

Note on Catalan Polynomials and Motzkin Polynomials

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In Section 1 we consider from the viewpoint of generating series the multiple Catalan polynomials introduced by Kreweras [3]. In Section 2 we give a definition of simple Motzkin polynomials which generalize the Motzkin numbers, and also give a combinatorial significance of the polynomials, with a remark on a definition of multiple Motzkin polynomials.

1. Multiple Catalan Polynomials

Simple Catalan polynomials and double Catalan polynomials were introduced by Kreweras [3]; they generalize the Catalan numbers in such a manner that they can be applied to certain characterization of tree types. In the last section of Kreweras [3], introduction of multiple Catalan polynomials was also considered with respect to a particular family of trees; however, explicit construction of the polynomials was not given there. We consider here multiple Catalan polynomials from the viewpoint of generating series.

We first set up some notation and definitions, primarily following Kreweras [3]. Let $t, u, x_1, x_2, \dots, x_n$ be indeterminates and we work with the powerseries algebra in these indeterminates. Put

$$(1) \quad \gamma(t) = \frac{2}{1 + \sqrt{1 - 4t}} = \sum_{i \geq 0} c_i t^i$$

with c_i the Catalan numbers,

$$(2) \quad \begin{aligned} \gamma_n(x_1, x_2, \dots, x_n) \\ &= (1 - x_1 \gamma(x_1) - x_2 \gamma(x_2) - \dots - x_n \gamma(x_n)) \\ &= 2 / (\sqrt{1 - 4x_1} + \sqrt{1 - 4x_2} + \dots + \sqrt{1 - 4x_n} + 2 - n), \end{aligned}$$

and

$$(3) \quad g_{n,u}(x_1, x_2, \dots, x_n) = \gamma_n(x_1, x_2, \dots, x_n)^{u+1}.$$

Note that $\gamma_1(x_1) = \gamma(x_1)$ and that $\gamma_n(x_1, x_2, \dots, x_n)$ is a powerseries with constant term 1; hence (3) can be expanded as powerseries in x_1, x_2, \dots, x_n with coefficients being polynomials in u :

$$(4) \quad g_{n,u}(x_1, x_2, \dots, x_n) = \sum_{i_1, i_2, \dots, i_n \geq 0} P_{i_1, i_2, \dots, i_n}(u) x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}.$$

We call $P_{i_1, i_2, \dots, i_n}(u)$ *multiple (n-ple in particular) Catalan polynomials*. ($P_i(u)$ are simple Catalan polynomials.)

The generating series $g_{n,u}$ satisfies the following:

$$(5) \quad g_{n,u} - g_{n,u-1} = g_{n,u} \cdot (x_1 \gamma(x_1) + x_2 \gamma(x_2) + \dots + x_n \gamma(x_n)),$$

which is equivalent to the coefficient relation:

$$\begin{aligned}
 (6) \quad P_{i_1, i_2, \dots, i_n}(u) - P_{i_1, i_2, \dots, i_n}(u-1) &= \sum_{j_1=0}^{i_1-1} c_{j_1} P_{i_1-1-j_1, i_2, \dots, i_n}(u) \\
 &+ \sum_{j_2=0}^{i_2-1} c_{j_2} P_{i_1, i_2-1-j_2, i_3, \dots, i_n}(u) + \dots \\
 &+ \sum_{j_n=0}^{i_n-1} c_{j_n} P_{i_1, i_2, \dots, i_{n-1}, i_n-1-j_n}(u).
 \end{aligned}$$

In the case that u takes positive integer values, putting

$$(7) \quad d(u; i_1, i_2, \dots, i_n) = u! P_{i_1, i_2, \dots, i_n}(u),$$

we see that the above coefficient relation gives:

$$\begin{aligned}
 (8) \quad d(u; i_1, i_2, \dots, i_n) &= u d(u-1; i_1, i_2, \dots, i_n) \\
 &+ \sum_{j_1=0}^{i_1-1} c_{j_1} d(u; i_1-1-j_1, i_2, \dots, i_n) + \dots \\
 &+ \sum_{j_n=0}^{i_n-1} c_{j_n} d(u; i_1, i_2, \dots, i_n-1-j_n).
 \end{aligned}$$

This formula shows that $d(u; i_1, i_2, \dots, i_n)$ is the descriptor (descripteur in French; see below for the definition) of a tree belonging to the family of trees each of which consists of $n+u$ open chains having a common end vertex, the first n ones with i_1, i_2, \dots, i_n edges and the other u ones with a single edge; see the first paragraph of this section.

