On the Order of Entire Solutions of a Differential Equation

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We consider the order of entire solutions of the differential equation

(1) $$P(z, w, w', \cdots, w^{(n)}) = Q(z, w)$$

where P is a polynomial in $w, w', \cdots, w^{(n)}$ with polynomial coefficients such that the degree of each term of P is not smaller than one and

$$Q(z, w) = a_0 w^d + a_1 w^{d-1} + \cdots + a_d.$$

Here, $d = \deg P$, $a_1, \cdots, a_d$ are polynomials at least one of which is not identically equal to zero and a_0 is a transcendental entire function.

This equation (1) contains several popular equations.

1. Introduction

Let P be a polynomial of $w, w', \cdots, w^{(n)}$ $(n \geq 1)$ with polynomial coefficients:

$$P(z, w, w', \cdots, w^{(n)}) = \sum_{\lambda \in I} c_\lambda(z) w^{i_0}(w')^{i_1} \cdots (w^{(n)})^{i_n},$$

where each c_λ is polynomial and I is a finite set of multi-indices $\lambda = (i_0, i_1, \cdots, i_n)$ for which $c_\lambda \neq 0$ and $i_0, i_1, \cdots, i_n$ are non-negative integers such that at least one of $i_1, \cdots, i_n$ is not equal to zero,

$$d = \max_{\lambda \in I} (i_0 + i_1 + \cdots + i_n)$$

and let

$$Q(z, w) = \sum_{j=0}^{\ell} a_{\ell-j}(z) w^j,$$

where $a_1, \cdots, a_\ell$ are polynomials at least one of which is not identically equal to zero and a_0 is a transcendental entire function.

We consider entire solutions of the differential equation

(1) $$P(z, w, w', \cdots, w^{(n)}) = Q(z, w).$$

It is known that if the differential equation (1) admits an admissible entire solution, then $\ell \leq d$.

From now on, we consider the differential equation (1) satisfying $\ell = d$.

Examples of (1).

(a) $-2EE'' + (E')^2 = 4AE^2 + c^2$ ($c \neq 0$, constant) (see [1]).

(b) $w^{(n)} + c_1(z) w^{(n-1)} + \cdots + c_{n-1}(z) w' + c_n(z) w = F(z)$ $(F \neq 0)$ (see [5]).

(c) $(w')^n = a_0(z) w^n + \cdots + a_{n-1}(z) w + a_n(z)$ (see [4]), etc.

For an entire function f, we denote the order of f by $\rho(f)$ and the lower order of f by $\mu(f)$, the order of $N(r, 1/f)$ by $\lambda(f)$ and the order of $\overline{N}(r, 1/f)$ by $\overline{\lambda}(f)$ respectively.

Recent results on the growth of solutions of (b) in which we are interested are the followings:

Theorem A ([11], Theorem 1). Suppose in (b) that $c_1, \cdots, c_n$, F are entire such that $\rho(c_n)$ and $\rho(F)$ are finite and that (i) or (ii) holds

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(i) $\rho(c_j) < \rho(c_n)$ $(j=1,\cdots,n-1)$;

(ii) $c_1,\cdots,c_{n-1}$ are polynomials and c_n is transcendental.

Then, every solution f of (b) satisfies $\rho(f) = \overline{\lambda}(f) = \lambda(f) = +\infty$ with at most one possible exceptional solution of finite order.

In [11] examples with an "exceptional solution" are given, but those without "exceptional solution" are not given. An equation of (b) without "exceptional solution" is given in [2]:

Theorem B ([2], Theorem 3). Suppose in (b) that $c_1,\cdots,c_n$ and F are entire such that

$$\max\{\rho(c_1),\cdots,\rho(c_{n-1}),\ \rho(F)\} < \rho(c_n) < 1/2.$$

Then, every solution of (b) has infinite order.

The purpose of this paper is to give a result on the order of entire solutions of the differential equation (1) similar to Theorem 3 in [9]. We use the standard notation of the Nevanlinna theory of meromorphic functions ([3],[6]).

