TRAJECTORIES FOR SASAKIAN MAGNETIC FIELDS

ON

HOMOGENEOUS REAL HYPERSURFACES

IN COMPLEX SPACE FORMS

2012

TUYA BAO

複素空間形内の等質実超曲面上の佐々木磁場による軌道

2012年

包　図雅

Contents

1. **Introduction**

It is needless to say that study of geodesics is one of important subjects in Riemannian geometry. Behaviors of geodesics are closely related with geometric and topological properties of base Riemannian manifolds. Global study on Riemannian manifolds by observing geodesics was started by Cohn-Vossen, H. Hopf, S.B. Myers and H.E. Rauch, and developed by M. Berger, W. Klingenberg, J. Cheeger, D. Gromoll, K. Shiohama, T. Sakai and some other geometers in 20 centuries. The reason why geodesics play an important role in the study of Riemannian geometry is that not only they have intuitive profile like the elementary Euclidean geometry but also they induce dynamical systems, which are called geodesic flows on unit tangent bundles. Particularly, for compact manifolds of negative sectional curvature their geodesic flows are of Anosov type (hyperbolic, in another word). Their ergodicity was studied by G.D. Birkhoff, M. Morse, E. Hopf, D.V. Anosov, A. Katok and some others.

We slightly change our viewpoint: If we consider families of curves containing geodesics, is it possible to get more information on base manifolds? We may say such study has been done in submanifold theory. In order to characterize isometric immersions, K. Sakamoto and J.S. Pak studied the behavior of geodesics through them, and S. Maeda studied the behavior of circles through them. Such study works because the existence of isometric immersions gives restrictions on Riemannian submanifolds. This suggests us that in order to go into our problem we need some restrictions on Riemannian manifolds. We hence consider Riemannian manifolds with some additional geometric structures, which are Kähler manifolds, contact manifolds and so on. Being furnished with geometric structures should give restrictions on base manifolds. Our problem then turns as follows: If we consider a family of curves associated with geometric structures, is it possible to investigate their properties form curve theoretic points of view? Since we consider Riemannian manifolds, this family should contain geodesics. Recalling the study of geodesics, we hope curves of this family are obtained by calculus of variations of some functional and induce a dynamical system.

Along such a consideration, the author's supervisor Adachi[1] introduced the notion of Kähler magnetic fields in order to study Kähler manifolds from Riemannian geometric point of view. As a generalization of static magnetic fields on a Euclidean 3-space, we say a closed 2-form on a manifold to be a magnetic field (see [26, 44], for example). He investigated motions of electric charged particles with unit speed under Kähler magnetic fields, and gave some results corresponding to classical results on geodesics; hyperbolicity of magnetic flows for complex hyperbolic spaces ([1]), comparison theorems on magnetic Jacobi fields ([2, 7]), theorems of Hopf-Rinow type and Hadamard-Cartan type ([8]), and so on.

Since Kähler manifolds are real even dimensional, we are interested in such an investigation on real odd dimensional manifolds. As a candidate we have a real hypersurface in a Kähler manifold. On real hypersurfaces in Kähler manifolds, we have almost contact metric structures induced by complex structures on Kähler manifolds. By the same way as for Kähler magnetic fields, we can define magnetic fields on real hypersurfaces which are associated with almost contact metric structures (see §3). We call them Sasakian magnetic fields. For study on magnetic fields on odd dimensional manifolds, Ikawa[32] chooses the class of homogeneous almost α-Sasakian manifolds and makes a trailblazing study on magnetic fields induced by their contact metric structures. Unfortunately, he does not make any mention on motions of electric charged particles on model spaces except for odd dimensional standard spheres.

Though definitions of Kähler and Sasakian magnetic fields are quite resemble and almost contact metric structures are induced by ambient complex structures, Kähler and Sasakian magnetic fields have many different properties. The force of a Kähler magnetic field is uniform, that is, it does not depend on the choice of places and directions of velocity vectors of charged particles. On the contrary, Sasakian magnetic fields are not uniform (see §6). This difference makes our treatment difficult but enrich our study on Sasakian magnetic fields. Trajectories, which are motions of electric unit charged particles of unit mass with unit speed, for Kähler magnetic fields are always circles, but not for Sasakian magnetic fields. Since circles are simplest curves next to geodesics in the sense of Frenet-Serre formula, we come to consider the following problems:

- Are there trajectories for Sasakian magnetic fields which are also circles on a real hypersurfaces?
- If exists, how many trajectories are also circles?
- Study properties of such trajectories.

In this paper, we take homogeneous Hopf hypersurfaces in nonflat complex space forms, especially take real hypersurfaces of type (A), and investigate some properties of motions of electric charged particles under Sasakian magnetic fields. The reason why we consider such hypersurfaces is that Sasakian space forms are represented as a odd dimensional standard sphere of radius 1 and real hypersurfaces of type (A_1), which are homogeneous Hopf hypersurfaces having two principal curvatures, in a nonflat complex space forms (see [21, 11]).

We here describe the organization and contents of this paper. There are 18 sections followed by this section. We devote some sections to explain some results and notations which will be used in the following sections. After brief summarization on some basic results in Riemannian geometry in section 2, we give a classification of smooth curves in the sense of Frenet-Serre in section 4, and introduce homogeneous Hopf hypersurfaces which have constant principal curvatures in section 5.

Such hypersurfaces are classified by Takagi[46, 47] and by Berndt[20]. In a complex projective space, they are classified into 5 classes; hypersurfaces of types (A), (B), (C), (D) and (E). Those hypersurfaces of types (C), (D) and (E) are so-called exceptional type. In a complex hyperbolic space, the are classified into 2 classes; hypersurfaces of types (A) and (B). Hypersurfaces of types (A) and (B) have at most 3 distinct principal curvatures. In this sense they are quite fundamental objects in submanifold theory.

In section 3, we define Sasakian magnetic fields by comparing the definition of Kähler magnteic fields. In section 6, we show that structure torsions of trajectories for Sasakian magnetic fields are important invariants. Structure torsions measure angles between characteristic vector fields of hypersurfaces and velocity vectors of trajectories. If a trajectory is a circle, then its structure torsion should be constant. In sections 7, 8, 9, 11, 12, 13 and 14, we restrict ourselves to real hypersurfaces of type (A) in nonflat complex space forms. In section 7, we give a condition that a trajectory to be a circle by the strength of a magnetic field, its structure torsion and its principal torsion on a real hypersurface in a complex projective space $\mathbb{C}P^n$.

In order to get more detail on circular trajectories on geodesic spheres, which are typical examples of real hypersurfaces of type (A) and are called real hypersurfaces of type (A_1), we investigate their extrinsic shapes in $\mathbb{C}P^n$ in section 8. In section 9, we take their horizontal lifts with respect to a Hopf fibration. If we regard these horizontal lifts as curves in a complex Euclidean space, we find that on geodesic spheres circular trajectories satisfy linear ordinary differential equations of order 3. Since it is known that circles on a complex projective space also satisfy linear ordinary differential equations of order 3, by comparing characteristic equations for these differential equations, we can get an algebraic information for them. As circles on $\mathbb{C}P^n$ are obtained as images of geodesics through a parallel isometric immersion of a torus, we have a geometric information on circles. Though we do not have such a geometric construction of circular trajectories, we can transplant geometric information on circles to circular trajectories through algebraic information on circles and circular trajectories. Under this consideration we can conclude that a circular trajectory is closed if and only if its invariant defined by its structure torsion and the strength of the magnetic field is expressed by a pair of mutually prime positive integers. As trajectories for Kähler magnetic fields on a complex projective space are always closed, this feature of trajectories for Sasakian magnetic fields is remarkable.

For about hypersurfaces of type (A) in a complex hyperbolic space $\mathbb{C}H^n$, we devote sections 11, 12, 13 and 14. We study trajectories for Sasakian magnetic fields along the same lines as in sections 7, 8 and 9. The difference between trajectories on hypersurfaces in $\mathbb{C}P^n$ and those on hypersurfaces in $\mathbb{C}H^n$ is the unbounded property because some hypersurfaces are not compact.

In sections 16, 17 and 18, we study trajectories on homogeneous Hopf hypersurfaces of types other than (A). On these hypersurfaces, being different from trajectories on real hypersurfaces of type (A), structure torsions of trajectories are functions in general. We study structure torsions in detail and show that on real hypersurfaces of type (B) in complex hyperbolic spaces there are no trajectories which are also curves of order two. As an application of our study of circular trajectories, we give some characterizations of real hypersurfaces of type (A) by the amount of circular trajectories in section 19. Our characterization of hypersurfaces of type (A_1) is a refinement of a characterization of real hypersurfaces of type (A) due to Maeda-Adachi.

The author would like to express her hearty gratitude to her supervisor Professor Toshiaki Adachi for his academic advice and encouragement during her stay in Japan.

2. **A short summary of notations and results in Riemannian geometry**

2.1. **Riemannian connections.** We shall begin by fixing some notations and recalling some standard facts about connections. Let M denote a smooth finite-dimensional manifold. Its tangent space at a point $p \in M$ is denoted by T_pM and its tangent bundle and unit tangent bundle by TM and UM, respectively. Let $\mathcal{X}(M)$ be the linear space of smooth vector fields on M and $C^\infty(M)$ be the ring of smooth functions of M. A *Riemannian metric* is an assignment to each $p \in M$ of a symmetric positive-definite bilinear form $\langle\ ,\ \rangle_p$ on T_pM such that for any $V, W \in \mathcal{X}(M)$, the function $p \mapsto \langle V, W\rangle_p$ is smooth on M. Also, $\langle V, V\rangle_p^{1/2}$ is denoted by $\|V\|_p$. A smooth manifold admitting a Riemannian metric is said to be a *Riemannian manifold*.

An affine connection is a bilinear map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ which has the following properties:

$$\nabla_{fV} W = f\nabla_V W,$$
$$\nabla_V(fW) = (Vf)W + f\nabla_V W,$$

for any $f \in C^\infty(M)$ and $V, W \in \mathcal{X}(M)$.

The fundamental theorem of Riemannian geometry states that for each Riemannian metric there is a unique affine connection, called the *Riemannian connection*, with the following two properties:

i) $X\langle V, W\rangle = \langle\nabla_X V, W\rangle + \langle V, \nabla_X W\rangle$,
ii) $\nabla_V W - \nabla_W V - [V, W] = 0$,

for arbitrary $X, V, W \in \mathcal{X}(M)$. Here $[\ ,\ ]$ denotes a Lie bracket, that is, $[V, W]f = (VW - WV)f$ for $V, W \in \mathcal{X}(M)$ and $f \in C^\infty(M)$. The property i) is a condition of compatibility between an affine connection and the metric, while the property ii) is a symmetry condition ∇ on the connection alone. In general, the quantity $\mathrm{Tor}(V, W) = \nabla_V W - \nabla_W V - [V, W]$ is called the torsion of an affine connection ∇.

It is a tensor of type $(1,2)$. Hence the fundamental theorem may be paraphrased as saying that there is a unique torsion-free connection compatible with any given metric.

2.2. **Distance function and geodesics.** For a smooth curve γ on a Riemannian manifold $(M, \langle\ ,\ \rangle)$, which is a smooth map $\gamma : I \to M$ of an interval $I \subset \mathbb{R}$, we define its length as

$$\mathrm{length}(\gamma) = \int_I \left\| \frac{d\gamma}{dt}(t) \right\| dt.$$

When M is connected, given two points $p, q \in M$ we set

$$d_M(p,q) = \inf\left\{ \mathrm{length}(\gamma) \;\middle|\; \begin{array}{c} \gamma : [a,b] \to M \text{ is a smooth curve} \\ \text{with } \gamma(a) = p,\ \gamma(b) = q \end{array} \right\}.$$

For a smooth curve $\gamma : [a,b] \to M$ we define $\gamma^{-1} : [a,b] \to M$ by $\gamma^{-1}(t) = \gamma(a+b-t)$. If γ is a curve from p to q, then γ^{-1} is a curve from q to p. Since $\mathrm{length}(\gamma^{-1}) = \mathrm{length}(\gamma)$, we have $d_M(p,q) = d_M(q,p)$. For curves $\gamma_1 : [a_1, b_1] \to M$ from p to q and $\gamma_2 : [a_2, b_2] \to M$ from q to r, we define a curve $\gamma_1 \cdot \gamma_2 : [a_1, b_1 + b_2 - a_2] \to M$ by

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(t), & \text{if } a_1 \le t \le b_1, \\ \gamma_2(t - b_1 + a_2), & \text{if } b_1 < t \le b_1 + b_2 - a_2. \end{cases}$$

Then it is a curve from p to r passing through q. As $\mathrm{length}(\gamma_1 \cdot \gamma_2) = \mathrm{length}(\gamma_1) + \mathrm{length}(\gamma_2)$, we see $d_M(p,r) \le d_M(p,q) + d_M(q,r)$. As it is clear that $d_M(p,q) = 0$ if and only if $p = q$, this d_M defines a distance function on M. We call this a distance associated with the Riemannian metric.

