a}$ be a point of $\boldsymbol{C}^{n+1}-V$. Then, we say that $\boldsymbol{a}$ is an asymptotic point of f if there exists a path $\Gamma : z=z(t)\,(0\leq t<1)$ in $|z|<\infty$ satisfying the following conditions:

$$\text{(i)} \lim_{t\to1} z(t)=\infty \quad \text{and} \quad \text{(ii)} \lim_{t\to1}\frac{|(\boldsymbol{a}, f(z(t)))|}{\|\boldsymbol{a}\|\|f(z(t))\|}=0,$$

where $V=\{\boldsymbol{a}\in\boldsymbol{C}^{n+1}:(\boldsymbol{a}, f)=0\}$.

The purpose of this paper is to give some sufficient conditions for $\boldsymbol{a}\in\boldsymbol{C}^{n+1}-V$ to be an asymptotic point of f. For example,

Theorem. If for some point $\boldsymbol{a}$ of $\boldsymbol{C}^{n+1}-V$

$$\lim_{r\to\infty}\{T(r, f)-\frac{r^{1/2}}{2}\int_0^\infty\frac{N_0(t,\boldsymbol{a},f)}{(t+r)^{3/2}}dt-n(0,\boldsymbol{a},f)\log r\}=\infty,$$

then $\boldsymbol{a}$ is an asymptotic point of f.

1 Introduction.

Let $f=[f_1,\cdots,f_{n+1}]$ be a transcendental holomorphic curve from $\boldsymbol{C}$ into the n dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1,\cdots,f_{n+1}) : \boldsymbol{C}\to\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\},$$

where n is a positive integer.

We use the following notation:

$$\|f(z)\|=(|f_1(z)|^2+\cdots+|f_{n+1}(z)|^2)^{1/2}$$

and for a point $\boldsymbol{a}=(a_1,\cdots,a_{n-1})$ in $\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$

$$\|\boldsymbol{a}\|=(|a_1|^2+\cdots+|a_{n+1}|^2)^{1/2},$$
$$(\boldsymbol{a}, f)=a_1f_1+\cdots+a_{n+1}f_{n-1},$$
$$(\boldsymbol{a}, f(z))=a_1f_1(z)+\cdots+a_{n+1}f_{n+1}(z).$$

The characteristic function $T(r, f)$ of f is defined as follows(see [10]):

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$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta-\log\|f(0)\|.$$

We put

$$\rho=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},\quad \lambda=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}$$

and we say that ρ is the order of f and λ the lower order of f. We note that

$$\lim_{r\to\infty}\frac{T(r,f)}{\log r}=\infty$$

since f is transcendental.

Let

$$V=\{\boldsymbol{a}\in \boldsymbol{C}^{n+1}:(\boldsymbol{a},f)=0\}.$$

Then, V is a subspace of $\boldsymbol{C}^{n+1}$ and $0\leq \dim V\leq n-1$. It is said that f is linearly nondegenerate when $\dim V=0$ and linearly degenerate otherwise.

For meromorphic functions in $|z|<\infty$ we shall use the standard notation and symbols of the Nevanlinna theory of meromorphic functions ([3],[5]).

For $\boldsymbol{a}\in \boldsymbol{C}^{n+1}-V$, we put for $r>0$

$$n(r,\boldsymbol{a},f)=n(r,1/(\boldsymbol{a},f)),$$
$$N(r,\boldsymbol{a},f)=N(r,1/(\boldsymbol{a},f)),$$
$$m(r,\boldsymbol{a},f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|\boldsymbol{a}\|\|f(re^{i\theta})\|}{|(\boldsymbol{a},f(re^{i\theta}))|}d\theta,$$
$$\delta(\boldsymbol{a},f)=\liminf_{r\to\infty}\frac{m(r,\boldsymbol{a},f)}{T(r,f)}$$
$$=1-\limsup_{r\to\infty}\frac{N(r,\boldsymbol{a},f)}{T(r,f)}.$$

The last equality holds since

$$N(r,\boldsymbol{a},f)+m(r,\boldsymbol{a},f)=T(r,f)+O(1)$$

(the first fundamental theorem (see [10])).