2. Lemma.

Lemma 1. Let f be an entire function of finite order. Then, for any positive number ε there exist $q=q(\varepsilon)$, $r_o>1$ and J(r) a subset of $[0, 2\pi)$ such that the inequality

$$(2)\qquad \left|\left(\sum_{i_0+i_1+\cdots+i_n=k} c_\lambda f^{i_0}(f')^{i_1}\cdots(f^{(n)})^{i_n}/f^k\right)(re^{i\theta})\right| \le r^q$$

holds for all $r \ge r_o$ and $\theta \notin J(r)$, where the angular measure of J(r), $m(J(r)) \le \varepsilon r$. (cf. Lemma 1 in [7])

Proof (see [7], Lemma 1). By the lemma of the logarithmic derivative and by the fact that f is of finite order and every c_λ is polynomial, there exist a constant K and an $r_o>1$ such that for all $r \ge r_o$

$$m(r, \sum_{i_0+\cdots+i_n=k} c_\lambda f^{i_0}(f')^{i_1}\cdots(f^{(n)})^{i_n}/f^k) \le K\log r.$$

For any positive number ε, let $q=2K/\varepsilon$ and

$$J(r) = \{\theta\in[0, 2\pi) : |\sum_{i_0+\cdots+i_n=k} c_\lambda f^{i_0}\cdots(f^{(n)})^{i_n}/f^k(re^{i\theta})| > r^q\}.$$

Then, $m(J(r)) \le \varepsilon\pi$. For $r\ge r_o$ and $\theta\notin J(r)$, the inequality (2) holds.

Lemma 2. Let f be a transcendental entire function, $R(\neq 0)$ a polynomial and set

$$G_o = \{z : |f(z)| > 1\}.$$

Then, for any positive number ε, there exist $q=q(\varepsilon)$, $r_o>1$ and J(r) a subset of $[0, 2\pi)$, $m(J(r)) \le \varepsilon r$ such that the inequality

$$\left|\frac{1}{R}\{(P(z, f, f',\cdots, f^{(n)})-(Q(z, f)-a_0f^d))/f^d\}(re^{i\theta})\right| \le r^q$$

holds for all $z=re^{i\theta}\in G_o$, $r\ge r_o$ and $\theta\notin J(r)$.

Proof. For any $z=re^{i\theta}\in G_o$,

$$(3)\qquad \begin{aligned}&\left|\frac{1}{R}\left\{\frac{P(z, f, f',\cdots, f^{(n)})-Q(z, f)+a_0f^d}{f^d}\right\}(z)\right|\\ &\qquad\le \frac{1}{|R(z)|}\left\{\sum_{k=1}^{d}\left|\sum_{i_0+\cdots+i_n=k} c_\lambda f^{i_0}(f')^{i_1}\cdots(f^{(n)})^{i_n}/f^k(z)\right| + \sum_{j=0}^{d-1}|a_j(z)|\right\}\end{aligned}$$

and we have a constant K and an $r_o>1$ such that for $r\ge r_o$

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|R|}\left\{\sum_{k=1}^{d}\left|\sum_{i_0+\cdots+i_n=k} c_\lambda f^{i_0}(f')^{i_1}\cdots(f^{(n)})^{i_n}/f^k\right| + \sum_{j=0}^{d-1}|a_j|\right\}(re^{i\theta})d\theta\\ &\le m(r, 1/R)+\sum_{k=1}^{d} m(r, \sum_{i_0+\cdots+i_n=k} c_\lambda f^{i_0}(f')^{i_1}\cdots(f^{(n)})^{i_n}/f^k)+\sum_{j=0}^{d-1} m(r, a_j)+O(1) \le K\log r\end{aligned}$$

as in the proof of Lemma 1. For any positive number ε, let $q=2K/\varepsilon$ and

$$J(r)=\{\theta\in[0,2\pi):\frac{1}{|R|}\{\sum_{k=1}^{d}|\sum_{i_0+\cdots+i_n=k}c_\lambda f^{i_0}(f')^{i_1}\cdots(f^{(n)})^{i_n}+\sum_{j=0}^{d-1}|a_j|\}(re^{i\theta})>r^q\}.$$

Then,

$$(2\pi)^{-1}m(J(r))q\log r\leq K\log r$$

and

(5) $$m(J(r))\leq\varepsilon r.$$

From (3),(4) and (5), we obtain our lemma.