For a smooth curve $\gamma : [a,b] \to M$ with $\gamma(a) = p,\ \gamma(b) = q$, we call a smooth map $\alpha : [a,b] \times (-\epsilon, \epsilon) \to M$ a smooth *variation* of curves for γ if it satisfies

i) $\alpha(t, 0) = \gamma(t)$ for $a \le t \le b$,

ii) $\alpha(a, s) = p$ and $\alpha(b, s) = q$ for $-\epsilon < s < \epsilon$.

For this α we define a vector field W along γ by $W(t) = \dfrac{\partial \alpha}{\partial s}(t, 0)$ and call it a variation vector field associated with α.

We put $\gamma' = \dfrac{d\gamma}{dt}$ and denote by $\nabla_{\gamma'}$ the covariant differentiation along γ with respect to the Riemannian connection ∇ on M.

Lemma 2.1 (First variation formula)**.** *Let α be a smooth variation of curves for a smooth curve $\gamma : [a, b] \to M$. We then have*

$$\frac{d}{ds}\mathrm{length}\big(\alpha(\cdot, s)\big)\Big|_{s=0} = -\int_a^b \Big\langle W(t), \nabla_{\gamma'}\Big(\frac{\gamma'}{\|\gamma'\|}\Big)(t)\Big\rangle\, dt.$$

Proof. By direct computation we have

$$\begin{aligned}
\frac{d}{ds}\mathrm{length}\big(\alpha(\cdot, s)\big)\Big|_{s=0} &= \frac{d}{ds}\int_a^b \Big\|\frac{\partial\alpha}{\partial t}\Big\|\, dt\Big|_{s=0} = \int_a^b \frac{d}{ds}\Big\langle\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\Big\rangle^{1/2}\Big|_{s=0} dt \\
&= \int_a^b \Big\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial\alpha}{\partial t}(t,0), \frac{\partial\alpha}{\partial t}(t,0)\Big\rangle \Big/ \Big\|\frac{\partial\alpha}{\partial t}(t,0)\Big\|\, dt \\
&= \int_a^b \Big\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}(t,0), \frac{\partial\alpha}{\partial t}(t,0) \Big/ \Big\|\frac{\partial\alpha}{\partial t}(t,0)\Big\|\Big\rangle\, dt \\
&= \int_a^b \Big\{\frac{d}{dt}\Big\langle W(t), \frac{\gamma'(t)}{\|\gamma'(t)\|}\Big\rangle - \Big\langle W(t), \frac{d}{dt}\Big(\frac{\gamma'(t)}{\|\gamma'(t)\|}\Big)\Big\rangle\Big\}\, dt \\
&= \Big\langle W(b), \frac{\gamma'(b)}{\|\gamma'(b)\|}\Big\rangle - \Big\langle W(a), \frac{\gamma'(a)}{\|\gamma'(a)\|}\Big\rangle - \int_a^b \Big\langle W(t), \frac{d}{dt}\Big(\frac{\gamma'(t)}{\|\gamma'(t)\|}\Big)\Big\rangle\, dt
\end{aligned}$$

Since $\alpha(a, s) = p,\ \alpha(b, s) = q$ for all s, we have $W(a) = 0$ and $W(b) = 0$, hence get the conclusion. □

We say a smooth curve γ satisfying the differential equation $\nabla_{\gamma'}\gamma' = 0$ to be a *geodesic*. As we have $\gamma'\big(\|\gamma'\|^2\big) = 2\langle\nabla_{\gamma'}\gamma', \gamma'\rangle = 0$ for a geodesic γ, we see it has constant speed $\|\gamma'\|$. Thus we see a geodesic is a stational curve for the functional length$(\cdot)$, that is, a curve which satisfies $\frac{d}{ds}\mathrm{length}\big(\alpha(\cdot, s)\big)\big|_{s=0} = 0$ for its arbitrary variation α of curves.

2.3. **Isometries of Riemannian manifolds.** Let $(M, \langle\ ,\ \rangle_M)$ and $(N, \langle\ ,\ \rangle_N)$ be two Riemannian manifolds. A homeomorphism $\varphi : M \to N$ is said to be an isometry if it satisfies $\varphi^*\langle\ ,\ \rangle_N = \langle\ ,\ \rangle_M$. Here $\varphi^*\langle\ ,\ \rangle_N$ denotes the pull back metric. That is, it is a Riemannian metric defined on M by $\langle d\varphi(u), d\varphi(v)\rangle_N$ for every $u, v \in T_pM$

at an arbitrary point $p \in M$. Hence, an isometry φ is a homeomorphism satisfying $\langle d\varphi(u), d\varphi(v)\rangle_N = \langle u, v\rangle_M$ for every $u, v \in T_pM$ at an arbitrary point $p \in M$. For two isometries $\varphi_1, \varphi_2 : M \to M$, it is clear that their composition $\varphi_2 \circ \varphi_1$ and the inverse φ_1^{-1} are also isometries of M. Therefore the set of all isometries of M forms a group. We call this set the isometry group of M and denote it by $\mathrm{Isom}(M)$.

When there is an immersion $\iota : N \to M$ of a differentiable manifold N to a Riemannian manifold M, we call N a submanifold of M. On a submanifold N we have an induced metric $\iota^*\langle\ ,\ \rangle$. A submanifold admitting this induced metric is called a Riemannian submanifold. We usually identify N with $\iota(N)$.

2.4. **Real space forms.** We define the curvature tensor R of M by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for $X, Y, Z \in \mathcal{X}(M)$. For a tangent vectors $v, w \in T_pM$ which span a 2-plane in T_pM, we denote by $\mathrm{Riem}(v, w)$ the sectional curvature of this plane. That is,

$$\mathrm{Riem}(v,w) = \langle R(v,w)w, v\rangle / \|v \wedge w\|^2.$$

A complete, simply connected Riemannian manifold of constant sectional curvature is called a *real space form*. It is known that a real space form $\mathbb{R}M^m$ of dimension m is congruent to one of a standard sphere S^m, a Euclidean space $\mathbb{R}^m$ and a real hyperbolic space H^m. A Euclidean space $\mathbb{R}^m$ with standard inner product is flat, that is, its sectional curvatures are zero. Sectional curvatures of a sphere of raius r

$$S^m(1/r^2) = \big\{x = (x_0, \dots, x_m) \in \mathbb{R}^{m+1} \bigm| x_0^2 + \cdots + x_m^2 = r^2\big\}$$

with the metric induced by the standard inner product on $\mathbb{R}^{m+1}$ are $1/r^2$. We can show this by considering the relationship of connections on S^m and $\mathbb{R}^{m+1}$. On $\mathbb{R}^{m+1}$, we consider a quadratic form $\langle\!\langle\ ,\ \rangle\!\rangle$ which is given by

$$\langle\!\langle v, w\rangle\!\rangle = -v_0w_0 + v_1w_1 + \cdots + v_mw_m$$

for $v=(v_0,\dots,v_m)$, $w=(w_0,\dots,w_m)\in\mathbb{R}^{m+1}$. When we consider this form on $\mathbb{R}^{m+1}$, we usually denote the space as $\mathbb{R}^{m+1}_1$. We consider a subset

$$H^m(-1/r^2)=\left\{x\in\mathbb{R}^{m+1}\;\middle|\;-x_0^2+x_1^2+\cdots+x_m^2=-r^2\right\}.$$

Its tangent space at $x\in H^m$ is

$$T_xH^m=\left\{v\in\mathbb{R}^{m+1}\;\middle|\;-v_0x_0+v_1x_1+\cdots+v_mx_m=0\right\}.$$

On this space, we have

$$\begin{aligned}\langle\!\langle v,v\rangle\!\rangle&=-v_0^2+v_1^2+\cdots+v_m^2=v_1^2+\cdots+v_m^2-(v_1x_1+\cdots+v_mx_m)^2x_0^{-2}\\&\geq-(v_1^2+\cdots+v_m^2)(x_1^2+\cdots+x_m^2-x_0^2)x_0^{-2}=r^2(v_1^2+\cdots+v_m^2)x_0^{-2}\geq0\end{aligned}$$

Thus $\langle\!\langle\ ,\ \rangle\!\rangle$ defines a Riemannian metric on H^m. With this metric, sectional curvatures of $H^m(-1/r^2)$ are $-1/r^2$. We note that a real hyperbolic space is sometimes denoted by $\mathbb{R}H^m$ to distinguish it from complex hyperbolic spaces, which will be given in below, and from quaternionic hyperbolic spaces clearly.

2.5. **Kähler manifolds.** A smooth $(1,1)$ tensor field $J:TM\to TM$ on a manifold M satisfying $J^2=-id_{TM}$ is said to be an almost complex structure on M. We call a manifold M an almost complex manifold if it admits an almost complex structure. A Riemannian metric $\langle\ ,\ \rangle$ on an almost complex manifold M is said to be a Hermitian metric if it satisfies $\langle JV,JW\rangle=\langle V,W\rangle$ for arbitrary $V,W\in\mathcal{X}(M)$. This means that J is an isometry with respect to this metric.

On a complex manifold M, an almost complex structure is naturally induced in the following manner. For a complex analytic chart (U,φ) we denote as $\varphi=(z_1,\dots,z_n)$ and $z_j=x_j+\sqrt{-1}y_j$ $(j=1,2,...,n)$, where x_j and y_j are real and imaginary part of z_j, respectively. At each point $p\in U$, the vectors $(\partial/\partial x_1)_p,(\partial/\partial y_1)_p,\dots,(\partial/\partial x_n)_p$, $(\partial/\partial y_n)_p$ form a basis of real linear space T_pM. If we define $J_p:T_pM\to T_pM$ by $(\partial/\partial x_j)_p\mapsto(\partial/\partial y_j)_p$ and $(\partial/\partial y_j)_p\mapsto-(\partial/\partial x_j)_p$, it is well-defined and is an almost complex structure on M. When a complex manifold M admits a Hermitian metric

$\langle\ ,\ \rangle$ with respect to this induced almost complex structure J, we define a 2-form Ω by $\Omega(X,Y) = \langle JX, Y\rangle$. It is called the fundamental form or the Kähler form associated with $\langle\ ,\ \rangle$. If this 2-form is closed, that is its exterior derivative $d\Omega$ is a null 3-form, the Hermitian metric is said to be a Kähler metric and the manifold is said to be a Kähler manifold. In other words, a Hermitian metric is Kähler if the induced almost complex structure is parallel (i.e. $\nabla J = 0$) with respect to the Riemannian connection.

2.6. **Complex space forms.** Complex space forms are typical examples of Kähler manifolds. For a nonzero tangent vector $v \in T_pM$ of an almost complex manifold M, we set $\mathrm{HRiem}(v) = \mathrm{Riem}(v, Jv)$ and call it the holomorphic sectional curvature of a complex line spanned by v. A complex space form $\mathbb{C}M^n(c)$ is a complex n-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature c. Hence, it is one of a complex projective space $\mathbb{C}P^n$, a complex Euclidean space $\mathbb{C}^n$ and a complex hyperbolic space $\mathbb{C}H^n$ according as c is positive, zero and negative.