As a natural generalization of asymptotic values for meromorphic functions, we gave a definition of asymptotic points for systems of entire functions (see Definition 3 in [7] or Definition 2 in [9]). The definition is also valid for holomorphic curves. We rewrite it for holomorphic curvres.

Definition 1 (asymptotic point). Let $\boldsymbol{a}$ be a point of $\boldsymbol{C}^{n+1}-V$. Then, we say that $\boldsymbol{a}$ is an asymptotic point of f if there exists a path $\Gamma: z=z(t)(0\leq t<1)$ in $|z|<\infty$ satisfying the following conditions:

(i) $\lim_{t\to 1} z(t)=\infty$;

(ii) $\lim_{t\to 1}\frac{|(\boldsymbol{a},f(z(t)))|}{\|\boldsymbol{a}\|\|f(z(t))\|}=0.$

We denote by $A(f)$ the set of asymptotic points of f.

We here give some theorems on asymptotic points obtained for systems of entire functions in [7] or [9], which are valid for holomorphic curves too.

Theorem A. If $\boldsymbol{a}\in \boldsymbol{C}^{n+1}-V$ is Picard exceptional for f; that is to say, the number of zeos of $(\boldsymbol{a},f)$ is at most finite, then $\boldsymbol{a}\in A(f)$ (see Theorem 1 in [7]).

Theorem B. Suppose that $\lim_{r\to\infty}T(2r,f)/T(r,f)=1$. If there exists a point $\boldsymbol{a}\in \boldsymbol{C}^{n+1}-V$ such that $\delta(\boldsymbol{a},f)>0$, then $\boldsymbol{a}\in A(f)$ (see Theorem 5 and Remark 4 in [9]).

The main purpose of this paper is to extend Theorem 2 in [4] or Lemma 1 in [8] to holomorphic curves to obtain a result containing Theorems A and B, and then to give some results for holomorphic curves with smooth growth.

2 General case.

The purpose of this section is to extend Theorem 2 in [4] or Lemma 1 in [8] to holomorphic curves. We shall first give some lemmas for later use. Let f be as in Section 1 and $\overline{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$.

Lemma 1. Let $h:\mathbf{C}\to\overline{\mathbf{C}}$ be continuous in the spherical metric. Then, at least one of the following possibilities must occur:

(a) h has ∞ as an asymptotic value at ∞;

(b) h is bounded on a path γ going to ∞;

(c) h is uniformly bounded on a sequence $\{\Gamma_k\}_{k=1}^{\infty}$ of closed curves which surround the origin and recede to ∞ with k (Theorem 2 in [2]).

Lemma 2. Every component of the complement of a continuum in $\overline{\mathbf{C}}$ is simply connected (Theorem 4.4 in [6]).

Lemma 3. Suppose that $D(\neq\mathbf{C})$ is a simply connected domain containing the origin in the complex plane and let d be the distance from the origin to the complement of D. Further let $g(z,a)$ be the Green function of D with pole at a. Then, for $a\neq 0$ in D

(I) $\log^{+}\frac{d}{|a|}\leq g(0,a)\leq \log\{1+\frac{2d}{|a|}+2\{\frac{d}{|a|}+(\frac{d}{|a|})^2\}^{1/2}\}$;

(II) $\frac{1}{2\pi}\int_0^{2\pi}g(de^{i\theta},a)d\theta=g(0,a)-\log^{+}\frac{d}{|a|}$.

Proof. (I) See Lemma 5 in [4] and its improvement in [8], p.492.

(II) The function

$$g(z,a)+\log|z-a|$$

is harmonic in D and continuous on $|z|=r$ $(0<r\leq d)$, so that we have

$$\frac{1}{2\pi}\int_0^{2\pi}\{g(de^{i\theta},a)+\log|de^{i\theta}-a|\}d\theta=g(0,a)+\log|a|.$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}\log|de^{i\theta}-a|d\theta=\log d+\log^{+}\frac{|a|}{d}$$

(see the formula (1.2) in [3], p.8), we easily have our equality.

