On Holomorphic Curves of Infinite Order with Maximal Deficiency Sum

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Let $f: \mathbf{C} \to P^n(\mathbf{C})$ be a transcendental holomorphic curve from $\mathbf{C}$ into the n-dimensional complex projective space $P^n(\mathbf{C})$ and X a subset of $\mathbf{C}^{n+1}$ in general position. Then, it is known that if f is non-degenerate, $(*)\ \Sigma_{a\in X}\delta(\mathbf{a},f)\leq n+1$. As in the case of meromorphic functions, the following problem is interesting:

Problem. What properties does f possess if the equality holds in $(*)$?

Concerning this problem, we gave several results for f of finite order in [9]. For example, "If the equality holds in $(*)$ and if $\delta(\mathbf{e}_j,f)=1$ $(j=1,\cdots,n)$, then f is of regular growth and the order of f is a positive integer."

In this paper, we shall give similar results for f of infinite order to those for holomorphic curves of finite order obtained in [9].

1. Introduction

Let

$$f:\mathbf{C} \to P^n(\mathbf{C})$$

be a holomorphic curve from $\mathbf{C}$ into the n-dimensional complex projective space $P^n(\mathbf{C})$, where n is a positive integer, and let

$$(f_1,\cdots,f_{n+1})\ :\ \mathbf{C} \to \mathbf{C}^{n+1}-\{0\}$$

be a reduced representation of f. We then write $f=[f_1,\cdots f_{n+1}]$.

For a vector $\mathbf{a}=(a_1,\cdots,a_{n+1})$ in $\mathbf{C}^{n+1}$, we write

$$(\mathbf{a},f)=\sum_{j=1}^{n+1}a_jf_j \quad \text{and} \quad \|\mathbf{a}\|=\{\sum_{j=1}^{n+1}|a_j|^2\}^{1/2}$$

and put

$$\|f(z)\|=\{\sum_{j=1}^{n+1}|f_j(z)|^2\}^{1/2}.$$

Then we define as usual the characteristic function of f as follows.

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta-\log\|f(0)\|.$$

In addition, put

$$U(z)=\max_{1\leq j\leq n+1}|f_j(z)|,$$

then

$$U(z)\leq\|f(z)\|\leq(n+1)^{1/2}U(z)$$

and we have

(1) $$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log U(re^{i\theta})d\theta+O(1) \qquad \text{(see [1])}.$$

We suppose that f is transcendental; that is to say,

$$\lim_{r\to\infty}\frac{T(r,f)}{\log r}=+\infty.$$

We denote the order of f by $\rho(f)$ and the lower order of f by $\mu(f)$ respectively:

$$\rho(f) = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \mu(f) = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log r}.$$

It is said that f is of regular growth if $\rho(f) = \mu(f)$.

We write for $\mathbf{a} = (a_1, \cdots, a_{n+1})$ in $\mathbf{C}^{n+1} - \{0\}$ such that $(\mathbf{a}, f) \neq 0$

$$m(r,\mathbf{a},f) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\|\mathbf{a}\|\|f\|}{|(\mathbf{a},f)|} d\theta \quad \text{and} \quad N(r,\mathbf{a},f) = N(r, \frac{1}{(\mathbf{a},f)}).$$

Then we have

(2) $$T(r,f) = N(r,\mathbf{a},f) + m(r,\mathbf{a},f) + O(1)$$

(the first fundamental theorem (see [10], p.76)).

We call the quantity

$$\delta(\mathbf{a},f) = 1 - \limsup_{r \to \infty} \frac{N(r,\mathbf{a},f)}{T(r,f)} = \liminf_{r \to \infty} \frac{m(r,\mathbf{a},f)}{T(r,f)}$$

the deficiency of $\mathbf{a}$ with respect to f. It is easy to see that

$$0 \leq \delta(\mathbf{a},f) \leq 1$$

by (2) since $m(r,\mathbf{a},f) \geq 0$. Put

$$\lambda = \dim\{(c_1, \cdots, c_{n+1}) \in \mathbf{C}^{n+1} : c_1 f_1 + \cdots + c_{n+1} f_{n+1} = 0\},$$

then it is easy to see that $0 \leq \lambda \leq n-1$. We say that f is (linearly) non-degenerate if $\lambda = 0$ and that f is (linearly) degenerate if $\lambda > 0$.

It is well-known that f is non-degenerate if and only if the Wronskian $W(f_1, \cdots, f_{n+1})$ of $f_1, \cdots, f_{n+1}$ is not identically equal to 0.

Let X be a subset of $\mathbf{C}^{n+1} - \{0\}$ in general position; that is to say, any n+1 vectors of X are linearly independent. The following inequality (the defect relation) is well-known (see [1]).

(3) $$\sum_{\mathbf{a} \in X} \delta(\mathbf{a},f) \leq n+1.$$

In [9], we gave several results for holomorphic curves of order finite for which the equality holds in (3). For example,

Theorem A. Suppose that f is non-degenerate and $\rho(f) < \infty$. If there are $\mathbf{a}_1, \cdots, \mathbf{a}_q$ in X $(n+1 \leq q \leq \infty)$ such that

(i) $\delta(\mathbf{a}_j, f) = 1$ $(j = 1, \cdots, n)$;

(ii) $\sum_{j=1}^{q} \delta(\mathbf{a}_j, f) = n+1$,

then f is of regular growth and $\rho(f)$ is equal to a positive integer.

The purpose of this paper is to give similar results to them when $\rho(f) = \infty$. From now on throughout the paper we suppose $\rho(f) = \infty$.

