On the Second Fundamental Inequality for Holomorphic Curves

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Let $f=[f_1,\cdots,f_{n+1}]$ be a transcendental holomorphic curve from $\boldsymbol{C}$ into the n-dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation $(f_1,\cdots,f_{n+1}):\boldsymbol{C}\to\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$, where n is a positive integer.

Let X be a subset of $\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$ in N-subgeneral position, where $N\geq n$, and $X(0)=\{\boldsymbol{a}=(a_1,\cdots,a_{n+1})\in X : a_{n+1}=0\}$. Then, we can improve the second fundamental inequality of Nochka ([5]) as follows.

Theorem. Let $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q$ be any elements of X $(2N-n+1<q<\infty)$ and let s be the maximum number of linearly independent vectors in $X(0)\cap\{\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q\}$. Then

$$\sum_{j=1}^{q} m(r,\boldsymbol{a}_j,f)\leq(N-n+n(s))T(r,f)+(N+1-n(s))t(r,f)+S(r,f),$$

where $n(s)=(s+1)(N+1)/(n+1)$.

Theorem. Let X be p-maximal $(1\leq p\leq n)$. Then,

$$\sum_{\boldsymbol{a}\in X}\delta(\boldsymbol{a},f)\leq 2N-n+1-(N+1)(n-p)(1-\Omega)/(n+1),$$

where $0\leq\Omega=\limsup_{r\to\infty}t(r,f)/T(r,f)\leq 1$.

1 Introduction

Let $f=[f_1,\cdots,f_{n+1}]$ be a holomorphic curve from $\boldsymbol{C}$ into the n-dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1,\cdots,f_{n+1}):\boldsymbol{C}\to\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\},$$

where n is a positive integer.

We use the following notation:

$$\|f(z)\|=(|f_1(z)|^2+\cdots+|f_{n+1}(z)|^2)^{1/2}$$

and for a vector $\boldsymbol{a}=(a_1,\cdots,a_{n+1})\in\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$

$$\|\boldsymbol{a}\|=(|a_1|^2+\cdots+|a_{n+1}|^2)^{1/2},$$
$$(\boldsymbol{a},f)=a_1f_1+\cdots+a_{n+1}f_{n+1},$$
$$(\boldsymbol{a},f(z))=a_1f_1(z)+\cdots+a_{n+1}f_{n+1}(z)$$

The characteristic function $T(r,f)$ of f is defined as follows(see [10]):

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta-\log\|f(0)\|.$$

On the other hand, put

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$$U(z) = \max_{1\leq j\leq n-1} |f_j(z)|,$$

then it is known([1]) that

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log U(re^{i\theta})d\theta + O(1). \tag{1}$$

We suppose throughout the paper that f is transcendental; that is to say,

$$\lim_{r\to\infty} \frac{T(r,f)}{\log r} = \infty$$

and f is linearly non-degenerate over $\boldsymbol{C}$; namely, $f_1,\cdots,f_{n+1}$ are linearly independent over $\boldsymbol{C}$.

It is well-known that f is linearly non-degenerate over $\boldsymbol{C}$ if and only if the Wronskian $W(f_1,\cdots,f_{n-1})$ of $f_1,\cdots,f_{n+1}$ is not identically equal to zero.

We denote by $\rho(f)$ the order of f and $\mu(f)$ the lower order of f:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

It is said that f is of regular growth if $\rho(f) = \mu(f)$.

For meromorphic functions in the complex plane we use the standard notation of the Nevanlinna theory of meromorphic functions([4]).

For $\boldsymbol{a} \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$, we write

$$m(r,\boldsymbol{a},f) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\|\boldsymbol{a}\|\|f(re^{i\theta})\|}{|(\boldsymbol{a},f(re^{i\theta}))|} d\theta,$$

$$N(r,\boldsymbol{a},f) = N(r, \frac{1}{(\boldsymbol{a},f)}).$$

We then have

$$T(r,f) = N(r,\boldsymbol{a},f) + m(r,\boldsymbol{a},f) + O(1) \tag{2}$$

([10], p.76). We call the quantity

$$\delta(\boldsymbol{a},f) = 1 - \limsup_{r\to\infty} \frac{N(r,\boldsymbol{a},f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,\boldsymbol{a},f)}{T(r,f)}$$

the deficiency of $\boldsymbol{a}$ with respect to f. We have

$$0 \leq \delta(\boldsymbol{a},f) \leq 1$$

by (2) since $N(r,\boldsymbol{a},f) \geq 0$ for $r \geq 1$ and $m(r,\boldsymbol{a},f) \geq 0$ for $r > 0$.

Further, let $\nu(c)$ be the order of zero of $(\boldsymbol{a},f(z))$ at $z=c$ and for a positive integer k let

$$n_k(r,\boldsymbol{a},f) = \sum_{|c|\leq r} \min\{\nu(c),k\}.$$

Then, we put for $r > 0$

$$N_k(r,\boldsymbol{a},f) = \int_0^r \frac{n_k(t,\boldsymbol{a},f) - n_k(0,\boldsymbol{a},f)}{t} dt + n_k(0,\boldsymbol{a},f)\log r$$

and put

$$\delta_k(\boldsymbol{a},f) = 1 - \limsup_{r\to\infty} \frac{N_k(r,\boldsymbol{a},f)}{T(r,f)}.$$

It is easy to see that $\delta(\boldsymbol{a},f) \leq \delta_k(\boldsymbol{a},f) \leq 1$ by definition.

Let X be a subset of $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ in N-subgeneral position; that is to say, $\#X \geq N+1$ and any $N+1$ elements of X generate $\boldsymbol{C}^{n+1}$, where N is an integer satisfying $N \geq n$.

We use $\boldsymbol{e}_1,\cdots,\boldsymbol{e}_{n+1}$ as the standard basis of $\boldsymbol{C}^{n+1}$.

Nochka([5]) gave the following

Theorem A. For any $q(>2N-n+1)$ elements $\boldsymbol{a}_j$ $(j=1,\cdots,q)$ of X,

$$(q-2N+n-1)T(r,f) < \sum_{j=1}^{q} N(r,\boldsymbol{a}_j,f) + S(r,f),$$

where $S(r,f)$ is any quantity satisfying

$$S(r,f) = o(T(r,f))$$

when r tends to ∞ outside a subset of r of at most a finite linear measure (see also [2] or [3]).

