On the Fine Cluster Set of Holomorphic Curves

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We shall give a definition of fine cluster set for holomorphic curves from $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$ as a generalization of the case of meromorphic functions and prove the followings.

"Let f be a transcendental holomorphic curve from $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$. Then the fine cluster set of f is either (i) $\boldsymbol{C}^{n+1}$ or (ii) a subset of $\boldsymbol{C}^{n+1}$ of dimension n.

In case of (ii),

f has neither Picard exceptional vectors nor deficient vectors nor Borel exceptional vectors."

1. Introduction

(a) Let V_o be the family of deleted fine neighbourhoods v_o of the origin of the complex plane and put

$$v=\{z : 1/z\in v_o\}$$

and put

$$V=\{v : v_o\in V_o\}.$$

For any function $\varphi(z)$ defined in the complex plane we put

$$\mathscr{F}C(\varphi)=\bigcap_{v\in V}\overline{\varphi(v)},$$

where $\varphi(v)=\{\varphi(z) : z\in v\}$ and $\overline{\varphi(v)}$ is the closure of $\varphi(v)$ in the extended complex plane.

The set $\mathscr{F}C(\varphi)$ is called the fine cluster set of φ at ∞. The set is non-empty closed set in the extended complex plane. φ is called to have a fine limit at ∞ when $\mathscr{F}C(\varphi)$ consists of a single point.

(b) In 1965, J.L.Doob([3]) applied the fine cluster set to meromorphic functions to obtain the following interesting

Theorem A. Let f be a transcendental meromorphic function in $|z|<\infty$. Then, one of the following situations must hold.

(i) $\mathscr{F}C(f)$ is the extended complex plane.

(ii) f has a fine limit at ∞. In this case f has no Picard exceptional value.

(Theorem 7.3 in [3]).

Theorem A(ii) was improved as follows.

Theorem B. Let f be a transcendental meromorphic function in $|z|<\infty$. If f has a fine limit at ∞, then

(i) f has no deficient value (Théorème 2 in [5]).

(ii) f has no Borel exceptional value (Théorème 3 in [5]).

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(c) These results were generalized to algebroid functions in $|z|<\infty$. In [6], we gave a definition of fine cluster set at ∞ for algebroid functions in $|z|<\infty$ and obtained some generalizations of Theorems A and B as follows.

Theorem C. Let f be a transcendental, n-valued algebroid function in $|z|<\infty$, where n is an integer not less than 1. Then, one of the following situations must hold.

(i) $\mathscr{F}C(f)$ is the extended complex plane.

(ii) $\mathscr{F}C(f)$ consists of at most n values. In this case, f has no Picard exceptional value, no deficient value or no Borel exceptional value (see Théorèmes 2,3,4 and 5 in [6]).

(d) The purpose of this paper is to give some results similar to Theorem C given above for holomorphic curves in $|z|<\infty$.

2 Preliminary and Lemma

Let $f=[f_1,\cdots,f_{n+1}]$ be a transcendental holomorphic curve from $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$ a reduced representation of which is

$$(f_1,\cdots,f_{n+1}) : \boldsymbol{C}\rightarrow \boldsymbol{C}^{n+1}-\{\boldsymbol{0}\},$$

where n is an integer not less than one. We may suppose without loss of generality that $f_j\neq 0$ $(j=1,\cdots,n+1)$. For any vector

$$\boldsymbol{a}=(a_1,\cdots,a_{n+1})\in \boldsymbol{C}^{n+1}-\{\boldsymbol{0}\},$$

we put

$$\varphi_a(z)=\frac{|(\boldsymbol{a},f(z))|}{\|\boldsymbol{a}\|\|f(z)\|},$$

where

$$(\boldsymbol{a},f(z))=\sum_{j=1}^{n+1} a_jf_j(z),$$

$$\|\boldsymbol{a}\|=(\sum_{j=1}^{n+1}|a_j|^2)^{1/2}$$

and

$$\|f(z)\|=(\sum_{j=1}^{n+1}|f_j(z)|^2)^{1/2}.$$

The characteristic function $T(r,f)$ of f is defined as follows (see [8]):

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta-\log\|f(0)\|.$$

As our f is transcendental, it holds that

$$\lim_{r\to\infty}\frac{T(r,f)}{\log r}=\infty.$$

We denote by $\rho(f)$ the order of f:

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

In the sequel, we use the standard notation of the Nevanlinna theory of meromorphic functions(see [4]).

