Another Look at the Generalized Ballot Problem

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We will give an alternative proof of a theorem concerning the generalized ballot problem. The proof assigns a combinatorial meaning to the binomial coefficients in the theorem.

The purpose of this note is to give another proof of Corollary 3.3 of Hilton-Pedersen [1]. The result has been announced in Ueno [2]. Before proceeding, we set some definitions and notation required following Hilton-Pedersen [1]; for further background material, see also Hilton-Pedersen [1].

We consider paths on the integral lattice in the coordinate plane, which we simply call paths: a *path from* $P=P_0$ *to* $Q=P_m$ is a sequence of points P_i $(0\leq i\leq m)$ with integer coordinates where P_{i+1} is obtained by stepping one unit east or one unit north of P_i.

Let p be a fixed integer greater than 1. A path from P to Q is said to be *p-good* if it lies entirely below the line $y=(p-1)x$. Let P_k be the point $(k, (p-1)k-1)$, $k\geq 0$ and let $q\leq p-1$. We define $d_{q,k}$ to be the number of p-good paths from $(1, q-1)$ to P_k $(k\geq 1)$ with $d_{q,0}=1$. ($d_{q,k}$ can be regarded as generalizations of the Catalan numbers; see Hilton-Pedersen [1, p. 70 and p.72].)

Corollary 3.3 of Hilton-Pedersen [1, p.73] in question states:

Theorem. *Under the assumption that*

$$2\leq k\leq n-p+1\leq p(k-1),$$

the number of p-good paths from $(1, q-1)$ *to* $(k, n-k)$ *is*

$$\sum_{j=m+1}^{k} d_{q,j}\binom{n-pj}{k-j},$$

where $m=[n/p]$ *i.e. the integral part of* n/p.

The assumption above rules out trivial cases; see Hilton-Pedersen [1, pp.72-73]. In particular, we have $n-k<(p-1)k$. *Note that* $n-pj$ $(j\geq m+1)$ *in the binomial coefficients are less than* 0, *so that combinatorial significance of the binomial coefficients in the formula is not obvious* (Hilton-Pedersen [1, p.74]); we will give an alternative proof of the theorem which assigns a combinatorial meaning to the binomial coefficients in the formula.

Let (x, y) be a lattice point with $y<(p-1)x$ and let $a_{x,y}$ be the number of p-good paths from $(1, q-1)$ to (x, y). If $y+1<(p-1)(x-1)$, then we have the recurrence relation

$$a_{x,y+1}=a_{x,y}+a_{x-1,y+1}; \tag{1}$$

if $(p-1)(x-1)\leq y+1<(p-1)x$, then we have instead

$$a_{x,y+1}=a_{x,y}; \tag{2}$$

and if $(p-1)x\leq y+1$, then the recurrence terminates.

The relations (1) and (2) can be rewritten as

(3) $$a_{x,y}=a_{x,y+1}-a_{x-1,\,y+1},$$

(4) $$a_{x,y}=a_{x,y+1}$$

under the same assumptions respectively. The x-coordinates of the right-hand sides of (3) and (4) are one unit less than or equal to those of the left-hand sides whereas the y-coordinates of the right-hand sides of (3) and (4) are one unit greater than those of the left-hand sides. Hence, if we start from $a_{k,n-k}$, then repeated applications of (3) and (4) with the conditions on the coordinates satisfied eventually yield a formula for $a_{k,n-k}$ which is a linear combination of $a_{j,(p-1)j-1}=d_{q,j}$ $(m+1\le j\le k;\ m=[n/p])$; note that the intersection of $y=(p-1)x$ and $x+y=n$ is $(n/p,\ (p-1)n/p)$, and that of $y=(p-1)x$ and $x=k$ is $(k,\ (p-1)k)$. Writing $a_{k,n-k}=\Sigma_{j=m+1}^{k}c_{q,j}\,d_{q,j}$, we will show that $c_{q,j}=\binom{n-pj}{k-j}$.

We consider the parallelogram with vertices $(k,n-k)$, $(k,pj-k-1)$, $(j,(p-1)j-1)$, $(j,n-j)$ and the lattice paths from $(k,\ n-k)$ to $(j,\ (p-1)j-1)$ with step vectors (0, 1) and (−1, 1); *the number of such paths is* $\binom{(p-1)j-1-n+k}{k-j}$.

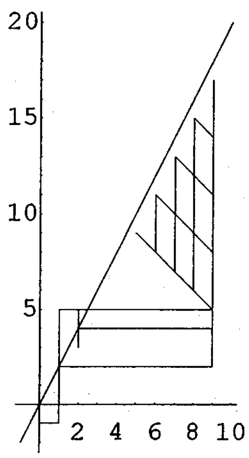
From (3) and (4) we see that $c_{q,j}$ is equal to $(-1)^{k-j}$ times the number of the lattice paths above. Hence

(5) $$c_{q,j}=(-1)^{k-j}\binom{(p-1)j-1-n+k}{k-j}=\binom{n-pj}{k-j};$$

the proof is completed.

The equality (5) can be taken to assign a combinatorial meaning to the binomial coefficients $\binom{n-pj}{k-j}$ appearing in the theorem.

Example. Let $q=0$, $p=3$, $k=9$, and $n=14$; $m=[14/3]=4$. The theorem tells us that the number of p-good paths from $(1,\ -1)$ to $(9,\ 5)$ is $\Sigma_{j=5}^{9}d_j\binom{14-3j}{9-j}$ with $d_j=d_{0,j}$. Note that $\binom{14-3j}{9-j}=(-1)^{9-j}\binom{2j-6}{9-j}$. One can visualize this example with the Figure, taking notice of the parallelograms. Note also that, for $j=1,2,3,4$, $\binom{14-3j}{9-j}$ is equal to $\binom{11}{8}$, $\binom{8}{7}$, 0, 0, respectively. The first two values correspond to the two rectangles depicted in the Figure; see Hilton-Pedersen [1, pp.73-74].



REFERENCES

1. Peter Hilton and Jean Pedersen, *Catalan Numbers, Their Generalization, and Their Uses,* The Math. Intelligencer **13** (1991), 64-75.
2. Kazuo Ueno, abstract at the fall meeting 1997 of the Mathematical Society of Japan.