Some Notes on the Fundamental Inequality for Holomorphic Curves and on Defects for Moving Targets

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Let $f: \mathbf{C} \to P^n(\mathbf{C})$ be a transcendental holomorphic curve from $\mathbf{C}$ into the n-dimensional complex projective space $P^n(\mathbf{C})$ and let $\mathbf{H}$ be a set of holomorphic curves A such that $T(r,A) = o(T(r,f))$ $(r \to \infty)$, $(A, f) \not\equiv 0$ and in general position.

When f is degenerate and $\mathbf{H} \subset P^n(\mathbf{C})$, we gave several results on the fundamental inequality for f in [11] and on defects with respect to f in [10].

In this paper, we shall extend those results to the case when $\mathbf{H}$ contains moving elements and apply one of them to estimate numbers of several kinds of exceptional targets A in $\mathbf{H}$.

1. Introduction

Let

$$f: \mathbf{C} \to P^n(\mathbf{C})$$

be a holomorphic curve from $\mathbf{C}$ into the n-dimensional complex projective space $P^n(\mathbf{C})$, where n is a positive integer, and let

$$(f_1, \cdots, f_{n+1}): \mathbf{C} \to \mathbf{C}^{n+1} - \{\mathbf{0}\}$$

be a reduced representation of f. We then write $f = [f_1, \cdots, f_{n+1}]$.

The characteristic function $T(r, f)$ of f is defined as follows:

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log\|f(re^{i\theta})\|d\theta - \log\|f(0)\|,$$

where

$$\|f(z)\| = \{\sum_{j=1}^{n+1} |f_j(z)|^2\}^{1/2}.$$

In addition, put

$$U(z) = \max_{1 \le j \le n+1} |f_j(z)|,$$

then

$$U(z) \le \|f(z)\| \le (n+1)^{1/2} U(z)$$

and we have

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log U(re^{i\theta})d\theta + O(1) \quad \text{(see [1])}. \tag{1}$$

We suppose throughout the paper that f is transcendental; that is to say,

$$\lim_{r\to\infty} \frac{T(r, f)}{\log r} = +\infty.$$

We denote by $\rho(f)$ the order of f:

$$\rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

Let $S_o(r, f)$ (resp. $S(r, f)$) be any quantity satisfying

$$S_o(r, f) = o(T(r, f)) \ (r \to \infty)$$

(resp. $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of r of finite linear measure), let $S_o(f)$ be the field of meromorphic functions α in $|z| < \infty$ such that $T(r, \alpha) = S_o(r, f)$ and Γ be a subfield of $S_o(f)$ containing $\mathbf{C}$.

The set

$$V = \{(\alpha_1, \alpha_2, \cdots, \alpha_{n+1}) : \sum_{j=1}^{n+1} \alpha_j f_j = 0,\ \alpha_j \in \Gamma\}$$

is a vector space over Γ. We denote by λ the dimension of V:

$$\lambda = \dim\ V.$$

We can easily prove that λ is independent of the choice of reduced representation of f and that

$$0 \leq \lambda \leq n-1.$$

Let

$$\Gamma(f) = \{A = [a_1, \cdots, a_{n+1}] : a_k/a_j \in \Gamma\ (k=1, \cdots, n+1) \text{ for an } a_j \neq 0\}$$

and for $A = [a_1, \cdots, a_{n+1}] \in \Gamma(f)$ we set

$$(A, f) = a_1 f_1 + \cdots + a_{n+1} f_{n+1}.$$

We put for any $A \in \Gamma(f)$ such that $(A, f) \not\equiv 0$

(2) $$m(r, A, f) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\|A\|\|f\|}{|(A, f)|}\, d\theta,$$

which is independent of the choice of reduced representations of f and A and non-negative since $\|A\|\|f\| \geq |(A, f)|$, and

$$N(r, A, f) = N(r, 1/(A, f)),$$

which is also independent of the choice of reduced representations of f and A.

The defect $\delta(A,f)$ of A with respect to f is defined as follows:

$$\delta(A, f) = \liminf_{r \to \infty} \frac{m(r, A, f)}{T(r, f)}.$$

The purpose of this paper is to extend some results for constant targets in [10] or [11] to moving targets. We shall use the standard notation of the Nevanlinna theory of meromorphic functions ([4],[5]).

