Symmetric Functions in Two Alphabets

Kazuo UENO*
Department of Mathematics
(Received August 31, 1994)

We study the algebra of symmetric functions in two sets of commutative indeterminates which we call alphabets. It was treated in some detail by MacMahon [5], [8] long ago; however, his expositions seem a little unclear in some points. We rearrange his theory and give a justification for its basis. We also deal with the generating series of the double-indexed Bell polynomials in connection to a formula for MacMahon operators, and give a Hopf algebra structure defined on the algebra of symmetric functions in two alphabets.

1. Introduction

The algebra of symmetric functions in two alphabets, denoted here by Sym(2, **Q**) (see Definitions 2.1), was treated in some detail by MacMahon [5], [8]; however, he did not give a proof of algebraic independence of powersums, elementary functions, or complete functions (see Definitions 2.1 and 3.1). To our knowledge, no proof has been given of their algebraic independence. In Section 2 we will give a proof of algebraic independence of powersums in two alphabets (Theorem 2.3), from which follows algebraic independence of elementary functions (or complete functions) immediately (Theorem 3.4).

We remark that the usual proof of algebraic independence of elementary symmetric functions in one alphabet (see [4, p.13]) does not extend in a straightforward manner to the two alphabet case; the matrix tables in [5, p.532] or [8, p.331] show that the transition matrices relating elementary functions and monomial functions in two alphabets (see Definitions 2.1) are rather complicated and we should work over **Q** rather than **Z**. Our proof of Theorem 2.3 is an extension of the proof of algebraic independence of powersums in one alphabet given by van der Corput [12].

In Section 4 we introduce a scalar product on Sym(2, **Q**); its definition and associated properties almost parallel those of the scalar product in the one alphabet case [4, pp.34-35] once the two **Q**-algebra bases of Sym (2, **Q**), complete functions and powersums, are given.

Section 5 develops the basic results of the operator calculus on Sym(2, **Q**), most of which originally appeared in [5] or [8] explicitly or implicitly. We rearrange them in the setting constructed in the preceding sections; in particular, we define MacMahon operators through the scalar product introduced in Section 4, following the line of reasoning of [4, pp.43-45, Ex.3] which deals with Hammond operators in the one alphabet case. The algebraic independence of powersums, elementary functions, or complete functions in two alphabets provides a justification for introducing the differential operators acting on Sym(2, **Q**) [5, pp.488-495], [8, pp. 297-310].

In Section 6 we first consider a general formula, which gives an exponential generating series expression of a set-partition operation given in form as a relation between two multiplication rules defined in a commutative algebra (Theorem 6.1). As an application of this formula, we show an exponential identity for certain operators acting on symmetric functions (Theorem 6.2). We note that MacMahon essentially observed both of Theorems 6.1 and 6.2 (see the references preceding these theorems); however, his proofs seem a little unclear. As particular cases of Theorem 6.2, we give two formulas for MacMahon operators (Theorem 6.3, Proposition

---

*Supported by Grant-in-Aid for Scientific Research No.04740030, Ministry of Education, Science and Culture

6.6), which give explicit relations between the two multiplications on the MacMahon operators defined in Section 5 (Corollary 6.4, Remarks 6.7 (i)). The formula of Theorem 6.3 links to the generating series of the double-indexed Bell polynomials that appear in the partial differentiation formulas for composite functions (Remark 6.5). We remark that Theorem 6.3 and Proposition 6.6 can be proved in a straightforward manner using Proposition 5.10 (Remarks 6.7 (iv)).

In Section 7 we give a Hopf algebra structure defined on Sym(2, $\mathbf{Q}$) (Definition 7.5), using the formula of Proposition 5.12; it is a generalization of the Hopf algebra of symmetric functions in one alphabet [2, pp.171-175], [3, pp.27-29].

We treat in this paper two alphabets instead of several alphabets for the sake of simplicity of notation (cf. [5; pp.482,536], [8, pp.280-281]). On the other hand, as remarked in [5, p.536], propositions and formulas in the one alphabet case [10] can be deduced by specialization from the corresponding ones in the two alphabet case; for example, [10, Proposition 2.1] from Theorem 6.3 by the specialization that $\beta_i := 0$ $(i \geq 1)$ and $y := 0$ (see Definitions 2.1 (iv)).

Part of the results obtained here have been announced in [11].

2. Powersums in Two Alphabets

We first give some notation and terminology:

Definitions 2.1. (cf. [5, pp.482-486], [8, pp.281-282])

(i) An *alphabet* is for us a countably infinite set of commutative indeterminates; see also (iv) below.

(ii) A *bipartite number*, or *binumber* for short, is $(i, j) \in \mathbf{Z}^2$ with $i, j \geq 0$ and $i + j \geq 1$.

(iii) A *bipartite partition*, or *bipartition* for short, is a partition of a binumber, i.e., a representation of a binumber as a sum of binumbers with order of summands not taken into account; the summands are called *biparts*.

(iv) We denote by $\mathbf{P}$ the set of positive integers. Let $\overleftarrow{\alpha} = \{\alpha_i | i \in \mathbf{P}\}$ and $\overleftarrow{\beta} = (\beta_i | i \in \mathbf{P}\}$ be two alphabets and let

$$(p_1q_1, p_2q_2, p_3q_3, \ldots)$$

be a bipartition with (p_1, q_1), (p_2, q_2), (p_3, q_3), ... its biparts. A *monomial symmetric function in two alphabets*, or *monomial function* for short, is a sum of the form

$$\sum \alpha_{i_1}^{p_1} \beta_{i_1}^{q_1} \alpha_{i_2}^{p_2} \beta_{i_2}^{q_2} \alpha_{i_3}^{p_3} \beta_{i_3}^{q_3} \cdots$$

where $(i_1, i_2, i_3, \ldots)$ is a finite permutation of (1,2,3, ...), i.e., permutation of $\mathbf{P}$ with $i_k = k$ for all sufficiently large k, and the summation is taken over all distinct such monomials. *We denote the monomial function associated with bipartition* $(p_1q_1, p_2q_2, p_3q_3, \ldots)$ *by the same symbol* $(p_1q_1, p_2q_2, p_3q_3, \ldots)$. Note that repetitions of biparts are allowed.

(v) Let $(p_1q_1^{\pi_1}, p_2q_2^{\pi_2}, \ldots)$ be a bipartition with bipart (p_i, q_i) repeated π_i times. An *augmented monomial function*, denoted by

$$[p_1q_1^{\pi_1}, p_2q_2^{\pi_2}, \ldots],$$

is defined to be $\pi_1! \pi_2! \cdots (p_1q_1^{\pi_1}, p_2q_2^{\pi_2}, \ldots)$.

(vi) A *powersum* is a monomial function associated with bipartition (ij) having one bipart. It is denoted by s_{ij}.

(vii) We denote by Sym(2, $\mathbf{Q}$) the $\mathbf{Q}$-linear space spanned by all monomial functions:

$$\mathrm{Sym}(2, \mathbf{Q}) = \bigoplus \mathbf{Q} \cdot (p_1q_1, p_2q_2, p_3q_3, \ldots)$$

with direct sum over all bipartitions including the *empty bipartition* ϕ *with which* 1 *is associated.* We call an element of $\mathrm{Sym}(2, \mathbf{Q})$ a *symmetric function in two alphabets.* Note that

$$\mathrm{Sym}(2, \mathbf{Q}) = \bigoplus \mathbf{Q}\cdot[p_1q_1, p_2q_2, p_3q_3, \dots].$$

Proposition 2.2. *We have* $\mathrm{Sym}(2, \mathbf{Q}) = \mathbf{Q}[s_{ij} \mid i+j \geq 1]$, i.e., $\mathrm{Sym}(2, \mathbf{Q})$ *is the* $\mathbf{Q}$*-algebra generated by all powersums.*

Proof. We first note that (cf. [12, p.224])

$$\begin{aligned}(1)\qquad & [p_1q_1, p_2q_2, \dots, p_rq_r][p_{r+1}q_{r+1}] \\ & = [p_1q_1, p_2q_2, \dots, p_{r+1}q_{r+1}] + [p_1+p_{r+1}\ q_1+q_{r+1}, p_2q_2, \dots, p_rq_r] + \dots \\ & \quad + [p_1q_1, \dots, p_{r-1}q_{r-1}, p_r+p_{r+1}\ q_r+q_{r+1}].\end{aligned}$$

Put $S := \mathbf{Q}[s_{ij} \mid i+j \geq 1]$. Clearly $s_{ij} = (ij) = [ij] \in \mathrm{Sym}(2, \mathbf{Q})$ and $s_{ij} \in S$.

To show that $\mathrm{Sym}(2, \mathbf{Q}) \subseteqq S$, we use (1) as

$$\begin{aligned}& [p_1q_1, p_2q_2, \dots, p_{r+1}q_{r+1}] \\ & = [p_1q_1, \dots, p_rq_r][p_{r+1}q_{r+1}] - [p_1+p_{r+1}\ q_1+q_{r+1}, p_2q_2, \dots, p_rq_r] - \cdots \\ & \quad - [p_1q_1, \dots, p_{r-1}q_{r-1}, p_r+p_{r+1}\ q_r+q_{r+1}];\end{aligned}$$

by induction on r, we see that $[p_1q_1, \dots, p_{r+1}q_{r+1}] \in S$. Thus for any bipartition, the corresponding monomial function belongs to S. Hence $\mathrm{Sym}(2, \mathbf{Q}) \subseteqq S$.

Conversely, assume that for any bipartition of the form $(p_1q_1, p_2q_2, \dots, p_rq_r)$, $s_{p_1q_1}s_{p_2q_2}\dots s_{p_rq_r}$ belongs to $\mathrm{Sym}(2, \mathbf{Q})$; $s_{p_1q_1}s_{p_2q_2}\dots s_{p_rq_r}$ is a $\mathbf{Q}$-linear combination of some augmented monomial functions. To show that $s_{p_1q_1}s_{p_2q_2}\dots s_{p_{r+1}q_{r+1}} \in \mathrm{Sym}(2, \mathbf{Q})$, we have only to consider terms of the form $[p_1'q_1', p_2'q_2', \dots, p_t'p_t'][p_{r+1}q_{r+1}]$; this expression belongs to $\mathrm{Sym}(2, \mathbf{Q})$ because of (1). Thus all the monomial expressions in powersums belong to $\mathrm{Sym}(2, \mathbf{Q})$. Hence $S \subseteqq \mathrm{Sym}(2, \mathbf{Q})$.