Lemma 4. Suppose that there is a simply connected domain $D(\neq\mathbf{C})$ containing the origin in the complex plane such that for a point $\boldsymbol{a}\in\mathbf{C}^{n+1}-V$

$$\frac{\|\boldsymbol{a}\|\|f(z)\|}{|(\boldsymbol{a},f(z))|}\leq M$$

on the finite boundary Γ_o of D for a finite positive number M. Let d be the distance from the origin to Γ_o. Then, we have

$$m(d,\boldsymbol{a},f)+N_o(d,\boldsymbol{a},f)\leq\frac{\sqrt{d}}{2}\int_0^{\infty}\frac{N_o(t,\boldsymbol{a},f)}{(t+d)^{3/2}}dt+\log M+n(0,\boldsymbol{a},f)\log(3+\sqrt{8}),\qquad(1)$$

where

$$n_o(t,\boldsymbol{a},f)=n(t,\boldsymbol{a},f)-n(0,\boldsymbol{a},f)\quad\text{and}\quad N_o(r,\boldsymbol{a},f)=\int_0^r\frac{n_o(t,\boldsymbol{a},f)}{t}dt.$$

Proof. We have only to prove the inequality (1) when

$$\int_0^{\infty}\frac{N_o(t,\boldsymbol{a},f)}{(t+d)^{3/2}}dt<\infty.\qquad(2)$$

Assume that (2) holds. Let $a_1, a_2, \cdots$ be the zeros of $(\boldsymbol{a},f)$ different from zero such that $|a_1|\leq|a_2|\leq\cdots$. In this sequence, each multiple zero appears as many times as its multiplicity. Then, (2) is equivalent to

$$\sum_{\nu} |a_\nu|^{-1/2} < \infty, \tag{3}$$

since

$$\int_0^\infty \frac{N_o(t, \boldsymbol{a}, f)}{(t+d)^{3/2}} dt = 2\int_0^\infty \frac{n_o(t, \boldsymbol{a}, f)}{t(t+d)^{1/2}} dt$$

and for $r > 0$

$$\int_0^\infty \frac{n_o(t, \boldsymbol{a}, f)}{t(t+d)^{1/2}} dt < \infty \quad \text{if and only if} \quad \sum_{\nu} |a_\nu|^{-1/2} < \infty.$$

Let $g(z, a_\nu)$ be the Green function of D with pole at a_ν. Then, since it follows from Lemma 3 and (3) that

$$\begin{aligned}\sum_{\nu} g(0, a_\nu) &\leq \sum_{\nu} \log\{1 + \frac{2d}{|a_\nu|} + 2\{\frac{d}{|a_\nu|} + (\frac{d}{|a_\nu|})^2\}^{1/2}\} \\ &= 2\sum_{\nu} \log\{(1+\frac{d}{|a_\nu|})^{1/2} + (\frac{d}{|a_\nu|})^{1/2}\} \\ &= 2\int_0^\infty \log\{(1+\frac{d}{|t|})^{1/2} + (\frac{d}{|t|})^{1/2}\} dn_o(t, \boldsymbol{a}, f) \\ &= d^{1/2} \int_0^\infty \frac{n_o(t, \boldsymbol{a}, f)}{t(t+d)^{1/2}} dt < \infty, \end{aligned} \tag{4}$$

the sum

$$g(z) = \sum_{\nu} g(z, a_\nu)$$

converges uniformly in any compact subset of D to a function harmonic in D except at the points a_ν and vanishes continuously on the finite boundary of D. The function

$$u(z) = \log \frac{\|\boldsymbol{a}\| \|f(z)\|}{|(\boldsymbol{a}, f(z))|} - n(0, \boldsymbol{a}, f) g(z, 0) - g(z) - \log M$$

is subharmonic in D and satisfies

$$u(z) \leq 0 \quad \text{on} \quad \Gamma_o.$$

This implies that $u(z) \leq 0$ in D, so that we have in D

$$\log \frac{\|\boldsymbol{a}\| \|f(z)\|}{|(\boldsymbol{a}, f(z))|} \leq n(0, \boldsymbol{a}, f) g(z, 0) + g(z) + \log M.$$