2. Preliminary and Lemma

I. Let f, $\Gamma(f)$ and λ etc. be as in the introduction. We shall give some lemmas in this section.

Lemma 1. $$T(r, f) < \sum_{k=1}^{n+1} T(r, a_k/a_j) + O(1) \quad (a_j \neq 0) \qquad ([8]).$$

Lemma 2. For any $A = [a_1, \cdots, a_{n+1}]$ and $B = [b_1, \cdots, b_{n+1}]$ of $\Gamma(f)$ such that $(A, f) \not\equiv 0$, $(B, f) \not\equiv 0$, $a_j \neq 0$, $b_k \neq 0$, we have

$$T(r, \frac{(A, f)/a_j}{(B, f)/b_k}) \leq 2nT(r, f) + S_o(r, f) \qquad ([12]).$$

Proposition 1. For any $A = [a_1, \cdots, a_{n+1}] \in \Gamma(f)$

(a) $T(r, A) = S_o(r, f)$, (b) $N(r, 1/a_j) = S_o(r, f)$ for $a_j \neq 0$.

Proof. (a) Applying Lemma 1 to A, we have for an $a_j \neq 0$

$$T(r, A) \leq \sum_{k=1}^{n+1} T(r, a_k/a_j) + O(1) = S_o(r, f)$$

since $a_k/a_j \in \Gamma$.

(b) Since $a_1, \cdots, a_{n+1}$ have no common zero,

$$N(r, 1/a_j) \leq \sum_{k=1}^{n+1} N(r, a_k/a_j) \leq \sum_{k=1}^{n+1} T(r, a_k/a_j) + O(1) = S_o(r, f)$$

as in (a).

Proposition 2. For any $A \in \Gamma(f)$ for which $(A, f) \not\equiv 0$

(3) $$T(r, f) = m(r, A, f) + N(r, A, f) + S_o(r, f)$$

(the first fundamental theorem).

Proof. From (2) we have

$$m(r, A, f) = T(r, f) + T(r, A) - N(r, A, f),$$

which reduces to (3) by Proposition 1.

Proposition 3. For any $A \in \Gamma(f)$ for which $(A, f) \neq 0$

$$\delta(A, f) = 1 - \limsup_{r \to \infty} \frac{N(r, A, f)}{T(r, f)} \quad \text{and} \quad 0 \leq \delta(A, f) \leq 1$$

We easily obtain these relations from Proposition 2 and the fact that $N(r, A, f) \geq 0$ for $r \geq 1$.

By the definition of λ, there are $n+1-\lambda$ functions in $\{f_1, \cdots, f_{n+1}\}$ (let them be $f_1, \cdots, f_{n+1-\lambda}$ without loss of generality) which are linearly independent over Γ such that the others (namely $f_{n+2-\lambda}, \cdots, f_{n+1}$) can be represented as linear combinations of $f_1, \cdots, f_{n+1-\lambda}$ with Γ-coefficients. It is easy to see from (1) that

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \max_{1 \leq j \leq n+1-\lambda} \log |f_j(re^{i\theta})| d\theta + S_o(r, f). \tag{4}$$

Let $\mathbf{H}$ be a subset of $\Gamma(f)$ in general position such that for any A in $\mathbf{H}$, $(A, f) \neq 0$. For $A = [a_1, \cdots, a_{n+1}]$ in $\mathbf{H}$, let a_{j_0} be the first element not identically equal to zero. Then, put

$$\tilde{A} = (a_1/a_{j_0}, \cdots, a_{n+1}/a_{j_0}) = (g_1, \cdots, g_{n+1}), \ \|\tilde{A}\| = \|A\|/|a_{j_0}|, \ \tilde{\mathbf{H}} = \{\tilde{A} : A \in \mathbf{H}\}$$

and for $(A, f) \equiv F$

$$\tilde{F} = F/a_{j_0} = (\tilde{A}, f) = \sum_{j=1}^{n+1} g_j f_j, \ N(r, \tilde{A}, f) = N(r, 1/(\tilde{A}, f)). \tag{5}$$

Then, $\tilde{\mathbf{H}}$ is in general position, $g_j = a_j/a_{j_0} \in \Gamma$ and by Proposition 1

$$N(r, A, f) = N(r, \tilde{A}, f) + S_o(r, f) \tag{6}$$

since $N(r, A, f) - N(r, 1/a_{j_0}) \leq N(r, \tilde{A}, f) \leq N(r, A, f)$ from (5).