Lemma 3. Let $f(z)$ be a transcendental entire function. Then, there exists a polygonal path $\Gamma: z=z(t)$ $(0\leq t<1)$ such that

(i) $\lim_{t\to1}z(t)=\infty$;

(ii) $\lim_{t\to1}\log|f(z(t))|/\log|z(t)|=+\infty$. ([8])

Let D be an unbounded plane domain the boundary of which consists of at most countable analytic curves clustering nowhere in $|z|<\infty$. We put

$$E(r)=\{\theta\in[0,2\pi): re^{i\theta}\in D\}$$

and

$$\theta(r)=\begin{cases}+\infty & \text{if } \{|z|=r\}\subset D\\ m(E(r)) & \text{otherwise.}\end{cases}$$

Let a be a positive number such that $\theta(r)>0$ for all $r\geq a$. Then, we have the following:

Lemma 4. Let $v(z)$ be a subharmonic function in D, continuous on the closure of D such that $v(z)\leq 0$. If there exists one point $z_0\in D$ such that $v(z_0)>0$, then

$$\log M(r,v,D)\geq\pi\int_a^{r/2}\frac{dt}{t\theta(t)}+O(1),$$

where $M(r,v,D)=\sup\{v(z):(|z|=r)\cap D\}$.

See the proof of Theorem III, 68 in [10], p.117. We can apply the method used there for $\log^+|f(z)|$ to our subharmonic function $v(z)$ and easily obtain this lemma.

Lemma 5. Let $A(z)$ be a transcendental entire function with $\mu(A)<+\infty$. Suppose that for some constant K_1 the set

$$\{z:|A(z)|>K_1\}$$

consists of at least N components $G_1,\cdots,G_N$, where $N\geq2$. Then, there exist a harmonic function $v_j(z)$ in G_j and an unbounded domain $D_j\subset G_j$ $(j=1,\cdots,N)$ such that

(i) $v_j(z)\geq|z|^{\{\rho(A)/(2\rho(A)+1-N)\}-\varepsilon_j(z)}$ in D_j ($\varepsilon_j(z)\to0$ as $z\to\infty$);

(ii) $\log|A(z)|-v_j(z)\begin{cases}>0 & \text{in } D_j\\ =0 & \text{on } \partial D_j\end{cases}$

Proof. For $j=1,\cdots,N$, let $u_j(z)$, δ and Γ_j be those given in the proof of Theorem 1 in [9], (We here use G_j in stead of D_j there.) For $j=1,\cdots,N$, put $v_j(z)=\delta u_j(z)+\log K_1$, which is positive in G_j, and let D_j be the unbounded component of $\{z\in G_j: \log|A(z)|-v_j(z)>0\}$ containing Γ_j. Then, it is easy to see from the proof of Theorem 1 in [9] that v_j satisfies (i) and (ii) of this lemma.
(We consider $\rho(A)/(2\rho(A)+1-N)=1/2$ if $\rho(A)=+\infty$.))

Lemma 6. Let f be any entire solution of the differential equation (1). Then f is transcendental and $\rho(f)\geq\rho(a_0)$.

Proof. From (1) we have

(6) $$a_0=\frac{P(z,f,f',\cdots,f^{(n)})-\sum_{j=0}^{d-1}a_jf^j}{f^d}.$$

Suppose that f is polynomial. Then the right-hand side of (1) is not transcendental, but a_0 is transcendental. This is a contradiction. f must be transcendental.

Next, Suppose that $\rho(f) < +\infty$. We then have from (6)

$$m(r, a_0) \leq dm(r, 1/f) + O(\log r) \leq dT(r, f) + O(\log r),$$

which implies $\rho(a_0) \leq \rho(f)$.