In this paper we frequently make use of Hopf fibrations to connect the geometry of complex projective or hyperbolic spaces and that of complex Euclidean spaces. We take a standard sphere S^{2n+1} of radius 1 in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2(n+1)}$. We define a equivalence relation on S^{2n+1} as follows: We define that $z, w \in S^{2n+1}$ are equivalent to each other if and only if there is $e^{\sqrt{-1}\theta}$ ($\theta \in \mathbb{R}$) with $w = e^{\sqrt{-1}\theta} z$. This means that the group $S^1 = \{e^{\sqrt{-1}\theta} \mid \theta \in \mathbb{R}\}$ acts freely on S^{2n+1}. A complex projective space $\mathbb{C}P^n$ is the quotient space of S^{2n+1} with respect to this equivalence relation. We call the quotient map $\varpi : S^{2n+1} \to \mathbb{C}P^n$ given by $S^{2n+1} \ni z \mapsto [z] \in \mathbb{C}P^n$, where $[z]$ denotes the equivalence class containing z, a *Hopf fibration*. Each point in $\mathbb{C}P^n$ is usually denoted as $[z_0, z_1, \ldots, z_n]$ with a point $(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. This expression is called the homogeneous coordinate of $\mathbb{C}P^n$. On $\mathbb{C}^{n+1}$ we define a metric by $\langle\ ,\ \rangle = \mathrm{Re}(\!(\ ,\ )\!)$ with the standard Hermitian inner product

$$(\!(u,v)\!) = u_0\bar{v}_0 + u_1\bar{v}_1 + \cdots + u_n\bar{v}_n$$

for $u = (u_0, u_1, \ldots, u_n),\ v = (v_0, v_1, \ldots, v_n) \in \mathbb{C}^{n+1}$. As we mentioned in §2.2, we see S^{2n+1} is of constant sectional curvature 1 with the induced metric.

We here induce a metric and a complex structure on $\mathbb{C}P^n$. For this sake we show the horizontal and vertical subbundles with respect to this Hopf fibration ϖ. We represent the tangent space T_zS^{2n+1} at point $z \in S^{2n+1}$ on unit sphere S^{2n+1} as

$$T_zS^{2n+1} = \big\{(z,u) \in \{z\} \times \mathbb{C}^{n+1} \bigm| \langle z,u\rangle = 0\big\}.$$

We set

$$\mathcal{V}_z = \big\{(z, \sqrt{-1}az) \in T_zS^{2n+1} \bigm| a \in \mathbb{R}\big\},$$
$$\mathcal{H}_z = \big\{(z,u) \in T_zS^{2n+1} \bigm| (\!(z,u)\!) = 0\big\}.$$

Since $\mathcal{V}_z$ is the tangent line of the curve $\mathbb{R} \ni \theta \mapsto e^{\sqrt{-1}\theta} \in S^{2n+1}$, we see it is the direction of the action of S^1. By the definitions of $\mathcal{V}_z$ and $\mathcal{H}_z$ we find that they form an orthogonal decomposition $T_zS^{2n+1} = \mathcal{V}_z \oplus \mathcal{H}_z$ of the tangent space T_zS^{2n+1}. Since $\mathcal{V}_z$ is the direction of the action of S^1, the tangent space $T_{[z]}\mathbb{C}P^n$ at $\varpi(z) = [z]$ in $\mathbb{C}P^n$ corresponds to $\mathcal{H}_z$, that is $d\varpi|_{\mathcal{H}_z} : \mathcal{H}_z \to T_{\varpi(z)}\mathbb{C}P^n$ is a lineary isometric map. We call $\mathcal{H}_z$ and $\mathcal{V}_z$ in the above decomposition of T_zS^{2n+1} the *horizontal part* and the *vertical part*, respectively. By the S^1-action we have a correspondence $(z,v) \mapsto (e^{\sqrt{-1}\theta}z, e^{\sqrt{-1}\theta}v)$ between tangent spaces. We denote by J the complex structure on $\mathbb{C}^{n+1}$ given by $Jw = \sqrt{-1}w$. When $(\!(z,u)\!) = 0$ then we have $(\!(z,Ju)\!) = 0$, hence the horizontal part $\mathcal{H}_z$ is invariant under the action of J. As we have $\big(e^{\sqrt{-1}\theta}z, J(e^{\sqrt{-1}\theta}u)\big) = \big(e^{\sqrt{-1}\theta}z, e^{\sqrt{-1}\theta}(Ju)\big)$, the action of J is compatible with the S^1-action. Therefore we can define a complex structure on $\mathbb{C}P^n$. As for a Riemannian metric on $\mathbb{C}P^n$, we define

$$\langle [z,u],[z,v]\rangle = \frac{4}{c}\,\langle u,v\rangle = \frac{4}{c}\,\mathrm{Re}(\!(u,v)\!)$$

for a positive constant c. Here, we denote as $[z,u] \in T_{\varpi(z)}\mathbb{C}P^n$ a tangent vector at $\varpi(z) = [z]$ under the identification of $\mathcal{H}_z$ and $T_{\varpi(z)}\mathbb{C}P^n$. Since $(\!(e^{\sqrt{-1}\theta}u, e^{\sqrt{-1}\theta}v)\!) = (\!(u,v)\!)$, we see it is well-defined. With this metric, we find $\mathbb{C}P^n$ is of constant holomorphic sectional curvature c. We denote this by $\mathbb{C}P^n(c)$.

We next consider a Hermitian form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathbb{C}^{n+1}$ defined by

$$\langle\!\langle u,v\rangle\!\rangle = -u_0\bar{v}_0 + u_1\bar{v}_1 + \cdots + u_n\bar{v}_n$$

for $u = (u_0, u_1, \dots, u_n)$, $v = (v_0, v_1, \dots, v_n) \in \mathbb{C}^{n+1}$. In order to clarify that we consider this form, we denote this complex Euclidean space by $\mathbb{C}_1^{n+1}$. We take an anti-de Sitter space H_1^{2n+1} which is given by

$$\begin{aligned} H_1^{2n+1} &= \big\{ z \in \mathbb{C}_1^{n+1} \bigm| \langle\!\langle z,z\rangle\!\rangle = -1 \big\} \\ &= \big\{ z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \bigm| -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = -1 \big\}. \end{aligned}$$

We define an equivalence relation on H_1^{2n+1} in the following way. We define that two points $z, w \in H_1^{2n+1}$ are equivalent to each other if there is $e^{\sqrt{-1}\theta}$ $(\theta \in \mathbb{R})$ with $w = e^{\sqrt{-1}\theta}z$. Thus, we see the group S^1 acts freely on H_1^{2n+1}. A complex hyperbolic space $\mathbb{C}H^n$ is the quotient space of H_1^{2n+1} with respect to this equivalence relation. We call the quatient map $\varpi : H_1^{2n+1} \to \mathbb{C}H^n$ given by $S^{2n+1} \ni z \mapsto [z] \in \mathbb{C}H^n$, where $[z]$ denotes the equivalence class containing z, a *canonical fibration* or sometimes call a Hopf fibration. Each point in $\mathbb{C}H^n$ is also denoted as $[z_0, z_1, \dots, z_n]$ with a point $(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. This expression is called the homogeneous coordinate of $\mathbb{C}H^n$.

We define a product $\langle\ ,\ \rangle$ on H_1^{2n+1} by $\langle\ ,\ \rangle = \mathrm{Re}\langle\!\langle\ ,\rangle\!\rangle$. The tangent space $T_zH_1^{2n+1}$ at point $z \in H_1^{2n+1}$ is expressed as

$$T_zH_1^{2n+1} = \big\{ (z,u) \in \{z\} \times \mathbb{C}^{n+1} \bigm| \langle z,u\rangle = 0 \big\}.$$

We set

$$\begin{aligned} \mathcal{V}_z &= \big\{ (z, \sqrt{-1}az) \in T_zH_1^{2n+1} \bigm| a \in \mathbb{R} \big\}, \\ \mathcal{H}_z &= \big\{ (z,u) \in T_zH_1^{2n+1} \bigm| \langle\!\langle z,u\rangle\!\rangle = 0 \big\}. \end{aligned}$$

By the definitions of $\mathcal{V}_z$ and $\mathcal{H}_z$ we find that for arbitrary $v \in \mathcal{V}_z$ and $u \in \mathcal{H}_z$ they satisfy $\langle v, u\rangle = 0$. Since these subspaces span $T_z H_1^{2n+1}$, we write as $T_z H_1^{2n+1} = \mathcal{H}_z \oplus \mathcal{V}_z$. As $\mathcal{V}_z$ is the direction of the action of S^1, the tangent space $T_{[z]}\mathbb{C}H^n$ at $\varpi(z) = [z]$ in $\mathbb{C}H^n$ corresponds to $\mathcal{H}_z$, that is $d\varpi|_{\mathcal{H}_z} : \mathcal{H}_z \to T_{\varpi(z)}\mathbb{C}H^n$ is a linearly isometric map. We call $\mathcal{H}_z$ and $\mathcal{V}_z$ in the above decomposition of $T_z H_1^{2n+1}$ the horizontal part and the vertical part, respectively. By the S^1-action we have a correspondence $(z, v) \mapsto (e^{\sqrt{-1}\theta}z, e^{\sqrt{-1}\theta}v)$ between tangent spaces. For a complex structure J on $\mathbb{C}_1^{n+1}$, which is given by $Jw = \sqrt{-1}w$, we have $\langle\!\langle z, Ju\rangle\!\rangle = 0$ if $\langle\!\langle z, u\rangle\!\rangle = 0$. We hence find that $\mathcal{H}_z$ is invariant under the action of J. Like the case of complex projective spaces, the action of J and the S^1-action on TH_1^{2n+1} are compatible each other. Moreover, if we consider the product $\langle\ ,\ \rangle$ on $\mathcal{H}_z$, as we have

$$\begin{aligned}\langle\!\langle u, u\rangle\!\rangle &= -|u_0|^2 + |u_1|^2 + \cdots + |u_n|^2\\ &= |u_1|^2 + \cdots + |u_n|^2 - |u_1\bar{z}_1 + \cdots + u_n\bar{z}_n|^2|z_0|^{-2}\\ &\geq |u_1|^2 + \cdots + |u_n|^2 - (|u_1||z_1| + \cdots + |u_n||z_n|)^2|z_0|^{-2}\\ &\geq -(|u_1|^2 + \cdots + |u_n|^2)(|z_1|^2 + \cdots + |z_n|^2 - |z_0|^2)|z_0|^{-2}\\ &= (|u_1|^2 + \cdots + |u_n|^2)|z_0|^{-2} \geq 0,\end{aligned}$$

it is positive-definite. Thus for tangent vectors $[z, u]$, $[z, v] \in T_{\varpi(z)}\mathbb{C}H^n$ we define

$$\langle [z, u], [z, v]\rangle = \frac{4}{|c|}\langle u, v\rangle = \frac{4}{|c|}\langle\!\langle u, u\rangle\!\rangle$$

for a negative constant c. We find it turns to a Riemannian metric on $\mathbb{C}H^n$. With this metric $\mathbb{C}H^n$ is of constant holomorphic sectional curvature c. We denote this by $\mathbb{C}H^n(c)$.

3. Magnetic fields

3.1. **Definition of magnetic fields.** A static magnetic field on $\mathbb{R}^3$ is a vector-valued function $\mathbb{B} = (B_1, B_2, B_3) : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying Gauss formula

$$\mathrm{div}(\mathbb{B}) = \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} = 0.$$

This gives the Lorentz force $v \times \mathbb{B} = \Omega_{\mathbb{B}} v$ on a unit charged particle when its velocity vector is v. Here $\Omega_{\mathbb{B}}$ is a skew-symmetric matrix given by

$$\begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix}.$$

When the mass of this unit charged particle is m, the equation of motion for this is hence $m\dfrac{dv}{dt} = v \times \mathbb{B}$. As we have $\dfrac{d}{dt}\|v\|^2 = 2\langle v, \dfrac{dv}{dt}\rangle = 2\langle v, \Omega_{\mathbb{B}} v\rangle = 0$, we see every electric charged particle has constant speed.

We define a 2-form $\boldsymbol{B}$ on $\mathbb{R}^3$ by $\boldsymbol{B}(u, v) = \langle u, \Omega_{\mathbb{B}} v\rangle$ with the standard inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^3$. Then this form is represented as

$$\boldsymbol{B} = B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2.$$

We then have

$$d\boldsymbol{B} = (\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3}) dx_1 \wedge dx_2 \wedge dx_3.$$

We therefore find that the Gauss formula $\mathrm{div}(\mathbb{B}) = 0$ is equivalent to the closedness of this 2-form $\boldsymbol{B}$.