Let $A_j = [a_{j1}, \cdots, a_{jn+1}]$ be any $n+1$ elements in $\mathbf{H}$ and put

$$\tilde{A}_j = (g_{j1}, \cdots, g_{jn+1}), \ (A_j, f) = F_j \quad \text{and} \quad (\tilde{A}_j, f) = \tilde{F}_j.$$

Then it is easy to see the following

Lemma 3. (a) The dimension of the vector space over Γ:

$$\{(\alpha_1, \cdots, \alpha_{n+1}) : \alpha_1 \tilde{F}_1 + \cdots + \alpha_{n+1} \tilde{F}_{n+1} = 0, \ \alpha_j \in \Gamma\}$$

is equal to λ.

(b) There are $n+1-\lambda$ elements in $\{\tilde{F}_1, \cdots, \tilde{F}_{n+1}\}$ (let them be $\tilde{F}_1, \cdots, \tilde{F}_{n+1-\lambda}$ without loss of generality) which are linearly independent over Γ such that for any A of $\mathbf{H}$, $(\tilde{A}, f) \equiv \tilde{F}$ can be represented as a linear combination of $\tilde{F}_1, \cdots, \tilde{F}_{n+1-\lambda}$ with Γ-coefficients. (We then say that $\tilde{F}_1, \cdots, \tilde{F}_{n+1-\lambda}$ form a basis of $\mathbf{H}$ over Γ.)

(c) $\alpha_1 \tilde{F}_1, \cdots, \alpha_{n+1-\lambda} \tilde{F}_{n+1-\lambda}$ ($\alpha_j \neq 0, \in \Gamma$) are linearly independent over $\mathbf{C}$.

(d) $$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \max_{1 \leq j \leq n+1-\lambda} \log |\tilde{F}_j(re^{i\theta})| d\theta + S_o(r, f) \qquad \text{(see (4))}.$$

Lemma 4. For any $\tilde{F}_{i_1}, \cdots, \tilde{F}_{i_m}$ $(1 \leq i_1 < \cdots < i_m \leq n+1-\lambda, \ 2 \leq m \leq n+1-\lambda)$ and $\alpha_1, \cdots, \alpha_m \in \Gamma$

$$m(r, W(\alpha_1 \tilde{F}_{i_1}, \cdots, \alpha_m \tilde{F}_{i_m})/\alpha_1 \tilde{F}_{i_1} \cdots \alpha_m \tilde{F}_{i_m}) = S(r, f),$$

where $\tilde{F}_1, \cdots, \tilde{F}_{n+1-\lambda}$ form a basis of $\mathbf{H}$ over Γ, $\alpha_j \neq 0$ $(j = 1, \cdots, m)$ and $W(f, \cdots, g)$ is the Wronskian of $f, \cdots, g$.

Proof. Applying Lemma 2, we can prove this lemma as in [1], p.14-p.15.

II. Let $\tilde{A}_j$ $(j = 1, \cdots, q; \ q \geq n+1)$ be elements of $\tilde{\mathbf{H}}$ and put

$$(\tilde{A}_j, f) = \tilde{F}_j \ (j = 1, \cdots, q).$$

We may suppose without loss of generality that $\tilde{F}_1, \cdots, \tilde{F}_{n+1-\lambda}$ form a basis of $\mathbf{H}$ over Γ. Let

$$Y = \{\tilde{F}_j : j = n+2-\lambda, \cdots, q\}.$$

As in [10], we introduce an equivalence relation into Y as follows.

Definition 1. (a) For $\tilde{H}_1$ and $\tilde{H}_2$ of Y such that

$$\tilde{H}_j = \alpha_{j1}\tilde{F}_1 + \cdots + \alpha_{jn+1-\lambda}\tilde{F}_{n+1-\lambda} \quad (j=1, 2),$$

$\tilde{H}_1 \simeq \tilde{H}_2$ if and only if there exists a k_o such that $\alpha_{1k_o} \cdot \alpha_{2k_o} \neq 0$.