Integrating both sides of this inequality with respect to θ ($z = de^{i\theta}$), we obtain

$$\begin{aligned} m(d, \boldsymbol{a}, f) &\leq n(0, \boldsymbol{a}, f) \frac{1}{2\pi} \int_0^{2\pi} g(de^{i\theta}, 0) d\theta + \frac{1}{2\pi} \int_0^{2\pi} g(de^{i\theta}) d\theta + \log M \\ &\leq n(0, \boldsymbol{a}, f) \log(3+\sqrt{8}) + g(0) - \sum_{|a_\nu| \leq d} \log^- \frac{d}{|a_\nu|} + \log M \end{aligned} \tag{5}$$

since by Lemma 3(I)

$$g(de^{i\theta}, 0) = g(0, de^{i\theta}) \leq \log(3+\sqrt{8})$$

and since from the equality

$$g(z) = \sum_{|a_\nu| \leq d} g(z, a_\nu) + \sum_{|a_\nu| > d} g(z, a_\nu),$$

where the second term of the right-hand side is harmonic in $|z| < d$ and continuous on $|z| = d$, and from Lemma 3(II) we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} g(de^{i\theta}) d\theta &= \sum_{|a_\nu| \leq d} \frac{1}{2\pi} \int_0^{2\pi} g(de^{i\theta}, a_\nu) d\theta + \frac{1}{2\pi} \int_0^{2\pi} \sum_{|a_\nu| > d} g(de^{i\theta}, a_\nu) d\theta \\ &= g(0) - \sum_{|a_\nu| \leq d} \log^- \frac{d}{|a_\nu|}. \end{aligned}$$

Using the inequality (4) and integrating by parts, from (5) we obtain

$$m(d,\boldsymbol{a},f)+N_o(d,\boldsymbol{a},f)\leq\sqrt{d}\int_0^\infty\frac{n_o(t,\boldsymbol{a},f)}{t(t+d)^{1/2}}dt+\log M+n(0,\boldsymbol{a},f)\log(3+\sqrt{8})$$
$$=\frac{\sqrt{d}}{2}\int_0^\infty\frac{N_o(t,\boldsymbol{a},f)}{(t+d)^{3/2}}dt+\log M+n(0,\boldsymbol{a},f)\log(3+\sqrt{8})$$

since

$$\sum_{|a_\nu|\leq d}\log^+\frac{d}{|a_\nu|}=N_o(d,\boldsymbol{a},f).$$

Theorem 1. If for some point $\boldsymbol{a}$ of $\boldsymbol{C}^{n+1}-V$

$$\lim_{r\to\infty}\{T(r,f)-\frac{r^{1/2}}{2}\int_0^\infty\frac{N_o(t,\boldsymbol{a},f)}{(t+r)^{3/2}}dt-n(0,\boldsymbol{a},f)\log r\}=\infty, \tag{6}$$

then $\boldsymbol{a}\in A(f)$.

Proof. Suppose that $\boldsymbol{a}$ does not belong to $A(f)$. Then, it follows from Lemma 1 with $h=\|\boldsymbol{a}\|\|f\|/|(\boldsymbol{a},f)|$ that there exists a positive constant M such that

$$\frac{\|\boldsymbol{a}\|\|f\|}{|(\boldsymbol{a},f)|}\leq M \tag{7}$$

(a) on a path γ going to ∞; or

(b) on the union of a sequence $\{\Gamma_k\}$ of closed curves surrounding the origin and receding to ∞ with k.

The case (a). Suppose first that (7) holds on γ and the path γ goes from $z_o(\neq 0)$ to ∞. Then, the path meets the circle $|z|=d$ for $d>|z_o|$. Hence there exists an arc γ_d of this path joining a point $z_1=de^{i\theta}$ to ∞ and lying otherwise in $|z|>d$. Let D be the component of $\boldsymbol{C}-\gamma_d$ containing the origin. Then D is simply connected by Lemma 2 and (7) is satisfied on the finite boundary of D, so that we obtain (1) of Lemma 4 with any $d>|z_o|$, which contradicts (6).

The case (b). Suppose that (7) holds on $\{\Gamma_k\}$. Let D_k be the component of $\boldsymbol{C}-\Gamma_k$ containing the origin and d_k be the distance from the origin to ∂D_k. Then, D_k is simply connected by Lemma 2, $d_k\to\infty\,(k\to\infty)$ and (7) holds on ∂D_k. Thus we obtain (1) of Lemma 4 with $d=d_k$ for any k, which contradicts (6).

Thus by Lemma 1, $\|\boldsymbol{a}\|\|f\|/|(\boldsymbol{a},f)|$ must have ∞ as an asymptotic value. This implies that $|(\boldsymbol{a},f)|/\|\boldsymbol{a}\|\|f\|$ has 0 as an asymptotic value, which means that $\boldsymbol{a}\in A(f)$.