With this consideration we introduce an object on a Riemannian manifold which is a generalization of a static magnetic field on a Euclidean 3-space. A closed 2-form $\boldsymbol{B}$ on a Riemannian manifold M is said to be a *magnetic field*. Given a magnetic field $\boldsymbol{B}$ on M, we define a bundle map $\Omega_{\boldsymbol{B}} : TM \to TM$ on the tangent bundle TM of M by $\boldsymbol{B}(u, v) = \langle u, \Omega_{\boldsymbol{B}}(v)\rangle$ for every $u, v \in T_pM$ at an arbitrary point $p \in M$ with Riemannian metric $\langle\ ,\ \rangle$ on M.

Lemma 3.1. (1) *This bundle map $\Omega_{\boldsymbol{B}}$ is well-defined and is skew symmetric.*

(2) *For two magnetic fields* $\boldsymbol{B}_1, \boldsymbol{B}_2$ *and constants* $\lambda_1, \lambda_2 \in \mathbb{R}$, *we have*

$$\Omega_{\lambda_1 \boldsymbol{B}_1 + \lambda_2 \boldsymbol{B}_2} = \lambda_1 \Omega_{\boldsymbol{B}_1} + \lambda_2 \Omega_{\boldsymbol{B}_2}.$$

Proof. (1) We consider on T_pM at an arbitrary point $p \in M$. If we take an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM, where n denotes the dimension of M, we find $\Omega_{\boldsymbol{B}}(v)$ is defined by

$$\Omega_{\boldsymbol{B}}(v) = \langle e_1, \Omega_{\boldsymbol{B}}(v)\rangle e_1 + \cdots + \langle e_n, \Omega_{\boldsymbol{B}}(v)\rangle e_n = \boldsymbol{B}(e_1, v)e_1 + \cdots + \boldsymbol{B}(e_n, v)e_n$$

for each $v \in T_pM$. Hence it is well-defined. As $\boldsymbol{B}$ is bilinear on T_pM at an arbitrary point $p \in M$, we see

$$\begin{aligned}\langle u, \Omega_{\mathbf{B}}(\lambda_1 v_1 + \lambda_2 v_2)\rangle &= \boldsymbol{B}(u, \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \boldsymbol{B}(u, v_1) + \lambda_2 \boldsymbol{B}(u, v_2) \\ &= \lambda_1 \langle u, \Omega_{\boldsymbol{B}}(v_1)\rangle + \lambda_2 \langle u, \Omega_{\boldsymbol{B}}(v_2)\rangle = \langle u, \lambda_1 \Omega_{\boldsymbol{B}}(v_1) + \lambda_2 \Omega_{\boldsymbol{B}}(v_2)\rangle,\end{aligned}$$

hence $\Omega_{\boldsymbol{B}}$ is linear. Similarly, we have

$$\langle u, \Omega_{\boldsymbol{B}}(v)\rangle = \boldsymbol{B}(u, v) = -\boldsymbol{B}(v, u) = -\langle v, \Omega_{\boldsymbol{B}}(u)\rangle = -\langle \Omega_{\boldsymbol{B}}(u), v\rangle,$$

we see $\Omega_{\boldsymbol{B}}$ is skew symmetric.

(2) As we have

$$\begin{aligned}\langle u, \Omega_{\lambda_1 \boldsymbol{B}_1 + \lambda_2 \boldsymbol{B}_2}(v)\rangle &= \big(\lambda_1 \boldsymbol{B}_1 + \lambda_2 \boldsymbol{B}_2\big)(u, v) = \lambda_1 \boldsymbol{B}_1(u, v) + \lambda_2 \boldsymbol{B}_2(u, v) \\ &= \lambda_1 \langle u, \Omega_{\boldsymbol{B}_1}(v)\rangle + \lambda_2 \langle u, \Omega_{\boldsymbol{B}_2}(v)\rangle = \big\langle u, \lambda_1 \Omega_{\boldsymbol{B}_1}(v) + \lambda_2 \Omega_{\boldsymbol{B}_2}(v)\big\rangle,\end{aligned}$$

for arbitrary $u, v \in T_pM$ at an arbitrary point $p \in M$, we get the conclusion. □

3.2. **Trajectories.** A motion of a unit electric charged particle of unit mass under this magnetic field $\boldsymbol{B}$ is a smooth curve which satisfies the equation $\nabla_{\gamma'}\gamma' = \Omega_{\boldsymbol{B}}(\gamma')$. We here give some basic properties of motions of electric charged particles.

Lemma 3.2. (1) *The speed of each motion of an electric charged particle under a magnetic field* $\boldsymbol{B}$ *is constant.*

(2) *Motions of electric charged particles under the trivial magnetic field* $\boldsymbol{B} = 0$ *are geodesics.*

(3) *When γ is a motion of an electric charged particle under a magnetic field $\boldsymbol{B}$, then the curve σ given by $\sigma(t) = \gamma(\lambda t)$ with some nonzero constant λ is a motion of an electric charged particle under a magnetic field $\lambda\boldsymbol{B}$.*

Proof. (1) Computing the derivite of the speed $\|\gamma'\|$ of a motion γ of an electric charged particle, we have

$$\gamma'(\|\gamma'\|^2) = \gamma'\langle\gamma', \gamma'\rangle = \langle\nabla_{\gamma'}\gamma', \gamma'\rangle + \langle\gamma', \nabla_{\gamma'}\gamma'\rangle = \langle\Omega_{\boldsymbol{B}}(\dot\gamma), \dot\gamma\rangle + \langle\dot\gamma, \Omega_{\boldsymbol{B}}(\dot\gamma)\rangle.$$

Since $\Omega_{\boldsymbol{B}}$ is skew symmetric, we find $\gamma'(\|\gamma'\|^2) = 0$, hence $\|\gamma'\|$ is constant along γ.

(2) For the trivial magnetic field, by the defnition of Ω_0, we find that it is the zero map. As a matter of fact, we take $u = \Omega_0(v)$, then

$$\langle\Omega_0(v), \Omega_0(v)\rangle = \langle u, \Omega_0(v)\rangle = 0(u, v) = 0.$$

Thus $\|\Omega_0(v)\|^2 = 0$, which means that $\Omega_0(v) = 0$ for an arbitrary $v \in TM$. Therefore from the definition of motions of electric charged particles, we have $\nabla_{\gamma'}\gamma' = 0$, hence γ is a geodesic.

(3) Since we have $\sigma'(t) = \lambda\gamma'(\lambda t)$ and $\Omega_{\lambda\boldsymbol{B}} = \lambda\Omega_{\mathbb{B}}$, we obtain

$$\nabla_{\sigma'(t)}\sigma'(t) = \lambda^2\nabla_{\gamma'}\gamma' = \lambda^2\Omega_{\boldsymbol{B}}(\gamma') = \Omega_{\lambda\boldsymbol{B}}(\lambda\gamma') = \Omega_{\lambda\boldsymbol{B}}(\sigma').$$

Therefore σ is a motion of an electric charged particle under a magnetic field $\lambda\boldsymbol{B}$. □

We say a motion of an electric charged particle to be a *trajectory* if it has unit speed. Therefore, a trajectory γ for a magnetic field $\boldsymbol{B}$ is a smooth curve which is parameterized by its arclength and satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = \Omega_{\boldsymbol{B}}(\dot\gamma)$. Here $\dot\gamma$ denotes the diferential with respect to the arclength parameter.

Lemma 3.3. *On a complete Riemannian manifold M, every trajectory is defined on $\mathbb{R}$.*

Proof. By the theorem on local existence of solutions for ordinary linear differential equations, we find that there is a trajectory $\gamma : (-\epsilon, \epsilon) \to M$ with given initial condition $\dot\gamma(0)$. We take the maximal interval I where γ is defined.

Suppose I is bounded from above. We set b the superimum of I. As $\|\dot\gamma\| \equiv 1$, we see the distance $d\big(\gamma(t_1), \gamma(t_2)\big)$ between two points $\gamma(t_1), \gamma(t_2)$ is not greater than $|t_1 - t_2|$. Therefore the set $\{\gamma(t) \mid 0 \le t < b\}$ is bounded. Since M is complete, we have a limit point $\lim_{t\uparrow b} \gamma(t) \in M$. Because $\dot\gamma(t)$ is a unit tangent vector for each t, we also have a limit unit tangent vector $\lim_{t\uparrow b} \dot\gamma(t) \in UM$ in the unit tangent space at $\lim_{t\uparrow b} \gamma(t)$. Thus we we find $b \in I$. Applying the theorem on local existence of solutions at $\gamma(b)$ we find γ is defined on an interval $I \cup [b, b + \epsilon_1)$ for some positive ϵ_1. As we chose I to be maximal, this is a contradiction.

If we suppose I is bounded from below, along the same lines as above we have a contradiction. Hence we get the conclusion. □

From now on we suppose Riemannian manifolds are complete. Hence we always consider that trajectories are defined on all part of the real line $\mathbb{R}$.

3.3. **Kähler magnetic fields and area magnetic fields.** We here give some examples of magnetic fields. We call a magnetic field $\boldsymbol{B}$ on M *uniform* if $\Omega_{\boldsymbol{B}}$ is parallel. That is, $\nabla\Omega_{\boldsymbol{B}} = 0$ with respect to the Riemannian connection ∇. Here, this covariant differential $\nabla\Omega_{\boldsymbol{B}}$ is given by $\big(\nabla_X \Omega_{\boldsymbol{B}}\big)Y = \nabla_X\big(\Omega_{\boldsymbol{B}}(Y)\big) - \Omega_{\boldsymbol{B}}(\nabla_X Y)$ for arbitrary vector fields X, Y on M. Thus the word "uniform" means that the influence of this mangetic field on unit vectors does not depend on their places and directions.

We take a Kähler manifold $(\widetilde{M}, J, \langle\ ,\ \rangle)$ with complex structure J and Riemannian metric $\langle\ ,\ \rangle$. We denote by $\boldsymbol{B}_J$ its Kähler form which is given by $\boldsymbol{B}_J(u, v) = \langle u, Jv\rangle$ for $u, v \in T\widetilde{M}$. We say a constant multiple $\boldsymbol{B}_\kappa = \kappa\boldsymbol{B}_J$ ($\kappa \in \mathbb{R}$) of this Kähler form to be a Kähler magnetic field. Since the complex structure is parallel, that is $\nabla J = 0$, Kähler magnetic fields are parallel. A trajectory γ for a Kähler magnetic field $\boldsymbol{B}_\kappa$ is a smooth curve which is parameterized by its arclength and satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$. For properties on trajectories for Kähler magnetic fields, see the works by Adachi ([1, 3, 8]).

Next we take a Riemann surface M. Since M is 2-dimensional, a 2-form on M is of the form $f vol_M$ with the area form vol_M on M and a function $f \in C^\infty(M)$. If f is not a constant function, this magnetic field is not uniform. When M is orientable, it admits a canonical complex structure J, which is given by $(\partial/\partial x)_p \mapsto (\partial/\partial y)_p$, $(\partial/\partial y)_p \mapsto -(\partial/\partial x)_p$ for each chart $\big(U, \varphi = (x, y)\big)$, we may regard it as a 1-dimensional Kähler manifold. Thus a trajectory γ for $f\boldsymbol{B}_J$ is a smooth curve which is parameterized by its arclength and satisfies $\nabla_{\dot\gamma}\dot\gamma = fJ\dot\gamma$.

Since Kähler manifolds and Riemann surfaces are real even dimensional, we next consider odd dimensional manifolds.