We gave a refinement of the second fundamental inequality of Cartan([1]) in [9]. The purpose of this paper is to give a result containing Theorem A for $N > n$.

2 Preliminary

Let $f = [f_1, \cdots, f_{n+1}]$ and X etc. be as in Section 1.

Definition 1([9]). We put

$$u(z) = \max_{1 \le j \le n} |f_j(z)|,$$

$$t(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \{\log u(re^{i\theta}) - \log u(e^{i\theta})\} d\theta.$$

and

$$\Omega = \limsup_{r \to \infty} \frac{t(r,f)}{T(r,f)} \quad \text{and} \quad \tau = \liminf_{r \to \infty} \frac{t(r,f)}{T(r,f)}.$$

It is easy to see the following

Proposition 1([9]). (a) $t(r,f)$ is independent of the choice of reduced representation of f.

(b) $t(r,f) \le T(r,f) + O(1)$.

(c) $N(r, 1/f_j) \le t(r,f) + O(1)$ $(j = 1, \cdots, n)$.

(d) $0 \le \tau \le \Omega \le 1$.

Definition 2([7]). We denote by f^* the holomorphic curve induced by the mapping

$$(f_1^{n+1}, \cdots, f_n^{n+1}, W(f_1, \cdots, f_n, f_{n+1})) : \boldsymbol{C} \to \boldsymbol{C}^{n+1}.$$

It is easy to see that f^* is independent of the choice of reduced representation of f. Let $d(z)$ be an entire function such that the functions

$$f_j^{n+1}/d \ (j = 1, \cdots, n) \quad \text{and} \quad W(f_1, \cdots, f_{n+1})/d$$

are entire functions without common zeros. Then

$$f^* = [f_1^{n+1}/d, \cdots, f_n^{n+1}/d, W(f_1, \cdots, f_{n+1})/d].$$

Proposition 2([7],[9]). (a) f^* is transcendental.

(b) $T(r,f^*) \le T(r,f) + nt(r,f) - N(r, 1/d) + S(r,f)$.

(c) $\rho(f^*) = \rho(f)$.

Example 1. Let a_j $(j = 1, \cdots, n)$ be real numbers satisfying $0 < a_1 < \cdots < a_{n-1} < a_n$ and put

$$f = [1, e^{a_1 z}, \cdots, e^{a_n z}].$$

Then, we easily have

$$T(r,f) = (a_n/\pi)r + O(1) \quad \text{and} \quad t(r,f) = (a_{n-1}/\pi)r + O(1)$$

(see [10], pp.94-95) and $\tau = \Omega = a_{n-1}/a_n (< 1)$. Further,

$$T(r,f^*) = (A/\pi)r + O(1),$$

where $A = \max\{(n+1)a_{n-1},\ a_1 + \cdots + a_n\}$.

We need the set

$$X(0)=\{\boldsymbol{a}=(a_1,\cdots,a_n,a_{n+1})\in X : a_{n+1}=0\}$$

to obtain an amelioration of Theorem A. Let p be the maximum number of linearly independent vectors in $X(0)$. Then, it is easy to see that

$$0\leq \#X(0)\leq N \quad \text{and} \quad 0\leq p\leq n.$$

since X is in N-subgeneral position.

Let q be an integer satisfying $2N-n+1<q<\infty$ and put $Q=\{1,2,\cdots,q\}$. Let $\{\boldsymbol{a}_j : j\in Q\}$ be a family of vectors in X. If $P\subset Q$, we denote

$$H(P)=\text{the vector space spanned by } \{\boldsymbol{a}_j : j\in P\} \quad \text{and} \quad d(P)=\dim H(P).$$

Lemma 1 (see [2], Theorem 0.3). For $\{\boldsymbol{a}_j : j\in Q\}$, there exist a Nochka weight function $\omega : Q\rightarrow(0,1]$ and a Nochka constant $\theta\geq 1$ such that

(a) $0<\omega(j)\theta\leq 1$ for all $j\in Q$;

(b) $q-2N+n-1=\theta(\Sigma_{j=1}^{q}\omega(j)-n-1)$;

(c) $(N+1)/(n+1)\leq\theta\leq(2N-n+1)/(n+1)$;

(d) If $P\subset Q$ and $0<\#P\leq N+1$, then $\Sigma_{j\in P}\omega(j)\leq d(P)$.

Lemma 2 ([2], Theorem 1.2). Let ω and θ be the same as in Lemma 1. Take $A\subset Q$ with $0<\#A\leq N+1$. Let

$$\{E_j\in \boldsymbol{R} : E_j\geq 1, j\in Q\}.$$

Then there exists a subset B of A such that

$$\{\boldsymbol{a}_j : j\in B\} \quad \text{is a basis of} \quad H(A)$$

and such that

$$\Pi_{j\in A}E_j^{\omega(j)}\leq\Pi_{j\in B}E_j.$$

Remark 1. If $\#A=N+1$, then $H(A)=\boldsymbol{C}^{n+1}$ and $\{\boldsymbol{a}_j : j\in B\}$ is a basis of $\boldsymbol{C}^{n-1}$.

3 Second fundamental inequality 1

Let $f=[f_1,\cdots,f_{n+1}]$ and X etc. be as in Section 1 or 2.

Theorem 1. For any $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q$ $(2N-n+1<q<\infty)$ in $X-X(0)$, we have the following inequalities:

(a) $\sum_{j=1}^{q}\omega(j)m(r,\boldsymbol{a}_j,f)\leq m(r,\boldsymbol{e}_{n+1},f^*)+S(r,f)$;

(b) $\sum_{j=1}^{q}m(r,\boldsymbol{a}_j,f)+\frac{N+1}{n+1}N(r,\frac{1}{W})\leq(N-n+\frac{N+1}{n+1})T(r,f)+(N+1-\frac{N+1}{n+1})t(r,f)+S(r,f)$,

where $W=W(f_1,\cdots,f_{n+1})$.