We prepare several lemmas in Section 2 and give a result for non-degenerate holomorphic curves in Section 3, which corresponds to Theorem A. In Section 4, we extend a result obtained in Section 3 to moving targets. In Section 5, we treat the degenerate case. We use the standard notation of the Nevanlinna theory of meromorphic functions ([2],[3]).

2. Lemma

We shall give some lemmas in this section for later use. Let f and X be as in Section 1. Note that $\rho(f) = \infty$.

We use the following notation to treat f of order infinite in the same way as in the case of holomorphic curves of finite order.

Let α be any positive number. Then, we put

$$T_\alpha(r,f)=\int_1^r \frac{T(t,f)}{t^{1+\alpha}}dt$$

and we difine $N_\alpha(r,a,f)$ and $m_\alpha(r,a,f)$ similarly. Further, put

$$\delta_\alpha(a,f)=\liminf_{r\to\infty}\frac{m_\alpha(r,a,f)}{T_\alpha(r,f)}$$

(see [7]). Let $S(r,f)$ be any quantity satisfying

$$S_\alpha(r,f)=\int_1^r \frac{S(t,f)}{t^{1+\alpha}}dt=o(T_\alpha(r,f)) \quad (r\to\infty)$$

for any positive number α.

Lemma 1. (a) For any $0<\alpha<\infty$

$$\limsup_{r\to\infty}\frac{\log T_\alpha(r,f)}{\log r}=\infty. \tag{4}$$

Conversely, if (4) holds for some finite $\alpha>0$, then $\rho(f)=\infty$.

(b) $$\mu(f)\geq \liminf_{r\to\infty}\frac{\log T_\alpha(r,f)}{\log r}\begin{cases}\geq \max(\mu(f)-\alpha,0) & \text{if } \mu(f)<\infty;\\ =\infty & \text{otherwise.}\end{cases}$$

In fact, (a) is given in Proposition 1([7]) and we can prove (b) as in the case of (a).

Lemma 2. $0\leq\delta(a,f)\leq\delta_\alpha(a,f)\leq 1$ (see [7], Proposition 3, 3)).

Lemma 3. Let $h(z)$ be a meromorphic function in $|z|<\infty$, then for any positive integer k

$$m_\alpha(r,\frac{h^{(k)}}{h})=O(\int_1^r\frac{\log^+T(t,h)}{t^{1+\alpha}}dt).$$

In fact, for $k=1$, this is Lemme II in [3], pp.62-63. For $k\geq 2$, we note first that

$$\frac{h^{(k)}}{h}=\prod_{j=1}^{k}\frac{h^{(j)}}{h^{(j-1)}}. \tag{5}$$

By using

$$T(r,h^{(j-1)})\leq 2T(r,h^{(j-2)})+m(r\frac{h^{(j-1)}}{h^{(j-2)}})+O(1)$$

we have

$$\log^+T(r,h^{(j-1)})\leq \log^+T(r,h^{(j-2)})+m(r\frac{h^{(j-1)}}{h^{(j-2)}})+O(1). \tag{6}$$

and by applying the case $k=1$ of this lemma to $h^{(j-1)}$, we have

$$m_\alpha(r,\frac{h^{(j)}}{h^{(j-1)}})=O(\int_1^r\frac{\log^+T(t,h^{(j-1)})}{t^{1+\alpha}}dt). \quad (j=1,\cdots,k). \tag{7}$$

From (6) and (7) we obtain by induction

$$m_\alpha(r,\frac{h^{(j)}}{h^{(j-1)}})=O(\int_1^r\frac{\log^+T(t,h)}{t^{1+\alpha}}dt). \quad (j=1,\cdots,k).$$

so that we have from (5)

$$m_\alpha(r,\frac{h^{(k)}}{h})=\leq\sum_{j=1}^{k}m_\alpha(r,\frac{h^{(j)}}{h^{(j-1)}})+O(1)=O(\int_1^r\frac{\log^+T(t,h)}{t^{1+\alpha}}dt)$$

Lemma 4. If there exist $n+1$ elements $a_1,\cdots,a_{n+1}$ in X such that $\delta_\alpha(a_j,f)=1$ $(j=1,\cdots,n+1)$ for any $0<\alpha<\infty$, then f is of regular growth ([7], Corollaire 5).

Lemma 5. (a) $T(r,f_k/f_j)<T(r,f)+O(1)$ $(k\neq j)$ ([1]).

(b) For any a, b in X such that $(a,f)\neq 0$ and $(b,f)\neq 0$,

$$T(r,(a,f)/(b,f))<T(r,f)+O(1) \quad ([1]).$$

Put for any $a_j\in X$ $(j=1,\cdots,n+1)$

$$K_\alpha(f)=\limsup_{r\to\infty}\frac{\sum_{j=1}^{n+1}N_\alpha(r,a_j,f)}{T_\alpha(r,f)}$$

(see [7], Definition 5). Then we have the following

Lemma 6. If $\mu(f) < \rho(f) = \infty$, then for any τ and α such that

(i) τ is not an integer and $\mu(f) < \tau$;

(ii) $0 < \alpha < \mu$ and $\tau + \alpha < \infty$;

(iii) $[\tau] = [\tau + \alpha]$,

$$K_\alpha(f) \geq \frac{n+1}{n} \cdot \frac{|\sin \pi(\tau + \alpha)|}{K(\alpha, \tau) + |\sin \pi(\tau + \alpha)|},$$

where $K(\alpha, \tau)$ is a positive number dependent only on α and τ ([7], Théorème 5).

Suppose now that f is non-degenerate. Let $d(z)$ be an entire function such that the functions

$$f_j^{n+1}/d \quad (j=1,\cdots,n) \quad \text{and} \quad W(f_1,\cdots,f_{n+1})/d$$

are entire functions without common zeros.