3.4. **Real hypersurfaces in Kähler manifolds.** As odd dimensional manifolds, we take real hypersurfaces in Kähler manifolds. For a Kähler manifold $(\widetilde{M}, J, \langle\ ,\ \rangle)$ of complex dimension n, a real submanifold M of real $(2n-1)$ dimension is called a *real hypersurface* of $\widetilde{M}$. It is well known that a real hypersurface M in a Kähler manifolds $(\widetilde{M}, J, \langle\ ,\ \rangle)$ admits an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$. We take a unit normal vector field $\mathcal{N}$ on M in $\widetilde{M}$. The quartet $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ is consists of the induced metric on M and a $(1,1)$-tensor ϕ, a vector field ξ and a function η on M defined by

$$\xi = -J\mathcal{N}, \quad \eta(v) = \langle v, \xi\rangle, \quad \phi(v) = Jv - \eta(v)\mathcal{N}$$

for arbitrary $v \in TM$. We call ϕ and ξ the *characteristic tensor* and the *characteristic vector field* on M, respectively. The characteristic tensor and the function η satisfy the following properties:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \langle \phi X, \phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y),$$

where I denotes the identity map of the tangent bundle TM of M. In particular, we have

$$\phi(\xi) = 0, \quad \phi^2(v) = -v$$

for arbitrary $v \in T^0M = \bigcup_{x\in M}\{w \in T_xM \mid \langle w, \xi\rangle = 0\}$.

We denote by ∇ and $\widetilde{\nabla}$ the Riemannian connections on M and $\widetilde{M}$, respectively. For vector fields $X, Y \in \mathcal{X}(M)$ we set $\sigma_M(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y$. We should note that we have to extend X, Y to vector fields $\widetilde{X}, \widetilde{Y}$ on some neighborhood of M in $\widetilde{M}$. But as one can see that $\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}$ does not depend on the choice of extensions, we denote it as $\widetilde{\nabla}_X Y$. Since the Riemannian connections do not have torsions, we see that σ_M is a simmetric tensor. It is called the *second fundamental form* of M in $\widetilde{M}$. We define $A = A_M : TM \to TM$ by

$$\langle Av, w\rangle = \langle \sigma_M(v,w), \mathcal{N}\rangle$$

for arbitrary $v, w \in T_xM$ at an arbitrary point $x \in M$. Since σ_M is symmetric and bilinear, we see that A is symmetric and linear. We call this the *shape operator* of M associated with $\mathcal{N}$. Eigenvalues and eigenvectors for A are called *principal curvatures* and *principal curvature vectors*, respectively. The shape operator A is characterized by the Gauss formula and the Weingarten formula. Both of these formula are given as follows:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle N, \tag{3.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX, \tag{3.2}$$

for all $X, Y \in \mathcal{X}(M)$.

Lemma 3.4. *The covariant derivatives of the characteristic vector field and the characteristic tensor are given as follows*:

$$\nabla_X \xi = \phi AX, \tag{3.3}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y\rangle\xi, \tag{3.4}$$

for arbitrary vector fields $X, Y \in \mathcal{X}(M)$.

Proof. By use of Gauss and Weingarten formulas, we have

$$\begin{aligned}\nabla_X\xi &= -\widetilde{\nabla}_X(J\mathcal{N}) - \langle AX,\xi\rangle\mathcal{N} = -J\widetilde{\nabla}_X\mathcal{N} - \langle AX,\xi\rangle\mathcal{N} \\ &= JAX - \langle AX,\xi\rangle\mathcal{N} = \phi AX.\end{aligned}$$

We also have

$$\begin{aligned}(\nabla_X\phi)Y &= \nabla_X(\phi Y) - \phi\nabla_X Y = \widetilde{\nabla}_X(\phi Y) - \langle AX,\phi Y\rangle\mathcal{N} - \phi\nabla_X Y \\ &= \widetilde{\nabla}_X\big(JY - \langle Y,\xi\rangle\mathcal{N}\big) - \langle AX,\phi Y\rangle\mathcal{N} - \phi\nabla_X Y \\ &= J\big(\nabla_X Y + \langle AX,Y\rangle\mathcal{N}\big) - X\langle Y,\xi\rangle\mathcal{N} + \langle Y,\xi\rangle AX + \langle \phi AX,Y\rangle\mathcal{N} - \phi\nabla_X Y \\ &= \phi\nabla_X Y + \langle\nabla_X Y,\xi\rangle\mathcal{N} - \langle AX,Y\rangle\xi - \langle\nabla_X Y,\xi\rangle\mathcal{N} - \langle Y,\phi AX\rangle\mathcal{N} \\ &\qquad + \langle Y,\xi\rangle AX + \langle\phi AX,Y\rangle\mathcal{N} - \phi\nabla_X Y \\ &= \langle Y,\xi\rangle AX - \langle AX,Y\rangle\xi,\end{aligned}$$

hence get the second equality. □

Let $\iota : M \to \widetilde{M}$ be an isometric immersion. We call an isometry φ of M *equivariant* if there exists an isometry $\tilde{\varphi}$ of $\widetilde{M}$ satisfying $\tilde{\varphi}\circ\iota = \iota\circ\varphi$.

Lemma 3.5. *Let M be a Riemannian submanifold of $\widetilde{M}$. Suppose an isometry φ of M is locally equivariant. This means that there is an open neighborhood U of M in $\widetilde{M}$ such that φ is equivariant on U. Then we have the following*:

(1) $\sigma_M\big(d\varphi(v), d\varphi(w)\big) = d\tilde{\varphi}\big(\sigma_M(v,w)\big)$ *for all* $v, w \in T_pM$ *at an arbitrary point* $p \in M$;

(2) $Ad\varphi(v) = d\varphi(Av)$ *for all* $v \in TM$, *in particular, if v is a principal curvature vector associated with a principal curvature λ, then $d\varphi(v)$ is also a principal curvature vector associated with λ.*

Proof. We take an isometry $\tilde{\varphi}$ of U satisfying $\tilde{\varphi}\circ\iota = \iota\circ\varphi$. By the definition of the second fundamental form, if we take vector fields V, W satisfying $V(p) = v$, $W(p) =$

w, we have the following by omitting to write $d\iota$:

$$\begin{aligned}\sigma_M\big(d\varphi(V), d\varphi(W)\big) &= \widetilde{\nabla}_{d\varphi(V)} d\varphi(W) - \nabla_{d\varphi(V)} d\varphi(W) \\ &= \widetilde{\nabla}_{d\tilde{\varphi}(V)} d\tilde{\varphi}(W) - \nabla_{d\varphi(V)} d\varphi(W) \\ &= d\tilde{\varphi}\big(\widetilde{\nabla}_V W\big) - d\varphi\big(\nabla_V W\big) \\ &= d\tilde{\varphi}\big(\widetilde{\nabla}_V W - \nabla_V W\big) = d\tilde{\varphi}\big(\sigma_M(V, W)\big).\end{aligned}$$

We therefore have

$$\begin{aligned}\langle A d\varphi(v), d\varphi(w)\rangle &= \big\langle \sigma_M\big(d\varphi(v), d\varphi(w)\big), \mathcal{N}_{\varphi(p)}\big\rangle = \big\langle d\tilde{\varphi}\big(\sigma_M(v, w)\big), d\tilde{\varphi}\big(\mathcal{N}_p\big)\big\rangle \\ &= \big\langle \sigma_M(v, w), \mathcal{N}_p\big\rangle = \langle Av, w\rangle = \big\langle d\varphi(Av), d\varphi(w)\big\rangle\end{aligned}$$

for arbitrary $v, w \in T_pM$ at an arbitrary point $p \in M$. As $d\varphi : T_pM \to T_{\varphi(p)}M$ is bijective, we get the conclusion. □

3.5. **Sasakian magnetic fields.** Let M be a real hypersurface in a Kähler manifold $(\widetilde{M}, J)$. Associated with the almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$, we have a canonical 2-form $\boldsymbol{F}_\phi$, which is defined by $\boldsymbol{F}_\phi(u, v) = \langle u, \phi v\rangle$.

Proposition 3.1. *The canonical* 2*-form* $\boldsymbol{F}_\phi$ *on a real hypersurface* M *in a Kähler manifold is a closed form.*

Proof. By direct computation we have

$$\begin{aligned}(\nabla_X \boldsymbol{F}_\phi)(Y, Z) &= X(\boldsymbol{F}_\phi(Y, Z)) - \boldsymbol{F}_\phi(\nabla_X Y, Z) - \boldsymbol{F}_\phi(Y, \nabla_X Z) \\ &= X\langle Y, \phi Z\rangle - \langle \nabla_X Y, \phi Z\rangle - \langle Y, \phi \nabla_X Z\rangle \\ &= \langle Y, (\nabla_X \phi) Z\rangle \\ &= \eta(Z)\langle AX, Y\rangle - \eta(Y)\langle AX, Z\rangle\end{aligned}$$

for vector fields $X, Y, Z \in \mathcal{X}(M)$ on M. Therefore we have

$$\begin{aligned}(d\boldsymbol{F}_\phi)(X, Y, Z) &= (\nabla_X \boldsymbol{F}_\phi)(Y, Z) - (\nabla_Y \boldsymbol{F}_\phi)(X, Z) + (\nabla_Z \boldsymbol{F}_\phi)(X, Y) \\ &= \eta(X)\langle AY, Z\rangle - \eta(Y)\langle AX, Z\rangle + \eta(Z)\langle AX, Y\rangle \\ &\quad - \eta(X)\langle Y, AZ\rangle + \eta(Y)\langle X, AZ\rangle - \eta(Z)\langle X, AY\rangle.\end{aligned}$$

As the shape operator A is symmetric, we see $(d\boldsymbol{F}_\phi)(X,Y,Z) = 0$ and get the conclusion. □

A constant multiple $\boldsymbol{F}_\kappa = \kappa \boldsymbol{F}_\phi$ ($\kappa \in \mathbb{R}$) of the above canonical closed 2-form $\boldsymbol{F}_\phi$ is said to be a *Sasakian magnetic field*. By (3.4) in Lemma 3.4, we find that Sasakian magnetic fields are not necessarily uniform.

A trajectory γ for a Sasakian magnetic field $\boldsymbol{F}_\kappa$ is hence a smooth curve which is parameterized by its arclength and satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma$. Though the equations of trajectories for Kähler magnetic fields and Sasakian magnetic fields are quite resemble, as Sasakian magnetic fields are not necessarily uniform, they have different properties. We shall discuss their properties in detail in the following sections.

For a real hypersurface M in a Kähler manifold $\widetilde{M}$ we say a diffeomorphism φ of M to be an isometry if it preserves the almost contact metric structure. This means that φ is an isometry of M as a Riemannian manifold and satisfies either $d\varphi \circ \phi = \phi \circ d\varphi$ or $d\varphi \circ \phi = -\phi \circ d\varphi$. These conditions correspond to the conditions on holomorphic isometries and anti-holomorphic isometries on a Kähler manifold.

Lemma 3.6. *Let γ be a trajectory for a Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a real hypersurface M in a Kähler manifold $\widetilde{M}$.*

(1) *If φ is an isometry of M satisfying $d\varphi \circ \phi = \phi \circ d\varphi$, then $\varphi \circ \gamma$ is also a trajectory for $\boldsymbol{F}_\kappa$.*

(2) *If φ is an isometry of M satisfying $d\varphi \circ \phi = -\phi \circ d\varphi$, then $\varphi \circ \gamma$ is a trajectory for $\boldsymbol{F}_{-\kappa}$.*

Proof. As we have

$$\nabla_{d\varphi(\dot\gamma)} d\varphi(\dot\gamma) = d\varphi\big(\nabla_{\dot\gamma}\dot\gamma\big) = d\varphi\big(\kappa\phi\dot\gamma\big) = \pm\kappa\phi d\varphi(\dot\gamma),$$

where the signature corresponds to the signature of the equality $d\varphi \circ \phi = \pm \phi \circ d\varphi$, we get the conclusion. □

Remark 3.1. More directly, by an isometry φ a Sasakian magnetic field $\boldsymbol{F}_\kappa$ corresponds to $\boldsymbol{F}_{\pm\kappa}$, because $\langle d\varphi(u), \phi d\varphi(v)\rangle = \langle d\varphi(u), \pm d\varphi(\phi v)\rangle = \pm\langle u, \phi v\rangle$.

4. **Helices and curves of order** 2

In order to progress our study on trajectories for Sasakian magnetic fields, we devote this and next sections to introduce some terminologies and to summarize up some results on curves and real hypersurfaces in a complex space form.