For $\boldsymbol{a}=(a_1,\cdots,a_{n+1})$ such that $(\boldsymbol{a},f)\neq 0$, we write

$$m(r,\boldsymbol{a},f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|\boldsymbol{a}\|\|f(re^{i\theta})\|}{|(\boldsymbol{a},f(re^{i\theta})|}d\theta$$

and

$$N(r, \boldsymbol{a}, f) = N(r, 1/(\boldsymbol{a}, f)).$$

We call the quantity

$$\delta(\boldsymbol{a}, f) = \liminf_{r\to\infty} \frac{m(r, \boldsymbol{a}, f)}{T(r, f)}$$

the deficiency of $\boldsymbol{a}$ with respect to f.

The vector $\boldsymbol{a}$ is said to be

(a) *Picard exceptional* for f when $(\boldsymbol{a}, f)$ has at most a finite number of zeros;

(b) *deficient* or *exceptional in the sense of Nevanlinna* for f when $\delta(\boldsymbol{a}, f) > 0$;

(c) *Borel exceptional* for f when

$$\limsup_{r\to\infty} \frac{\log N(r, \boldsymbol{a}, f)}{\log r} < \rho(f).$$

Now, we shall give a definition of fine cluster set $\mathscr{F}(f, \infty)$ for f.

Definition. A vector $\boldsymbol{a} \neq \boldsymbol{0}$ is contained in $\mathscr{F}(f, \infty)$ if and only if 0 is contained in $\mathscr{F}C(\varphi_a)$. We call $\mathscr{F}(f, \infty)$ the fine cluster set of f at ∞.

Remark. Put

$$\mathscr{H} = \{\boldsymbol{a} \in \boldsymbol{C}^{n+1} : (\boldsymbol{a}, f) = 0\},$$

then $\mathscr{H}$ is a subspace of $\boldsymbol{C}^{n+1}$. Put $\lambda = \dim \mathscr{H}$, then $0 \leq \lambda \leq n-1$. A holomorphic curve f is said to be

(a) (linearly) nondegenerate if $\lambda = 0$;

(b) (linearly) degenerate if $\lambda > 0$.

In case f is degenerate

$$\mathscr{F}(f, \infty) \cup \{\boldsymbol{0}\} \supset \mathscr{H}$$

since for $\boldsymbol{a}(\neq \boldsymbol{0}) \in \mathscr{H}$, $\varphi_a(z) \equiv 0$ and $\mathscr{F}C(\varphi_a) = \{0\}$.

Now we give some lemmas for later use.

Lemma 1. For any $v \in V$, there is a positive sequence $\{r_k\}$ tending to ∞ such that the circles $|z| = r_k$ $(k = 1, 2, \cdots)$ are contained in v ([2], p.89).

Lemma 2. Let $u(z)$ be a superharmonic function in a domain D such that $D^c \in V$. If $u(z)/\log|z|$ is bounded below in a set $v \in V$, then $u(z)/\log|z|$ has a finite fine limit at ∞ (Théorème 2 in [1]).

Lemma 3. Let X be a subset of $\boldsymbol{C}^{n+1}$ any $n+1$ vectors in which are linearly independent. Then, there are at most $2n$ Borel exceptional vectors for f in X when $\rho(f) > 0$ (see Théorème 4 in [7]).

3 Theorem

We shall prove some theorems for holomorphic curves corresponding to Theorem C in Section 1.

Theorem 1. Let f be a transcendental holomorphic curve from $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$. Then, one of the following situations must hold.

(i) $\mathscr{F}(f, \infty) \cup \{\boldsymbol{0}\} = \boldsymbol{C}^{n+1}$

(ii) $\mathscr{F}(f, \infty) \cup \{\boldsymbol{0}\}$ is a subspace of $\boldsymbol{C}^{n+1}$ of dimension n.