Proof. (a) Put

$$(\boldsymbol{a}_j,f)=F_j \quad \text{and} \quad E_j=\|\boldsymbol{a}_j\|\|f\|/|F_j|(\geq 1)\ (j=1,\cdots,q).$$

For any $z(\neq 0)$ arbitrarily fixed, let

$$|F_{j_1}(z)|\leq|F_{j_2}(z)|\leq\cdots\leq|F_{j_q}(z)|,$$

where $j_1,\cdots,j_q$ are distinct integers satisfying $1\leq j_1,\cdots,j_q\leq q$.

Then there is a positive constant K such that

$$\|f(z)\| \leq K|F_{j_\nu}(z)| \quad (\nu=N+1,\cdots,q), \tag{3}$$

$$|F_{j_\nu}(z)| \leq K\|f(z)\| \quad (\nu=1,\cdots,q) \tag{4}$$

and for any j_ν

$$\begin{aligned}\|f(z)\| &\leq K(|f_1(z)|^2+\cdots+|f_n(z)|^2+|F_{j_\nu}(z)|^2)^{1/2}\\ &\leq \begin{cases} K(n+1)^{1/2}u(z) & \text{if } |F_{j_\nu}(z)|\leq u(z)\\ K(n+1)^{1/2}|F_{j_\nu}(z)| & \text{otherwise}\end{cases}\end{aligned} \tag{5}$$

since the $n+1-$th elements of vectors $\boldsymbol{a}_j$ are different from zero. (From now on we denote by K a positive constant, which may be different from each other when it appears.)

(I) The case when $u(z)<|F_{j_1}(z)|$. We have $\|f(z)\|\leq K|F_{j_1}(z)|$ from (5) in this case and the following inequality holds:

$$\Pi_{j=1}^q\left(\frac{\|\boldsymbol{a}_j\|\|f(z)\|}{|(\boldsymbol{a}_j,f(z))|}\right)^{\omega(j)} \leq K. \tag{6}$$

(II) The case when $|F_{j_1}(z)|\leq u(z)$. We have $\|f(z)\|\leq Ku(z)$ from (5) in this case, and so we obtain by (3), Lemma 2 and Remark 1 that

$$\begin{aligned}\Pi_{j=1}^q\left(\frac{\|\boldsymbol{a}_j\|\|f(z)\|}{|(\boldsymbol{a}_j,f(z))|}\right)^{\omega(j)} &\leq K\Pi_{\nu=1}^{N+1}\left(\frac{\|\boldsymbol{a}_{j_\nu}\|\|f(z)\|}{|F_{j_\nu}(z)|}\right)^{\omega(j_\nu)}\\ &\leq K\Pi_{j_\nu\in B}\frac{\|\boldsymbol{a}_{j_\nu}\|\|f(z)\|}{|F_{j_\nu}(z)|}\\ &\leq K\frac{u(z)^{n+1}}{\Pi_{j_\nu\in B}|F_{j_\nu}(z)|}\\ &= K\frac{u(z)^{n+1}}{|W(z)|}\cdot\frac{|W_B(z)|}{\Pi_{j_\nu\in B}|F_{j_\nu}(z)|},\end{aligned} \tag{7}$$

where W_B is the Wronskian of $\{F_{j_\nu}: j_\nu\in B\}$. We know that $W_B=cW$ ($c\neq 0$, constant).

From (6) and (7) we have the inequality

$$\sum_{j=1}^q \omega(j)\log\frac{\|\boldsymbol{a}_j\|\|f(z)\|}{|(\boldsymbol{a}_j,f(z))|}\leq \log^+\frac{u(z)^{n+1}}{|W(z)|}+\sum_{(j_1,\cdots,j_q)}\log^+\frac{|W_B(z)|}{\Pi_{j_\nu\in B}|F_{j_\nu}(z)|}+O(1), \tag{8}$$

where the summation $\Sigma_{(j_1,\cdots,j_q)}$ is taken over all permutations of the elements of Q which appear when z varies in $\boldsymbol{C}-\{0\}$.

Integrating both sides of (8) with respect to φ ($z=re^{i\varphi}$) from 0 to 2π, we obtain the inequality

$$\sum_{j=1}^q \omega(j)m(r,\boldsymbol{a}_j,f)\leq m(r,\boldsymbol{e}_{n+1},f^*)+S(r,f)$$

since

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{u(re^{i\varphi})^{n+1}}{|W(re^{i\varphi})|}d\varphi=m(r,\boldsymbol{e}_{n+1},f^*)+O(1); \tag{9}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{|W_B(re^{i\varphi})|}{\Pi_{j_\nu\in B}|F_{j_\nu}(re^{i\varphi})|}d\varphi=S(r,f). \tag{10}$$

We obtain (9) from the definition of $m(r,\boldsymbol{e}_{n-1},f^*)$ and (10) as in [1], pp.12-15.

(b) By using Proposition 2 and (2), we obtain from (a) that

$$\sum_{j=1}^q \omega(j)m(r,\boldsymbol{a}_j,f)+N(r,\frac{1}{W})\leq T(r,f)+nt(r,f)+S(r,f)$$

and so

$$\sum_{j=1}^{q} \theta\,\omega(j) m(r,\boldsymbol{a}_j,f) + \theta N(r,\frac{1}{W}) \leq \theta T(r,f) + n\theta t(r,f) + S(r,f). \tag{11}$$

Adding

$$\sum_{j=1}^{q} (1-\theta\,\omega(j)) m(r,\boldsymbol{a}_j,f)$$

to both sides of (11) and using (2), we obtain

$$\sum_{j=1}^{q} m(r,\boldsymbol{a}_j,f) \ + \ \theta N(r,\frac{1}{W}) + \sum_{j=1}^{q}(1-\theta\,\omega(j))N(r,\boldsymbol{a}_j,f)$$
$$\leq \ (q+\theta-\sum_{j=1}^{q}\theta\,\omega(j))T(r,f) + n\theta t(r,f) + S(r,f). \tag{12}$$

By (a) and (b) of Lemma 1 and from (12) we obtain

$$\sum_{j=1}^{q} m(r,\boldsymbol{a}_j,f) \ + \ \theta N(r,\frac{1}{W}) \leq (2N-n+1)T(r,f) - n\theta\{T(r,f)-t(r,f)\} + S(r,f)$$

and by using Lemma 1 (c) and Proposition 1 (b)

$$\sum_{j=1}^{q} m(r,\boldsymbol{a}_j,f) \ + \ \frac{N+1}{n+1}N(r,\frac{1}{W}) \ \leq \ (2N-n+1)T(r,f) - n\frac{N+1}{n+1}\{T(r,f)-t(r,f)\} + S(r,f)$$
$$= \ (N-n+\frac{N+1}{n+1})T(r,f) + (N+1-\frac{N+1}{n+1})t(r,f) + S(r,f).$$

4 Second fundamental inequality 2

In this section we suppose that $X(0)$ is not empty.