Theorem 2.3. s_{ij} $(i+j \geq 1)$ *are algebraically independent over* $\mathbf{Q}$.

Proof. It suffices to show that s_{ij} $(i+j \geq 1,\ 0 \leq i, j \leq p)$ are algebraically independent over $\mathbf{Q}$ for $p \in \mathbf{P}$. Put $r := (1+p)^2 - 1$. Let $f(X_1, X_2, \dots, X_r) \in \mathbf{Q}[X_1, X_2, \dots, X_r]$ with $X_1, X_2, \dots, X_r$ indeterminates and assume that $f(s_{01}, s_{02}, \dots, s_{0p}, s_{10}, s_{11}, \dots, s_{1p}, \dots, s_{p0}, s_{p1}, \dots, s_{pp}) = 0$.

Let $u_1, u_2, \dots, u_r \in \mathbf{P}$ and put $U_k := \sum_{\ell=1}^{k} u_\ell$. Consider the specialization $\varphi : \mathrm{Sym}(2, \mathbf{Q}) \longrightarrow \mathbf{Q}$ such that (cf. [12, pp.223-224]):

$$\begin{aligned}\varphi(\alpha_k) := {} & 0 \quad \text{for} \quad 1 \leq k \leq U_p \quad \text{and} \quad k \geq U_r + 1, \\ & 1 \quad \text{for} \quad U_p + 1 \leq k \leq U_{2p+1}, \\ & \dots, \\ & i \quad \text{for} \quad U_{i(1+p)-1} + 1 \leq k \leq U_{(i+1)p+i}, \\ & \dots, \\ & p \quad \text{for} \quad U_{p(1+p)-1} + 1 \leq k \leq U_{(p+1)p+p} = U_r; \\ \varphi(\beta_k) := {} & 0 \quad \text{for} \quad U_p + 1 \leq k \leq U_{p+1},\ U_{2(1+p)-1} \leq k \leq U_{2(1+p)}, \dots, U_{p(1+p)-1} \leq k \leq U_{p(1+p)}, \text{ and} \\ & \qquad k \geq U_r + 1, \\ & 1 \quad \text{for} \quad 1 \leq k \leq u_1 = U_1,\ U_{1+p} + 1 \leq k \leq U_{(1+p)+1},\ U_{2(1+p)} + 1 \leq k \leq U_{2(1+p)+1}, \dots, \text{ and} \\ & \qquad U_{p(1+p)} + 1 \leq k \leq U_{p(1+p)+1}, \\ & \dots,\end{aligned}$$

j for $U_{j-1}+1 \le k \le U_j$, $U_{(1+p)+j-1}+1 \le k \le U_{1+p+j}$,
$U_{2(1+p)+j-1}+1 \le k \le U_{2(1+p)+j}$, ..., and $U_{p(1+p)+j-1}+1 \le k \le U_{p(1+p)+j}$,

...,

p for $U_{p-1}+1 \le k \le U_p$, $U_{2p}+1 \le k \le U_{2p+1}$, ..., and
$U_{(1+p)^2-2}+1 \le k \le U_{(1+p)^2-1}+1 = U_r$.

Put next the matrix

$$A := [j^i]_{0 \le i,j \le p} \text{ with the convention that } 0^0 := 1$$

and consider also the matrix

$$A \otimes A := [j^i A]_{0 \le i,j \le p}.$$

Since $\det A \neq 0$, we have $\det(A \otimes A) = (\det A)^{2(1+p)} \neq 0$, so that

$$A \otimes A \in GL((1+p)^2, \mathbf{Q}).$$

Let C be the matrix $A \otimes A$ with the first row and column deleted. Since the first row of $A \otimes A$ is $(1,1, ..., 1)$ and the first column ${}^t(1,0, ..., 0)$, we have

$$C \in GL((1+p)^2 - 1, \mathbf{Q}) = GL(r, \mathbf{Q}).$$

Put

$$\begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix} := C \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}$$

Then we have:

$$\varphi(s_{ij}) = v_{i(1+p)+j} \quad (0 \le i, j \le p, \ i+j \ge 1).$$

The proof of these equalities is by straightforward computation; note that the case $i=0$ and $1 \le j \le p$ should be treated separately. Thus we obtain:

$${}^t(\varphi(s_{01}), \varphi(s_{02}), ..., \varphi(s_{0p}), \varphi(s_{10}), ..., \varphi(s_{1p}), ..., \varphi(s_{p0}), ..., \varphi(s_{pp})) = v = Cu,$$

where $v = {}^t(v_1, ..., v_r)$ and $u = {}^t(u_1, ..., u_r)$. It follows from the assumption $f(s_{01}, s_{02}, ..., s_{0p}, s_{10}, ..., s_{1p}, ..., s_{p0}, ..., s_{pp}) = 0$ that, by an application of φ on both sides,

$$0 = f(\varphi(s_{01}), ..., \varphi(s_{pp})) = f(v_1, ..., v_r)$$

for any $u \in \mathbf{P}^r$. Note that C is independent of choice of u. Take another set of indeterminates $Y = (Y_1, ..., Y_r)$ and denote by $\widetilde{C}$ the $\mathbf{Q}$-algebra isomorphism induced by $X \mapsto CY$, i.e., $X_i \mapsto$ (ith row of C)$\cdot {}^t(Y_1, ..., Y_r)$ $(i=1, ..., r)$. Put

$$g(Y) := \widetilde{C}(f(X)) = f(CY) \in \mathbf{Q}[Y].$$

Then, for any $u \in \mathbf{P}^r$, we have $g(u) = f(Cu) = f(v) = 0$. Hence (cf. [12, p.224]) $g(Y) = 0$ in $\mathbf{Q}[Y]$, so that

$$f(X) = \widetilde{C}^{-1}(g(Y)) = 0 \text{ in } \mathbf{Q}[X].$$

Thus we see that s_{ij} $(0 \le i, j \le p, \ i+j \ge 1)$ are algebraically independent over $\mathbf{Q}$.

Combining Proposition 2.2 and Theorem 2.3, we arrive at:

Theorem 2.4. *The* s_{ij} $(i+j \ge 1)$ *form a* $\mathbf{Q}$*-algebra basis of* $\mathrm{Sym}(2, \mathbf{Q})$.

3. Elementary Functions and Complete Functions in Two Alphabets

Definitions 3.1. (cf. [5, pp.482-483,487], [8, pp.280,283-284])

(i) The *elementary symmetric functions in two alphabets*, or *elementary functions* for short, denoted by a_{ij} $(i+j \geq 1)$ are defined by the generating series

$$\sum_{i,j\geq 0} a_{ij} x^i y^j := \prod_{k\geq 1} (1+\alpha_k x+\beta_k y)$$

with $a_{00}=1$.

(ii) The *complete symmetric functions in two alphabets*, or *complete functions* for short, denoted by h_{ij} $(i+j \geq 1)$ are defined by the generating series

$$\sum_{i,j\geq 0} h_{ij} x^i y^j := \prod_{k\geq 1} (1-\alpha_k x-\beta_k y)^{-1}$$

with $h_{00}=1$.

We will show that each of the sets $\{a_{ij} \mid i+j \geq 1\}$, $\{h_{ij} \mid i+j \geq 1\}$ forms a $\mathbf{Q}$-algebra basis of $\mathrm{Sym}(2, \mathbf{Q})$. We need two lemmas.

Lemma 3.2. *Let K be a field and let $X_1, \ldots, X_n$ be algebraically independent over K. If $Y_1, \ldots, Y_n \in K[X_1, \ldots, X_n]$ are such that $K[Y_1, \ldots, Y_n]=K[X_1, \ldots, X_n]$, then $Y_1, \ldots, Y_n$ are algebraically independent over K.*

Proof. Use the definition of height for prime ideals; see books on commutative algebra.

Lemma 3.3. ([5, pp.486-488], [8, p.283-284]) *Let (p,q) be a binumber. Then we have:*

(i) $(-1)^{p+q-1}((p+q-1)!/p!q!)s_{pq} = \sum (-1)^{\ell-1}((\ell-1)!/\pi_1!\pi_2!\cdots) a_{p_1q_1}{}^{\pi_1} a_{p_2q_2}{}^{\pi_2}\cdots,$

(ii) $(-1)^{p+q-1}a_{pq}$
$= \sum ((p_1+q_1-1)!/p_1!q_1!)^{\pi_1}((p_2+q_2-1)!/p_2!q_2!)^{\pi_2}\cdots((-1)^{\ell-1}/\pi_1!\pi_2!\cdots) s_{p_1q_1}{}^{\pi_1} s_{p_2q_2}{}^{\pi_2}\cdots,$

(iii) $(-1)^{p+q-1}h_{pq} = \sum (-1)^{\ell-1}(\ell!/\pi_1!\pi_2!\cdots) a_{p_1q_1}{}^{\pi_1} a_{p_2q_2}{}^{\pi_2}\cdots,$

(iv) $(-1)^{p+q-1}a_{pq} = \sum (-1)^{\ell-1}(\ell!/\pi_1!\pi_2!\cdots) h_{p_1q_1}{}^{\pi_1} h_{p_2q_2}{}^{\pi_2}\cdots,$

where $\ell := \sum_i \pi_i$ and the summations range over all bipartitions $(p_1q_1{}^{\pi_1}, p_2q_2{}^{\pi_2}, \ldots)$ with $\sum_i p_i\pi_i=p$ and $\sum_i q_i\pi_i=q$.

Proof. By simple generating series computations; see [8, pp.282-284] for details.