(b) For $\tilde{F}$ and $\tilde{G}$ of Y, $\tilde{F} \sim \tilde{G}$ if and only if $\tilde{F} \simeq \tilde{G}$ or there exist $\tilde{H}_1, \cdots, \tilde{H}_s$ in Y such that $\tilde{F} \simeq \tilde{H}_1$, $\tilde{H}_1 \simeq \tilde{H}_2, \cdots, \tilde{H}_{s-1} \simeq \tilde{H}_s$, $\tilde{H}_s \simeq \tilde{G}$.

Proposition 4. The relation "$\sim$" is an equivalence relation in Y.

This is trivial from the definition.

We classify Y by this equivalence relation. Let

$$Y/\sim = \{Y_1, \cdots, Y_p\} \quad (1 \leq p \leq n+1-\lambda)$$

and put for $t = 1, \cdots, p$

$X_t = \{\tilde{F}_i$: there is at least one element $\tilde{F}$ in Y_t such that $\alpha_i \neq 0\}$, where $\tilde{F} = \sum_{i=1}^{n+1-\lambda} \alpha_i \tilde{F}_i$ $(\alpha_i \in \Gamma)$;

$X_0 = \{\tilde{F}_1, \cdots, \tilde{F}_{n+1-\lambda}\} - \bigcup_{t=1}^{p} X_t$;

ν_t = the number of elements of X_t $(t=0, \cdots, p)$.

Lemma 5. (a) If $t \neq s$, then $X_t \cap X_s = \phi$.

(b) $\sum_{t=0}^{p} \nu_t = n+1-\lambda$..

(c) If $q \geq n+\lambda+2$, then $p = 1$ and $\nu_0 = 0$.

It is easy to see (a) and (b) by definition. We can prove (c) as in the proof of Lemma 3 in [11].

3. Theorem

Let f, Γ and λ etc. be as in Section 1 or 2. For a positive integer μ and $A \in \Gamma(f)$ such that $(A, f) \not\equiv 0$, we denote by $n_\mu(r, A, f)$ the number of zeros of (A, f) in $|z| \leq r$, where for a zero z_o of (A, f) of order k, we count it k times if $k \leq \mu$ and μ times if $k > \mu$ and put for $r > 0$

$$N_\mu(r, A, f) = \int_0^r \frac{n_\mu(u, A, f) - n_\mu(0, A, f)}{u} du + n_\mu(0, A, f) \log r.$$

As an extension of Theorem 1 in [11], we can prove the following theorem.

Theorem 1. Let $A_1, \cdots, A_{n+\lambda+2}$ be any elements of **H**. Then we have the following inequality:

$$T(r, f) \leq \sum_{j=1}^{n+\lambda+2} N_{n-\lambda}(r, A_j, f) + S(r, f).$$

Proof. Put as in Section 2

$$(A_j, f) = F_j \quad \text{and} \quad (\tilde{A}_j, f) = \tilde{F}_j \quad (j=1, \cdots, n+\lambda+2).$$

We may suppose without loss of generality that $\tilde{F}_1, \cdots, \tilde{F}_{n+1-\lambda}$ form a basis of **H** over Γ. We represent $\tilde{F}_j$ $(j=n+2-\lambda, \cdots, n+\lambda+2)$ by $\tilde{F}_1, \cdots, \tilde{F}_{n+1-\lambda}$ with Γ-coefficients. For simplicity we put

$$\tilde{F}_{n+1-\lambda+k} = \tilde{H}_k \quad (k=1, \cdots, 2\lambda+1)$$

and

(7) $$\tilde{H}_k = \alpha_{k1}\tilde{F}_1 + \cdots + \alpha_{kn+1-\lambda}\tilde{F}_{n+1-\lambda} \quad (k=1, \cdots, 2\lambda+1;\ \alpha_{kj} \in \Gamma).$$

Due to Lemma 5(c), there is at least one element in $\{\tilde{H}_1, \cdots, \tilde{H}_{2\lambda+1}\}$ such that the number of coefficients different from zero in (7) is at least two. We may suppose without loss of generality that $\tilde{H}_1$ is such an element. Let

$$\alpha_{1i_1} \neq 0, \cdots, \alpha_{1i_m} \neq 0,\ \alpha_{1i} = 0\ (i \neq i_1, \cdots, i_m)\ (2 \leq m \leq n+1-\lambda).$$

Then,

(8) $$\tilde{H}_l = \alpha_{li_1}\tilde{F}_{i_1} + \cdots + \alpha_{li_m}\tilde{F}_{i_m}.$$