Theorem 2. Let $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q$ $(2N-n+1<q<\infty)$ be any elements of X and suppose that

$$X(0)\cap\{\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q\} = \{\boldsymbol{a}_1,\cdots,\boldsymbol{a}_\ell\},$$

where $1\leq \ell \leq \sharp X(0)$. Let s be the maximum number of linearly independent vectors in $\{a_1,\cdots,a_\ell\}$. Then,

(a) $\sum_{j=1}^{q} \omega(j) m(r,\boldsymbol{a}_j,f) + N(r,\frac{1}{W}) \leq (s+1)T(r,f) + (n-s)t(r,f) + S(r,f)$;

(b) $\sum_{j=1}^{q} m(r,\boldsymbol{a}_j,f) + \frac{N+1}{n+1}N(r,\frac{1}{W}) \leq (N-n+n(s))T(r,f) + (N+1-n(s))t(r,f) + S(r,f)$,

where W is the Wronskian of $f_1,\cdots,f_{n+1}$ and $n(s)=(s+1)(N+1)/(n+1)$.

Proof. (a) Put

$$(\boldsymbol{a}_j,f)=F_j \quad \text{and} \quad E_j=\|\boldsymbol{a}_j\|\|f\|/|F_j|(\geq 1)\ (j=1,\cdots,q).$$

For any $z(\neq 0)$ arbitrarily fixed, let

$$|F_{j_1}(z)| \leq |F_{j_2}(z)| \leq \cdots \leq |F_{j_q}(z)|,$$

where $1\leq j_1,\cdots,j_q\leq q$ and $j_1,\cdots,j_q$ are distinct. Then there is a positive constant K such that

$$\|f(z)\| \ \leq \ K|F_{j_\nu}(z)| \ \ (\nu=N+1,\cdots,q), \tag{13}$$
$$|F_{j_\nu}(z)| \ \leq \ K\|f(z)\| \ \ (\nu=1,\cdots,q) \tag{14}$$

and for any $j_\nu \geq \ell+1$

$$\begin{aligned}\|f(z)\| &\leq K(|f_1(z)|^2+\cdots+|f_n(z)|^2+|F_{j_\nu}(z)|^2)^{1/2},\\ &\leq \begin{cases} K(n+1)^{1/2}u(z) & \text{if } |F_{j_\nu}(z)|\leq u(z)\\ K(n+1)^{1/2}|F_{j_\nu}(z)| & \text{otherwise}\end{cases}\end{aligned} \tag{15}$$

since the $n+1-$th element of $\boldsymbol{a}_{j_\nu}$ is different from zero if $j_\nu \geq \ell+1$. (From now on we denote by K a positive constant, which may be different from each other when it appears.)

We have by (13) and by Lemma 2

$$\begin{aligned}\Pi_{j=1}^{q}\left(\frac{\|\boldsymbol{a}_j\|\|f(z)\|}{|(\boldsymbol{a}_j,f(z))|}\right)^{\omega(j)} &\leq K\Pi_{\nu=1}^{N+1}\left(\frac{\|\boldsymbol{a}_{j_\nu}\|\|f(z)\|}{|F_{j_\nu}(z)|}\right)^{\omega(j_\nu)}\\ &\leq K\Pi_{j_\nu\in B}\frac{\|\boldsymbol{a}_{j_\nu}\|\|f(z)\|}{|F_{j_\nu}(z)|}\equiv I.\end{aligned} \tag{16}$$

(A) The case when $\{\boldsymbol{a}_{j_\nu}: j_\nu\in B\}\cap X(0)=\phi$

(A-1) If for any $j_\nu\in B$

$$u(z)<|F_{j_\nu}(z)|,$$

we have by (15)

$$I\leq K. \tag{17}$$

(A-2) If for some $j_\nu \in B$

$$|F_{j_\nu}(z)|\leq u(z),$$

we have by (15)

$$I\leq K\frac{u(z)^{n+1}}{\Pi_{j_\nu\in B}|F_{j_\nu}(z)|}=K\frac{u(z)^{n+1}}{|W(z)|}\cdot\frac{|W_B(z)|}{\Pi_{j_\nu\in B}|F_{j_\nu}(z)|}, \tag{18}$$

where W_B is the Wronskian of $\{F_{j_\nu}: j_\nu\in B\}$. Note that $W_B=cW$ ($c\neq 0$, constant).

(B) The case when $\{\boldsymbol{a}_{j_\nu}: j_\nu\in B\}\cap X(0)\neq\phi$. Suppose without loss of generality that

$$X(0)\cap\{\boldsymbol{a}_{j_\nu}: j_\nu\in B\}=\{\boldsymbol{a}_1,\cdots,\boldsymbol{a}_k\},$$

Then, $1\leq k\leq s$. We suppose without loss of generality that $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_s$ are linearly independent.

(B-1) If for any $j_\nu\in B$, $j_\nu\neq 1,\cdots,k$

$$u(z)<|F_{j_\nu}(z)|,$$

we have by (14) and (15)

$$I\leq K\frac{\|f(z)\|^k}{|F_1(z)\cdots F_k(z)|}\leq K\frac{\|f(z)\|^s}{|F_1(z)\cdots F_s(z)|}. \tag{19}$$

We can find $\boldsymbol{e}_{i_{s+1}},\cdots,\boldsymbol{e}_{i_n}$ ($i_{s+1},\cdots,i_n\leq n$) such that the vectors

$$\boldsymbol{a}_1,\cdots,\boldsymbol{a}_s,\ \boldsymbol{e}_{i_{s+1}},\cdots,\boldsymbol{e}_{i_n}$$

are linearly independent.