We cite here the definition of the descriptor $d(T)$ of a tree T from Kreweras [3, p.393]. $d(T)$ is defined by recurrence on the number of edges of T using the following two conventions:

(i) $d(\text{the empty tree}) = 1$, where the empty tree means the tree with 0 edge and 1 vertex.

(ii) Let T be a tree with n edges numbered 1 through n , and let T'_k and T''_k be the two trees obtained by suppression of the edge numbered k ; then $d(T) = \sum_{k=1}^n d(T'_k) d(T''_k)$.

We have $d(i \text{ edges in chain}) = c_i$ (Note the recurrence relation for the Catalan numbers $c_i : c_i = c_0 c_{i-1} + c_1 c_{i-2} + \dots + c_{i-1} c_0$.) and $d(u \text{ edges in star}) = u!$. (See Kreweras [3, p.393].) The formula (8) in the case of the family of trees mentioned just after the formula follows from these conventions and properties.

For the cases $n=1, 2$, we recover some of the results of Kreweras [3].

2. Motzkin Polynomials

We refer to Donaghey-Shapiro [1] for basic information on Motzkin numbers m_i ; they are defined by

$$(9) \quad m_i = \sum_{j \geq 0} \binom{i}{2j} c_j$$

with c_j Catalan numbers and $\binom{i}{2j} = 0$ if $2j > i$. (For Catalan numbers, see Hilton-Pedersen [2].) m_i are interpreted to be the ballot numbers modified to allow abstentions; $k = i - 2j$ is the number of abstentions over which the summation is done. (See Donaghey-Shapiro [1, p.293].) Using the generating series γ of the Catalan numbers (1) we can write the generating series μ of the Motzkin numbers as follows:

$$(10) \quad \mu(t) = (1-t)^{-1} \gamma(t^2(1-t)^{-2}),$$

which can be proved by straightforward powerseries computation. (See Riordan [4, p.217].) Expanding $\mu(t)^{u+1} = (1-t)^{-u-1} \gamma(t^2(1-t)^{-2})^{u+1}$ as powerseries in t , we have by $\gamma(t)^{u+1} = \gamma_1(t)^{u+1} = \sum_{i \geq 0} P_i(u) t^i$ (see (4) with $n=1$) and by straightforward powerseries computation that

$$(11) \quad \mu(t)^{u+1} = \sum_{i \geq 0} \left(\sum_{2j \leq i} \binom{i+u}{i-2j} P_j(u) \right) t^i.$$

We denote by $Q_i(u)$ the coefficient of t^i in (11); we call $Q_i(u)$ (*simple*) *Motzkin polynomials*.

Before giving a combinatorial significance of $Q_i(u)$, we first consider what $P_j(u)$ means in the context of lattice path counting with horizontal steps and vertical steps in the case that u takes positive integer values. As Kreweras [3, pp.389-390] gave, we have

$$(12) \quad P_j(u) = \binom{2j+u}{j} - \binom{2j+u}{j-1}.$$

The integer value (12) counts the number of 2-good paths from $(0, -1)$ to $(j+u, j-1)$; for the definition, see Hilton-Pedersen [2, p.69] and Ueno [5]. With u positive integer values, we can write

$$(13) \quad Q_i(u) = \sum_{2j \leq i} \binom{i+u}{2j+u} P_j(u).$$

Let us remember the interpretation of the Motzkin numbers m_i as the ballot numbers modified to allow abstentions, which is also equivalent to a lattice path counting with diagonal steps representing abstentions introduced; see Donaghey-Shapiro [1, p.294]. Having this in mind and comparing (13) with (9), we arrive at the following setting of a combinatorial interpretation of $Q_i(u)$:

Consider a series of competitions between two players A and B in which each game yields one point to the victor with ties allowed, and in which player A is never behind player B in points with the result that player A is eventually ahead of player B by u points. (Note that ties give no point to either player and that ties in this setting correspond to abstentions in the ballot problem setting.) Then $Q_i(u)$ counts the number of such series of competitions with $i+u$ games played.

We conclude by remarking that we can give a definition of *multiple Motzkin polynomials* by expanding

$$(14) \quad \begin{aligned} & \left((1-t_1)^{-1}(1-t_2)^{-1} \cdots (1-t_n)^{-1} \right)^{u+1} \\ & \times \left(\gamma_n(t_1^2(1-t_1)^{-2}, t_2^2(1-t_2)^{-2}, \dots, t_n^2(1-t_n)^{-2}) \right)^{u+1} \end{aligned}$$

as powerseries in t_1, t_2, \dots, t_n (see (2), (10), and (11)) and picking out the coefficients of monomials in t_1, t_2, \dots, t_n .

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