Theorem 3.4. *The a_{ij} $(i+j \geq 1)$ form a $\mathbf{Q}$-algebra basis of* $\mathrm{Sym}(2, \mathbf{Q})$, *and so do the h_{ij} $(i+j \geq 1)$.*

Proof. It follows from Lemma 3.3 (i) and (ii) that, for any $p \in \mathbf{P}$,

$$\mathbf{Q}[s_{ij} \mid 0 \leq i, j \leq p,\ i+j \geq 1] = \mathbf{Q}[a_{ij} \mid 0 \leq i, j \leq p,\ i+j \geq 1].$$

Since s_{ij} $(0 \leq i, j \leq p,\ i+j \geq 1)$ are algebraically independent over $\mathbf{Q}$, so are a_{ij} $(0 \leq i, j \leq p,\ i+j \geq 1)$ by Lemma 3.2. Hence, by Theorem 2.4, $\mathrm{Sym}(2, \mathbf{Q}) = \mathbf{Q}[s_{ij} \mid i+j \geq 1] = \mathbf{Q}[a_{ij} \mid i+j \geq 1]$ and a_{ij} $(i+j \geq 1)$ are algebraically independent over $\mathbf{Q}$. The result for h_{ij} $(i+j \geq 1)$ follows similarly by using Lemma 3.3 (iii) and (iv).

4. Scalar Product

Let $\lambda=(p_1q_1, p_2q_2, p_3q_3, \ldots)$ be a bipartition; λ *also denotes the monomial function associated with the bipartition denoted by the same symbol* (cf. Definitions 2.1 (iv)). For such a bipartition, we put

$$h_\lambda := h_{p_1q_1}h_{p_2q_2}h_{p_3q_3}\cdots$$

with $h_\phi := 1$. Note that, by Theorem 3.4, h_λ for all bipartitions λ form a $\mathbf{Q}$-linear basis of $\mathrm{Sym}(2, \mathbf{Q})$.

Definition 4.1. (cf. [4, p.34]) We define a scalar product on $\mathrm{Sym}(2, \mathbf{Q})$, i.e., a $\mathbf{Q}$-valued bilinear form $<\cdot,\cdot>$, by requiring that

$$<h_\lambda, \mu> = \delta_{\lambda\mu}$$

for all bipartitions λ, μ, where $\delta_{\lambda\mu}$ is the Kronecker delta.

Lemma 4.2. (cf. [4, p.33 (4.2)]) *For alphabets* $\overleftarrow{\alpha}, \overleftarrow{\beta}, \overleftarrow{x}, \overleftarrow{y}$, *we have*

$$\prod_{i,j\geq 1}(1-\alpha_i x_j-\beta_i y_j)^{-1}=\sum_\lambda h_\lambda(\alpha, \beta)\cdot\lambda(x,y),$$

where the summation ranges over all bipartitions λ, *and* λ *also denotes the associated monomial function* (*see the first paragraph of this section*).

Proof. We compute:

$$\prod_{i,j\geq 1}(1-\alpha_i x_j-\beta_i y_j)^{-1}=\prod_{j\geq 1}\sum_{k,\ell\geq 0}h_{k\ell}(\overleftarrow{\alpha}, \overleftarrow{\beta})x_j{}^k y_j{}^\ell$$
$$=\sum_{\substack{k_1,k_2,\ldots,\\ \ell_1,\ell_2,\ldots\geq 0}} h_{k_1\ell_1}(\overleftarrow{\alpha}, \overleftarrow{\beta})h_{k_2\ell_2}(\overleftarrow{\alpha}, \overleftarrow{\beta})\cdots x_1{}^{k_1}y_1{}^{\ell_1}x_2{}^{k_2}y_2{}^{\ell_2}\cdots$$
$$=\sum_\lambda h_\lambda(\overleftarrow{\alpha}, \overleftarrow{\beta})\cdot\lambda(\overleftarrow{x}, \overleftarrow{y}).$$

Theorem 4.3. (cf. [4, p.34 (4.6)] *The following two conditions are equivalent*:

(i) $\prod_{i,j\geq 1}(1-\alpha_i x_j-\beta_i y_j)^{-1}=\sum_\lambda u_\lambda(\overleftarrow{\alpha}, \overleftarrow{\beta})v_\lambda(\overleftarrow{x}, \overleftarrow{y})$,

the summation being over all bipartions λ,

(ii) $<u_\lambda, v_\mu>=\delta_{\lambda\mu}$ *for all bipartitions* λ, μ,

where (u_λ), (v_λ) *are* $\mathbf{Q}$-*linear bases of* $\bigoplus_{(p,q)}\mathbf{Q}\cdot\lambda$ (*direct sum over all bipartitions* λ *of binumber* (p, q)), *indexed by the bipartitions* λ *of the binumber* (p, q).

Proof. With Lemma 4.2 being taken into consideration, the proof goes along almost the same line as that of [4, p.34, (4.6)].

For a bipartition $\lambda=(p_1q_1, p_2q_2, \ldots)$, we put

$$s_\lambda := s_{p_1q_1}s_{p_2q_2}$$

with $s_\phi := 1$. (*Do not confuse this with the Schur function notation of* [4].) Note that, by Theorem 2.4, s_λ for all bipartitions λ form a $\mathbf{Q}$-linear basis of $\mathrm{Sym}(2, \mathbf{Q})$.

Lemma 4.4. (cf. [4, p.17, (2.14)]) *For alphabets* $\overleftarrow{\alpha}, \overleftarrow{\beta}$, *and indeterminates t,u, we have*

$$\prod_{i\geq 1}(1-\alpha_i t-\beta_i u)^{-1}=\sum_\lambda t^{|\lambda|_1}u^{|\lambda|_2}s_\lambda(\overleftarrow{\alpha}, \overleftarrow{\beta})c_\lambda{}^{-1},$$

the summation being over all bipartitions; here, for a bipartition $\lambda=(p_1q_1{}^{\pi_1}, p_2q_2{}^{\pi_2}, \ldots)$, *we put*

$$|\lambda|_1 := \pi_1 p_1 + \pi_2 p_2 + \cdots,$$
$$|\lambda|_2 := \pi_1 q_1 + \pi_2 q_2 + \cdots,$$
$$c_\lambda := \prod_{i\geq 1} (p_i! q_i!/(p_i+q_i-1)!)^{\pi_i} \cdot \pi_i!.$$

Proof. We compute:

$$\prod_{i\geq 1}(1-\alpha_i t - \beta_i u)^{-1} = \exp(\sum_{i\geq 1}\log((1-\alpha_i t-\beta_i u)^{-1}))$$
$$=\exp(\sum_{i,j\geq 1}(\alpha_i t + \beta_i u)^j/j)$$
$$= \prod_{p+q\geq 1}\exp(((p+q-1)!/p!q!)s_{pq}(\overleftarrow{\alpha},\overleftarrow{\beta})t^p u^q)$$
$$= \prod_{p+q\geq 1}\prod_{\pi_{pq}\geq 0}(((p+q-1)!/p!q!)s_{pq}(\overleftarrow{\alpha},\overleftarrow{\beta})t^p u^q)^{\pi_{pq}}/\pi_{pq}!$$
$$=\sum_\lambda t^{|\lambda|_1}u^{|\lambda|_2}s_\lambda(\overleftarrow{\alpha},\overleftarrow{\beta})c_\lambda^{-1}.$$

Proposition 4.5. (cf. [4, p.33,(4.1)]) *For alphabets $\overleftarrow{\alpha}, \overleftarrow{\beta}, \overleftarrow{x}, \overleftarrow{y}$, we have*

$$\prod_{i,j\geq 1}(1-\alpha_i x_j - \beta_i y_j)^{-1} = \sum_\lambda c_\lambda^{-1}s_\lambda(\overleftarrow{\alpha},\overleftarrow{\beta})s_\lambda(\overleftarrow{x},\overleftarrow{y}),$$

the summation being over all bipartitions.

Proof. Putting $\overleftarrow{\alpha x} := \{\alpha_i x_j \mid i, j\geq 1\}$ and $\overleftarrow{\beta y} := \{\beta_i y_j \mid i,j\geq 1\}$, we have

$$s_{pq}(\overleftarrow{\alpha x},\ \overleftarrow{\beta y}) = \sum_{i,j\geq 1}(\alpha_i x_j)^p(\beta_i y_j)^q = \sum_{i\geq 1}\alpha_i^p\beta_i^q\cdot\sum_{j\geq 1}x_j^p y_j^q = s_p(\overleftarrow{\alpha},\overleftarrow{\beta})s_q(\overleftarrow{x},\overleftarrow{y}).$$

It follows from Lemma 4.4 that

$$\prod_{i,j\geq 1}(1-\alpha_i x_j t - \beta_i y_j u)^{-1} = \sum_\lambda t^{|\lambda|_1}u^{|\lambda|_2}s_\lambda(\overleftarrow{\alpha x},\ \overleftarrow{\beta y})c_\lambda^{-1} = \sum_\lambda t^{|\lambda|_1}u^{|\lambda|_2}s_\lambda(\overleftarrow{\alpha},\overleftarrow{\beta})s_\lambda(\overleftarrow{x},\overleftarrow{y})c_\lambda^{-1}.$$

Substituting $t=1$ and $u=1$ gives the result.

Theorem 4.6. (cf. [4, p.35, (4.7)]) *For bipartitions λ, μ, we have*

$$\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}c_\lambda.$$

Proof. By virtue of Proposition 4.5 we can apply Theorem 4.3 with $u_\lambda = c_\lambda^{-1}s_\lambda$ and $v_\lambda = s_\lambda$ to obtain the desired identity.

Theorem 4.7. (cf. [4, p.35, (4.9)]) *The bilinear form $\langle\cdot,\cdot\rangle$ is symmetric and positive definite.*

Proof. This follows immediately from Theorem 4.6.

5. MacMahon Operator Algebra

By Theorem 3.4 we can define the algebra endomorphism ω of Sym(2, $\mathbf{Q}$) by setting $\omega(a_{ij}) := h_{ij}$ $(i+j\geq 1)$ with $a_{00}=h_{00}=1$.

Proposition 5.1. (cf. [4, p.14 (2.7)], [5, p.488], [8,p.284]) *ω is an involution:* $\omega^2=\mathrm{id}$.