We differentiate (8) j-times $(0 \leq j \leq m-1)$. From these m relations, we have

(9) $$\alpha_{li_k}\tilde{F}_{i_k} = \tilde{H}_l \Delta_{lk}/\Delta_l \quad (k = 1, \cdots, m),$$

where

$$\Delta_l = W(\alpha_{li_1}\tilde{F}_{i_1}, \cdots, \alpha_{li_m}\tilde{F}_{i_m})/\alpha_{li_1}\tilde{F}_{i_1} \cdots \alpha_{li_m}\tilde{F}_{i_m}$$

and Δ_{lk} is one obtained by exchanging $\alpha_{li_k}\tilde{F}_{i_k}$ for $\tilde{H}_l$ in Δ_l. We note that $\Delta_l \neq 0$ and $\Delta_{ik} \neq 0$ since $\alpha_{li_1}\tilde{F}_{i_1}, \cdots, \alpha_{li_m}\tilde{F}_{i_m}$ are linearly independent over **C** (Lemma 3(c)).

From (9) we have

$$\max_{\alpha_{li} \neq 0} \log|\tilde{F}_i| \leq \log|\tilde{H}_l| + \log^+\left|\frac{1}{\Delta_l}\right| + \sum_{k=1}^{m}\left(\log^+|\Delta_{lk}| + \log^+\left|\frac{1}{\alpha_{li_k}}\right|\right) + O(1)$$

(I) When $m = n+1-\lambda$, integrating this inequality with respect to θ from 0 to 2π $(z = re^{i\theta})$, we obtain the following inequality due to Lemmas 2, 3(d) and 4.

(10) $$\begin{aligned} T(r, f) &\leq \frac{1}{2\pi}\int_0^{2\pi}\log|\tilde{H}_l|d\theta + m(r, 1/\Delta_l) + \sum_{k=1}^{m} m(r, \Delta_{lk}) + S_o(r, f) \\ &\leq N(r, 0, \tilde{H}_l) + N(r, \Delta_l) - N(r, 1/\Delta_l) + S(r, f) \end{aligned}$$

since

$$m(r, \Delta_l) = S(r, f), \qquad m(r, \Delta_{lk}) = S(r, f)$$

and

$$m(r, 1/\alpha_{li_k}) \leq T(r, \alpha_{li_k}) + O(1) = S_o(r, f).$$

Next,

(11) $$\begin{aligned} N(r, \Delta_l) - N(r, 1/\Delta_l) &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{1}{\Delta_l}\right|d\theta + O(1) \\ &= \sum_{k=1}^{n+1-\lambda}\frac{1}{2\pi}\int_0^{2\pi}\{\log|\tilde{F}_k| + \log|\alpha_{lk}|\}d\theta \\ &\quad - \frac{1}{2\pi}\int_0^{2\pi}\log|W(\alpha_{l1}\tilde{F}_1, \cdots, \alpha_{ln+1-\lambda}\tilde{F}_{n+1-\lambda})|d\theta + O(1) \\ &\leq \sum_{k=1}^{n+1-\lambda} N(r, 0, \tilde{F}_k) - N(r, 1/W(\alpha_{l1}\tilde{F}_1, \cdots, \alpha_{ln+1-\lambda}\tilde{F}_{n+1-\lambda})) + S_o(r, f) \end{aligned}$$

since $\alpha_{lk} \in \Gamma$. From (10) and (11), we obtain

(12) $$T(r, f) \leq N(r, 0, \tilde{H}_l) + \sum_{k=1}^{n+1-\lambda} N(r, 0, \tilde{F}_k) - N(r, 1/W(\alpha_{l1}\tilde{F}_1, \cdots, \alpha_{ln+1-\lambda}\tilde{F}_{n+1-\lambda})) + S(r, f).$$

Here, by (6)

(13) $$N(r, 0, \tilde{H}_l) = N(r, 0, \tilde{F}_{n+2-\lambda}) = N(r, \tilde{A}_{n+2-\lambda}, f) = N(r, A_{n+2-\lambda}, f) + S_o(r, f),$$