Definition ([8]). We call the holomorphic curve induced by the mapping

$$(f_1^{n+1},\cdots,f_n^{n+1}, W(f_1,\cdots,f_n^{n+1})): \mathbf{C} \to \mathbf{C}^{n+1}$$

the derived holomorphic curve of f and we write it by f^*:

$$f^* = [f_1^{n+1}/d,\cdots,f_n^{n+1}/d, W(f_1,\cdots,f_{n+1})/d].$$

Lemma 7 ([8]). $T(r,f^*) \leq (n+1)T(r,f) - N(r,1/d) + S(r,f)$.

Proof (see [8], Lemma 3). Put $h_j = f_j/f_1$. Then,

$$S(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{W(h_2',\cdots,h_{n+1}')}{h_2 \cdots h_{n+1}}\right| d\theta + O(1)$$

$$\leq \sum_{k=1}^{n-1}\sum_{j=2}^{n+1} m(r, h_j^{(k)}/h_j) + O(1)$$

and by Lemmas 3 and 5 (a) we have

$$S_\alpha(r,f) = o(T_\alpha(r,f)) \quad (r \to \infty).$$

In addition, f^* has the following properties:

Proposition 1 ([8]). (a) f^* is transcendental. (b) $\rho(f^*) = \rho(f)$. (c) f^* is not always non-degenerate.

3. Non-Degenerate Case

Let $f = [f_1,\cdots,f_{n+1}]$ and X be as in Section 1. We shall give similar results to those obtained in [9], Section 3 when f is non-degenerate and $\rho(f) = \infty$ in this section. We need another lemma.

Lemma 8. Suppose that f is non-degenerate and $\rho(f) = \infty$. For any $\mathbf{a}_1,\cdots\mathbf{a}_q$ $(n+1 \leq q < \infty)$ of X, we have

$$(q-n-1)T(r,f) < \sum_{j=1}^{q} N(r,\mathbf{a}_j,f) - N(r,1/W(f_1,\cdots,f_{n+1})) + S(r,f) \qquad \text{(see [1])}.$$

Proof. We have only to change slightly the proof of the fundamental inequality of Cartan ([1], p.12-p.15). We make use of the formula

$$\frac{1}{2\pi}\int_0^{2\pi} \log \left|\frac{F_1 \cdots F_q}{W(f_1,\cdots,f_{n+1})}\right| d\theta = \sum_{j=1}^{q} N(r, \frac{1}{F_j}) - N(r, \frac{1}{W(f_1,\cdots,f_{n+1})}) + O(1),$$

instead of the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} \log \left|\frac{F_1 \cdots F_q}{W(f_1,\cdots,f_{n+1})}\right| d\theta \leq \sum_{j=1}^{q} N_{n-1}(r, F_j) + O(1),$$

used in [1], where $F_j = (\mathbf{a}_j, f)$.

Since the error term $S(r)$ used in [1] is equal to a finite sum of integrals of the from

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{W(h_1',\cdots,h_n')}{h_1 \cdots h_n}\right| d\theta + O(1) \leq \sum_{j=1}^{n}\sum_{k=1}^{n-1} m(r, \frac{h_j^{(k)}}{h_j}) + O(1)$$

where h_j is a ratio of the form F_{j_1}/F_{j_2} $(j_1 \neq j_2)$, it is easy to see that

$$S(r) = S(r,f)$$

by Lemma 3 as in the proof of Lemma 7 since

$$T(r,h_j) < T(r,f) + O(1)$$

by Lemma 5(a).

Corollary 1. Under the same condition as in Lemma 8, if the equality holds in (3), then for any $0<\alpha<\infty$

$$\lim_{r\to\infty} = \frac{N_\alpha(r,0,W(f_1,\cdots,f_{n+1}))}{T_\alpha(r,f)} = 0. \tag{8}$$

Proof. By integrating both sides divided by $r^{1+\alpha}$ of the inequality of Lemma 8 from 1 to r with respect to r, we obtain the inequality

$$(q-n-1)T_\alpha(r,f) \le \sum_{j=1}^{q} N_\alpha(r,a_j,f) - N_\alpha(r,0,W) + S_\alpha(r,f),$$

where $W=W(f_1,\cdots,f_{n+1})$, from which we obtain

$$\sum_{a\in X} \delta_\alpha(a,f) + \limsup_{r\to\infty} \frac{N_\alpha(r,0,W)}{T_\alpha(r,f)} \le n+1. \tag{9}$$

Note that

$$\delta_\alpha(a,f) = 1 - \limsup_{r\to\infty} N_\alpha(r,a,f)/T_\alpha(r,f)$$

since $T_\alpha(r,f)=N_\alpha(r,a,f)+m_\alpha(r,a,f)+O(1)$.
If the equality holds in (3), then by Lemma 2 and (9) we have

$$\sum_{a\in X} \delta_\alpha(a,f) = n+1$$

and so (8) from (9).

Let $\{e_1,\cdots,e_{n+1}\}$ be the standard basis of C^{n+1} and put

$$X_0 = \{a=(a_1,\cdots,a_{n+1}) \in X : a_{n+1}=0\}.$$

Then, $\# X_0 \le n$ since X is in general position.