Proof. Suppose that the set $\mathscr{F}(f, \infty)$ is not equal to $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$. Then, there exists a vector $\boldsymbol{a} = (a_1, \cdots, a_{n+1}) \neq \boldsymbol{0}$ such that

$$0 \notin \mathscr{F}C(\varphi_{a}).$$

In this case, there is an element v of V and positive number M such that

$$1 \leq \frac{\|\boldsymbol{a}\|\|f(z)\|}{|(\boldsymbol{a}, f(z))|} \leq M \quad (z \in v). \tag{1}$$

Put

$$(\boldsymbol{a}, f(z)) \equiv a_1 f_1(z) + \cdots + a_{n+1} f_{n+1}(z) = F(z)$$

and we have from (1) for $j = 1, \cdots, n+1$

$$\frac{|f_j(z)|}{|F(z)|} \leq \left\{\sum_{j=1}^{n+1}\left(\frac{|f_j(z)|}{|F(z)|}\right)^2\right\}^{1/2} \leq \frac{M}{\|\boldsymbol{a}\|}$$

in v. This means that all meromorphic functions $f_j(z)/F(z)(j=1,\cdots,n+1)$ are bounded in v and by theorem A each $f_j(z)/F(z)$ has a finite fine limit at ∞. Let the limits be $\alpha_j(j=1,\cdots,n+1)$. Then, there is an element v_1 of V such that

$$\frac{f_j(z)}{F(z)} \to \alpha_j \ (j=1,\cdots,n+1), \tag{2}$$

when $v_1 \ni z \to \infty$ in the usual sense in $|z| < \infty$ (see [2], p.91). Put $\boldsymbol{p} = (\alpha_1, \cdots, \alpha_{n+1})$, then, from (1) we have the inequality

$$\frac{1}{\|\boldsymbol{a}\|} \leq (\sum_{j=1}^{n+1} |\alpha_j|^2)^{1/2} = \|\boldsymbol{p}\|$$

and so $\boldsymbol{p} \neq \boldsymbol{0}$. Further, from (1) and (2) we have

$$\alpha_1 a_1 + \cdots + \alpha_{n+1} a_{n+1} \neq 0.$$

Put

$$W = \{(b_1, \cdots, b_{n+1}) \in \boldsymbol{C}^{n+1} : \alpha_1 b_1 + \cdots + \alpha_{n+1} b_{n+1} = 0\}.$$

Then, W is a subspace of $\boldsymbol{C}^{n+1}$ and since $\boldsymbol{p} \neq \boldsymbol{0}$,

$$\dim W = n.$$

For any vector $\boldsymbol{b} = (b_1, \cdots, b_{n+1})$ in $W - \{\boldsymbol{0}\}$, the function

$$\frac{|(\boldsymbol{b}, f(z))|}{\|\boldsymbol{b}\|\|f(z)\|} = \frac{|\Sigma_{j=1}^{n+1} b_j f_j(z)/F(z)|}{\|\boldsymbol{b}\|(\Sigma_{j=1}^{n+1}|f_j(z)/F(z)|^2)^{1/2}}$$

tends to

$$\frac{|\Sigma_{j=1}^{n+1} b_j \alpha_j|}{\|\boldsymbol{b}\|\|\boldsymbol{p}\|} = 0$$

when $v_1 \ni z \to \infty$ in the usual sense in $|z| < \infty$. This means that

$$\boldsymbol{b} \in \mathscr{F}(f, \infty);$$

that is to say,

$$W - \{\boldsymbol{0}\} \subset \mathscr{F}(f, \infty).$$

Let $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n$ be a system of basis of W. Then, $\boldsymbol{a}, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n$ are linearly independent over $\boldsymbol{C}$. In fact, suppose to the contrary that $\boldsymbol{a}, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n$ are linearly dependent over $\boldsymbol{C}$. Then, there are constants $\beta, \beta_1, \cdots, \beta_n$ satisfying

$$\beta \boldsymbol{a} + \beta_1 \boldsymbol{b}_1 + \cdots + \beta_n \boldsymbol{b}_n = 0, \tag{3}$$

where at least one of $\beta, \beta_1, \cdots, \beta_n$ is not equal to zero. Now, β is not equal to zero, because if $\beta = 0$, we have

$$\beta_1\boldsymbol{b}_1+\cdots+\beta_n\boldsymbol{b}_n=\boldsymbol{0},$$

and so, as $\boldsymbol{b}_1,\cdots,\boldsymbol{b}_n$ are linearly independent

$$\beta_1=\cdots=\beta_n=0,$$

which is a contradiction. From (3) we have

$$\boldsymbol{a}=-\frac{1}{\beta}(\beta_1\boldsymbol{b}_1+\cdots+\beta_n\boldsymbol{b}_n),$$

which implies that $\boldsymbol{a}\in W$. This is a contradiction, that is to say, $\boldsymbol{a}$, $\boldsymbol{b}_1,\cdots,\boldsymbol{b}_n$ must be linearly independent over $\boldsymbol{C}$.