3. Result.

We suppose that the differential equation (1) admits at least one entire solution. Let f be any entire solution of (1). Let D_0 be a component of the set

$$\{z : |f(z)| > 1\},$$

which is a non-empty unbounded domain since f is transcendental by Lemma 6. Put

$$E_0(r) = \{\theta \in [0, 2\pi) : re^{i\theta} \in D_0\},$$

$$\theta_0(r) = \begin{cases} +\infty & \text{if } \{|z| = r\} \subset D_0 \\ m(E_0(r)) & \text{otherwise} \end{cases}$$

and

$$\ell_0(r) = m(E_0(r)) = \begin{cases} 2\pi & \text{if } \theta_0(r) = +\infty \\ \theta_0(r) & \text{otherwise.} \end{cases}$$

Applying the method used in the proof of Theorem 3 in [9], we shall give a theorem on the order of f in this section.

Theorem. Suppose in (1) that $a_0(z) = \alpha(z)A(z) + \beta(z)$, where A is transcendental, $\rho(A) < +\infty$ and $\alpha(\neq 0)$, β are polynomials. If for some constant K_1 the set

$$\{z : |A(z)| > K_1\},$$

consists of at least N components $(1 \leq N < +\infty)$, then either $\rho(f) = +\infty$ or

$$\frac{N}{\mu(A)} + \frac{1}{\rho(f)} \leq 2 \qquad (\text{resp. } \frac{N}{\rho(A)} + \frac{1}{\mu(f)} \leq 2).$$

Proof. Suppose that $\rho(f) < +\infty$. By Lemma 2, for any positive ε, the inequality

$$(7) \qquad \left|\frac{1}{\alpha}\left\{\frac{P(z, f, f', \cdots, f^{(n)})}{f^d} - \frac{Q(z, f) - a_0 f^d}{f^d}\right\}(z)\right| \leq r^q$$

holds for $q = q(\varepsilon) > 0$, $z = re^{i\theta} \in G_0$, $r \geq r_0 > 1$ except for $\theta \in J(r)$ satisfying $m(f(r)) \leq \varepsilon r$.

For $K = \max\{K_1, 1, M(1, A)\}$ and a positive integer $p > q$, the set

$$\{z : \log|A(z)| - p\log|z| - \log K > 0, \ |z| > 1\}$$

consists of at least N unbounded components $D_1, \cdots, D_N$ by Lemma 3 in case of $N = 1$ or by Lemma 5 in case of $N \geq 2$. For $j = 1, \cdots, N$ put

$$E_j(r) = \{\theta \in [0, 2\pi) : re^{i\theta} \in D_j\},$$

$$\theta_j(r) = \begin{cases} +\infty & \text{if } \{|z| = r\} \subset D_j \\ m(E_j(r)) & \text{otherwise} \end{cases}$$

and

$$\ell_j(r) = m(E_j(r)) = \begin{cases} 2\pi & \text{if } \theta_j(r) = +\infty \\ \theta_j(r) & \text{otherwise.} \end{cases}$$

Then, there is a positive number a such that $\theta_j(r) > 0$ for all $r \geq a$ and for $j = 0, 1, \cdots, N$. By Lemma 4 we have

(8) $$\pi\int_a^{r/2}\frac{dt}{t\theta_0(t)}\leq \log\log M(r,f)+O(1)$$

and

(9) $$\pi\int_a^{r/2}\frac{dt}{t\theta_j(t)}\leq \log\{\log M(r,A)-p\log r-\log K\}+O(1)$$
$$\leq \log\log M(r,A)+O(1)\quad (j=1,\cdots,N)$$

since $\log|A(z)|-p\log|z|-\log K$ is positive harmonic and $p\log|z|+\log K$ is positive in D_j by the choice of K. Further we have from (7) and by the choice of p

(10) $$\sum_{j=0}^{N}\ell_j(r)\leq(2+\varepsilon)\pi\quad (r\geq b=\max\{r_o, a\}).$$

From (10) we have

(11) $$\sum_{j=0}^{N}\int_b^r\frac{\ell_j(t)}{t}dt\leq(2+\varepsilon)\pi\log(r/b).$$