Corollary 1. If for some point $\boldsymbol{a}$ of $\boldsymbol{C}^{n+1}-V$

$$\lim_{r\to\infty}\{T(r,f)-\frac{r^{1/2}}{2}\int_r^\infty\frac{N(t,\boldsymbol{a},f)}{t^{3/2}}dt\}=\infty, \tag{8}$$

then $\boldsymbol{a}\in A(f)$.

Proof. Since

$$\frac{r^{1/2}}{2}\int_r^\infty\frac{N(t,\boldsymbol{a},f)}{t^{3/2}}dt=\frac{r^{1/2}}{2}\int_r^\infty\frac{N_o(t,\boldsymbol{a},f)}{t^{3/2}}dt+n(0,\boldsymbol{a},f)\frac{r^{1/2}}{2}\int_r^\infty\frac{\log t}{t^{3/2}}dt$$
$$\geq\frac{r^{1/2}}{2}\int_r^\infty\frac{N_o(t,\boldsymbol{a},f)}{(t+r)^{3/2}}dt+n(0,\boldsymbol{a},f)(\log r+2),$$

if (8) holds, then (6) holds.

Remark 1. 1) We can easily obtain Theorem A from Corollary 1. In fact, let $\boldsymbol{a}$ be Picard exceptional for f, then

$$\frac{r^{1/2}}{2}\int_r^\infty\frac{N(t,\boldsymbol{a},f)}{t^{3/2}}dt=O(\log r)$$

and (8) holds as f is transcendental.

2) If for some point $\boldsymbol{a}$ of $\boldsymbol{C}^{n+1}-V$

$$N(r,\boldsymbol{a},f)=O(\gamma^\alpha)\quad\text{and}\quad\liminf_{r\to\infty}\frac{T(r,f)}{\gamma^\beta}>0,$$

where $\alpha < 1/2$, $\alpha < \beta$, then $\boldsymbol{a} \in A(f)$.

In fact, in this case

$$\frac{r^{1/2}}{2}\int_r^\infty \frac{N(t, \boldsymbol{a}, f)}{t^{3/2}}dt = O(r^\alpha)$$

and we have (8).

Corollary 2. Suppose

$$\limsup_{r\to\infty} \frac{r^{1/2}}{2}\int_r^\infty \frac{T(t, f)}{(t+r)^{3/2}}dt / T(r, f) = K < \infty.$$

Then if for some point $\boldsymbol{a}$ of $\boldsymbol{C}^{n+1} - V$

$$\delta(\boldsymbol{a}, f) > 1 - K^{-1},$$

$\boldsymbol{a} \in A(f)$.

We can prove this corollary as in Lemma 2([8], p.493) by applying Theorem 1.

3 Holomorphic curves with smooth growth.

Let f be as in Section 1. In [9] we gave some results on asymptotic properties of f satisfying

$$\lim_{r\to\infty} \frac{T(2r, f)}{T(r, f)} = 1. \tag{9}$$

A holomorphic curve satisfying (9) is of order zero(Theorem 1 in [9]). The purpose of this section is to generalize Theorem B to holomorphic curves of order positive applying the method used in [8].

Definition 2. Let ρ be a positive number. We say that f is of ρ-smooth growth if and only if $T(r, f)$ satisfies

$$\lim_{r\to\infty} \frac{T(xr, f)}{x^\rho T(r, f)} = 1 \quad \text{for any} \quad x > 0 \tag{10}$$

(see [8], p.495).

For example, it is easy to see that if f has perfectly regular growth of order $\rho > 0$ (see [4]), then $T(r, f)$ satisfies (10).

Remark 2. (10) is equivalent to

$$\lim_{r\to\infty} \frac{T(xr, f)}{x^\rho T(r, f)} = 1 \quad \text{for any} \quad x > 1 \tag{11}$$

(see [8], Remark 1).

Let f' be the holomorphic curve induced by $(f_1', \cdots, f_{n+1}')$ and $[f, f']$ be a bivector determined by f and f' with the components $f_i f_j' - f_i' f_j$ $(1 \leq i < j \leq n+1)$. Put

$$S(r, f) = \frac{1}{\pi}\int_0^r\int_0^{2\pi} \frac{\|[f(te^{i\theta}), f'(te^{i\theta})]\|^2}{\|f(te^{i\theta})\|^4} tdtd\theta$$

Then we have the following relation between $T(r, f)$ and $S(r, f)$.