Let $\boldsymbol{c}=(c_1,\cdots,c_{n+1})$ be any vector not contained in W. Then for some constants $\gamma,\gamma_1,\cdots,\gamma_n$

$$\boldsymbol{c}=\gamma\boldsymbol{a}+\gamma_1\boldsymbol{b}_1+\cdots+\gamma_n\boldsymbol{b}_n$$

and when $v_1\ni z\to\infty$ in the usual sense in $|z|<\infty$ the function

$$\frac{|(\boldsymbol{c}, f(z))|}{\|\boldsymbol{c}\|\|f(z)\|}$$

tends to

$$\frac{|\gamma||(\Sigma_{j=1}^{n+1}a_j\alpha_j|}{\|\boldsymbol{c}\|\|\boldsymbol{p}\|}\neq 0,$$

which means that $\boldsymbol{c}\notin\mathscr{F}(f,\infty)$. We have

$$\mathscr{F}(f,\infty)\cup\{\boldsymbol{0}\}=W,$$

which is a subspace of $\boldsymbol{C}^{n+1}$ of dimension n. This completes the proof.

Theorem 2. Let f be a transcendental holomorphic curve from $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$ such that $\mathscr{F}(f,\infty)\cup\{\boldsymbol{0}\}$ is a subspace of $\boldsymbol{C}^{n+1}$ of dimension n. Then, for any $\boldsymbol{c}=(c_1,\cdots,c_{n+1})\in\boldsymbol{C}^{n+1}-\mathscr{H}$,

(i) $\boldsymbol{c}$ is not Picard exceptional for f.

(ii) $\delta(\boldsymbol{c},f)=0$.

(iii) $\boldsymbol{c}$ is not Borel exceptional for f.

Proof. Let $\boldsymbol{a}=(a_1,\cdots,a_{n+1})$ be a vector in $\boldsymbol{C}^{n+1}-\mathscr{F}(f,\infty)\cup\{\boldsymbol{0}\}$. Then,

$$0\notin\mathscr{F}C(\varphi_a).$$

In this case, there is an element v of V and a positive number M satisfying

$$1\leq\frac{\|\boldsymbol{a}\|\|f(z)\|}{|(\boldsymbol{a}, f(z))|}\leq M\qquad(z\in v).$$

Put

$$(\boldsymbol{a}, f(z))\equiv a_1f_1(z)+\cdots+a_{n+1}f_{n+1}(z)=F(z).$$

As in the proof of Theorem 1, for each $j=1,\cdots,n+1$, $f_j(z)/F(z)$ has a finite fine limit α_j at ∞ and there exists v_1 in V such that

$$\frac{f_j(z)}{F(z)}\to\alpha_j\ (j=1,\cdots,n+1)\tag{4}$$

when $v_1\ni z\to\infty$ in the usual sense in $|z|<\infty$. Put

$$\boldsymbol{p}=(\alpha_1,\cdots,\alpha_{n+1}),$$

then $\boldsymbol{p}\neq 0$.

(a) $F(z)$ is transcendental.

In fact, suppose to the contrary that $F(z)$ is polynomial. As f is transcendental, at least one of $f_j(z)$ is transcendental. We suppose without loss of generality that $f_1(z)$ is transcendental. Then, $f_1(z)/F(z)$ is transcendental and it has a finite fine limit α_1 by (4). This means from Theorem A(ii) that $f_1(z)/F(z)$ has no Picard exceptional value. This is a contradiction since $F(z)$ has only a finite number of zeros and $f_1(z)/F(z)$ has only a finite number of poles. This implies that $F(z)$ is transcendental.