Theorem 1. Suppose that f is non-degenerate and $\rho(f)=\infty$. For any $a_1,\cdots,a_q$ $(1\le q<\infty)$ in $X-X_0$, we have the following inequality:

$$\sum_{j=1}^{q} m(r,a_j,f) \le m(r,e_{n+1},f^*) + S(r,f).$$

Proof. We have only to prove this theorem when $q\ge n+1$. We put

$$(a_j,f)=F_j \ (j=1,\cdots,q) \quad \text{and} \quad u(z)=\max_{1\le j\le n} |f_j(z)|$$

and for any $z(\neq 0)$ arbitrarily fixed, let

$$|F_{j_1}(z)| \le |F_{j_2}(z)| \le \cdots \le |F_{j_q}(z)| \ (1\le j_1,\cdots,j_q\le q).$$

Then, as in the proof of Theorem 1 in [9], we obtain the inequality

$$\sum_{j=1}^{q} m(r,a_j,f) \le \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{u(z)^{n+1}}{W(f_1,\cdots,f_{n+1})}\right| d\theta$$

$$+ \sum_{(j_1,\cdots,j_{n+1})} m(r, \frac{W(F_{j_1},\cdots,F_{j_{n+1}})}{F_{j_1}\cdots F_{j_{n+1}}}) + O(1) \le m(r,e_{n+1},f^*) + S(r),$$

where $\sum_{(j_1,\cdots,j_{n+1})}$ is the summation taken over all combinations $(j_1,\cdots,j_{n+1})$ chosen from $\{1,\cdots,q\}$ and

$$S(r) = \sum_{(j_1,\cdots,j_{n+1})} m(r, \frac{W(F_{j_1},\cdots,F_{j_{n+1}})}{F_{j_1}\cdots F_{j_{n+1}}}) + O(1).$$

$$= S(r,f)$$

as in the case of Lemma 8. Thus, our proof is complete.

Corollary 2. Let f be as in Theorem 1. Then we have

$$\frac{1}{n+1} \sum_{a\in X-X_0} \delta_\alpha(a,f) \le \delta_\alpha(e_{n+1},f^*),$$

$$\sum_{a\in X-X_0} \delta_\alpha(a,f) \le \liminf_{r\to\infty} \frac{T_\alpha(r,f^*)}{T_\alpha(r,f)} \le \limsup_{r\to\infty} \frac{T_\alpha(r,f^*)}{T_\alpha(r,f)} \le n+1. \tag{10}$$

We can easily prove this corollary by Lemma 7 and Theorem 1.

Now, we can prove one of main results of this paper.

Theorem 2. Suppose that f is non-degenerate, $\rho(f)=\infty$ and

(i) $\delta(\mathbf{e}_j, f)=1$ $(j=1,\cdots,n)$.

If there exist $\mathbf{a}_1,\cdots,\mathbf{a}_q$ $(n+1\leq q\leq\infty)$ in X such that

(ii) $\sum_{j=1}^{q}\delta(\mathbf{a}_j,f)=n+1$,

then f is of regular growth.

Proof. Suppose that X_0 consists of $\mathbf{a}_1,\cdots,\mathbf{a}_\ell$. Then, $0\leq\ell\leq n$. Let α be any positive number. Then, by Corollary 1, we have from (ii)

$$\lim_{r\to\infty}\frac{N_\alpha(r,1/W(f_1,\cdots,f_{n+1}))}{T_\alpha(r,f)}=0. \tag{11}$$

By (10) and (ii), we have

$$1\leq n+1-\ell\leq\sum_{j=\ell+1}^{q}\delta_\alpha(\mathbf{a}_j,f)\leq\liminf_{r\to\infty}\frac{T_\alpha(r,f^*)}{T_\alpha(r,f)}\leq\limsup_{r\to\infty}\frac{T_\alpha(r,f^*)}{T_\alpha(r,f)}\leq n+1. \tag{12}$$

By Proposition 1 (b), $\rho(f^*)=\rho(f)=\infty$ and (12) implies that

$$\liminf_{r\to\infty}\frac{\log T_\alpha(r,f)}{\log r}=\liminf_{r\to\infty}\frac{\log T_\alpha(r,f^*)}{\log r} \tag{13}$$

From (11) and (12), we have

$$\delta_\alpha(\mathbf{e}_{n+1},f^*)=1 \tag{14}$$

and from (i) and (12)

$$\delta_\alpha(\mathbf{e}_j,f^*)=1\quad(j=1,\cdots,n). \tag{15}$$

By Lemma 4, (14) and (15) imply that f^* is of regular growth since the set $\{\mathbf{e}_j\}_{j=1}^{n+1}$ is in general position. So, (13) and Lemma 1 (b) imply that $\mu(f)=\infty$. That is, f is of regular growth.

Corollary 3. Suppose that f is non-degenerate and $\rho(f)=\infty$. If there are $\mathbf{a}_1,\cdots,\mathbf{a}_q$ in X $(n+1\leq q\leq\infty)$ such that

(i) $\delta(\mathbf{a}_j,f)=1$ $(j=1,\cdots,n)$;

(ii) $\sum_{j=1}^{q}\delta(\mathbf{a}_j,f)=n+1$,

then f is of regular growth.

We can prove this corollary as in the case of Corollary 3 in [9].

4. Extension

Let $f=[f_1,\cdots,f_{n+1}]$ be a transcendental holomorphic curve from $\mathbf{C}$ into $P^n(\mathbf{C})$. We use the same notation as in Sections 1 and 2. Let Γ be the field consisting of meromorphic functions a in $|z|<\infty$ such that $T(r,a)=S(r,f)$.

Throughout the section we suppose that f is non-degenerate over Γ and we note that f is of order infinite. Let

$$\mathsf{S}(f)=\{\mathbf{A}=[a_1,\cdots,a_{n+1}]: \text{holomorphic curve from } \mathbf{C} \text{ into } P^n(\mathbf{C}) \text{ such that } T(r,\mathbf{A})=S(r,f)\}$$

and let H be a subset of $\mathsf{S}(f)$ in general position. It is clear that $\mathsf{S}(f)\supset P^n(\mathbf{C})$. For $\mathbf{A}=[a_1,\cdots,a_{n+1}]\in\mathsf{S}(f)$ we set

$$(\mathbf{A},f)=a_1f_1+\cdots+a_{n+1}f_{n+1}.$$

Then, we have the following

Proposition 2. (a) $a_k/a_j\in\Gamma$ if $a_j\neq0$. (b) $(\mathbf{A},f)\neq0$.