Proof. Put

$$A(x, y) := \prod_{i\geq 1}(1+\alpha_i x+\beta_i y) = \sum_{j,k\geq 0} a_{jk}x^j y^k,$$

$$H(x, y) := \prod_{i\geq 1}(1-\alpha_i x-\beta_i y)^{-1} = \sum_{j,k\geq 0} h_{jk}x^j y^k;$$

we have $H(x, y)A(-x, -y)=1$. ω naturally acts on $\mathrm{Sym}(2, \mathbf{Q})[[x, y]]$ and we see that

$$\omega(A(x, y)) = \sum_{j,k\geq 0}\omega(a_{jk})x^j y^k = H(x, y),$$

$$\omega(H(x, y)) = \omega(A(-x, -y))^{-1} = H(-x, -y)^{-1} = A(x, y);$$

hence $\omega(h_{jk}) = a_{jk}$, so that $\omega^2(a_{jk}) = a_{jk}$ $(j+k\geq 1)$.

Definitions 5.2. (cf. [4; p.43, Ex.3], [10, p.2])
(i) We define D: $\mathrm{Sym}(2, \mathbf{Q}) \longrightarrow \mathrm{End}_{\mathbf{Q}}(\mathrm{Sym}(2, \mathbf{Q}))$ by $<D(f)g,h> := <g,fh>$ $(f,g,h \in \mathrm{Sym}(2, \mathbf{Q}))$. By Theorem 4.7 D is a $\mathbf{Q}$-algebra injection.
(ii) We denote by Mah the composition $D\circ\omega$: $\mathrm{Sym}(2, \mathbf{Q}) \longrightarrow \mathrm{End}_{\mathbf{Q}}(\mathrm{Sym}(2, \mathbf{Q}))$. By (i) and Proposition 5.1, Mah is a $\mathbf{Q}$-algebra injection.
(iii) We denote $\mathrm{Mah}(2, \mathbf{Q}) := \mathrm{Mah}(\mathrm{Sym}(2, \mathbf{Q}))$ for short and call it the *MacMahon operator algebra*; an element of $\mathrm{Mah}(2, \mathbf{Q})$ is called a *MacMahon operator*. Since $\mathrm{Mah} : \mathrm{Sym}(2, \mathbf{Q}) \xrightarrow{\sim} \mathrm{Mah}(2, \mathbf{Q})$ as $\mathbf{Q}$-algebras, $\mathrm{Mah}(2, \mathbf{Q})$ is a *commutative* $\mathbf{Q}$-algebra isomorphic to $\mathrm{Sym}(2, \mathbf{Q})$.

Proposition 5.3. (cf. [5, p.488], [8, p.284], [4, p.16]) *We have*

$$\omega(s_{ij}) = (-1)^{i+j-1}s_{ij} \quad (i+j\geq 1).$$

Proof. Generating series computation gives

$$\sum_{i+j\geq 1}((-1)^{i+j-1}(i+j-1)!/i!j!)s_{ij}x^i y^j$$
$$= \log(1+\sum_{i+j\geq 1}a_{ij}x^i y^j) = -\log(1+\sum_{i+j\geq 1}h_{ij}(-1)^{i+j}x^i y^j).$$

Apply ω (see the proof of Proposition 5.1) to compute:

$$\sum_{i+j\geq 1}((-1)^{i+j-1}(i+j-1)!/i!j!)\,\omega(s_{ij})x^i y^j$$
$$= \log(1+\sum_{i+j\geq 1}\omega(a_{ij})x^i y^j) = \log(1+\sum_{i+j\geq 1}h_{ij}x^i y^j)$$
$$= \sum_{i+j\geq 1}((i+j-1)!/i!j!)s_{ij}x^i y^j.$$

Taking out the coefficients of $x^i y^j$ gives the result.

Proposition 5.4. (cf. [5, p.495], [4, p.44]) *Let d_{ij} denote* $\mathrm{Mah}(s_{ij})$ $(i+j\geq 1)$. *Then we have*

$$d_{ij} = (-1)^{i+j-1}(i!j!/(i+j-1)!)\,\partial_{s_{ij}} \quad (i+j\geq 1)$$

with $\partial_{s_{ij}} := \partial/\partial s_{ij}$. *In particular, d_{ij} is a derivation of* $\mathrm{Sym}(2, \mathbf{Q})$.

Proof. For bipartitions λ, μ, we compute

$$<D(s_{ij})s_\lambda,\ s_\mu> = <s_\lambda,\ s_{ij}s_\mu>,$$

using Theorem 4.6. If λ has no bipart (ij), then $<s_\lambda, s_{ij}s_\mu>=0$, so that $D(s_{ij})s_\lambda=0$ by Theorem 4.7. Otherwise, let $\tilde{\mu}$ be μ with a bipart (ij) added and let $\underset{\sim}{\lambda}$ be λ with a bipart (ij) deleted. Then:

$$<s_\lambda,\ s_{ij}s_\mu> = \delta_{\lambda\tilde{\mu}}c_\lambda = \delta_{\underset{\sim}{\lambda}\mu}c_{\underset{\sim}{\lambda}}(c_\lambda/c_{\underset{\sim}{\lambda}}) = <s_{\underset{\sim}{\lambda}},\ s_\mu>(c_\lambda/c_{\underset{\sim}{\lambda}}) = <(c_\lambda/c_{\underset{\sim}{\lambda}})s_{\underset{\sim}{\lambda}},\ s_\mu>.$$

Hence by Theorem 4.7,

$$D(s_{ij})s_\lambda=(c_\lambda/c_{\underline{\lambda}})s_{\underline{\lambda}}=(i!j!/(i+j-1)!)\,\pi_{ij}s_{\underline{\lambda}}=(i!j!/(i+j-1)!)\,\partial_{s_{ij}}(s_\lambda),$$

where π_{ij} is the multiplicity of bipart (ij) in λ ; see Lemma 4.4. Thus we see that

$$D(s_{ij})=(i!j!/(i+j-1)!)\,\partial_{s_{ij}}$$

and

$$d_{ij}=\mathrm{Mah}(s_{ij})=D\circ\omega(s_{ij})=(-1)^{i+j-1}(i!j!/(i+j-1)!)\,\partial_{s_{ij}}$$

by Proposition 5.3.

Proposition 5.5. (cf. [5, pp.490, 495], [4, p.44]) *We have*

$$d_{ij}=\sum_{k,\ell\geq 0}a_{k\ell}\,\partial_{a_{i+k\,j+\ell}}\quad(i+j\geq 1)$$

with $\partial_{a_{pq}}:=\partial/\partial a_{pq}$.

Proof. By Lemma 3.3 (ii) we have

$$(1)\qquad \begin{aligned}&(-1)^{p+q-1}a_{pq}\\ &=\Sigma((p_1+q_1-1)!/p_1!q_1!)^{\pi_1}((p_2+q_2-1)!/p_2!q_2!)^{\pi_2}\cdots((-1)^{\ell-1}/\pi_1!\,\pi_2!\cdots)s_{p_1q_1}{}^{\pi_1}s_{p_2q_2}{}^{\pi_2}\cdots,\end{aligned}$$

where the summation ranges over all bipartitions $(p_1q_1{}^{\pi_1},\ p_2q_2{}^{\pi_2},\ \ldots)$ of the binumber (pq) and $\ell:=\sum_i\pi_i$. Since $d_{ij}=(-1)^{i+j-1}(i!j!/(i+j-1)!)\,\partial_{s_{ij}}$(Proposition 5.4), it follows from (1) that, if $p<i$ or $q<j$, then $d_{ij}(a_{pq})=0$. If $p\geq i$ and $q\geq j$, then, for a bipartition λ of (pq) having no bipart equal to (ij) we have $\partial_{s_{ij}}(s_\lambda)=0$, and for μ having bipart (ij) with multiplicity π_{ij} we have $\partial_{s_{ij}}(s_\mu)=\pi_{ij}s_\mu/s_{ij}$. There is a bijection between the bipartitions of (pq) with some bipart equal to (ij) and the bipartitions of $(p-i,\ q-j)$, i.e., one that deletes or adds a bipart (ij). Thus for (pq) with $p\geq i$ and $q\geq j$ we compute using (1):

$$d_{ij}(a_{pq})=(-1)^{i+j-1}(i!j!/(i+j-1)!)\,\partial_{s_{ij}}(a_{pq})=a_{p-i,q-j}.$$

We arrive at:

$$d_{ij}(a_{pq})=0\text{ if }p<i\text{ or }q<j\,;\ =a_{p-i,q-j}\text{ if }p\geq i\text{ and }q\geq j.$$

Since d_{ij} acts on $\mathrm{Sym}(2,\mathbf{Q})=\mathbf{Q}[a_{ij}\mid i+j\geq 1]$ as derivation, it follows that $d_{ij}=\sum_{k,\ell\geq 0}a_{k\ell}\,\partial_{a_{i+k,j+\ell}}$ $(a_{00}=1)$.

Remark 5.6. Since $\mathrm{Sym}(2,\mathbf{Q})=\mathbf{Q}[s_{ij}\mid i+j\geq 1]$ and $\mathrm{Mah}:\mathrm{Sym}(2,\mathbf{Q})\xrightarrow{\sim}\mathrm{Mah}(2,\mathbf{Q})$ as $\mathbf{Q}$-algebras (see Definitions 5.2), we see that

$$\mathrm{Mah}(2,\mathbf{Q})=\mathbf{Q}[d_{ij}\mid i+j\geq 1]$$

(see Proposition 5.4) is the commutative $\mathbf{Q}$-algebra generated by d_{ij} $(i+j\geq 1)$ that are algebraically independent over $\mathbf{Q}$.

Definition 5.7. (cf. [5, pp.489-491], [8, pp.297-299]) For $d_{i_tj_t}=\sum_{k_t,\ell_t\geq 0}a_{k_t\ell_t}\partial_{a_{i_t+k_tj_t+\ell_t}}$ $(i_t+j_t\geq 1,\ 1\leq t\leq m)$ (see Proposition 5.5), we define the $\#$*-product* by

$$d_{i_1j_1}\#d_{i_2j_2}\#\cdots\#d_{i_mj_m}:=\sum_{\substack{k_1,\ldots,k_m,\\ \ell_1,\ldots,\ell_m\geq 0}}a_{k_1\ell_1}\cdots a_{k_m\ell_m}\partial^m/\partial a_{i_1+k_1,j_1+\ell_1}\cdots\partial a_{i_m+k_m,j_m+\ell_m},$$

which extends by $\mathbf{Q}$-linearity (*symbolic multiplication* in the terminology of [5] and [8]); we also write

$$d_{ij}{}^{\#\pi}=d_{ij}\#\cdots\#d_{ij}\quad(\pi\text{ times}).$$

Note that #-product is commutative.