From (19) and by the inequalities

$$|f_{n+1}(z)|\leq\|f(z)\|,\qquad |f_{i_j}(z)|\leq u(z)\ (j=s+1,\cdots,n)$$

and

$$W(F_1,\cdots,F_s,f_{i_{s+1}},\cdots,f_{i_n},f_{n+1})(z)=KW(z)$$

we have

$$I \leq K\frac{\|f(z)\|^{s+1}u(z)^{n-s}}{|W(z)|}\cdot\frac{|W(F_1,\cdots,F_s,f_{i_{s+1}},\cdots,f_{i_n},f_{n+1})(z)|}{|F_1(z)\cdots F_s(z)f_{i_{s+1}}(z)\cdots f_{i_n}(z)f_{n+1}(z)|}. \tag{20}$$

(B-2) If for some $j_\nu \in B$, $j_\nu \neq 1,\cdots,k$

$$|F_{j_\nu}(z)| \leq u(z),$$

we have by (15)

$$I \leq K\frac{u(z)^{n+1}}{\Pi_{j_\nu\in B}|F_{j_\nu}(z)|} = K\frac{u(z)^{n+1}}{|W(z)|}\cdot\frac{|W_B(z)|}{\Pi_{j_\nu\in B}|F_{j_\nu}(z)|} \tag{21}$$

Since

$$u(z)^{n+1} \leq \|f(z)\|^{s+1}u(z)^{n-s},$$

we have from (16),(17),(18),(20) and (21)

$$\sum_{j=1}^{q}\omega(j)\log\frac{\|\boldsymbol{a}_j\|\|f(z)\|}{|F_j(z)|} \leq \log^+\frac{\|f(z)\|^{s+1}u(z)^{n-s}}{|W(z)|} + \sum_{(j_1,\cdots,j_q)}\log^+\frac{|W_B(z)|}{\Pi_{j_\nu\in B}|F_{j_\nu}(z)|}$$
$$+\sum_{(j_1,\cdots,j_q)}\log^+\frac{|W(F_1,\cdots,F_s,f_{i_{s+1}},\cdots,f_{i_n},f_{n+1})(z)|}{|F_1(z)\cdots F_s(z)f_{i_{s+1}}(z)\cdots f_{i_n}(z)f_{n+1}(z)|} + O(1), \tag{22}$$

where $\Sigma_{(j_1,\cdots,j_q)}$ is taken over all permutations of the elements of Q which appear when z varies in $\boldsymbol{C}-\{0\}$. Since

$$\log^+\frac{\|f(z)\|^{s+1}u(z)^{n-s}}{|W(z)|} = \log\max\{\|f(z)\|^{s+1}u(z)^{n-s}, |W(z)|\} - \log|W(z)|$$

and

$$|W(z)| = |f_1(z)\cdots f_{n+1}(z)|\frac{|W(z)|}{|f_1(z)\cdots f_{n+1}(z)|} \leq \|f(z)\|^{s+1}u(z)^{n-s}\frac{|W(z)|}{|f_1(z)\cdots f_{n+1}(z)|},$$

integrating both sides of (22) with respect to φ from 0 to 2π ($z=re^{i\varphi}$) we obtain

$$\sum_{j=1}^{q}\omega(j)m(r,\boldsymbol{a}_j,f) + N(r,\frac{1}{W}) \leq (s+1)T(r,f) + (n-s)t(r,f) + S(r,f).$$

(b) We obtain (b) of this theorem from (a) as in the case of Theorem 1.

5 Subset of $\boldsymbol{C}^{n+1}$ in subgeneral position

To obtain a refinement of the defect relation for holomorphic curves (see, for example, Theorem 3.3.8 and Theorem 3.3.10 in [3]), we need some new notions on subsets of $\boldsymbol{C}^{n+1}$ in subgeneral position.

Let X be a subset of $\boldsymbol{C}^{n+1}$ in N-subgeneral position such that the number of elements of X is not smaller than $N+1$ as in Section 1 and

$$X(0) = \{\boldsymbol{a}=(a_1,a_2,\cdots,a_n,a_{n+1})\in X : a_{n+1}=0\}$$

as in Section 2. Suppose that $N>n$ in this section.

Let $V_{X(0)}$ be the vector space generated by the elements of $X(0)$. Remember that the number of elements of $X(0)$ is not greater than N and the dimension of $V_{X(0)}$ is not greater than n.

Definition 3. We say that

(i) X is maximal in the sense of subgeneral position if for any Y in N-subgeneral position such that $X\subset Y\subset \boldsymbol{C}^{n+1}$, $X=Y$.

(ii) X is p-maximal (in the sense of subgeneral position) if X is maximal in the sense of subgeneral position and dim $V_{X(0)}=p$.

Note that X is p-maximal in the sense of general position if $N=n$ ([8], Definition 1).

The purpose of this section is to give examples of p-maximal subsets of $\boldsymbol{C}^{n+1}$ in the sense of subgeneral position.

We shall use the following lemma for our purpose.

Lemma 3. For any vector $(\alpha_1,\cdots,\alpha_n,\alpha_{n+1})$ of $\boldsymbol{C}^{n-1}$ which is neither equal to $\boldsymbol{0}$, $\alpha\boldsymbol{e}_1$ nor $\beta\boldsymbol{e}_{n-1}$, there exist complex numbers $a_1,\cdots,a_n$ different from each other for which the vectors

$$(\alpha_1,\cdots,\alpha_{n+1}),\ (a_1^n,a_1^{n-1},\cdots,a_1,1),\cdots,(a_n^n,a_n^{n-1},\cdots,a_n,1)$$

are linearly dependent, where α and β are any complex numbers ([8], Lemma 3).

Proposition 3. For $n\geq 2$, the set

$$A=\{(a^n,a^{n-1},\cdots,a,1):a\in\boldsymbol{C}\}\cup\{k\boldsymbol{e}_1:k=1,\cdots,N-n+1\}\cup\{\boldsymbol{e}_2\}$$

is 2-maximal.