(14) $$N(r, 0, \tilde{F}_k) = N(r, \tilde{A}_k, f) = N(r, A_k, f) + S_o(r, f) \quad (k = 1, \cdots, n+1-\lambda)$$

and

(15) $$\begin{aligned} &N(r, 1/W(\alpha_{l1}\tilde{F}_1, \cdots, \alpha_{ln+1-\lambda}\tilde{F}_{n+1-\lambda})) \\ &\geq \sum_{k=1}^{n+1-\lambda} N(r, 0, \tilde{F}_k) + N(r, 0, \tilde{H}_l) - \sum_{k=1}^{n+1-\lambda} N_{n-\lambda}(r, 0, \tilde{F}_k) - N_{n-\lambda}(r, 0, \tilde{H}_l) + S_o(r, f). \end{aligned}$$

From (12),(13),(14) and (15), we obtain

$$\begin{aligned} T(r, f) &\leq N_{n-\lambda}(r, A_{n+2-\lambda}, f) + \sum_{k=1}^{n+1-\lambda} N_{n-\lambda}(r, A_k, f) + S(r, f) \\ &\leq \sum_{j=1}^{n-\lambda+2} N_{n-\lambda}(r, A_j, f) + S(r, f) \end{aligned}$$

since for any $A \in \mathbf{H}$

$$N_{n-\lambda}(r, A, f) + O(\log r) \geq 0.$$

(II) When $m<n+1-\lambda$, modifying II, III and IV of the proof of Theorem 1 in [11], we obtain our theorem.

Definition 2. For a positive integer μ and any A in $\Gamma(f)$ such that $(A, f)\neq 0$

$$\delta_\mu(A, f) = 1 - \limsup_{r\to\infty} \frac{N_\mu(r, A, f)}{T(r, f)}.$$

It is easy to see that

$$0 \leq \delta(A, f) \leq \delta_\mu(A, f) \leq 1.$$

Corollary 1. For any $A_1, \cdots, A_{n+\lambda+2}$ in **H**

$$\sum_{j=1}^{n+\lambda+2} \delta_{n-\lambda}(A_j, f) \leq n+\lambda+1.$$

(cf. Theorem 1 in [6] or Theorem 3.2 in [7]).

Theorem 2. Suppose that there exist $n+\tau+1$ elements $A_1, \cdots, A_{n+\tau+1}$ $(1\leq\tau\leq n-1)$ in **H** such that

$$\sum_{i=1}^{n+1} \delta_{n-\lambda}(A_i, f) + \delta_{n-\lambda}(A_{n+1+j}, f) > n+1 \quad (j=1, \cdots, \tau).$$

Then, we have $\lambda \geq \tau$.

Proof. Modifying the proof of Lemma 8([10]) to our case as in the proof of Theorem 1, we can prove this theorem.

Corollary 2. Suppose that **H** contains n+1 elements $A_1, \cdots, A_{n+1}$ satisfying $\delta_{n-\lambda}(A_j, f) = 1$ $(j=1, \cdots, n+1)$. Then, $\mathbf{H}-\{A_1, \cdots, A_{n+1}\}$ contains at most λ elements A satisfying $\delta_{n-\lambda}(A, f) > 0$.

Corollary 3. Suppose that **H** contains $n+\lambda+2$ elements $A_1, \cdots, A_{n+\lambda+2}$ such that

$$\sum_{j=1}^{n+\lambda+2} \delta_{n-\lambda}(A_j, f) = n+\lambda+1 \tag{16}$$

(resp. (16)′ $$\sum_{j=1}^{n+\lambda+2} \delta(A_j, f) = n+\lambda+1).$$

Then, there exists a j_o such that

$$\delta_{n-\lambda}(A_{j_o}, f) = 0 \quad \text{and} \quad \delta_{n-\lambda}(A_j, f) = 1 \quad (j \neq j_o)$$
$$(\text{resp. } \delta(A_{j_o}, f) = 0 \quad \text{and} \quad \delta(A_j, f) = 1 \quad (j \neq j_o)).$$

Proof. We may suppose without loss of generality that

$$\delta_{n-\lambda}(A_1, f) \geq \delta_{n-\lambda}(A_2, f) \geq \cdots \geq \delta_{n-\lambda}(A_{n+\lambda+2}, f)$$
$$(\text{resp. } \delta(A_1, f) \geq \delta(A_2, f) \geq \cdots \geq \delta(A_{n+\lambda+2}, f)).$$