Proposition 5.8. (cf. [5, pp.493-494], [8, pp.303], [10,Proposition 1.1]) *We have*

$$\mathrm{Mah}\,([p_1q_1^{\pi_1}, p_2q_2^{\pi_2}, \ldots]) = d_{p_1q_1}^{\#\pi_1} \# d_{p_2q_2}^{\pi_2} \# \cdots$$

(*see* Definitions 2.1 (v) *and* 5.7).

Proof. We show by induction on k that $\mathrm{Mah}([p_1q_1, p_2q_2, \ldots, p_kq_k]) = d_{p_1q_1} \# \cdots \# d_{p_kq_k}$, where $(p_1q_1), \ldots, (p_kq_k)$ are allowed to have repetitions of binumbers. The case $k=1$ is obvious: $\mathrm{Mah}([pq]) = \mathrm{Mah}(s_{pq}) = d_{pq}$. Assume the case k. Since

$$\begin{aligned}&[p_1q_1, \ldots, p_{k+1}q_{k+1}]\\ &=[p_1q_1][p_2q_2, \ldots, p_{k+1}q_{k+1}] - \sum_{\ell=1}^{k}[p_2q_2, \ldots, p_\ell+p_1\ q_\ell+q_1, \ldots, p_{k+1}q_{k+1}]\end{aligned}$$

(cf. Proof of Proposition 2.2), applying Mah on both sides gives

$$\begin{aligned}&\mathrm{Mah}([p_1q_1, \ldots, p_{k+1}q_{k+1}])\\ &=\mathrm{Mah}([p_1q_1]) \circ \mathrm{Mah}([p_2q_2, \ldots, p_{k+1}q_{k+1}]) - \sum_{\ell=1}^{k}\mathrm{Mah}([p_2q_2, \ldots, p_\ell+p_1\ q_\ell+q_1, \ldots, p_{k+1}\ q_{k+1}])\\ &=d_{p_1q_1} \circ (d_{p_2q_2} \# \cdots \# d_{p_{k+1}q_{k+1}}) - \sum_{\ell=1}^{k} d_{p_2q_2} \# \cdots \# d_{p_\ell+p_1, q_\ell+q_1} \# \cdots \# d_{p_{k+1}q_{k+1}},\end{aligned}$$

where $\circ$ (which may be omitted) denotes composition of maps in $\mathrm{End}_{\mathbf{Q}}(\mathrm{Sym}(2, \mathbf{Q}))$. We see by direct computation that the last expression is equal to $d_{p_1q_1} \# \cdots \# d_{p_{k+1}q_{k+1}}$, completing the proof.

Since $\mathrm{Sym}(2, \mathbf{Q}) = \bigoplus \mathbf{Q}\cdot[p_1q_1, p_2q_2, \ldots]$ (direct sum over all bipartitions), from Proposition 5.8 we have:

Corollary 5.9. (cf. [10, Remark 1.2], [5, p.494], [8, p.303])

$$\mathrm{Mah}(2, \mathbf{Q}) = \bigoplus \mathbf{Q}\cdot d_{p_1q_1} \# d_{p_2q_2} \# \cdots$$

with direct sum over all bipartitions $(p_1q_1, p_2q_2, \ldots)$; *in particular,* d_{ij} $(i+j \geq 1)$ *are algebraically independent over* $\mathbf{Q}$ *with respect to* #*-product.*

Proposition 5.10. (cf. [10, Proposition 1.3]) *We have in* $\mathrm{End}_{\mathbf{Q}}(\mathrm{Mah}(2, \mathbf{Q}))$ *that*

(1) $$d_{ij} \circ = d_{ij} \# + \sum_{k+\ell \geq 1} d_{i+k, j+\ell} \# \partial^{\#}_{d_{k\ell}},$$

where $\circ$ *denotes the composition of maps and* $\partial^{\#}_{d_{k\ell}}$ *the derivation with respect to* $d_{k\ell}$ *acting on* #*-product expressions.*

Proof. By Corollary 5.9, the right-hand side of (1) is well-defined in $\mathrm{End}_{\mathbf{Q}}(\mathrm{Mah}(2, \mathbf{Q}))$. We compute:

$$\begin{aligned}&d_{ij} \circ (d_{p_1q_1}^{\#\pi_1} \# \cdots \# d_{p_kq_k}^{\#\pi_k})\\ &= d_{ij} \# d_{p_1q_1}^{\#\pi_1} \# \cdots \# d_{p_kq_k}^{\#\pi_k} + \pi_1 d_{i+p_1, j+q_1} \# d_{p_1q_1}^{\#\pi_1-1} \# \cdots \# d_{p_kq_k}^{\#\pi_k} + \cdots\\ &\quad + \pi_k d_{i+p_k, j+q_k} \# d_{p_1q_1}^{\#\pi_1} \# \cdots \# d_{p_kq_k}^{\#\pi_k-1}\\ &= (d_{ij} \# + \sum_{k+\ell \geq 1} d_{i+k, j+\ell} \# \partial^{\#}_{d_{k\ell}})(d_{p_1q_1}^{\#\pi_1} \# \cdots \# d_{p_kq_k}^{\#\pi_k});\end{aligned}$$

by Corollary 5.9 we obtain (1).

For a use of identity (1) of Proposition 5.10, see Remarks 6.7 (iv).

Proposition 5.11. (cf. [5, pp.489,493], [8, pp.298,302]) *We have*

$$\mathrm{Mah}(a_{ij}) = d_{10}{}^{\#i} \# d_{01}{}^{\#j}/i!j! \quad (i+j \geq 1).$$

Proof. It follows from Proposition 5.8 that

$$\mathrm{Mah}(a_{ij}) = \mathrm{Mah}((10^i, 01^j)) = \mathrm{Mah}([10^i, 01^j]/i!j!) = d_{10}{}^{\#i} \# d_{01}{}^{\#j}/i!j!.$$

Proposition 5.12. (cf. [5, p.492], [8, p.301]) *We have in* Mah $(2, \mathbf{Q})[[x, y]]$ *that*

$$\exp^{\#}((x\rho + y\sigma)(1)) = \exp((\log(1 + x\rho + y\sigma))(1)),$$

where $\exp^{\#}$ *denotes the exponential series with respect to* # *-product;* ρ *resp.* σ *denotes the linear operation* $\rho(d_{ij}) = d_{i+1,j}$ *resp.* $\sigma(d_{ij}) = d_{i,j+1}$ *with* $d_{00} = 1$ *(note that* $\rho\sigma = \sigma\rho$*), so that the above formula reads:*

$$\exp^{\#}(d_{10}x + d_{01}y) = \exp\left(\sum_{i \geq 1}((-1)^{i-1}/i) \sum_{j+k=i} \binom{i}{j} d_{jk}\, x^j y^k\right).$$

Proof. A simple generating series computation gives a generalization of Waring's formula:

$$\sum_{i,j \geq 0} a_{ij} x^j y^j = \exp\left(\sum_{i \geq 1}((-1)^{i-1}/i) \sum_{j+k=i} \binom{i}{j} s_{jk}\, x^j y^k\right)$$

(cf. [5, p.486], [8, pp.282-283]). Applying Mah (which naturally extends to Sym$(2, \mathbf{Q})[[x, y]]$) on both sides, we compute:

$$\begin{aligned}
&\exp^{\#}((x\rho + y\sigma)(1)) \\
&= \exp^{\#}(d_{10}x + d_{01}y) = \exp^{\#}(d_{10}x) \# \exp^{\#}(d_{01}y) = \sum_{i,j \geq 0} (d_{10}{}^{\#i}/i!) \# (d_{01}{}^{\#j}/j!) x^i y^j \\
&= \sum_{i,j \geq 0} \mathrm{Mah}(a_{ij}) x^i y^j \quad \text{(by Proposition 5.11)} \\
&= \exp\left(\sum_{i \geq 1}((-1)^{i-1}/i) \sum_{j+k=i} \binom{i}{j} d_{jk}\, x^j y^k\right) = \exp\left(\sum_{i \geq 1}((-1)^{i-1}/i)\ (x\rho + y\sigma)^i(1)\right) \\
&= \exp\left((\log(1 + x\rho + y\sigma))(1)\right).
\end{aligned}$$

Remarks 5.13. (i) As stated in [5, pp.488-489] and [8, pp.297-298], the expression $\exp^{\#}(d_{10}x + d_{01}y)$ in Proposition 5.12 acts as operator introducing a new indeterminate x into the alphabet $\overleftarrow{\alpha}$ and y into $\overleftarrow{\beta}$; see also Proposition 7.1.
(ii) Another proof of Proposition 5.12 will be given in Remarks 6.7 (iii).

6. MacMahon Operators and the Double-Indexed Bell Polynomials

We first give a general formula essentially observed by MacMahon [6, pp.246-248], [7, pp.30-31]:

Theorem 6.1. *Let* K *be a field of characteristic zero and let* $L := K[u_i \mid i \geq 1]$ *be the* K*-algebra generated by* u_i $(i \geq 1)$, *which are commutative but not necessarily algebraically independent. Assume that, besides the original multiplication denoted by juxtaposition as usual,* L *has another* K*-linear commutative multiplication denoted by* \$ *satisfying for* $i \geq 1$:

(1) $$u_1(u_{t_1} \$ \cdots \$ u_{t_i}) = u_1 \$ u_{t_1} \$ \cdots \$ u_{t_i} + \sum_{j=1}^{i} u_{t_1} \$ \cdots \$ u_{1+t_j} \$ \cdots \$ u_{t_i}.$$

Then we have in $L[[c]]$ *with* c *being an indeterminate that:*

(2) $$\exp(u_1 c) = \exp^{\$}\left(\sum_{i \geq 1} u_i c^i/i!\right)$$

where $\exp^{\$}$ *denotes the exponential series with respect to* \$ *-multiplication.*

Proof. We first note that identity (1) can be interpreted as follows. To $u_{t_1}\$\cdots\$u_{t_i}$ corresponds a partition of a set with $t_1+\cdots+t_i$ objects into i nonempty subsets each containing $t_1, \ldots, t_i$ objects respectively. The left-hand side of (1) signifies addition of another object, and the right-hand side of (1) expresses all the possible set-partitions after the addition of the new object.