Proof. (a) Applying Lemma 5 (a) to $\mathbf{A}$, we have

$$T(r,a_k/a_j)<T(r,\mathbf{A})+O(1)=S(r,f).$$

(b) Since there is at least one $a_j\neq0$ $(1\leq j\leq n+1)$,

$$\frac{(A,f)}{a_j}=\frac{a_1}{a_j}f_1+\cdots+\frac{a_{n+1}}{a_j}f_{n+1}$$

is a linear combination of $f_1,\cdots,f_{n+1}$ with Γ-coefficients. As f is non-degenerate over Γ, $(A,f)/a_j\neq 0$. That is, $(A,f)\neq 0$.

We put

$$m(r,A,f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|A\|\ \|f\|}{|(A,f)|}d\theta,$$

which is non-negative as in Section 1 and independent of the choice of reduced representations of f and A, and

$$N(r,A,f)=N(r,1/(A,f)).$$

Then we have the first fundamental theorem:

$$T(r,f)=m(r,A,f)+N(r,A,f)+S(r,f).$$

The defect of A with respect to f is defined as follows:

$$\delta(A,f)=\liminf_{r\to\infty}\frac{m(r,A,f)}{T(r,f)}.$$

We define $m_\alpha(r,A,f)$, $N_\alpha(r,A,f)$ and $\delta_\alpha(A,f)$ as in Section 2.

Then, by the first fundamental theorem

$$\delta_\alpha(A,f)=1-\limsup_{r\to\infty}\frac{N_\alpha(r,A,f)}{T_\alpha(r,f)}.$$

and Lemma 2 holds for $A\in S(f)$. We can prove the following defect relation by using the inequality (19) given below:

The defect relation (cf.[6], see also [4]):

$$\sum_{A\in H}\delta(A,f)\le\sum_{A\in H}\delta_\alpha(A,f)\le n+1. \tag{16}$$

The purpose of this section is to give a similar result to Theorem 3 in [9] when $\rho(f)=\infty$ and the equality holds in (16). We need the following lemma.

Lemma 9. For any $A=[a_1,\cdots,a_{n+1}]$ and $B=[b_1,\cdots,b_{n+1}]$ of $S(f)$ such that $a_j\neq 0$, $b_k\neq 0$, put $(A,f)=F$ and $(B,f)=G$. Then,

$$T(r,\frac{F/a_j}{G/b_k})\le 2nT(r,f)+S(r,f).$$

Proof. Since

$$\frac{F/a_j}{G/b_k}=\{\sum_{\nu=1}^{n+1}(a_\nu/a_j)f_\nu\}/\{\sum_{\nu=1}^{n+1}(b_\nu/b_k)f_\nu)$$
$$=\{\sum_{\nu=1}^{n+1}(a_\nu/a_j)f_\nu/f_1\}/\{\sum_{\nu=1}^{n+1}(b_\nu/b_k)f_\nu/f_1)\},$$

$$T(r,\frac{F/a_j}{G/b_k})\le\sum_{\nu=1}^{n+1}\{2T(r,\frac{f_\nu}{f_1})+T(r,\frac{a_\nu}{a_j})+T(r,\frac{b_\nu}{b_k})\}+O(1)\le 2nT(r,f)+S(r,f)$$

by Lemma 5 (a) and Proposition 2 (a).

For $A=[a_1,\cdots,a_{n+1}]$ of H, let a_{j_0} be the first element not identically equal to zero. Then we put

$$\tilde{A}=(\frac{a_\nu}{a_{j_0}},\cdots,\frac{a_{n+1}}{a_{j_0}})=(g_1,\cdots,g_{n+1}),\ \|\tilde{A}\|=\|A\|/|a_{j_0}|,\ \tilde{H}=\{\tilde{A}:A\in H\}$$

and for $(A,f)=F$

$$\tilde{F}=F/a_{j_0}=(\tilde{A},f)=\sum_{j=1}^{n+1}g_jf_j.$$

Then, it is clear that $\tilde{H}$ is in general position and $g_j=\dfrac{a_j}{a_{j_0}}\in\Gamma$ by Proposition 2 (a).

Put

$$H_0=\{A=[a_1,\cdots,a_{n+1}]\in H:a_{n+1}=0\}.$$

Then we have

Theorem 3. Suppose that $\rho(f)=\infty$ and that

(i) $\delta(e_j,f)=1$ $(j=1,\cdots,n)$.

If there exist $A_1,\cdots,A_q$ $(n+1 \leq q \leq \infty)$ in H such that

(ii) $\sum_{j=1}^{q} \delta(A_j,f)=n+1$,

then, f is of regular growth.

Proof. We may suppose without loss of generality that $q \geq 2n+1$ as in the case of Theorem 3 in [9].

Let ε be any positive number smaller than 1/4. Then, there exists a finite number ν $(\geq 2n+1)$ such that

(17) $$\sum_{j=1}^{\nu} \delta(A_j,f)>n+1-\varepsilon.$$

Put for $j=1,\cdots \nu$

$$A_j=[a_{j1},\cdots,a_{jn+1}] \quad \text{and} \quad \tilde{A}_j=(g_{j1},\cdots g_{jn+1}).$$

For any integer p, Let V(p) be the vector space generated by

$$\{\prod_{k=1}^{n+1}\prod_{j=1}^{\nu} g_{jk}^{p(j,k)}: \sum_{k=1}^{n+1}\sum_{j=1}^{\nu} p(j,k)\leq p,\ p(j,k)\geq 0 \text{ and integer}\}$$

over **C** and

$$d(p)=\dim V(p).$$

Then, V(p) is a subspace of V(p+1) and

$$\liminf_{p\to\infty} d(p+1)/d(p)=1$$

since $d(p)\leq\binom{(n+1)\nu+p}{p}$ (see [5], see also [6]), so that we can choose p so large that the following inequality holds:

(18) $$d(p+1)/d(p)<1+\varepsilon/(n+1).$$

Note that any element of V(p) belongs to Γ since $g_{jk}\in\Gamma$.