Proof. It is easy to see that A is in N-subgeneral position and dim $V_{A(0)}=2$. We have only to prove that for any vector

$$\boldsymbol{x}=(\alpha_1,\cdots,\alpha_n,\alpha_{n+1})(\neq\boldsymbol{0})$$

which does not belong to A, $A\cup\{\boldsymbol{x}\}$ is not in N-subgeneral position.

(a) The case when $\alpha_1\neq 0$, $\alpha_2=\cdots=\alpha_{n+1}=0$.

For any distinct numbers $a_1,\cdots,a_{n-1}$, we can not find $n+1$ linearly independent vectors in the following $N+1$ vectors

$$(a_j^n,a_j^{n-1},\cdots,a_j,1)\ (j=1,\cdots,n-1),\ k\boldsymbol{e}_1\ (k=1,\cdots,N-n+1),\ \boldsymbol{x}.$$

(b) The case when $\alpha_1=\cdots=\alpha_n=0$, $\alpha_{n+1}\neq 0$.

For any distinct numbers $a_1,\cdots,a_{n-2}$ different from zero, we can not find $n+1$ linearly independent vectors in the following $N+1$ vectors

$$(a_j^n,a_j^{n-1},\cdots,a_j,1)\ (j=1,\cdots,n-2),\ k\boldsymbol{e}_1\ (k=1,\cdots,N-n+1),\ \boldsymbol{e}_{n+1},\ \boldsymbol{x}.$$

(c) The case when $\alpha_1=\alpha_3=\cdots=\alpha_{n+1}=0$ and $\alpha_2\neq 0$.

For any distinct numbers $a_1,\cdots,a_{n-2}$, we can not find $n+1$ linearly independent vectors in the following $N+1$ vectors

$$(a_j^n,a_j^{n-1},\cdots,a_j,1)\ (j=1,\cdots,n-2),\ k\boldsymbol{e}_1\ (k-1,\cdots,N-n+1),\ \boldsymbol{e}_2,\ \boldsymbol{x}.$$

(d) The case when $\boldsymbol{x}\neq\alpha\boldsymbol{e}_1,\ \beta\boldsymbol{e}_2,\ \gamma\boldsymbol{e}_{n+1}$.

By Lemma 3, there are $n-1$ distinct numbers $a_1,\cdots,a_{n-1}$ such that the vectors

$$(\alpha_2,\cdots,\alpha_{n+1}),\ (a_j^{n-1},\cdots,a_j,1)\ (j=1,\cdots,n-1)$$

are linearly dependent. Then, we can not find $n+1$ linearly independent vectors in the following $N+1$ vectors

$$(a_j^n,a_j^{n-1},\cdots,a_j,1)\ (j=1,\cdots,n-1),\ k\boldsymbol{e}_1\ (k=1,\cdots,N-n+1),\ \boldsymbol{x}.$$

since $(a_j^n,a_j^{n-1},\cdots,a_j,1)\ (j=1,\cdots,n-1)$, $\boldsymbol{e}_1$ and $\boldsymbol{x}$ are linearly dependent.

From (a),(b),(c) and (d) it is proved that $A\cup\{\boldsymbol{x}\}$ is not in N-subgeneral position.

Proposition 4. For $n\geq 2$, the set

$$A_p=\{(a^n,a^{n-1},\cdots,a,a^p+1):a\in\boldsymbol{C}\}\cup\{k\boldsymbol{e}_1:k=1,\cdots,N-n+1\}\cup\{\boldsymbol{e}_2\}$$

is p+2-maximal, where $1\leq p\leq n-2$.

Proof. It is easy to see that A_p is in N-subgeneral position and dim $V_{A_p(0)}=p+2$ since the p+2 vectors

$$\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \{(a^n,a^{n-1},\cdots,a,a^p+1):a^p+1=0\}$$

are linearly independent.

We have only to prove that for any vector

$$\boldsymbol{x}=(\alpha_1,\ \alpha_2,\cdots,\alpha_{n-1})(\neq \boldsymbol{0})$$

which does not belong to A_p, $A_p \cup \{\boldsymbol{x}\}$ is not in N-subgeneral position. In fact the vector

$$\boldsymbol{y}=(\alpha_1,\cdots,\alpha_n,\alpha_{n-1}-\alpha_{n-1-p})$$

does not belong to A and $A\cup\{\boldsymbol{y}\}$ is not in N-subgeneral position by Proposition 3 and so $A_p\cup\{\boldsymbol{x}\}$ is not in N-subgeneral position.

For $n\geq 2$, let

$$B=\{(a^n,a^{n-1},\cdots,a,1):a\in \boldsymbol{C}\}\cup\{k\boldsymbol{e}_1:k=1,\cdots,N-n+1\}\cup\{\boldsymbol{e}_1+\boldsymbol{e}_2\}.$$

Then,

Proposition 5. The set B is 2-maximal.

Proof. It is easy to see that B is in N-subgeneral position and dim $V_{B(0)}=2$. We have only to prove that for any vector

$$(\alpha_1,\cdots,\alpha_n,\alpha_{n+1})(\neq \boldsymbol{0})$$

which does not belong to B, $B\cup\{\boldsymbol{x}\}$ is not in N-subgeneral position.

(a) The case when $\alpha_1\neq 0$, $\alpha_2=\cdots=\alpha_{n-1}=0$.

(b) The case when $\alpha_1=\cdots=\alpha_n=0$, $\alpha_{n+1}\neq 0$.

In these two cases we can prove that $A\cup\{\boldsymbol{x}\}$ is not in N-subgeneral position as in the proof of Proposition 3.

(c) The case when $\boldsymbol{x}=\alpha_1\boldsymbol{e}_1+\alpha_2\boldsymbol{e}_2$ $(\alpha_2\neq 0)$.