With this interpretation in mind, we see by an inductive argument that, for $k\geq 1$,

$$(3)\qquad u_1{}^k=\Sigma(k!/a!^{\ell}b!^m\cdots\ell!m!\cdots)u_a{}^{\$\ell}\$u_b{}^{\$m}\$\cdots$$

where $u_a{}^{\$\ell}=u_a\$\cdots\$u_a$($\ell$ times), and the summation is over all partitions $(a^{\ell}b^m\cdots)$ of k, since $k!/a!^{\ell}b!^m\cdots\ell!m!\cdots$ is the number of partitions of a set with k objects into ℓ a-subsets, m b-subsets, ...

The formula (2) follows from (3) by computation:

$$\begin{aligned}\exp(u_1c) &= \sum_{k\geq 0}u_1{}^kc^k/k!\\ &=\Sigma(u_ac^a/a!)^{\$\ell}(u_bc^b/b!)^{\$m}\cdots/\ell!m!\cdots \text{ (summation over all partitions } (a^{\ell}b^m\cdots))\\ &=\sum_{j\geq 0}(1/j!)\Sigma(j!/\ell!m!\cdots)(u_ac^a/a!)^{\$\ell}(u_bc^b/b!)^{\$m}\cdots\\ &\qquad\text{(inner summation over all partitions } (a^{\ell}b^m\cdots) \text{ with } \ell+m+\cdots=j)\\ &=\sum_{j\geq 0}(1/j!)(\sum_{i\geq 1}u_ic^i/i!)^{\$j}\\ &=\exp^{\$}(\sum_{i\geq 1}u_ic^i/i!).\end{aligned}$$

Before proceeding, we set some notation to be used. For K a field of characteristic zero, we denote by Sym $(2, K)$ the symmetric function algebra in two alphabets with coefficients in K:

$$\mathrm{Sym}(2, K)=K[s_{ij}\mid i+j\geq 1].$$

The definitions and results in Sections 2 and 3 are all valid when K replaces $\mathbf{Q}$. We see that the operators

$$\sum_{k,\ell\geq 0}a_{k\ell}\,\partial_{a_{i+k,j+\ell}}\quad(i+j\geq 1)$$

belong to $\mathrm{End}_K(\mathrm{Sym}(2, K))$, and we denote them also by d_{ij} as in the case of $\mathbf{Q}$ (see Proposition 5.5). Note that, since $\mathrm{Sym}(2, K)=K[a_{ij}\mid i+j\geq 1]$, infinite linear combinations of the form

$$\sum_{i+j\geq 1}k_{ij}d_{ij}\quad(k_{ij}\in K)$$

belong to $\mathrm{End}_K(\mathrm{Sym}(2, K))$.

As an application of Theorem 6.1, we next show an exponential formula for the operators of the form $\sum_{i+j\geq 1}k_{ij}d_{ij}$ $(k_{ij}\in K)$, which was also essentially observed by MacMahon [5, pp.490-491], [8, pp.299-300]:

Theorem 6.2. *Let K and c be as in* Theorem 6.1, *let $\dot{\rho}$ and σ be as in* Proposition 5.12, *and put*

$$\varphi:=\sum_{i+j\geq 1}k_{ij}\,\rho^i\,\sigma^j\ (k_{ij}\in K),$$

so that

$$\varphi^{\ell}(1)\in\mathrm{End}_K(\mathrm{Sym}(2, K))\quad(\ell\geq 1).$$

Then we have in $\mathrm{End}_K(\mathrm{Sym}(2, K))[[c]]$ *that*

$$\exp(c\,\varphi(1))=\exp^{\#}((\exp(c\,\varphi)-1)(1))$$

(*for* $\exp^{\#}$ *see* Proposition 5.12 *and* Definition 5.7).

Proof. Putting $u_{\ell}:=\varphi^{\ell}(1)$ $(\ell\geq 1)$, we see that they are commutative with respect to composition of maps in $\mathrm{End}_K(\mathrm{Sym}(2, K))$, and that they satisfy (1) in Theorem 6.1 with \$ replaced by # by virtue of the definition of d_{ij}. We can apply Theorem 6.1 to obtain:

$$\exp(c\,\varphi(1))=\exp^{\#}(\sum_{i\geq 1} c^i\,\varphi^i(1)/i!)=\exp^{\#}((\exp(c\,\varphi)-1)(1)).$$

As a particular case of Theorem 6.2, we show:

Theorem 6.3. (cf. [10, Proposition 2.1]) *We have in* $\mathrm{Mah}(2, \mathbf{Q})[[x, y]]$ *that*

$$\exp((x\rho+y\sigma)(1))=\exp^{\#}((\exp(x\rho+y\sigma)-1)(1)),$$

where ρ *and* σ *are the same as those in* Proposition 5.12 ; *the formula reads:*

$$\exp(d_{10}x+d_{01}y)=\exp^{\#}(\sum_{i+j\geq 1} d_{ij}\,x^iy^j/i!j!).$$

Proof. We take K as the fractional field of $\mathbf{Q}[[x, y]]$, put $\varphi:=x\rho+y\sigma$, and apply Theorem 6.2 to have:

$$\exp(c(x\rho+y\sigma)(1))=\exp^{\#}((\exp(c(x\rho+y\sigma))-1)(1)).$$

This identity can specialize at $c:=1$ and gives the desired formula, both sides of which belong to $\mathrm{Mah}(2, \mathbf{Q})[[x, y]]$.

Corollary 6.4. (cf. [10, Corollary 2.2]) *We have*

(i) $$d_{10}{}^i d_{01}{}^j/i!j!=\sum d_{p_1q_1}{}^{\#\pi_1}\# d_{p_2q_2}{}^{\#\pi_2}\#\cdots/\pi_1!\,\pi_2!\cdots p_1!^{\pi_1}p_2!^{\pi_2}\cdots q_1!^{\pi_1}\,q_2!^{\pi_2}\cdots,$$

(ii) $$s_{10}{}^i s_{01}{}^j/i!j!=\sum[p_1q_1{}^{\pi_1}, p_2q_2{}^{\pi_2}, \ldots]/\pi_1!\,\pi_2!\cdots p_1!^{\pi_1}p_2!^{\pi_2}\cdots q_1!^{\pi_1}\,q_2!^{\pi_2}\cdots,$$

where the summations range over all bipartitions $(p_1q_1{}^{\pi_1}, p_2q_2{}^{\pi_2}, \ldots)$ *of* (ij).

Proof. Expanding both sides of the identity of Theorem 6.3 as powerseries in x and y, and picking out the coefficients of x^iy^j, we obtain (i). Applying Mah^{-1}: $\mathrm{Mah}(2, \mathbf{Q})\longrightarrow \mathrm{Sym}(2, \mathbf{Q})$ to (i) gives (ii) immediately.

Remark 6.5. (cf. [10, Remark 2.3], [1, pp.89-90], [9, pp.105-106]) Defining the *double-indexed Bell polynomials* $Y_{ij}=Y_{ij}(g_{k\ell}\mid k+\ell\geq 1)$ of indeterminates $g_{k\ell}$ $(k+\ell\geq 1)$ by

(1) $$1+\sum_{i+j\geq 1} Y_{ij}x^iy^j/i!j! := \exp(\sum_{i+j\geq 1} g_{ij}x^iy^j/i!j!),$$

we see that

$$1+\sum_{i+j\geq 1} Y^{\#}{}_{ij}x^iy^j/i!j! := \exp^{\#}(\sum_{i+j\geq 1} d_{ij}\,x^iy^j/i!j!)=\exp(d_{10}x+d_{01}y),$$

where $Y^{\#}{}_{ij}$ denotes the Y_{ij} in terms of $d_{k\ell}$ $(k+\ell\geq 1)$ with respect to #-product, and the last equality follows from Theorem 6.3. Thus we have

$$Y^{\#}{}_{ij}=d_{10}{}^i d_{01}{}^j;$$

see Corollary 6.4 (i). Note that, as in the case of the usual Bell polynomials, the double-indexed Bell polynomials appear in the differential formulas for composite functions. For example, we have

(2) $$(\partial_t{}^i\partial_u{}^j)\exp(g(t, u))=\exp(g(t, u))\cdot Y_{ij}(g^{(k,\ell)}(t, u)\mid k+\ell\geq 1)$$

with $\partial_t=\partial/\partial t$, $\partial_u=\partial/\partial u$, and $g^{(k,\ell)}(t, u)=\partial_t{}^k\partial_u{}^\ell g(t, u)$; to see this, we compute:

$$\begin{aligned}
&(\sum_{i,j\ge 0} x^i y^j \partial_t{}^i \partial_u{}^j / i!j!)\exp(g(t, u)) = \exp(x\partial_t + y\partial_u)(\exp(g(t, u)))\\
&= \exp(\exp(x\partial_t + y\partial_u)g(t, u)) = \exp((1 + \sum_{i+j\ge 0} x^i y^j \partial_t{}^i \partial_u{}^j / i!j!)g(t, u))\\
&= \exp(g(t, u))\exp(\sum_{i+j\ge 1} x^i y^j g^{(i,j)}(t, u)/i!j!)\\
&= \exp(g(t, u))(1 + \sum_{i+j\ge 1} Y_{ij}(g^{(k,\ell)}(t, u) \mid k + \ell \ge 1)x^i y^j / i!j!),
\end{aligned}$$

where $\exp(x\partial_t + y\partial_u)$ is the Maclaurin expansion map acting as algebra morphism, and the last equality follows from (1). Taking out the coefficients of $x^i y^j$ on the leftmost and rightmost sides gives (2).