By the Cauchy-Schwarz inequality

(12) $$\int_b^r\frac{\ell_j(t)}{t}dt\int_b^r\frac{dt}{t\ell_j(t)}\geq\left(\int_b^r\frac{dt}{t}\right)^2=\left(\log\frac{r}{b}\right)^2\quad (j=0,\cdots,N).$$

From (11) and (12) we obtain the inequality

(13) $$\sum_{j=0}^{N}\frac{\log(r/b)}{\pi\int_b^r\frac{dt}{t\ell_j(t)}}\leq 2+\varepsilon.$$

Define

$$B_o=\{r:\theta_0(r)=+\infty\}.$$

Then, B_o is a sum of intervals. Let

$$\chi_o(r)=\begin{cases}1 & \text{if } r \text{ belongs to } B_o\\ 0 & \text{otherwise.}\end{cases}$$

If r belongs to B_o and $r\geq b$, we have

$$\theta_j(r)=\ell_j(r)\quad (j=1,\cdots,N)$$

and

$$\theta_1(r)+\cdots+\theta_N(r)\leq\varepsilon\pi$$

from (10). Thus, if we set

$$F_j(r)=\{r:\theta_j(r)\leq\varepsilon\pi\},$$

then

(14) $$B_o\subset\bigcup_{j=1}^{N}F_j.$$

Define

$$\phi_j(r)=\begin{cases}1 & \text{if } r \text{ belongs to } F_j\\ 0 & \text{otherwise.}\end{cases}$$

We then have from (14)

(15) $$\int_b^r\frac{\chi_o(t)}{t}dt\leq\sum_{j=0}^{N}\int_b^r\frac{\phi_j(t)}{t}dt\leq N\varepsilon\log\log M(2r,A)+O(1)$$

since $\varepsilon^{-1}\phi_j(t)\leq\pi/\theta_j(t)$ and

$$\varepsilon^{-1}\int_b^r\frac{\phi_j(t)}{t}dt\leq\pi\int_b^r\frac{dt}{t\theta_j(t)}\leq\log\log M(2r,A)+O(1)$$

by (9). Further we have

$$\pi\int_b^r \frac{dt}{t\theta_0(t)} = \pi\int_b^r \frac{dt}{t\ell_0(t)} - \frac{1}{2}\int_b^r \frac{\chi_0(t)}{t}dt. \tag{16}$$

(I) The case when $N=1$. Let $B_1 = \{r : \theta_1(r) = +\infty\}$. Then, B_1 is a sum of intervals. Define

$$\chi_1(r) = \begin{cases} 1 & \text{if } r \in B_1 \\ 0 & \text{otherwise.} \end{cases}$$

If $r \in B_1$ and $r \geq b$, then, by (10) $\theta_1(r) \leq \varepsilon\pi$. Put

$$F_0 = \{r : \theta_0(r) \leq \varepsilon\pi\}$$

and

$$\phi_0(r) = \begin{cases} 1 & \text{if } r \in F_0 \\ 0 & \text{otherwise.} \end{cases}$$

We then have

$$\int_b^r \frac{\chi_1(t)}{t}dt \leq \int_b^r \frac{\phi_0(t)}{t}dt \leq \varepsilon \log\log M(2r, f) + O(1) \tag{17}$$

since $B_1 \subset F_0$, $\varepsilon^{-1}\phi_0(t) \leq \pi/\theta_0(t)$ and

$$\varepsilon^{-1}\int_b^r \frac{\phi_0(t)}{t}dt \leq \pi\int_b^r \frac{dt}{t\theta_0(t)} \leq \log\log M(2r, f) + O(1)$$

by (8). Since

$$\pi\int_b^r \frac{dt}{t\theta_1(t)} = \pi\int_b^r \frac{dt}{t\ell_1(t)} - \frac{1}{2}\int_b^r \frac{\chi_1(t)}{t}dt,$$

from (8), (9), (13) and (15) for $N=1$, (16) and (17), we have

$$\frac{\log(r/b)}{\log\log M(2r, A) + \varepsilon\log\log M(2r, f) + O(1)} + \frac{\log(r/b)}{\log\log M(2r, f) + (\varepsilon/2)\log\log M(2r, A) + O(1)} \leq 2 + \varepsilon. \tag{18}$$