4.1. **Helices.** We say a smooth curve γ parameterized by its arclength on a Riemannian manifold M to be a *helix of proper order d* if there are positive constants $\kappa_1, \kappa_2, \dots, \kappa_{d-1}$ and a field of orthonormal frame $\{Y_1 = \dot\gamma, Y_2, ..., Y_d\}$ along γ satisfying the following system of ordinary differential equations:

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma = k_1 Y_2, \\ \nabla_{\dot\gamma} Y_2 = -k_1\dot\gamma + k_2 Y_3, \\ \nabla_{\dot\gamma} Y_3 = -k_2 Y_2 + k_3 Y_4, \\ \quad\vdots \qquad\qquad \ddots \qquad \ddots \\ \nabla_{\dot\gamma} Y_{d-1} = -k_{d-2} Y_{d-2} + k_{d-1} Y_d, \\ \nabla_{\dot\gamma} Y_d = -k_{d-1} Y_{d-1}. \end{cases}$$

This system of equations is called the Frenet-Serre formula of γ. In order to simplify the expression of the above system, we usually denote it as

$$\nabla_{\dot\gamma} Y_i = -k_{i-1} Y_{i-1} + k_i Y_{i+1} \qquad (i = 1, 2, \dots, d), \tag{4.1}$$

where we set $k_0 = k_d = 0$ and Y_0, Y_{d+1} as null vector fields. We call these constants $k_1, k_2, ..., k_{d-1}$ its *geodesic curvatures* and $\{Y_1, ..., Y_d\}$ its *Frenet frame*, respectively. In particular, a helix of proper order 1 is a geodesic, and a helix of proper order not greater than 2 is said to be a *circle*. We say a helix of proper order not greater than d to be a helix of order d.

Lemma 4.1 (Nomizu-Yano[42])**.** *A smooth curve γ parameterized by its arclength on a Riemannian manifold M is a circle if and only if it satisfies* $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \|\nabla_{\dot\gamma}\dot\gamma\|^2\dot\gamma = 0$.

Proof. Suppose γ is a circle. As we have $\nabla_{\dot\gamma}\dot\gamma = kY$, $\nabla_{\dot\gamma}Y = -k\dot\gamma$ with some nonnegative constant k and a unit vector field Y along γ, we obtain that

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}(kY) = k\nabla_{\dot\gamma}Y = -k^2\dot\gamma.$$

Since we have $k = \|kY\| = \|\nabla_{\dot\gamma}\dot\gamma\|$, we find that γ satisfies $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \|\nabla_{\dot\gamma}\dot\gamma\|^2\dot\gamma = 0$.

On the contrary, we suppose γ satisfies $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \|\nabla_{\dot\gamma}\dot\gamma\|^2\dot\gamma = 0$. Since $\|\dot\gamma\| \equiv 1$, we see $0 = \dfrac{d}{dt}\|\dot\gamma\|^2 = 2\langle\nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle$. Therefore we have

$$\frac{d}{dt}\|\nabla_{\dot\gamma}\dot\gamma\|^2 = 2\langle\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle = -2\|\nabla_{\dot\gamma}\dot\gamma\|^2\langle\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle = 0.$$

Thus we see $\|\nabla_{\dot\gamma}\dot\gamma\|$ is constant along γ. We put $k = \|\nabla_{\dot\gamma}\dot\gamma\|$. When $k = 0$, we have $\nabla_{\dot\gamma}\dot\gamma \equiv 0$, which shows that γ is a geodesic. When $k > 0$, we set $Y = (1/k)\nabla_{\dot\gamma}\dot\gamma$. We then find that γ satisfies

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma = kY, \\ \nabla_{\dot\gamma}Y = \dfrac{1}{k}\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = -\dfrac{1}{k}\|\nabla_{\dot\gamma}\dot\gamma\|^2\dot\gamma = -k\dot\gamma, \end{cases}$$

hence find that it is a circle of geodesic curvature k. This completes the proof. □

We here consider helices on a Euclidean space $\mathbb{R}^m$. Since the covariant differentiation on $\mathbb{R}^n$ is the usual differential, the equation for a geodesic is $\gamma'' = 0$, hence is a line $\gamma(t) = At + B$ with some $A, B \in \mathbb{R}^m$. These A, B are determined by initial condition $\gamma(0)$ and $\gamma'(0)$. When γ is a geodesic of unit speed, we have $A = \gamma'(0)$ and $B = \gamma(0)$. The equation of a circle of positive geodesic curvature k is $\gamma''' + k^2\gamma' = 0$. hence it is of the form $\gamma(t) = A\sin(kt) + B\cos(kt) + C$ with some $A, B, C \in \mathbb{R}^m$. These A, B, C are determined by initial point $\gamma(0)$ and initial frame $\{\gamma'(0), (1/k)\gamma''(0)\}$. By direct computation, we have $A = (1/k)\gamma'(0)$, $B = -(1/k^2)\gamma''(0)$ and $C = \gamma(0) + (1/k^2)\gamma''(0)$. Similarly, the equation for a helix of proper order 3 with geodesic curvatures k_1, k_2 is as follows:

$$\gamma^{(4)} = k_1Y_2'' = -k_1^2\gamma'' + k_1k_2Y_3' = -k_1^2\gamma'' + k_1k_2^2Y_2 = -(k_1^2 + k_2^2)\gamma''.$$

Thus it is of the form $\gamma(t) = A\sin(\sqrt{k_1^2+k_2^2}\,t) + B\cos(\sqrt{k_1^2+k_2^2}\,t) + Ct + D$ with some $A, B, C, D \in \mathbb{R}^m$ which are obtained by initial condition.

We now consider to classify helices on a Riemannian manifold M. We say two smooth curves γ_1, γ_2 on M parameterized by its arclengths are *congruent* to each other if there exist an isometry φ of M and a constant t_0 with $\gamma_2(t) = \varphi \circ \gamma_1(t + t_0)$ for all t. If we can take $t_0 = 0$, we say that they are congruent to each other in strong sense. On a real space form $\mathbb{R}M^m(c)$ of constant sectional curvature c, which is one of a standard sphere $S^m(c)$, a Euclidean space $\mathbb{R}^m$ and a real hyperbolic space $H^m(c)$ according as c is positive, zero and negative, for arbitrary orthonormal frames $\{v_1, \dots, v_m\}$ of $T_p\mathbb{R}M^m(c)$ and $\{w_1, \dots, w_m\}$ of $T_q\mathbb{R}M^m(c)$ there is an isometry φ of $\mathbb{R}M^m(c)$ satisfying $\varphi(p) = q$ and $d\varphi(v_i) = w_i$, $i = 1, \dots, m$. For example, on $\mathbb{R}^m$ we can obtain φ by a composition of a parallel translation from p to q and a motion obtained by some orthogonal matrix. On $S^m(c)$, if we represent it as a sphere of radius $1/\sqrt{c}$ in $\mathbb{R}^{m+1}$, we find that $\{\sqrt{c}p, v_1, \dots, v_m\}$ and $\{\sqrt{c}q, w_1, \dots, w_m\}$ are orthonormal frames of $\mathbb{R}^{m+1}$. Hence we have a motion $\tilde{\varphi}$ of $\mathbb{R}^{m+1}$ obtained by some orthogonal matrix which transform the former to the latter. Since $\tilde{\varphi}$ preserves $S^m(c)$ because it is obtained by some orthogonal matrix, we can take φ as the restriction of $\tilde{\varphi}$ onto S^m. Thus, the uniqueness of the solutions for linear differential equations guarantees the following:

Lemma 4.2. *Two helices on $\mathbb{R}M^m(c)$ are congruent to each other in strong sense if and only if they are of the same proper order and have the same series of geodesic curvatures.*

A smooth curve γ on a Riemannian manifold M is said to be *Killing* if it is generated by some Killing vector field on M. That is, γ is a Killing curve if there is a one-parameter family $\{\varphi_t\}_{t\in I}$ of isometries of M and a point $p \in M$ with

$\gamma(t) = \varphi_t(p)$. Such a curve is also said to be *homogeneous*. By the property of isometries on $\mathbb{R}M^m(c)$ mentioned above, we have

Lemma 4.3. *Every helices on $\mathbb{R}M^m(c)$ is Killing.*

4.2. **Complex torsions of helices.** In order to classify helices on a Kähler manifold $(\widetilde{M}, J)$, we need another quantity associated with the complex structure. For a helix γ of proper order d with its Frenet frame $Y_1, ..., Y_d$, we define its *complex torsions* by $\tau_{ij} = \langle Y_i, JY_j \rangle$ for $1 \le i < j \le d$. We should note that complex torsions are not necessarily constant. By the equations (4.1) we have

$$\begin{aligned}\tau'_{ij} &= \nabla_{\dot\gamma}\langle Y_i, JY_j\rangle = \langle \nabla_{\dot\gamma} Y_i, JY_j\rangle + \langle Y_i, J\nabla_{\dot\gamma} Y_j\rangle \\ &= \langle k_{i-1}Y_{i-1} + k_i Y_{i+1}, JY_j\rangle + \langle Y_i, J(k_{j-1}Y_{j-1} + k_j Y_{j+1})\rangle,\end{aligned}$$

hence obtain

$$\tau'_{ij} = -k_{i-1}\tau_{i-1j} + k_i\tau_{i+1j} - k_{j-1}\tau_{ij-1} + k_j\tau_{ij+1}. \tag{4.2}$$

It is known that every isometry φ of a nonflat complex space form $\mathbb{C}M^n(c)$ is either holomorphic or anti-holomorphic, that is φ preserves the complex structure ($d\varphi \circ J = J \circ d\varphi$) or reverses it ($d\varphi \circ J = -J \circ d\varphi$). Since a nonflat complex space form $\mathbb{C}M^n(c)$ is a rank one symmetric space, for arbitrary unit tangent vectors $v \in U_p\mathbb{C}M^n(c)$ and $w \in U_q\mathbb{C}M^n(c)$ there are a holomorphic isometry φ_+ and an anti-holomorphic isometry φ_- satisfying $\varphi_\pm(p) = q$ and $d\varphi_\pm(v) = w$. Therefore we have the following:

Lemma 4.4 (Maeda-Ohnita[38])**.** *On a nonflat complex space form, two helices γ_1, γ_2 are congruent to each other if and only if they have the following properties*:

i) *they are of the same proper order*;

ii) *they have the same series of geodesic curvatures, i.e. $\kappa_i^{(1)} = \kappa_i^{(2)}$ for all i*;

iii) *there exists a constant t_0 such that their complex torsions $\tau_{ij}^{(k)}$ satisfy either $\tau_{ij}^{(1)}(0) = \tau_{ij}^{(2)}(t_0)$ for all i, j, or $\tau_{ij}^{(1)}(0) = -\tau_{ij}^{(2)}(t_0)$ for all i, j.*

Proof. "Only if part". If γ_1, γ_2 are congruent to each other, we have an isometry φ and $t_0 \in \mathbb{R}$ satisfying $\gamma_2(t+t_0) = \varphi \circ \gamma_1(t)$. Thus conditions i) and ii) hold, and their Frenet frames satisfy $Y_j^{(1)}(t) = Y_j^{(2)}(t+t_0),\ j = 1, \ldots, d$. When φ is a holomorphic isometry we find that their complex torsions satisfy

$$\begin{aligned}\tau_{ij}^{(1)}(t) &= \langle Y_i^{(1)}(t), JY_j^{(1)}(t)\rangle = \langle d\varphi\big(Y_i^{(1)}(t)\big), d\varphi\big(JY_j^{(1)}(t)\big)\rangle \\ &= \langle Y_i^{(2)}(t+t_0), Jd\varphi\big(Y_j^{(1)}(t)\big)\rangle = \tau_{ij}^{(2)}(t+t_0),\end{aligned}$$

and when φ is an anti-holomorphic isometry we find that they satisfy

$$\begin{aligned}\tau_{ij}^{(1)}(t) &= \langle d\varphi\big(Y_i^{(1)}(t)\big), d\varphi\big(JY_j^{(1)}(t)\big)\rangle \\ &= \langle Y_i^{(2)}(t+t_0), -Jd\varphi\big(Y_j^{(1)}(t)\big)\rangle = -\tau_{ij}^{(2)}(t+t_0).\end{aligned}$$

"If part". There are a holomorphic isometry φ_+ and an anti-holomorphic isometry φ_- satisfying $\varphi_\pm\big(\gamma_1(0)\big) = \gamma_2(t_0)$ and $d\varphi_\pm\big(\dot\gamma_1(0)\big) = \dot\gamma_2(t_0)$. Then $\varphi_+ \circ \gamma_1$ is a helix of proper order d_1 whose geodesic curvatures are $k_1, \ldots, k_{d_1-1}$ and whose complex torsions are $\pm\tau_{ij}^{(1)}(t-t_0),\ 1 \le i < j \le d_1$. Therefore the uniquness of the solutions for linear differential equations guarantees $\gamma_2(t+t_0) = \varphi_\pm \circ \gamma_1(t)$ for all t corresponding to the signatures of the relations of complex torsions. □

Corollary 4.1. *On a nonflat $\mathbb{C}M^n(c)$ the following hold on Killing helices.*

(1) *A helix is Killing if and only if all its complex torsions are constant function.*

(2) *Two Killing helices γ_1, γ_2 are congruent to each other in strong sense if and only if*

  i) *they are of the same proper order*;

  ii) *they have the same series of geodesic curvatures*;

  iii) *their complex torsions satisfy either $\tau_{ij}^{(1)} = \tau_{ij}^{(2)}$ for all (i,j) or $\tau_{ij}^{(1)} = -\tau_{ij}^{(2)}$ for all (i,j).*

We here make mention of the condition that all complex torsions of helices are constant. Since we set $k_0 = k_d = 0$, we have more information on complex torsions. By (4.2) we have the following.