As another particular case of Theorem 6.2, we give a generalization of Theorem 6.3:

Proposition 6.6. (cf. [10, Proposition 2.4]) *We have in* Mah(2, $\mathbf{Q}$)[[x, y]] *that*

$$\exp((x\rho^p\sigma^q + y\rho^r\sigma^s)(1)) = \exp^{\#}((\exp(x\rho^p\sigma^q + y\rho^r\sigma^s) - 1)(1)),$$

where ρ *and* σ *are the same as those in* Proposition 5.12; *the formula reads*:

$$\exp(d_{pq}x + d_{rs}y) = \exp^{\#}(\sum_{i+j\ge 1} d_{pi+rj,qi+sj} x^i y^j / i!j!)$$

$(p + q \ge 1,\ r + s \ge 1)$. *Note that with* $p = s = 1$ *and* $q = r = 0$ *we recover* Theorem 6.3.

Proof. Same as that of Theorem 6.3 except that we put here $\varphi := x\rho^p\sigma^q + y\rho^r\sigma^s$.

Remarks 6.7. (i) (cf. [10, Corollary 2.5]) By Proposition 6.6 and Remark 6.5, we have:

$$\exp(d_{pq}x + d_{rs}y) = 1 + \sum_{i+j\ge 1} Y^{\#}{}_{ij}(d_{pk+r\ell,qk+s\ell} \mid k + \ell \ge 1)x^i y^j / i!j!,$$

so that

$$d_{pq}{}^i d_{rs}{}^j = Y^{\#}{}_{ij}(d_{pk+r\ell,qk+s\ell} \mid k + \ell \ge 1) \quad (i + j \ge 1),$$

i.e., Corollary 6.4 (i) with $d_{k\ell}$ replaced by $d_{pk+r\ell,qk+s\ell}$. Applying Mah^{-1}: Mah(2, $\mathbf{Q}$) $\longrightarrow$ Sym(2, $\mathbf{Q}$) on both sides, we obtain an expression for $s_{pq}{}^i s_{rs}{}^j$ as linear combination of augmented monomial functions, i.e., Corollary 6.4 (ii) with necessary modifications.

(ii) The formulas of Theorems 6.1, 6.2, and 6.3 have a certain resemblance to the generating series of the Bell numbers B_i:

$$\sum_{i\ge 0} B_i t^i / i! = \exp(e^t - 1)$$

(see books on combinatorics for details of B_i).

(iii) (cf. [5, pp.490, 492], [8, pp.299, 301]) Proposition 5.12 can be obtained from Theorem 6.2; the proof is the same as that of Theorem 6.3 except that we put in this case $\varphi := \log(1 + x\rho + y\sigma)$.

(iv) We can prove Theorem 6.3 and Proposition 6.6 in a straightforward manner using Proposition 5.10. As Theorem 6.3 is a particular case of Proposition 6.6, we give here the proof of Proposition 6.6 using Proposition 5.10 (cf. Proof of [10, Proposition 2.4]). Consider the system of partial differential equations in Mah(2, $\mathbf{Q}$)[[x, y]]:

$$\partial_x(f) = d_{pq}f,\ \partial_y(f) = d_{rs}f$$

with $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$, and

$$f = 1 + \sum_{i+j\ge 1} f_{ij} x^i y^j,\ f_{ij} \in \mathrm{Mah}(2, \mathbf{Q});$$

we easily see that the unique solution is

$f = \exp(d_{pq}x + d_{rs}y)$.

Putting $g = \exp^{\#}(\sum_{i+j\geq 1} d_{pi+rj,qi+sj}x^i y^j/i!j!)$, we compute:

$$\partial_x(g) = g\#(\sum_{\substack{i+j\geq 1,\\ i\geq 1}} d_{pi+rj,qi+sj}x^{i-1}y^j/(i-1)!j!) = d_{pq}\#g + g\#(\sum_{i+j\geq 1} d_{p+pi+rj,q+qi+sj}x^i y^j/i!j!);$$

note that $pi + rj + qi + sj = (p+q)i + (r+s)j \geq i+j \geq 1$. Besides we have:

$$\begin{aligned}
&\sum_{k+\ell\geq 1} d_{p+k,q+\ell}\#\partial^{\#}_{d_{k\ell}}(g)\\
&= \sum_{k+\ell\geq 1} d_{p+k,q+\ell}\#g\#(\sum_{i+j\geq 1}\partial^{\#}_{d_{k\ell}}(d_{pi+rj,qi+sj})x^i y^j/i!j!)\\
&= g\#(\sum_{\substack{i+j\geq 1,\\ k+\ell\geq 1}} d_{p+k,q+\ell}\#\partial^{\#}_{d_{k\ell}}(d_{pi+rj,qi+sj})\,x^i y^j/i!j!)\\
&= g\#(\sum_{i+j\geq 1} d_{p+pi+rj,q+qi+sj}x^i y^j/i!j!).
\end{aligned}$$

Hence:

$$\begin{aligned}
\partial_x(g) &= d_{pq}\#g + \sum_{k+\ell\geq 1} d_{p+k,q+\ell}\,\#\partial^{\#}_{d_{k\ell}}(g)\\
&= (d_{pq}\# + \sum_{k+\ell\geq 1} d_{p+k,q+\ell}\,\#\partial^{\#}_{d_{k\ell}})\ (g)\\
&= d_{pq}g \ \ \text{(by Proposition 5.10)};
\end{aligned}$$

similarly we have $\partial_y(g) = d_{rs}g$. Since the constant term of g is 1, we obtain $f = g$.

In fact, identity (1) of Proposition 5.10 is considered as an expression of (1) of Theorem 6.1 in the case of $\mathrm{End}_{\mathbf{Q}}(\mathrm{Mah}(2, \mathbf{Q}))$.

7. Hopf Algebra Structure

We put

$$\begin{aligned}
E(x, y) &:= \exp^{\#}(d_{10}x + d_{01}y) = \exp((\log(1 + x\rho + y\sigma))(1))\\
&= \exp(\sum_{i\geq 1}((-1)^{i-1}/i)\sum_{j+k=i}\binom{i}{j} d_{jk}x^j y^k)
\end{aligned}$$

(see Proposition 5.12), and recall that

$$A(t, u) = \sum_{i,j\geq 0} a_{ij}t^i u^j = \prod_{i\geq 1}(1 + \alpha_i t + \beta_i u)$$

(see the proof of Proposition 5.1).

Proposition 7.1. (cf. Remarks 5.13 (i)) *For indeterminates $t, u, x_1, \ldots, x_n, y_1, \ldots, y_n$, we have*

(1) $\quad \prod_{i=1}^{n} E(x_i, y_i)(A(t, u)) = (\prod_{i=1}^{n}(1 + x_i t + y_i u))A(t, u)$

(note that $E(x_i, y_i)$ are mutually commutative).

Proof. First observe that

$$E(x, y)(a_{ij}) = \sum_{k\geq 1}((d_{10}x + d_{01}y)^{\#k}/k!)(a_{ij}) = a_{ij} + x\,a_{i-1,j} + y\,a_{i,j-1}$$

with the convention that $a_{k\ell} = 0$ ($k < 0$ or $\ell < 0$); see Definition 5.7. Hence we see that

$$E(x, y)(A(t, u)) = \sum_{i,j\geq 0} E(x, y)(a_{ij})t^i u^j = (1 + xt + yu)A(t, u)$$

Induction on n gives (1).

Proposition 7.2. (cf. [3, pp.21-23]) *We have*

$$E(\overleftarrow{x}, \overleftarrow{y})(a_{ij}(\overleftarrow{\alpha}, \overleftarrow{\beta})) = \sum_{\substack{k+p=i,\\ \ell+q=j}} a_{k\ell}(\overleftarrow{x}, \overleftarrow{y}) a_{pq}(\overleftarrow{\alpha}, \overleftarrow{\beta}) \quad (i, j \geq 0),$$

where $E(\overleftarrow{x}, \overleftarrow{y}) := \prod_{i \geq 1} E(x_i, y_i)$.

Proof. Taking the limit $n \to \infty$ on both sides of (1) of Proposition 7.1 gives

$$E(\overleftarrow{x}, \overleftarrow{y})(A(t, u)) = \widetilde{A}(t, u) A(t, u)$$

with $\widetilde{A}(t, u) = \sum_{i,j \geq 0} a_{ij}(\overleftarrow{x}, \overleftarrow{y}) t^i u^j = \prod_{i \geq 1} (1 + x_i t + y_i u)$. Taking out the coefficients of $t^i u^j$ from both sides of the above identity, we obtain the result.

Remarks 7.3. (i) For disjoint unions of alphabets $\overleftarrow{x} \cup \overleftarrow{\alpha}$, $\overleftarrow{y} \cup \overleftarrow{\beta}$, we see that

$$a_{ij}(\overleftarrow{x} \cup \overleftarrow{\alpha}, \overleftarrow{y} \cup \overleftarrow{\beta}) = \sum_{\substack{k+p=i,\\ \ell+q=j}} a_{k\ell}(\overleftarrow{x}, \overleftarrow{y}) a_{pq}(\overleftarrow{\alpha}, \overleftarrow{\beta}),$$

since

$$\sum_{i,j \geq 0} a_{ij}(\overleftarrow{x} \cup \overleftarrow{\alpha}, \overleftarrow{y} \cup \overleftarrow{\beta}) t^i u^j = \widetilde{A}(t, u) A(t, u)$$

(see the proof of Proposition 7.2). Hence we can restate Proposition 7.2 as

$$E(\overleftarrow{x}, \overleftarrow{y})(a_{ij}(\overleftarrow{\alpha}, \overleftarrow{\beta})) = a_{ij}(\overleftarrow{x} \cup \overleftarrow{\alpha}, \overleftarrow{y} \cup \overleftarrow{\beta}) \quad (i, j \geq 0).$$

(ii) Since $\mathrm{Sym}(2, \mathbf{Q}) = \mathbf{Q}[a_{ij} \mid i+j \geq 1]$ with a_{ij} $(i+j \geq 1)$ forming a $\mathbf{Q}$-algebra basis, we have the natural identification:

$$\begin{array}{ccc} \mathrm{Sym}(2, \mathbf{Q}) \otimes \mathrm{Sym}(2, \mathbf{Q}) & \cong & \mathbf{Q}[a_{ij}(\overleftarrow{x}, \overleftarrow{y}), a_{ij}(\overleftarrow{\alpha}, \overleftarrow{\beta}) \mid \mathrm{i}+\mathrm{j} \geq 1] \\ a_{ij} \otimes a_{k\ell} & \longleftrightarrow & a_{ij}(\overleftarrow{x}, \overleftarrow{y}) a_{k\ell}(\overleftarrow{\alpha}, \overleftarrow{\beta}), \end{array}$$

where $\otimes$ denotes the tensor product over $\mathbf{Q}$.