For any distinct numbers $a_1,\cdots,a_{n-2}$. we can not find $n+1$ linearly independent vectors in the following $N+1$ vectors

$$(a_j^n,a_j^{n-1},\cdots,a_j,1)\ (j=1,\cdots,n-2),\ k\boldsymbol{e}_1\ (k=1,\cdots,N-n+1),\ \boldsymbol{e}_1+\boldsymbol{e}_2,\ \boldsymbol{x}.$$

(d) The case when $\boldsymbol{x}\neq\alpha\boldsymbol{e}_1+\beta\boldsymbol{e}_2$ $(|\alpha|+|\beta|\neq 0)$, $\gamma\boldsymbol{e}_{n+1}$.

By Lemma 3, there are $n-1$ distinct numbers $a_1,\cdots,a_{n-1}$ such that the vectors

$$(\alpha_2,\cdots,\alpha_{n+1}),\ (a_1^{n-1},\cdots,a_1,1),\cdots,(a_{n-1}^{n-1},\cdots,a_{n-1},1)$$

are linearly dependent. Then, we can not find $n+1$ linearly independent vectors in the following $N+1$ vectors

$$(a_j^n,a_j^{n-1},\cdots,a_j,1)\ (j=1,\cdots,n-1),\ k\boldsymbol{e}_1\ (k=1,\cdots,N-n+1),\ \boldsymbol{x}$$

since $\boldsymbol{e}_1$, $\boldsymbol{x}$ and $(a_j^n,a_j^{n-1},\cdots,a_j,1)$ $(j=1,\cdots,n-1)$ are linearly dependent.

From (a),(b),(c) and (d), $A\cup\{\boldsymbol{x}\}$ is not in N-subgeneral position.

Proposition 6. For $n\geq 2$, the set

$$B_1=\{(1,a^{n-1},\cdots,a,a^n):a\in \boldsymbol{C}\}\cup\{k\boldsymbol{e}_{n+1}:k=1,\cdots,N-n+1\}\cup\{\boldsymbol{e}_2+\boldsymbol{e}_{n+1}\}$$

is 1-maximal.

Proof. It is easy to see that B_1 is in N-subgeneral position and dim $V_{B_1(0)}=1$. We have only to prove that for any vector

$$\boldsymbol{x}=(\alpha_1,\alpha_2,\cdots,\alpha_n,\alpha_{n+1})(\neq \boldsymbol{0})$$

which does not belong to B_1, $B_1\cup\{\boldsymbol{x}\}$ is not in N-subgeneral position. Put

$$\boldsymbol{y}=(\alpha_{n+1},\alpha_2,\cdots,\alpha_n,\alpha_1).$$

Then, $\boldsymbol{y}$ is not equal to $\mathbf{0}$ and does not belong to B given just before Proposition 5. By Proposition 5, $B \cup \{\boldsymbol{y}\}$ is not in N-subgeneral position, so that $B_1 \cup \{\boldsymbol{x}\}$ is not in N-subgeneral position.

Theorem 3. Suppose $N > n \geq 2$. For any p $(1 \leq p \leq n)$, there is a p-maximal subset of $\boldsymbol{C}^{n+1}$ in the sense of subgeneral position.

Remark 2. It is easy to see that any maximal subset of $\boldsymbol{C}^2$ in the sense of subgeneral position is 1-maximal.

Problem. Is there a 0-maximal subset of $\boldsymbol{C}^n$ $(n \geq 3)$ in the sense of subgeneral position?

6 Defect relation

Let f, X and $X(0)$ etc. be as in Section 1,2,3 or 4.

Theorem 4 (defect relation). For any q elements $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q \in X$ $(2N-n+1 < q < \infty)$,

(a-1) $\sum_{j=1}^{q} \omega(j)\delta_n(\boldsymbol{a}_j, f) \leq p+1+(n-p)\Omega$;

(a-2) $\sum_{j=1}^{q} \omega(j)\delta(\boldsymbol{a}_j, f) + \xi \leq p+1+(n-p)\Omega$;

(b-1) $\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) \leq 2N-n+1-\frac{N+1}{n+1}(n-p)(1-\Omega)$;

(b-2) $\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) + \frac{N+1}{n+1}\xi \leq 2N-n+1-\frac{N+1}{n+1}(n-p)(1-\Omega)$,

where p is the maximum number of linearly independent vectors in $X(0) \cap \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q\}$ $(0 \leq p \leq n)$ and

$$\xi = \begin{cases} \limsup_{r \to \infty} \frac{N(r, 1/W)}{T(r, f)} & \text{if f has finite order,} \\ \liminf_{r \to \infty} \frac{N(r, 1/W)}{T(r, f)} & \text{otherwise.} \end{cases}$$

We easily obtain this theorem from Theorem 1 when $p=0$ or from Theorem 2 when p is positive. We obtain (a-1) and (b-1) by applying Lemma 3.2.13 in [3],p.102.

Remark 3. $p+1+(n-p)\Omega \leq n+1$ and $2N-n+1-(N+1)(n-p)(1-\Omega)/(n+1) \leq 2N-n+1$. The equalities hold if and only if $p=n$ or $\Omega=1$ in these two inequalities.

The number "$2N-n+1-(N+1)(n-p)(1-\Omega)/(n+1)$" increases with p $(0 \leq p \leq n)$ when $\Omega < 1$. If p increases to n when q tends to ∞, the bound "$2N-n+1-(N+1)(n-p)(1-\Omega)/(n+1)$" of Theorem 4 (b-1),(b-2) increases to $2N-n+1$ for any $\Omega < 1$. But, as Theorem 3 shows, there exist examples of X for which p does not increase to n even when q tends to ∞. By the way, Example 1 gives a holomorphic curve for which $\Omega < 1$.

Theorem 5(Defect relation). Let X be a p-maximal subset of $\boldsymbol{C}^{n+1}$ in N-subgeneral position. Then, we have

(I) $\sum_{\boldsymbol{a} \in X} \delta_n(\boldsymbol{a}, f) \leq 2N-n+1-\frac{N+1}{n+1}(n-p)(1-\Omega)$;

(II) $\sum_{\boldsymbol{a} \in X} \delta(\boldsymbol{a}, f) + \frac{N+1}{n+1}\xi \leq 2N-n+1-\frac{N+1}{n+1}(n-p)(1-\Omega)$.