Lemma 4.5 (Maeda-Adachi[36]). *Complex torsions of a Killing helices of odd proper order d on a nonflat $\mathbb{C}M^n(c)$ satisfy the following relations*:

$$\begin{cases} \tau_{i\,i+2k} = 0, & i = 1, 2, \ldots, d-2k,\ k = 1, 2, \ldots, (d-1)/2, \\ k_1\tau_{2\,d} = k_{d-1}\tau_{1\,d-1}, & \\ k_1\tau_{2\,j} + k_j\tau_{1\,j+1} = k_{j-1}\tau_{1\,j-1}, & j = 3, 5, \ldots, d-2, \\ k_{i-1}\tau_{i-1\,d} + k_{d-1}\tau_{i\,d-1} = k_i\tau_{i+1\,d}, & i = 3, 5, \ldots, d-2, \\ k_{i-1}\tau_{i-1\,j} + k_{j-1}\tau_{i\,j-1} = k_i\tau_{i+1\,j} + k_j\tau_{i\,j+1}, & \\ & i = 2, 3, \ldots, d-3,\ j = i+2, i+4, \ldots, d-1. \end{cases}$$

Lemma 4.6 (Maeda-Adachi[36]). *Complex torsions of a Killing helices of even proper order d on a nonflat $\mathbb{C}M^n(c)$ satisfy the following relations*:

$$\begin{cases} \tau_{i\,i+2k} = 0, & i = 1, 2, \ldots, d-2k,\ k = 1, 2, \ldots, (d-2)/2, \\ k_1\tau_{2\,d} = k_{d-1}\tau_{1\,d-1}, & \\ k_1\tau_{2\,j} + k_j\tau_{1\,j+1} = k_{j-1}\tau_{1\,j-1}, & j = 3, 5, \ldots, d-1, \\ k_{i-1}\tau_{i-1\,d} + k_{d-1}\tau_{i\,d-1} = k_i\tau_{i+1\,d}, & i = 2, 4, \ldots, d-2, \\ k_{i-1}\tau_{i-1\,j} + k_{j-1}\tau_{i\,j-1} = k_i\tau_{i+1\,j} + k_j\tau_{i\,j+1}, & \\ & i = 2, 3, \ldots, d-3,\ j = i+2, i+4, \ldots, d-1. \end{cases}$$

Since complex torsions of Killing helices seem quite complicated, we study Killing helices of low orders. When γ is a helix of proper order 2, if we rewrite the computation of the derivative of its complex torsion, we have

$$\tau_{12}' = \langle k_1Y_2, JY_2\rangle + \langle\dot{\gamma}, J(-k_1\dot{\gamma})\rangle = 0,$$

we find that it is Killing. But the situation is not the same for helices of proper order greater than 2. Since each vector field of the Frenet frame of a helix of proper order d is a unit vector field, we see

$$\sum_{j=1}^{i-1}\tau_{ij}^2 + \sum_{j=i+1}^{d}\tau_{ij}^2 \leq 1.$$

We therefore find that there are infinitely many helices of proper order greater than 2 all of whose complex torsions are not constant. This point is quite different between helices on real space forms and those on nonflat complex space forms. If we write down the case $d = 3$, we have the following.

Lemma 4.7 (Maeda-Adachi[36]). (1) *Geodesic curvatures and complex torsions of a Killing helix γ of proper order 3 on a nonflat $\mathbb{C}M^n(c)$ satisfy*

$$k_1\tau_{23} = k_2\tau_{12},\ \tau_{13} = 0,\ |\tau_{12}| \le k_1/\sqrt{k_1^2 + k_2^2}.$$

(2) *On the other hand, if positive constants k_1, k_2 and a constant τ satisfy $|\tau| \le k_1/\sqrt{k_1^2 + k_2^2}$, then there is a Killing helix of proper order 3 whose geodesic curvatures are k_1, k_2 and whose complex torsions are $\tau_{12} = \tau, \tau_{13} = 0$ and $\tau_{23} = k_2\tau/k_1$.*

This Lemma suggests us that geodesic curvatures and complex torsions of Killing helices which lie on some low dimensional totally geodesic submanifolds have strong relationships. A helix of proper order $2\ell - 1$ or of 2ℓ on a nonflat $\mathbb{C}M^n(c)$ is called *essential* if it lies on some totally geodesic $\mathbb{C}M^\ell(c)$. Since $\mathbb{C}M^\ell(c)$ is real 2ℓ dimensional, such an essential Killing helix is not contained in some totally geodesic $\mathbb{C}M^{\ell-1}(c)$. If a helix γ of proper order 2 is essential, then as its Frenet frame $\dot\gamma, Y_2$ spans a complex line in the tangent space $T_{\gamma(t)}\mathbb{C}M^n(c)$ at each point t, the normal Y_2 should be parallel to $J\dot\gamma$, therefore it is a trajectory for some Kähler magnetic field. For about essential Killing helices of proper order 3 and 4, we can express their complex torsions by use of their geodesic curvatures.

Lemma 4.8 (Adachi[6]). *A helix of proper order 3 on $\mathbb{C}M^n$ is essential and Killing if and only if its geodesic curvatures and complex torsions satisfy*

$$\tau_{12} = \pm k_1/\sqrt{k_1^2 + k_2^2},\ \tau_{13} = 0,\ \tau_{23} = \pm k_2/\sqrt{k_1^2 + k_2^2},$$

where double signs take the same signatures.

In particular, the vector fields in the Frenet frame $\{\dot\gamma, Y_2, Y_3\}$ *of an essential Killing helix of proper order* 3 *on* $\mathbb{C}M^n$ *satisfy* $Y_3 = (k_1/k_2)\dot\gamma \mp \big(\sqrt{k_1^2+k_2^2}/k_2\big)JY_2$.

Lemma 4.9 (Adachi[6])**.** *A helix of proper order* 4 *on* $\mathbb{C}M^n$ *is essential and Killing if and only if their geodesic curvatures and complex torsions satisfy one of the following*;

(I) $\tau_{12} = \tau_{34} = \pm\dfrac{k_1 + k_3}{\sqrt{k_2^2 + (k_1 + k_3)^2}}$, $\tau_{23} = \tau_{14} = \pm\dfrac{\kappa_2}{\sqrt{k_2^2 + (k_1+k_3)^2}}$,
$\tau_{13} = \tau_{24} = 0$,

(II) $\tau_{12} = -\tau_{34} = \pm\dfrac{k_1 - k_3}{\sqrt{k_2^2 + (k_1-k_3)^2}}$, $\tau_{23} = -\tau_{14} = \pm\dfrac{k_2}{\sqrt{k_2^2 + (k_1-k_3)^2}}$,
$\tau_{13} = \tau_{24} = 0$.

In each of the above conditions double signs take the same signatures.

In particular, the vector fields in the Frenet frame $\{\dot\gamma, Y_2, Y_3, Y_4\}$ *of an essential Killing helix of proper order* 4 *on* $\mathbb{C}M^n$ *satisfy*

$$\text{(I)}\qquad \begin{cases} k_2Y_3 = (k_1+k_3)\dot\gamma \mp \sqrt{k_2^2 + (k_1+k_3)^2}\,JY_2, \\ \kappa_2 Y_4 = \mp\sqrt{k_2^2 + (k_1+k_3)^2}\,J\dot\gamma - (k_1+k_3)Y_2, \end{cases}$$

$$\text{(II)}\qquad \begin{cases} k_2Y_3 = (k_1-k_3)\dot\gamma \mp \sqrt{k_2^2 + (k_1-k_3)^2}\,JY_2, \\ k_2 Y_4 = \pm\sqrt{k_2^2 + (k_1-k_3)^2}\,J\dot\gamma + (k_1-k_3)Y_2, \end{cases}$$

corresponding to these cases.

In view of Lemma 4.9, we find that complex torsions of essential Killing helices of proper order 4 take extremum values if their geodesic curvatures and complex torsions satisfy $k_1 = k_3$ and the condition (II) in Lemma 4.9. They are called moderate Killing helices of proper order 4 (see [5]).

4.3. **Curves of order 2.** We here extend the notion of helices. A smooth curve γ on a Riemannian manifold M which is parameterized by its arc-length is said to be a *Frenet curve* of proper order d if it satisfies the system of differential equations

$$\nabla_{\dot\gamma} Y_i(t) = -k_{i-1}(t)Y_{i-1}(t) + k_i(t)Y_{i+1}(t) \qquad (i = 1, 2, \ldots, d)$$

with some positive functions $k_1(t), \dots, k_{d-1}(t)$ and a frame of orthonormal vector fields $\{Y_1 = \dot\gamma, Y_2, \dots, Y_d\}$. Here, we take $k_0(t), k_d(t)$ to be null functions (i.e. $k_0(t) = k_d(t) \equiv 0$) and Y_0, Y_{d+1} to be null vector fields. Like helices, we call those functions $k_1(t), \dots, k_{d-1}(t)$ its geodesic curvature functions. Helices are hence Frenet curves having constant geodesic curvature functions. We say a smooth curve a Frenet curve of order d if it is a Frenet curve of proper order not greater than d.

We call a smooth curve γ on a Riemannian manifold M which is parameterized by its arc-length a *curve of order* 2 if it satisfies the differential equation

$$\|\nabla_{\dot\gamma}\dot\gamma\|^2\big(\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \|\nabla_{\dot\gamma}\dot\gamma\|^2\dot\gamma\big) = \langle \nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma\rangle \nabla_{\dot\gamma}\dot\gamma. \tag{4.3}$$

We here consider the difference between the notion of curves of order two and that of Frenet frame of order 2.

Lemma 4.10 (Suizu-Maeda-Adachi[45]). (1) *A Frenet curve of order* 2 *is a curve of order* 2*.*

(2) *If a curve γ of order* 2 *has non-vanishing $\nabla_{\dot\gamma}\dot\gamma$, then it is a Frenet curve of order* 2*.*

Proof. (1) Suppose γ is a Frenet curve of order 2 and satisfies $\nabla_{\dot\gamma}\dot\gamma(t) = k(t)Y(t)$ and $\nabla_{\dot\gamma}Y(t) = -k(t)\dot\gamma$. We then have

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma(t) = \nabla_{\dot\gamma}\big(k(t)Y(t)\big) = k'(t)Y(t) - k^2(t)\dot\gamma(t).$$

As $\{\dot\gamma, Y\}$ is orthonormal, hence we have $\|Y(t)\| = 1$ and $\langle\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle = 0$, we see $k^2(t) = \|\nabla_{\dot\gamma}\dot\gamma\|^2$ and $\langle \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma(t), \nabla_{\dot\gamma}\dot\gamma(t)\rangle = k'(t)k(t)$. By multiplying $k^2(t)$ to the both sides of the equality $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma(t) = k'(t)Y(t) - k^2(t)\dot\gamma(t)$, we obtain

$$\begin{aligned}\|\nabla_{\dot\gamma}\dot\gamma(t)\|^2\,\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma(t) &= \kappa'(t)k(t)\cdot k(t)Y(t) - k^4(t)\dot\gamma(t)\\ &= \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma(t), \nabla_{\dot\gamma}\dot\gamma(t)\rangle \nabla_{\dot\gamma}\dot\gamma(t) - \|\nabla_{\dot\gamma}\dot\gamma(t)\|^4\dot\gamma(t),\end{aligned}$$

and get the first assertion.