Let

$$b_1,\cdots,b_{d(p)},b_{d(p)+1},\cdots,b_{d(p+1)}$$

be a basis of V(p+1) such that

$$b_1,\cdots,b_{d(p)}$$

form a basis of V(p). Then, it is clear that the functions

$$\{b_t f_k : t=1,\cdots,d(p+1)\ ;\ k=1,\cdots,n+1\}$$

are linearly independent over **C**. We put for convenience

$$W=W(b_1f_1,b_2f_1,\cdots,b_{d(p+1)}f_{n+1}).$$

Then, we can prove the following inequality as in [4]:

(19) $$N(r,1/W)+d(p)(\nu-n-1)T(r,f)$$
$$\leq d(p)\sum_{j=1}^{\nu} N(r,A_j,f)+(n+1)\{d(p+1)-d(p)\}T(r,f)+S(r,f).$$

Suppose without loss of generality that H_0 consists of $A_1,\cdots,A_\ell$, where $0\leq\ell\leq n$. As in the proof of Theorem 3 in [9], by making use of Proposition 2 (a) and Lemma 9, we have

(20) $$d(p)\sum_{j=\ell+1}^{\nu} m(r,A_j,f)\leq\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{|u(re^{i\theta})|^{(n+1)d(p+1)}}{|W|}d\theta+S(r,f).$$

Let g(z) be a meromorphic function such that the functions

$$\frac{1}{g(z)}\{f_j(z)\}^{(n+1)d(p+1)}\ (j=1,\cdots,n) \quad \text{and} \quad \frac{1}{g(z)}W$$

are entire functions without common zeros.

We put

$$h^*=[\frac{1}{g}(f_1)^{(n+1)d(p+1)},\cdots,\frac{1}{g}(f_n)^{(n+1)d(p+1)},\frac{1}{g}W].$$

Then, we have the inequality

(21) $$T(r,h^*)\leq(n+1)d(p+1)T(r,f)+S(r,f)$$

(cf. Lemma 7) by using

$$N(r,g)\leq(n+1)d(p+1)\sum_{t=1}^{d(p+1)}N(r,b_t)=S(r,f).$$

From (20) and (21), we have the following as in the case of (10):

$$(22)\qquad d(p)\{\sum_{j=\ell+1}^{p}\delta_\alpha(A_j,f)\}\le \liminf_{r\to\infty}\frac{T_\alpha(r,h^*)}{T_\alpha(r,f)}$$

$$\le \limsup_{r\to\infty}\frac{T_\alpha(r,h^*)}{T_\alpha(r,f)}\le (n+1)d(p+1)$$

(cf. (29) in [9]) and from (17), (18) and (19)

$$(23)\qquad \limsup_{r\to\infty}\frac{N_\alpha(r,1/W)}{T_\alpha(r,f)}<2\varepsilon\, d(p).$$

From (17) and (22) by the facts "$0\le \ell\le n$" and "$\delta(A,f)\le\delta_\alpha(A,f)$", we have for any $0<\alpha<\infty$

$$(24)\qquad \limsup_{r\to\infty}\frac{\log T_\alpha(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log T_\alpha(r,h^*)}{\log r}$$

and

$$(25)\qquad \liminf_{r\to\infty}\frac{\log T_\alpha(r,f)}{\log r}=\liminf_{r\to\infty}\frac{\log T_\alpha(r,h^*)}{\log r}.$$

Since $\rho(f)=\infty$, (24) and Lemma 1 (a) imply that $\rho(h^*)=\infty$.

Suppose now that $\mu(f)<\infty$. Then, (25) and Lemma 1 (b) imply that $\mu(h^*)<\infty$. Let ε satisfy

$$(26)\qquad 0<4\varepsilon<\min\{1,\sup_{\alpha,\tau}\frac{n+1}{n}\cdot\frac{|\sin\pi(\tau+\alpha)|}{K(\alpha,\tau)+|\sin\pi(\tau+\alpha)|}\},$$

where α and τ satisfy the conditions (i), (ii) and (iii) of Lemma 6 for h^*. By the hypothesis (i), (17), (22) and (23), we have

$$(27)\qquad K_\alpha(h^*)=\limsup_{r\to\infty}\frac{\sum_{j=1}^{n+1}N_\alpha(r,e_j,h^*)}{T_\alpha(r,h^*)}\le\frac{2\varepsilon}{1-2\varepsilon}<4\varepsilon.$$

since $\varepsilon<1/4$. (26) and (27) contradict with Lemma 6. This shows that $\mu(f)=\infty$. That is, f is of regular growth.

Our proof is complete.

5. Degenerate Case

Let f, X and λ be as in Section 1. Throuout the section we suppose that $\lambda>0$.

By the definition of λ, there are $n+1-\lambda$ functions in $\{f_1,\cdots f_{n+1}\}$ which are linearly independent over C. We suppose without loss of generality that $f_1,\cdots,f_{n+1-\lambda}$ are linearly independent over C. Then $f_{n+2-\lambda},\cdots,f_{n+1}$ can be represented as linear combinations of $f_1,\cdots,f_{n+1-\lambda}$ with constant coefficients.