If $\delta_{n-\lambda}(A_{n+\lambda+2}, f) > 0$ (resp. $\delta(A_{n+\lambda+2}, f) > 0$),
then

$$0 < \delta_{n-\lambda}(A_{n+\lambda+2}, f) < 1 \quad (\text{resp. } 0 < \delta(A_{n+\lambda+2}, f) < 1).$$

From (16) (resp. (16)′) we obtain the inequalities

$$\sum_{i=1}^{n+1} \delta_{n-\lambda}(A_i, f) + \delta_{n-\lambda}(A_{n+1+j}, f) > n+1 \quad (j=1, \cdots, \lambda+1)$$

$$(\text{resp. } \sum_{i=1}^{n+1} \delta(A_i, f) + \delta(A_{n+1+j}, f) > n+1 \quad (j=1, \cdots, \lambda+1)).$$

Then, we have

$$\lambda \geq \lambda + 1$$

due to Theorem 2, which is absurd. This means that

$$\delta_{n-\lambda}(A_{n+\lambda+2}, f) = 0 \quad (\text{resp. } \delta(A_{n+\lambda+2}, f) = 0)$$

and

$$\delta_{n-\lambda}(A_j, f) = 1 \ (\text{resp. } \delta(A_j, f) = 1) \ (j=1, \cdots, n+\lambda+1).$$

Remark. If $(16)'$ holds, $\rho(f)$ is positive integer or $+\infty$ and f is of regular growth ([12], Theorem 6 and [13], Theorem 6).

Let $A_1, \cdots, A_{n+\nu+1}$ $(1 \leq \nu \leq \lambda - 1)$ and $B_1, \cdots, B_\tau$ be in **H** and put

$$(\tilde{A}_i, f) = \tilde{F}_i \ \ (i = 1, \cdots, n+\nu+1) \quad \text{and} \quad (\tilde{B}_j, f) = \tilde{G}_j \ \ (j = 1, \cdots, \tau).$$

We apply the discussion in II of Section 2 to

$$\{\tilde{F}_1, \cdots, \tilde{F}_{n+\nu+1}, \tilde{G}_1, \cdots, \tilde{G}_\tau\}.$$

We may suppose without loss of generality that $\tilde{F}_1, \cdots, \tilde{F}_{n+1-\lambda}$ form a basis of **H** over Γ. Let

$$Y^0 = \{\tilde{F}_{n+2-\lambda}, \cdots, \tilde{F}_{n+\nu+1}\}, \ Y^j = Y^0 \cup \{\tilde{G}_j\} \quad (j = 1, \cdots, \tau)$$

and p_o be the number of equivalence classes of $Y^0/\sim$. Then, we have the following theorem.

Theorem 3. (**I**) Suppose

$$\sum_{i=1}^{n+\nu+1} \delta_{n-\lambda}(A_i, f) > n+\nu. \tag{17}$$

Then, $2 \leq p_o \leq n+1-\lambda$ and $\nu(p_o - 1) \leq \lambda$.

(**II**) Suppose that

$$\sum_{i=1}^{n+\nu+1} \delta_{n-\lambda}(A_i, f) + \delta_{n-\lambda}(B_j, f) > n+\nu+1 \quad (j = 1, \cdots, \tau). \tag{18}$$

Then, $2 \leq p_o \leq n+1-\lambda$ and $\nu(p_o - 1) + \tau \leq \lambda$.

Proof. We first note that we get (17) from (18). Let

$$Y^0/\sim = \{Y_1^0, \cdots, Y_{p_o}^0\}, \ Y^j/\sim = \{Y_1^j, \cdots, Y_{p_j}^j\} \quad \{j = 1, \cdots, \tau\},$$

$X_t^j = \{\tilde{F}_i :$ there is at least one element $\tilde{F}$ in Y_t^j such that $\alpha_i \neq 0\}$, where $\tilde{F} = \sum_{i=1}^{n+1-\lambda} \alpha_i \tilde{F}_i$ $(\alpha_i \in \Gamma)$;

$\nu_t^j =$ the number of elements in X_t^j $(j = 0, 1, \cdots, \tau;\ t = 1, \cdots, p_j)$. Then,

(a) $X_t^j \cap X_s^j = \phi$ if $t \neq s$.

(b) $\sum_{t=1}^{p_j} \nu_t^j = n+1-\lambda$.