Proof. (I) When $\sharp\{\boldsymbol{a} \in X : \delta_n(\boldsymbol{a}, f) > 0\} < \infty$, there is nothing to prove by Theorem 4 (b-1). When $\sharp\{\boldsymbol{a} \in X : \delta_n(\boldsymbol{a}, f) > 0\} = \infty$, it is countable by Theorem 4 (b-1). Let

$$\{\boldsymbol{a} \in X : \delta_n(\boldsymbol{a}, f) > 0\} = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots\},$$

and without loss of generality we put

$$X(0) \cap \{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots\} = \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k\} \ (0 \leq k \leq N).$$

Let

$$\dim V_{\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k\}} = s \ (0 \leq s \leq p).$$

Then, by Theorem 4 (b-1), for any q

$$\sum_{j=1}^{q}\delta_n(\boldsymbol{a}_j,f)\leq 2N-n+1-\frac{N+1}{n+1}(n-s)(1-\Omega)\leq 2N-n+1-\frac{N+1}{n+1}(n-p)(1-\Omega)$$

and letting q tend to ∞ we have

$$\sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a},f)=\sum_{j=1}^{\infty}\delta_n(\boldsymbol{a}_j,f)\leq 2N-n+1-\frac{N+1}{n+1}(n-p)(1-\Omega)$$

since p is independent of q.

(II) We obtain (II) of Theorem 5 by using Theorem 4 (b-2) instead of (b-1) as in the case of (I).

7 Holomorphic curves with maximal deficiency sum

Let $f=[f_1,\cdots,f_{n-1}]$, X and $X(0)$ etc. be as in the previous sections.

Lemma 4. If

$$\delta(\boldsymbol{e}_j,f^*)=1\ (j=1,\cdots,n+1),$$

then f^* is of regular growth and $\rho(f^*)$ is either ∞ or a positive integer (see [6], Théorème 3).

Lemma 5. For any $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q$ $(2N-n+1<q<\infty)$ in $X-X(0)$ and for $r\geq 1$

$$\sum_{j=1}^{q}m(r,\boldsymbol{a}_j,f)\leq\frac{2N-n+1}{n+1}m(r,\boldsymbol{e}_{n+1},f^*)+(N-n)T(r,f)+S(r,f).$$

Proof. From Theorem 1 (a), we have

$$\sum_{j=1}^{q}\theta\,\omega(j)m(r,\boldsymbol{a}_j,f)\leq\theta\,m(r,\boldsymbol{e}_{n+1},f^*)+S(r,f).$$

Adding $\Sigma_{j=1}^{q}(1-\theta\,\omega(j))T(r,f)$ to both sides of this inequality, we obtain

$$\sum_{j=1}^{q}m(r,\boldsymbol{a}_j,f)+\sum_{j=1}^{q}(1-\theta\,\omega(j))N(r,\boldsymbol{a}_j,f)\leq\theta\,m(r,\boldsymbol{e}_{n+1},f^*)+T(r,f)\sum_{j=1}^{q}(1-\theta\,\omega(j))+S(r,f).$$

Since $N(r,\boldsymbol{a}_j,f)\geq 0$ for $r\geq 1$ and by (a),(b),(c) of Lemma 1, we obtain our lemma.

Theorem 6. Suppose that X is p-maximal in the sense of N-subgeneral position, $\rho(f)<\infty$ and

$$\sum_{\boldsymbol{a}\in X}\delta(\boldsymbol{a},f)=2N-n+1.$$

Then, the following statements hold:

(a) $p=n$ or $\Omega=1$.

(b) $\xi=0$.

(c) $\dfrac{n+1}{2N-n+1}\leq\liminf_{r\to\infty}\dfrac{T(r,f^*)}{T(r,f)}\leq\limsup_{r\to\infty}\dfrac{T(r,f^*)}{T(r,f)}\leq 1+n\Omega.$

(d) In particular, if

$$\delta(\boldsymbol{e}_j,f)=1\ (j=1,\cdots,n),$$

then $\rho(f)$ is a positive integer and f is of regular growth.

Proof. (a) and (b). These are trivial by Theorem 5 (II).

(c). Since $\sharp X(0)\leq N$,

$$\sum_{\boldsymbol{a}\in X-X(0)}\delta(\boldsymbol{a},f)\geq N-n+1. \tag{23}$$

From (23) and Lemma 5, we have

$$1\leq\frac{2N-n+1}{n+1}\liminf_{r\to\infty}\frac{T(r,f^*)}{T(r,f)}$$

and from Proposition 2,

$$\limsup_{r\to\infty}\frac{T(r,f^*)}{T(r,f)}\leq 1+n\Omega.$$

Combining these two inequalities we obtain (c). Note that

$$S(r,f)=O(\log r)\quad (r\to\infty)$$

since $\rho(f)<\infty$.

(d). Since for $j=1,\cdots,n$

$$\begin{aligned}0\leq \limsup_{r\to\infty}\frac{N(r,\boldsymbol{e}_j,f^*)}{T(r,f^*)} &\leq \limsup_{r\to\infty}\frac{(n+1)N(r,\boldsymbol{e}_j,f)}{T(r,f^*)}\\ &= (n+1)\limsup_{r\to\infty}\frac{N(r,\boldsymbol{e}_j,f)}{T(r,f)}\cdot\frac{T(r,f)}{T(r,f^*)}\\ &\leq (2N-n+1)\limsup_{r\to\infty}\frac{N(r,\boldsymbol{e}_j,f)}{T(r,f)}=0\end{aligned}$$

by (c) and by the assumption that $\delta(\boldsymbol{e}_j,f)=1$ $(j=1,\cdots,n)$ and since

$$0\leq \limsup_{r\to\infty}\frac{N(r,\boldsymbol{e}_{n+1},f^*)}{T(r,f^*)} \leq \limsup_{r\to\infty}\frac{N(r,1/W)}{T(r,f)}\cdot\frac{T(r,f)}{T(r,f^*)}\leq\frac{2N-n+1}{n+1}\xi=0$$

by (b), we have

$$\delta(\boldsymbol{e}_j,f^*)=1-\limsup_{r\to\infty}\frac{N(r,\boldsymbol{e}_j,f^*)}{T(r,f^*)}=1\ (j=1,\cdots,n+1).$$

Then, we have (d) by Lemma 4.

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