Let $\{r_n\}$ be a sequence tending to $+\infty$ such that

$$\lim_{n\to\infty} \frac{\log\log M(2r_n, A)}{\log r_n} = \mu(A) \qquad \left(\text{resp. } \lim_{n\to\infty} \frac{\log\log M(2r_n, f)}{\log r_n} = \mu(f)\right).$$

Put $r = r_n$ in (18) and let n tend to ∞. We then have

$$\frac{1}{\mu(A)} + \frac{1}{\rho(f) + N\varepsilon\mu(A)} \leq 2 + \varepsilon \qquad \left(\text{resp. } \frac{1}{\rho(A)} + \frac{1}{\mu(f) + N\varepsilon\rho(A)} \leq 2 + \varepsilon\right).$$

Tending $\varepsilon \to 0$, we have

$$\frac{1}{\mu(A)} + \frac{1}{\rho(f)} \leq 2 \tag{19}$$

(resp. (19)′
$$\frac{1}{\rho(A)} + \frac{1}{\mu(f)} \leq 2).$$

(II) The case when $N \geq 2$. In this case it is clear that for $j = 1, \cdots, N$

$$0 < \theta_j(r) < 2\pi \quad \text{and} \quad \theta_j(r) = \ell_j(r) \quad (r \geq b).$$

From (8),(9),(13),(15) and (16) we obtain for $r \geq b$

$$\frac{N\log(r/b)}{\log\log M(2r, A) + O(1)} + \frac{\log(r/b)}{\log\log M(2r, f) + (N\varepsilon/2)\log\log M(2r, A) + O(1)} \leq 2 + \varepsilon. \tag{20}$$

Then as in the case of $N=1$, we obtain the inequality

$$\frac{N}{\mu(A)} + \frac{1}{\rho(f)} \leq 2 \qquad \left(\text{resp. } \frac{N}{\rho(A)} + \frac{1}{\mu(f)} \leq 2\right).$$

from (20).

Corollary. Under the same assumption as in Theorem, if

$$\mu(A) \leq 1/2 \quad \text{or} \quad \mu(A) = N/2 \quad \text{or} \quad \rho(A) = N/2 \ (N = 2, 3, \cdots),$$

then $\rho(f) = +\infty$.

Example 1. For a polynomial $p(z)$ of degree 1, put $f(z) = e^{p(z)}$. Then, $p_j = f^{(j)}(z)/f(z) = (p')^j$ $(j = 1, 2, \cdots)$ are constants. For any polynomials $a_1, \cdots, a_{n-1}$ and $F(\neq 0)$, we set

$$q(z) = \sum_{j=1}^{n-1} p_j a_j(z) \quad \text{and} \quad a_0(z) = F(z)e^{-p(z)} - q(z).$$

Then, $f(z) = e^{p(z)}$ is a solution of the differential equation

$$f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f' + a_0 f = F.$$

It is easy to see that for some sufficiently large K the sets

$$\{z : |e^{p(z)}| > K\} \quad \text{and} \quad \{z : |e^{-p(z)}| > K\}$$

have one unbounded component. Further $\rho(a_0) = \mu(a_0) = \rho(e^p) = 1$. This shows that $1/\mu(a_0) + 1/\rho(f) = 2$.

Example 2. The function $A(z) = \frac{1}{2}\{\exp(z^{m/2}) + \exp(z^{-m/2})\}$ $(m = 2, 3, \cdots)$ is of order $\rho(A) = m/2$ and $N = m$ (see [9], example 1). For this $A(z)$, any entire solution of (1) under the assumption of Theorem is of order $+\infty$.

Remark. By a well-known theorem of Ahlfore (see [10], p.236),

$$N = 1 \quad \text{when} \quad \mu(A) < 1 \quad \text{and} \quad N \leq 2\mu(A) \quad \text{when} \quad 1 \leq \mu(A) < +\infty$$

for any non-constant entire function A.

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