(b) At least one of $f_j(z)/F(z)$ $(j=1,\cdots,n+1)$ is transcendental.

In fact, suppose to the contrary that any one of $f_j(z)/F(z)$ is rational. Then, $F(z)$ has at most a finite number of zeros since, if it has an infinite number of zeros, $f_1(z),\cdots,f_{n+1}(z)$ have an infinite number of common zeros, which is a contradiction. Put

$$\frac{f_j(z)}{F(z)}=R_j(z) \quad (j=1,\cdots,n+1)$$

and

$$F(z)=P(z)e^{g(z)},$$

where $P(z)$ is a polynomial and $g(z)$ is a nonconstant entire function. Then,

$$f_j(z)=P(z)R_j(z)e^{g(z)} \quad (j=1,\cdots,n+1)$$

and as $f_j(z)$ is entire, $P(z)R_j(z)$ must be polynomial, which reduces to

$$T(r,f)=O(\log r) \quad (r\to\infty).$$

This is a contradiction. The assertion (b) holds.

(c) $F(z)$ has an infinite number of zeros.

In fact, we may suppose without loss of generality that $f_1(z)/F(z)$ is transcendental by (b) and, from (4), the transcendental function $f_1(z)/F(z)$ has a fine limit α_1 at ∞, so that by Theorem A(ii), $F(z)$ has an infinite number of zeros.

Now, denote by

$$\boldsymbol{b}_1,\cdots,\boldsymbol{b}_n$$

a system of basis of $\mathscr{F}(f,\infty)\cup\{\boldsymbol{0}\}$. Then, there are constants $\gamma,\gamma_1,\cdots\gamma_n$ such that

$$\boldsymbol{c}=\gamma\boldsymbol{a}+\gamma_1\boldsymbol{b}_1+\cdots+\gamma_n\boldsymbol{b}_n$$

and we have

$$(\boldsymbol{c},f)=\gamma(\boldsymbol{a},f)+\gamma_1(\boldsymbol{b}_1,f)+\cdots+\gamma_n(\boldsymbol{b}_n,f). \tag{5}$$

As $\boldsymbol{b}_1,\cdots,\boldsymbol{b}_n$ are in $\mathscr{F}(f,\infty)$ and

$$\left(\sum_{j=1}^{n+1}\left|\frac{f_j(z)}{F(z)}\right|^2\right)^{1/2}\to\|\boldsymbol{p}\|(\neq 0)$$

when $v_1\ni z\to\infty$ in the usual sense in $|z|<\infty$,

$$\frac{(\boldsymbol{b}_1,f(z))}{F(z)}\to 0 \tag{6}$$

when $v_1\ni z\to\infty$ in the usual sense in $|z|<\infty$. (5) and (6) imply that

$$\frac{(\boldsymbol{c},f(z))}{F(z)}\to\gamma$$

when $v_1\ni z\to\infty$ in the usual sense in $|z|<\infty$. This means that the function $(\boldsymbol{c},f(z))/F(z)$ has a fine limit γ at ∞.

(I) Proof of (i). When $(\boldsymbol{c},f(z))/F(z)$ is transcendental, $(\boldsymbol{c},f(z))$ has an infinite number of zeros by Theorem A(ii) and when $(\boldsymbol{c},f(z))/F(z)$ is rational, $(\boldsymbol{c},f(z))$ has also an infinite number of zeros since $F(z)$ has an infinite number of zeros.

(II) Proof of (ii). For $\mathbf{c}=(c_1,\cdots,c_{n+1})$ given in the statement of this theorem, we estimate

$$m(r,\mathbf{c},f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|\mathbf{c}\|\|f(re^{i\theta})\|}{|(\mathbf{c},f(re^{i\theta}))|}\,d\theta$$

as follows.