(iii) We have

$$\begin{aligned} E(\overleftarrow{x}, \overleftarrow{y}) &= \prod_{i \geq 1} \exp\Big(\sum_{j \geq 1} ((-1)^{j-1}/j) \sum_{k+\ell=j} \binom{j}{k} d_{k\ell}\, x_i^k y_i^\ell\Big) \\ &= \exp\Big(\sum_{j \geq 1} ((-1)^{j-1}/j) \sum_{k+\ell=j} \binom{j}{k} s_{k\ell}(\overleftarrow{x}, \overleftarrow{y})\, d_{k\ell}\Big) \end{aligned}$$

(see the first paragraph of this section), so that $E(\overleftarrow{x}, \overleftarrow{y})$ is an exponential of a derivation; hence it gives a $\mathbf{Q}$-algebra morphism from $\mathrm{Sym}(2, \mathbf{Q})$ into $\mathrm{Sym}(2, \mathbf{Q}) \otimes \mathrm{Sym}(2, \mathbf{Q})$ through the natural identification of (ii). We denote this simply by:

$$\begin{array}{rccc} E: & \mathrm{Sym}(2, \mathbf{Q}) & \longrightarrow & \mathrm{Sym}(2, \mathbf{Q}) \otimes \mathrm{Sym}(2, \mathbf{Q}) \\ & a_{ij} & \longmapsto & \sum_{\substack{k+p=i,\\ \ell+q=j}} a_{k\ell} \otimes a_{pq}. \end{array}$$

Lemma 7.4. (cf. [2, pp.171-172], [3, pp.27-29]) *Let E be as defined in* Remarks 7.3 (iii), *and put*:

$$\begin{aligned} &\varepsilon : \mathrm{Sym}(2, \mathbf{Q}) \longrightarrow \mathbf{Q}, \ \varepsilon(a_{ij}) := 0 \ (i+j \geq 1), \ \varepsilon(1) := 1; \\ &\gamma : \mathrm{Sym}(2, \mathbf{Q}) \longrightarrow \mathrm{Sym}(2, \mathbf{Q}), \ \gamma(a_{ij}) := (-1)^{i+j} h_{ij} \ (i, j \geq 0), \end{aligned}$$

where ε *and* γ *are required to be* $\mathbf{Q}$*-algebra morphisms. Then we have*:

$$(E \otimes \mathrm{id}) \circ E = (\mathrm{id} \otimes E) \circ E, \ (\varepsilon \otimes \mathrm{id}) \circ E = \mathrm{id},$$

with the identification $\mathbf{Q} \otimes \mathrm{Sym}(2, \mathbf{Q}) \cong \mathrm{Sym}(2, \mathbf{Q})$, *and*

$$\sum_{\substack{k+p=i,\\ \ell+q=j}} \gamma(a_{k\ell})a_{pq} = \varepsilon(a_{ij}) \quad (i, j \geq 0)$$

with the natural injection $\mathbf{Q} \subset \mathrm{Sym}(2, \mathbf{Q})$.

Proof. By straightforward computations.

Definition 7.5. (cf. [2, pp.171-172], [3, pp.27-29]) By virtue of Lemma 7.4 we can define the Hopf algebra $(\mathrm{Sym}(2, \mathbf{Q}); E, \varepsilon, \gamma)$ with comultiplication E, counit ε, and antipode γ; it is commutative and cocommutative.

Remark 7.6. Defining on the polynomial algebra $\mathbf{Q}[X, Y]$ the comultiplication $\Delta : \mathbf{Q}[X, Y] \longrightarrow \mathbf{Q}[X, Y] \otimes \mathbf{Q}[X, Y]$ by $\Delta(X^i Y^j) := (X \otimes 1 + 1 \otimes X)^i (Y \otimes 1 + 1 \otimes Y)^j$, and putting $W_{ij} := X^i Y^j / i!j!$, we have

$$\Delta(W_{ij}) = \sum_{\substack{k+p=i,\\ \ell+q=j}} W_{k\ell} \otimes W_{pq} \quad (i, j \geq 0).$$

Such a double-indexed sequence as W_{ij} $(i, j \geq 0)$ in a coalgebra might be called a sequence of *double-divided powers* (cf. [2, p.170]). Using this term we could say that a_{ij} $(i, j \geq 0)$ form a sequence of double-divided powers in $(\mathrm{Sym}(2, \mathbf{Q}); E, \varepsilon, \gamma)$; see Remarks 7.3 (iii).

Note that, putting $x := X \otimes 1$, $x_1 := 1 \otimes X$, $y := Y \otimes 1$, $y_1 := 1 \otimes Y$, we can write as

$$\Delta = \exp(x_1 \partial/\partial x + y_1 \partial/\partial y),$$

which is the Maclaurin expansion map in two indeterminates.

Proposition 7.7. (cf. [2; pp.170, 172], [3; pp.21, 23]) *We have*
(i) *for a monomial function* λ,

$$E(\lambda) = \sum \mu \otimes \nu$$

with summation over all pairs of bipartitions (μ, ν) *such that* $\mu \cup \nu = \lambda$, *where* $\mu \cup \nu$ *denotes the bipartition whose biparts are those of* μ *and* ν *(see also the first paragraph of* Section 4);
(ii) *for a powersum* s_{ij} $(i + j \geq 1)$,

$$E(s_{ij}) = s_{ij} \otimes 1 + 1 \otimes s_{ij},$$

so that s_{ij} $(i + j \geq 1)$ *are primitive elements in* $(\mathrm{Sym}(2, \mathbf{Q}); E, \varepsilon, \gamma)$.

Proof. For any $f(\overleftarrow{\alpha}, \overleftarrow{\beta}) \in \mathrm{Sym}(2, \mathbf{Q}) = \mathbf{Q}[a_{ij} \mid i + j \geq 1]$, we have by Remarks 7.3 (i) and (iii) that

$$E(\overleftarrow{x}, \overleftarrow{y})(f(\overleftarrow{\alpha}, \overleftarrow{\beta})) = f(\overleftarrow{x} \cup \overleftarrow{\alpha}, \overleftarrow{y} \cup \overleftarrow{\beta}).$$

Hence we see by Definitions 2.1 (iii) and (iv) that

$$E(\overleftarrow{x}, \overleftarrow{y})(\lambda(\overleftarrow{\alpha}, \overleftarrow{\beta})) = \lambda(\overleftarrow{x} \cup \overleftarrow{\alpha}, \overleftarrow{y} \cup \overleftarrow{\beta}) = \sum \mu(\overleftarrow{x}, \overleftarrow{y})\, \nu(\overleftarrow{\alpha}, \overleftarrow{\beta})$$

with summation over all pairs of bipartitions (μ, ν) such that $\mu \cup \nu = \lambda$. By the natural identification (see Remarks 7.3 (ii) and (iii)) we obtain (i). In particular, if λ is a bipartition (ij) having one bipart, then the summation reduces to be over the two pairs $((ij), \phi)$ and $(\phi, (ij))$, so that

$$E(\overleftarrow{x}, \overleftarrow{y})(s_{ij}(\overleftarrow{\alpha}, \overleftarrow{\beta})) = s_{ij}(\overleftarrow{x} \cup \overleftarrow{\alpha}, \overleftarrow{y} \cup \overleftarrow{\beta}) = s_{ij}(\overleftarrow{x}, \overleftarrow{y}) + s_{ij}(\overleftarrow{\alpha}, \overleftarrow{\beta}),$$

since $s_{\phi} = 1$. We obtain (ii) through the natural identification.

Remark 7.8. Proposition 5.4 (d_{ij} are derivations) can be related to Proposition 7.7 (ii) (s_{ij} are primitive elements); see [2, p.174].

References

[1] D. Fujiwara, Generalized Bell polynomials, *Sûgaku* 42 (1990), 89-90 (in Japanese).

[2] L. Geissinger, Hopf algebras of symmetric functions and class functions, *Springer LNM* 579 (1977), 168-181.

[3] D. E. Loeb, Sequences of symmetric functions of binomial type, *Stud. Appl. Math.* 83 (1990), 1-30.

[4] I. G. Macdonald, "Symmetric Functions and Hall Polynomials," Clarendon, Oxford, 1979.

[5] P. A. MacMahon, Memoir on symmetric functions of the roots of systems of equations, *Phil. Trans.* 181 (1890), 481-536 (in "Collected Papers II," MIT Press, 1986, pp.32-87).

[6] P. A. MacMahon, A theorem in the calculus of linear partial differential operations, *Quart. J. Math.* 24 (1890), 246-250 (in "Collected Papers II," MIT Press, 1986, pp.673-677).

[7] P. A. MacMahon, "Combinatory Analysis I (reprint with II in one volume)," Chelsea, New York, 1960 (original 1917).

[8] P. A. MacMahon, "Combinatory Analysis II (reprint with I in one volume)," Chelsea, New York, 1960 (original 1918).

[9] S. Mase, Surname problem — Birthday problem with unequal occurrence probabilities, *RIMS (Kyôto Univ.) Kôkyûroku* 777 (1992), 103-125 (in Japanese).

[10] Ka. Ueno, The Hammond operator algebra and the Bell polynomials, NIT Seminar Report on Mathematics No.66, 1991.

[11] Ka. Ueno, MacMahon symmetric function operators, in "Proceedings of the 16th Symposium on Semigroups, Languages, and their Related Fields, " Tôhô University, 1993.

[12] J. G. van der Corput, Sur les fonctions symétriques, *Indag. math.* 12 (1950), 216-230.