From now on we put $n-\lambda=\ell$ for simplicity.

For any $a=(a_1,\cdots,a_{n+1})$ of C^{n+1} such that $(a,f)\neq 0$, there exists only one vector $a'=(a'_1,\cdots,a'_{\ell+1},0,\cdots,0)$ of C^{n+1} such that

$$(a,f)=(a',f)$$

since $f_{\ell+2},\cdots,f_{n+1}$ can be uniquely represented as linear combinations of $f_1,\cdots,f_{\ell+1}$ with constant coefficients. We map a to a′. In this mapping, we put

$$X'_0=\{a\in X:\ a'_{\ell+1}=0\}.$$

Lemma 10. (I) The number of vectors of X'_0 is at most n.

(II) For any vectors $a_{j_1},\cdots,a_{j_m}$ $(1\le m\le\ell)$ of $X-X'_0$ such that $a'_{j_1},\cdots,a'_{j_m}$ are linearly independent over C, we can choose $e_{i_1},\cdots,e_{i_{\ell+1-m}}$ from $\{e_1,\cdots e_\ell\}$ such that

$$e'_{i_1},\cdots,e'_{i_{\ell+1-m}},\ a'_{j_1},\cdots,a'_{j_m}$$

are linearly independent over C.

(III) There is a subset X''_0 of X'_0 such that $\#X''_0\le\lambda$ and such that from any n+1 vectors $a_1,\cdots,a_{n+1}$ of $X-X''_0$, we can find $\ell+1$ vectors $a_{j_1},\cdots,a_{j_{\ell+1}}$ for which

$$(\mathbf{a}_{j_1},f),\cdots,(\mathbf{a}_{j_{\ell+1}},f)$$

are linearly independent over **C** and $\mathbf{a}_{j_{\ell+2}},\cdots,\mathbf{a}_{j_{n+1}}$, do not belong to X'_o. ([9], Lemma 8).

Lemma 11. Suppose that $f_1,\cdots,f_{\ell+1}$ ($\ell=n-\lambda$) are linearly independent over **C** and $\rho(f)=\infty$. Then for any $\mathbf{a}_1,\cdots,\mathbf{a}_q$ $(n+1\leq q<\infty)$ of $X-X''_o$, we have

$$\sum_{j=1}^{q} m(r,\mathbf{a}_j,f)\leq(n+1)T(r,f)+\lambda\sum_{j=1}^{\ell} N(r,\mathbf{e}_j,f)-(\lambda+1)N(r,1/W)+S(r,f),$$

where $W=W(f_1,\cdots,f_{\ell+1})$.

We can prove this lemma as in the case of Lemma 9 in [9].

Theorem 4. Suppose that $f_1,\cdots,f_{\ell+1}$ are linearly independent over **C** and $\rho(f)=\infty$. Let $\mathbf{a}_1,\cdots,\mathbf{a}_q$ $(n+\lambda+1\leq q<\infty)$ be any elements of X such that $X''_o\cap\{\mathbf{a}_1,\cdots,\mathbf{a}_q\}=\{\mathbf{a}_1,\cdots,\mathbf{a}_k\}$. Then we have

$$\sum_{j=1}^{q} m(r,\mathbf{a}_j,f)\leq(n+\lambda+1)T(r,f)+\lambda\sum_{j=1}^{\ell} N(r,\mathbf{e}_j,f)-(\lambda+1)N(r,\frac{1}{W})+S(r,f), \tag{28}$$

where $W=W(f_1,\cdots,f_{\ell+1})$ and X''_o is the set obtained in Lemma 10.

Further if $\delta(\mathbf{e}_j,f)=1$ $(j=1,\cdots,\ell)$, then

$$\sum_{j=1}^{q}\delta(\mathbf{a}_j,f)\leq n+1+\sum_{j=1}^{k}\delta(\mathbf{a}_j,f)\leq n+\lambda+1. \tag{29}$$

Proof. We first note that $0\leq k\leq\lambda$ by Lemma 10 (III). Applying Lemma 11 to $\{\mathbf{a}_{k+1},\cdots,\mathbf{a}_q\}$, we have

$$\sum_{j=k+1}^{q} m(r,\mathbf{a}_j,f)\leq(n+1)T(r,f)+\lambda\sum_{j=1}^{\ell} N(r,\mathbf{e}_j,f)-(\lambda+1)N(r,\frac{1}{W})+S(r,f). \tag{30}$$

Adding $\sum_{j=1}^{k} m(r,\mathbf{a}_j,f)$ to both sides of (30), using

$$m(r,\mathbf{a}_j,f)\leq T(r,f)+O(1)$$

and noting $k\leq\lambda$, we have (28).

If $\delta(\mathbf{e}_j,f)=1$ $(j=1,\cdots,\ell)$, then from (27) we have

$$\sum_{j=k+1}^{q}\delta(\mathbf{a}_j,f)\leq n+1.$$

Adding $\sum_{j=1}^{k}\delta(\mathbf{a}_j,f)$ to both sides of this inequality, we obtain (29).

Corollary 4. Suppose that $f_1,\cdots,f_{\ell+1}$ are linearly independent over **C**, $\rho(f)=\infty$ and that

(i) $\delta(\mathbf{e}_j,f)=1$ $(j=1,\cdots,\ell)$.