(a) When $\gamma\neq 0$, the function

$$\frac{\|\mathbf{c}\|\|f(z)\|}{|(\mathbf{c},f(z))|}=\frac{\|\mathbf{c}\|(\Sigma_{j=1}^{n+1}|f_j(z)/F(z)|^2)^{1/2}}{|(\mathbf{c},f(z))/F(z)|}\tag{7}$$

tends to $\|\mathbf{c}\|\|\mathbf{p}\|/|\gamma|$ as $v_1\ni z\to\infty$ in the usual sense in $|z|<\infty$. Then, by Lemma 1, there is a sequence $\{r_k\}$ tending to ∞ such that the circles $|z|=r_k(k=1,2,\cdots)$ are contained in v_1 and

$$\frac{\|\mathbf{c}\|\|f(r_ke^{i\theta})\|}{|(\mathbf{c},f(r_ke^{i\theta}))|}=O(1)\ \ (k\to\infty),$$

which shows that

$$m(r_k,\mathbf{c},f)=O(1)\ \ (k\to\infty).\tag{8}$$

(b) When $\gamma=0$, from (7) we have

$$\log\frac{\|\mathbf{c}\|\|f(z)\|}{|(\mathbf{c},f(z))|}=\log\|\mathbf{c}\|+\frac{1}{2}\log(\sum_{j=1}^{n+1}|f_j(z)/F(z)|^2)-\log|(\mathbf{c},f(z))/F(z)|\tag{9}$$

Here, for $v_1\ni z\to\infty$

$$\frac{1}{2}\log(\sum_{j=1}^{n+1}|f_j(z)/F(z)|^2)\to\log\|\mathbf{p}\|$$

and

$$-\log|(\mathbf{c},f(z))/F(z)|\to\infty.$$

Let D be a unbounded component contained in

$$\{z\in v:-\log|(\mathbf{c},f(z))/F(z)|\geq 0;\ |z|>1\}.$$

Then, $D\in V$ and $-\log|(\mathbf{c},f(z))/F(z)|$ is a positive superharmonic function in D. Since

$$\frac{-\log|(\mathbf{c},f(z))/F(z)|}{\log|z|}$$

is bounded below it has a finite fine limit at ∞ by Lemma 2. This implies that by Lemma 1 and by Cartan's theorem ([2], p.91) there is a sequence $\{r_k\}$ tending to ∞ such that the circles $|z|=r_k$ $(k=1,2,\cdots)$ are contained in D and

$$-\log|(\mathbf{c},f(r_ke^{i\theta}))/F(r_ke^{i\theta})|=O(\log r_k)\ \ (k\to\infty),$$

so that we have from (9) that

$$m(r_k,\mathbf{c},f)=O(\log r_k)\ \ (k\to\infty).\tag{10}$$

From (8) and (10), we have

$$\delta(\mathbf{c},f)=\liminf_{r\to\infty}\frac{m(r,\mathbf{c},f)}{T(r,f)}=0$$

since f is transcendental.

(III) Proof of (iii). For $\mathbf{c}=(c_1,\cdots,c_{n+1})$ given in the statement of this theorem, when $(\mathbf{c},f(z))/F(z)$ is transcendental, by Theorem B (ii)

$$\limsup_{r\to\infty}\frac{\log N(r,\mathbf{c},f)}{\log r}=\limsup_{r\to\infty}\frac{\log N(r,1/F)}{\log r}\tag{11}$$

and when $(\mathbf{c},f(z))/F(z)$ is rational, (11) also holds since

$$N(r, \boldsymbol{c}, f) = N(r, 1/F) + O(\log r).$$

By Lemma 3, the equality

$$\limsup_{r\to\infty} \frac{\log N(r, 1/F)}{\log r} = \rho(f) \tag{12}$$

must hold. In fact, suppose to the contrary that

$$\limsup_{r\to\infty} \frac{\log N(r, 1/F)}{\log r} < \rho(f).$$

Then, from (11) for any vector $\boldsymbol{c} \in \boldsymbol{C}^{n+1} - \mathscr{H}$,

$$\limsup_{r\to\infty} \frac{\log N(r, \boldsymbol{c}, f)}{\log r} < \rho(f). \tag{13}$$

On the contrary, take

$$X = \{(a^n, \cdots, a, 1) : a \in \boldsymbol{C}\}$$

in Lemma 3. Then $X \cap \mathscr{H}$ contains at most $n-1$ elements and $X - \mathscr{H}$ contains at most $2n$ Borel exceptional vectors for f, and so there are vectors in $X - \mathscr{H}$ not Borel exceptional for f, which contradicts with (13).

From (11) and (12) we have (iii) of this theorem.

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