Lemma 5. Let r_o be a positive number. Then for $r \geq r_o$

$$T(r, f) - T(r_o, f) = \int_{r_o}^r \frac{S(t, f)}{t}dt$$

(see formula (14) in [1] and pp.142-143 in [10]).

Lemma 6. For a positive number ρ, the following three statements are equivalent:

(i) f has ρ-smooth growth;

(ii)

$$\lim_{r\to\infty} \frac{S(r, f)}{T(r, f)} = \rho; \tag{12}$$

(iii) For any positive $\epsilon < \rho$, there is an R_o such that the following inequality holds:

$$\left(\frac{t}{r}\right)^{\rho-\epsilon}T(r,f)\le T(t,f)\le\left(\frac{t}{r}\right)^{\rho+\epsilon}T(r,f)\ (R_o\le r\le t). \tag{13}$$

Proof. (i) Suppose that f has ρ-smooth growth. For $x>1$

$$S(r,f)\log x\le\int_r^{xr}\frac{S(t,f)}{t}dt=T(xr,f)-T(r,f),$$

so that we have

$$\frac{S(r,f)}{T(r,f)}\le\left\{\frac{T(xr,f)}{T(r,f)}-1\right\}/\log x.$$

This inequality and (11) yield

$$\limsup_{r\to\infty}\frac{S(r,f)}{T(r,f)}\le\frac{x^\rho-1}{\log x}$$

and letting $x\to1$, we have

$$\limsup_{r\to\infty}\frac{S(r,f)}{T(r,f)}\le\rho. \tag{14}$$

On the other hand, for $x<1$

$$S(r,f)\ge\int_{xr}^{r}\frac{S(t,f)}{t}dt=T(r,f)-T(xr,f),$$

so that we have

$$\frac{S(r,f)}{T(r,f)}\ge\left\{\frac{T(xr,f)}{T(r,f)}-1\right\}/\log x.$$

This inequality with (11) yields

$$\liminf_{r\to\infty}\frac{S(r,f)}{T(r,f)}\ge\frac{x^\rho-1}{\log x}$$

and letting $x\to1$, we have

$$\liminf_{r\to\infty}\frac{S(r,f)}{T(r,f)}\ge\rho. \tag{15}$$

From (14) and (15) we have (12).

(ii) Suppose that (12) holds. Let ϵ be any positive number smaller than ρ. Then, there exists an R_o such that for $R_o\le r\le t$ we have the inequality

$$(\rho-\epsilon)\log\frac{t}{r}\le\log\frac{T(t,f)}{T(r,f)}=\int_r^t\frac{S(u,f)}{uT(u,f)}du\le(\rho+\epsilon)\log\frac{t}{r},$$

which reduces to

$$\left(\frac{t}{r}\right)^{\rho-\epsilon}T(r,f)\le T(t,f)\le\left(\frac{t}{r}\right)^{\rho+\epsilon}T(r,f)\ (R_o\le r\le t).$$

(iii) Suppose that (13) holds. Let $x\ge1$, $r\ge R_o$ and put $t=xr$. Then from (13) we have

$$x^{-\epsilon}\le\frac{T(xr,f)}{x^\rho T(r,f)}\le x^\epsilon.$$

Letting $r\to\infty$ first and then $\epsilon\to0$, we obtain

$$\lim_{r\to\infty}\frac{T(xr,f)}{x^\rho T(r,f)}=1\quad\text{for any}\quad x\ge1.$$

Considering Remark 2, we have (10).

Remark 3. As is easily seen from (13), if f has ρ-smooth growth, f has regular growth of order ρ.

Remark 4. As in Lemma 6, we can prove the following.