This is because each Y^j $(j = 0, \cdots, \tau)$ contains at least $\lambda + 1$ elements and

$$\{\tilde{F}_1, \cdots, \tilde{F}_{n+1-\lambda}\} - \bigcup_{t=1}^{p_j} X_t^j = \phi.$$

(c) $\nu_t^j \leq n - \lambda$ $\quad (j = 0, \cdots, \tau;\ t = 1, \cdots, p_j)$.

We can prove these inequalities as in the proof of Lemma 6 in [10] by applying the method used in the proof of Theorem 1.

Next, we suppose without loss of generality that $\tilde{G}_j$ belongs to Y_1^j $(j = 1, \cdots, \tau)$. Then, we have

(d) For each j $(j = 1, \cdots, \tau)$, there exist a t_1 and a t_2 such that

$$X_{t_1}^0 \subset X_1^j \quad \text{and} \quad X_{t_2}^0 \cap X_1^j = \phi.$$

We can prove this fact as in the proof of Lemma 7, i) in [10].

(e) When we represent $\tilde{F}_{n+1-\lambda+k}$ $(k = 1, \cdots, \nu+\lambda)$ as linear combinations of $\tilde{F}_1, \cdots, \tilde{F}_{n+1-\lambda}$ with Γ-coefficients, there are $p_o - 1$ classes in $\{X_1^0, \cdots, X_{p_o}^0\}$ such that all coefficients of elements in those classes are equal to zero.

(f) When we represent $\tilde{G}_j$ as a linear combination of $\tilde{F}_1, \cdots, \tilde{F}_{n+1-\lambda}$ with Γ-coefficients, because of (d), there is at least one class $X_{t(j)}^0$ such that all coefficients of elements in that class are equal to zero.

Proof of (I). From the definition of λ and due to (e), we have

$$(\nu+\lambda)(p_o - 1) \leq p_o \lambda,$$

which reduces to $\nu(p_o - 1) \leq \lambda$.

Because of (b) and (c) for $j=0$, it is trivial that $2 \leq p_o \leq n+1-\lambda$.

Proof of (II). From the definition of λ, due to (e) and (f), we have

$$(\nu+\lambda)(p_o-1)+\tau\leq p_o\lambda,$$

which reduces to $\nu(p_o-1)+\tau\leq\lambda$.

As the number p_o is the same one as in (I), we have

$$2\leq p_o\leq n+1-\lambda.$$

From this theorem, we can deduce many well-known results on the number of exceptional elements in **H**. We use λ_c, λ_p, or λ_f instead of λ when $\Gamma=\mathbf{C}$, $\Gamma=$the field of rational functions or $\Gamma=S_o(f)$ respectively.

Corollary 4. 1°. When $\Gamma=\mathbf{C}$, let N_1 be the number of elements A of **H** satisfying the condition

1] (A, f) has no zero.

Then, $N_1\leq n+1+\lambda_c/(n-\lambda_c)$ ([2]).

2°. When $\Gamma=\mathbf{C}$, let N_2 be the number of elements A of **H** satisfying the condition

2] (A, f) has at most a finite number of zeros.

Then, $N_2\leq n+1+\lambda_c/(n-\lambda_p)$ ([3]).

3°. When $\Gamma=$the field of rational functions, let N_3 be the number of elements A in **H** satisfying the condition

3] (A, f) has at most a finite number of zeros.

Then, $N_3\leq n+1+\lambda_p/(n-\lambda_p)$ ([9]).

4°. When Γ is any subfield of $S_o(f)$ containing **C**, let N_4 be the number of elements A in **H** satisfying the condition

4] $\delta(A, f)=1$.

If $\rho(f)<+\infty$, then, $N_4\leq n+1+\lambda/(n-\lambda_f)$ ([9]).

Proof. For each $i(=1,2,3,4)$, we have only to prove our inequality when $N_i\geq n+2$. Let $A_1,\cdots,A_{n+\nu+1}$ $(\nu\geq 1)$ be in **H** satisfying the condition i] (i = 1, 2, 3 or 4). Then, by applying Theorem in [5], p.116 to each case, we can prove the followings.

Case 1°. $p_o=n+1-\lambda_c$.

Case 2°. $p_o\geq n+1-\lambda_p$.

Case 3°. $p_o=n+1-\lambda_p$.

Case 4°. $p_o\geq n+1-\lambda_f$.

Due to Theorem 3, (I), we have our inequalities.

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