If there exist $\mathbf{a}_1,\cdots,\mathbf{a}_q$ $(n+\lambda+1\leq q\leq\infty)$ in X such that

(ii) $\sum_{j=1}^{q}\delta(\mathbf{a}_j,f)=n+\lambda+1$

and such that

$$X''_o\cap\{\mathbf{a}_1,\cdots,\mathbf{a}_q\}=\{\mathbf{a}_1,\cdots,\mathbf{a}_k\},$$

then

(a) $k=\lambda$ and $\delta(\mathbf{a}_j,f)=1$ $(j=1,\cdots,\lambda)$;

(b) $\lim_{r\to\infty}\frac{N_\alpha(r,1/W)}{T_\alpha(r,f)}=0.$

Proof. (a) From the hypothesis (ii) and (30), we have

$$n+\lambda+1=\sum_{j=1}^{q}\delta(\mathbf{a}_j,f)\leq n+1+\sum_{j=1}^{k}\delta(\mathbf{a}_j,f)\leq n+\lambda+1,$$

so that we have

$$k=\lambda \quad\text{and}\quad \delta(\mathbf{a}_j,f)=1\ (j=1,\cdots,\lambda).$$

(b) From (28) of Theorem 4 and the hypothesis (i), we have

$$\sum_{j=1}^{q}\delta_\alpha(\mathbf{a}_j,f)+(\lambda+1)\limsup_{r\to\infty}\frac{N_\alpha(r,1/W)}{T_\alpha(r,f)}\leq n+\lambda+1,$$

so that by the hypothesis (ii) and Lemma 2 we obtain

$$\lim_{r\to\infty}\frac{N_\alpha(r,1/W)}{T_\alpha(r,f)}=0.$$

Suppose that $f_1,\cdots,f_{\ell+1}$ are linearly independent over **C**. Let f^* be the holomorphic curve induced by the mapping

$$(f_1^{\ell+1},\cdots,f_\ell^{\ell+1},W): \mathbf{C} \to \mathbf{C}^{\ell+1}$$

where $W=W(f_1,\cdots,f_{\ell+1})$ is the Wronskian of $f_1,\cdots,f_{\ell+1}$.

Note that there is an entire function $d(z)$ such that the functions $f_j^{\ell+1}/d$ $(j=1,\cdots,\ell)$ and W/d have no common zeros.

Let $\{\tilde{e}_j\}_{j=1}^{\ell+1}$ be the standard basis of $\mathbf{C}^{\ell+1}$. Then, we have

Theorem 5. Suppose that $\rho(f)=\infty$. For any $a_1,\cdots,a_q$ $(n+1\le q<\infty)$ in $X-X_0'$, we have

$$\sum_{j=1}^{q} m(r,a_j,f)\le(\lambda+1)m(r,\tilde{e}_{\ell+1},f^*)+S(r,f).$$

We can prove this theorem as in the case of Theorem 5 in [9] with a slight change in estimating the error term.

Corollary 5. Under the same assumption as in Theorem 5, we have

$$\frac{1}{(\lambda+1)(\ell+1)}\sum_{a\in X-X_0'}\delta_o(a,f)\le\delta_o(\tilde{e}_{\ell+1},f^*); \tag{31}$$

$$\frac{1}{(\lambda+1)}\sum_{a\in X-X_0'}\delta_o(a,f)\le\liminf_{r\to\infty}\frac{T_o(r,f^*)}{T_o(r,f)}\le\limsup_{r\to\infty}\frac{T_o(r,f^*)}{T_o(r,f)}\le\ell+1. \tag{32}$$

We can prove this corollary by Theorem 5 and Lemma 7 as in the case of Corollary 2 in Section 3.

Theorem 6. Suppose that $f_1,\cdots,f_{\ell+1}$ are linearly independent over $\mathbf{C}$, $\rho(f)=\infty$ and that

(i) $\delta(e_j,f)=1$ $(j=1,\cdots,\ell)$.

If there exist $a_1,\cdots,a_q$ $(n+\lambda+1\le q\le\infty)$ in X such that

(ii) $\sum_{j=1}^{q}\delta(a_j,f)=n+\lambda+1$,

then f is of regular growth.

Proof. By Lemma 10 (I), X_0' contains at most n vectors. We may suppose without loss of generality that

$$X_0'=\{a_1,\cdots,a_p\}\ (0\le p\le n).$$

Then from the hypothesis (ii), we have

$$\lambda+1\le n+\lambda+1-p\le\sum_{j=p+1}^{q}\delta(a_j,f)\le\sum_{j=p+1}^{q}\delta_o(a_j,f) \tag{33}$$

by Lemma 2. (32) and (33) imply that

$$\limsup_{r\to\infty}\frac{\log T_o(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log T_o(r,f^*)}{\log r} \tag{34}$$

and

$$\liminf_{r\to\infty}\frac{\log T_o(r,f)}{\log r}=\liminf_{r\to\infty}\frac{\log T_o(r,f^*)}{\log r} \tag{35}$$

By Lemma 1 (a) and (34) we have $\rho(f^*)=\infty$. The hypothesis (i), (32) and (33) imply that

$$\delta_o(\tilde{e}_j,f^*)=1.\quad (j=1,\cdots,\ell). \tag{36}$$

Further, Corollary 4 (b), (32) and (33) imply that

$$\delta_o(\tilde{e}_{\ell+1},f^*)=1. \tag{37}$$

By Lemma 4, (36) and (37) imply that f^* is of regular growth; that is to say, $\mu(f^*)=\rho(f^*)=\infty$. Then, by Lemma 1 (b) and (35) we have $\mu(f)=\infty$. Namely, f is of regular growth.

As in Corollary 3, we have the following

Corollary 6. Suppose that $\rho(f)=\infty$. If there exist $a_1,\cdots a_q$ $(n+\lambda+1\le q\le\infty)$ in X such that

(i) $\delta(a_j,f)=1$ $(j=1,\cdots,n)$,

(ii) $\sum_{j=1}^{q}\delta(a_j,f)=n+\lambda+1$,

then f is of regular growth.

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