"The following four statements are equivalent:

(i) $\lim_{r\to\infty}T(2r,f)/T(r,f)=1$;

(ii) $\lim_{r\to\infty}T(xr,f)/T(r,f)=1$ for any $x>0$;

(iii) $\lim_{r\to\infty}S(r,f)/T(r,f)=0$;

(iv) For any positive number ϵ there exists an R_o such that the following inequality holds:

$$T(t,f)\le(\frac{t}{r})^{\epsilon}T(r,f)\ \ (R_o\le r\le t)."$$

Theorem 2. Suppose that f has ρ-smooth growth, where $0<\rho<1/2$. If there exists a point $\boldsymbol{a}$ in $\boldsymbol{C}^{n-1}-V$ such that

$$\delta(\boldsymbol{a},f)>1-\frac{\sqrt{\pi}}{\Gamma(\rho+1)\Gamma(1/2-\rho)},$$

then $\boldsymbol{a}\in A(f)$.

Proof. As in the proof of Theorem 1 in [8], we have

$$\limsup_{r\to\infty}\frac{r^{1/2}}{2}\int_0^{\infty}\frac{T(t,f)}{(t+r)^{3/2}}dt/T(r,f)\le\frac{\Gamma(\rho+1)\Gamma(1/2-\rho)}{\sqrt{\pi}}.$$

Using Corollary 2, we obtain this theorem.

Remark 5. $1-\frac{\sqrt{\pi}}{\Gamma(\rho+1)\Gamma(1/2-\rho)}<2\rho$ if $0<\rho<1/2$ (see Remark 2 in [8]).

Theorem 3. Suppose that f satisfies

$$\int_1^{\infty}\frac{T(t,f)}{t^{3/2}}dt<\infty.$$

If there exists a point $\boldsymbol{a}\in\boldsymbol{C}^{n-1}-V$ such that

$$\lim_{r\to\infty}\{m(r,\boldsymbol{a},f)-2S(r,f)\}=\infty, \tag{16}$$

then $\boldsymbol{a}\in A(f)$.

Proof. We apply Corollary 1. For any sufficiently large r

$$T(r,f)-\frac{r^{1/2}}{2}\int_r^{\infty}\frac{N(t,\boldsymbol{a},f)}{t^{3/2}}dt=T(r,f)-\frac{r^{1/2}}{2}\int_r^{\infty}\frac{T(t,f)-m(t,\boldsymbol{a},f)}{t^{3/2}}dt+O(1)$$

$$=\frac{r^{1/2}}{2}\int_r^{\infty}\frac{m(t,\boldsymbol{a},f)-2S(t,f)}{t^{3/2}}dt+O(1).$$

Thus (16) implies (8), so that $\boldsymbol{a}\in A(f)$.

Corollary 3. Under the same assumption as in Theorem 3, if there exists a point $\boldsymbol{a}\in\boldsymbol{C}^{n-1}-V$ such that

$$\liminf_{r\to\infty}\frac{m(r,\boldsymbol{a},f)}{S(r,f)}>2, \tag{17}$$

then $\boldsymbol{a}\in A(f)$.

This is a direct consequence of Theorem 3.

Application of Corollary 3.

I. Suppose that

$$\limsup_{r\to\infty}\frac{T(r,f)}{(\log r)^2}=A<\infty.$$

If there exists a point $\boldsymbol{a}$ in $\boldsymbol{C}^{n-1}-V$ such that

$$\liminf_{r\to\infty}\frac{m(r,\boldsymbol{a},f)}{\log r}>8A, \tag{18}$$

then $\boldsymbol{a}\in A(f)$

Proof. As

$$S(r,f)\log r\le\int_r^{r^2}\frac{S(t,f)}{t}dt\le T(r^2,f)=(4A+o(1))(\log r)^2$$

that is,

$$S(r,f) \leq (4A+o(1))\log r$$

for $r \to \infty$, (18) implies (17), and $\boldsymbol{a} \in A(f)$.

Remark 6. We can replace $8A$ by $8A\log(\sqrt{2}+1)$ as in the case of meromorphic functions (see[8], pp.502-503).

II. Suppose

$$\lim_{r\to\infty} \frac{T(2r,f)}{T(r,f)} = 1.$$

Then (17) is weaker than $\delta(\boldsymbol{a},f)>0$.

In fact, if $\delta(\boldsymbol{a},f)>0$,

$$\frac{m(r,\boldsymbol{a},f)}{S(r,f)} = \frac{m(r,\boldsymbol{a},f)}{T(r,f)} \frac{T(r,f)}{S(r,f)} \to \infty$$

for $r \to \infty$ by Remark 4.

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