(2) Since $\nabla_{\dot\gamma}\dot\gamma \neq 0$, we put $Y = \nabla_{\dot\gamma}\dot\gamma/\|\nabla_{\dot\gamma}\dot\gamma\|$, which is a unit vector field along γ. As γ is parameterized by its arclength, differentiate the both side of equality $\|\dot\gamma\|^2 = 1$, we have $2\langle\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle = 0$, hence we find $\{\dot\gamma, Y\}$ is an orthonormal frame. As $\|Y(t)\| = 1$, we similarly have $\langle Y(t), \nabla_{\dot\gamma}Y(t)\rangle = 0$. By putting $k(t) = \|\nabla_{\dot\gamma}\dot\gamma\|$, we see

$$\begin{aligned}
\nabla_{\dot\gamma}\dot\gamma(t) &= k(t)Y(t),\\
\nabla_{\dot\gamma}Y(t) &= \frac{1}{k(t)}\big\{\nabla_{\dot\gamma}\big(k(t)Y(t)\big) - k'(t)Y(t)\big\} = \frac{1}{k(t)}\big\{\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma - k'(t)Y(t)\big\}\\
&= \frac{1}{k(t)}\big\{\langle Y(t), \nabla_{\dot\gamma}\big(k(t)Y(t)\big)\rangle Y(t) - k(t)^2\dot\gamma(t) - k'(t)Y(t)\big\}\\
&= \langle Y(t), \nabla_{\dot\gamma}Y(t)\rangle Y(t) - k(t)\dot\gamma(t) = -k(t)\dot\gamma(t)
\end{aligned}$$

Thus we find it is a Frenet curve of order 2. □

In order to study curves of order 2, we here introduce a related definition. A smooth curve γ is said to be a *plane curve* if it lies on some totally geodesic Riemann surface S of M.

Lemma 4.11. *Under a reparameterization, a plane curve γ on a Riemannian manifold M is a Frenet curve of proper order 2 if it satisfies $\gamma'(s) \neq 0$ for all s.*

Proof. By the definition of plane curves, we have a totally geodesic 2-dimensional submanifold S of M which contains γ. Since S is totally geodesic, we have

$$\gamma'(s) \in T_\gamma S \hookrightarrow T_\gamma M, \quad \big(\nabla_{\gamma'}\gamma'\big)(s) \in T_\gamma S \hookrightarrow T_\gamma M$$

for all s. We take a unit vector field Y along γ which is orthogonal to γ'. Since $\{Y(s), \gamma'(s)\}$ is an orthonormal basis of $T_{\gamma(s)}S$, we find that $\nabla_{\gamma'}\gamma'(s)$ and $\nabla_{\gamma'}Y(s)$ are expressed by linear combinations of $Y(s)$ and $\gamma'(s)$ at each point s. We reparameterized this curve by its arclength parameter t, and denote it as $\gamma(t)$. It is clear that

$$\dot\gamma(t)//\gamma'(s(t)), \quad \nabla_{\dot\gamma}\dot\gamma(t)//Y(s(t)) \quad \text{hence} \quad \dot\gamma(t) \perp Y(s(t)),$$

because from the property $\langle\dot\gamma,\dot\gamma\rangle = 1$ we have $\langle\nabla_{\dot\gamma}\dot\gamma,\dot\gamma\rangle = (1/2)\dot\gamma\langle\dot\gamma,\dot\gamma\rangle = 0$, which shows $\nabla_{\dot\gamma}\dot\gamma(t)//Y(s(t))$. We can therefore write $\nabla_{\dot\gamma}\dot\gamma(t) = k(t)Y(s(t))$ with some function $k(t)$. Similarly, from the property $\langle Y, Y\rangle = 1$, we see $\langle\nabla_{\dot\gamma}Y, Y\rangle = 0$, hence find that $\nabla_{\dot\gamma}Y(t) \perp Y(t)$. We can therefore write $\nabla_{\dot\gamma}Y(t) = \alpha(t)\dot\gamma(t)$ with some function $\alpha(t)$.

On the other hand, as we have $\langle Y,\dot\gamma\rangle = 0$, by taking its covariant differentiation along γ, we find that it turns to

$$0 = \langle\nabla_{\dot\gamma}Y(t),\dot\gamma(t)\rangle + \langle Y(t),\nabla_{\dot\gamma}\dot\gamma(t)\rangle = \alpha(t) + \kappa(t).$$

This means that $\alpha(t) = -\kappa(t)$. Thus, γ is the Frenet curve of proper order 2. □

In order to show that the notion of curves of order 2 is different from the notion of Frenet curves of order 2, we here give an example.

Example 4.1. Let γ be a smooth curve in a Euclidean space $\mathbb{R}^3$ defined by

$$\gamma(s) = \begin{cases} (s, e^{-1/s^2}, 0) & \text{if } s < 0,\\ (0,0,0), & \text{if } s = 0,\\ (s, 0, e^{-1/s^2}), & \text{if } s > 0. \end{cases}$$

It is a curve of order 2, but not a Frenet curve of order 2.

Proof. The curve $\gamma : (-\infty, 0) \to \mathbb{R}^3$ lies on the xy-plane $\mathbb{R}^2$, and the curve γ : $(0,\infty) \to \mathbb{R}^2$ lies on the xz-plane $\mathbb{R}^2$. They are hence plane curves. We see

$$\frac{d\gamma}{ds}(s) = \begin{cases} \big(1, 2s^{-3}e^{-1/s^2}, 0\big), & \text{if } s < 0,\\ (1,0,0), & \text{if } s = 0,\\ \big(1, 0, 2s^{-3}e^{-1/s^2}\big), & \text{if } s > 0, \end{cases} \tag{4.4}$$

in particular, find $\gamma'(s) \neq 0$. By Lemma 4.11, we know that they are Frenet curves of proper order 2 under reparameterizations. We reparameterize the curve $\gamma(s)$ by the arclength parameter t satisfying that $\gamma(0)$ is the origin. Since $\gamma(t)$ is a curve of order 2 for $t \neq 0$, it satisfies (4.3) for $t \neq 0$. When $t = 0$, by taking the limits of both sides of (4.3) as $t \downarrow 0$ and $t \uparrow 0$, we see that γ also satisfies (4.3) at $t = 0$. Thus $\gamma(t)$ is a curve of order 2.

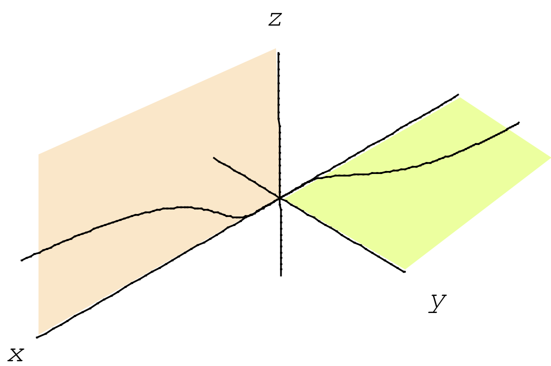
We next show that $\gamma(t)$ is not a Frenet curve of order 2. Since $\gamma(t)$ is a Frenet curve of order 2 for $t \neq 0$, we see the normal vector field Y along γ is of the form

$$Y(s(t)) = \begin{cases} \dfrac{\pm 1}{\sqrt{1+3s^{-6}e^{-2/s^2}}}\big(2s^{-3}e^{-1/s^2}, -1, 0\big), & \text{if } t < 0, \\ \dfrac{\pm 1}{\sqrt{1+3s^{-6}e^{-2/s^2}}}\big(2s^{-3}e^{-1/s^2}, 0, -1\big), & \text{if } t > 0, \end{cases}$$

by (4.4), because $Y(s(t))$ is orthogonal to $\gamma'(s)$. We therefore have

$$\lim_{t\uparrow 0} Y(s(t)) = (0, \pm 1, 0), \quad \lim_{t\downarrow 0} Y(s(t)) = (0, 0, \pm 1).$$

They are different directions. This means that we can not take a smooth normal vector field Y which consists of a Frenet frame $\{\dot{\gamma}, Y\}$ of γ. We then find that it is not a Frenet curve of order 2. □



5. **Real hypersurfaces in nonflat complex space forms**

In this section we summarize some results on real hypersurfaces in nonflat complex space forms.

5.1. **Hopf hypersurfaces.** A real hypersurface M in a Kähler manifold $(\widetilde{M}, J)$ is said to be a *Hopf hypersurface* if its characteristic vector field ξ is a principal curvature vector at each point. In another word, a real hypersurface is Hopf if and only if its shape operator makes the holomorphic distribution $T^0M = \{v \in TM \mid v \perp \xi\}$ invariant. In this paper, we denote by $\nu : M \to \mathbb{R}$ the function of the principal curvature of the characteristic vector field on a Hopf hypersurface M. We here consider integral curves of the vector field ξ. We say a smooth curve σ is an integral curve of ξ if it satisfies $\sigma'(t) = \xi_{\sigma(t)}$ for every t.

Lemma 5.1. *On a Hopf hypersurface, every integral curve σ for ξ is a geodesic of unit speed.*

Proof. As $\sigma'(t) = \xi_{\sigma(t)}$, we have by (3.3) in Lemma 3.4 that

$$\nabla_{\sigma'}\sigma' = \phi A\xi = \phi(\nu\xi) = 0.$$

As we have $\|\sigma'\| = \|\xi_\sigma\| = 1$, we get the conclusion. □

On the other hand, this property on integral curves characterize Hopf hypersurfaces.

Lemma 5.2. *A real hypersurface M is Hopf if and only if all its integral curves of the charateristic vector field are geodesics.*

Proof. By Lemma 5.1, we only need to show the "if" part. As we have $0 = \nabla_{\sigma'}\sigma' = \phi A\xi$, we see $A\xi$ is parallel to ξ. In fact, if we decompose a vector $v \in TM$ as $v = w + a\xi \in T^0M \oplus \mathbb{R}\xi$, we have $\phi v = \phi w\ (= Jw)$. Thus, $\phi v = 0$ means that $v \in \mathbb{R}\xi$. We therefore obtain that ξ is principal and that M is a Hopf hypersurface. □

For principal curvatures of Hopf hypersurfaces in nonflat complex space forms the following properties are known (see [41], for example).

Lemma 5.3. *Let M be a Hopf hypersurface in a nonflat complex space form $\mathbb{C}M^n(c)$.*

(1) *The function of principal curvature $\nu : M \to \mathbb{R}$ associated with the characteristic vector field ξ is locally constant.*

(2) *If $v \in TM$ is a principal curvature vector which is orthogonal to ξ and satisfies $Av = \lambda v$, then $(2\lambda - \nu)A\phi v = \{\lambda\nu + (c/2)\}\phi v$ holds. In particular, when $c > 0$, the vector ϕv is principal with principal curvature $\{\lambda\nu + (c/2)\}/(2\lambda - \nu)$.*

5.2. **Standard hypersurfaces in a complex projective space.** We here give standard real hypersurfaces in a nonflat complex space form which are Hopf and homogeneous. Here, a homogeneous space is a smooth manifold admitting a transitive action of a Lie group.

In his paper [46], R. Takagi classified homogenous real hypersurfaces in a complex projective space. We note that in a complex projective space it is known that all homogeneous real hypersurfaces are Hopf hypersurfaces. In his paper [34], Kimura studied Hopf real hypersurfaces in a complex projective space all of whose principal curvatures are constant.

Proposition 5.1 (Kimura[34], Takagi[46]). *In a complex projective space $\mathbb{C}P^n(c)$, a Hopf hypersurface all of whose principal curvatures are constant function is congruent to one of the following homogeneous hypersurfaces*:

(A_1) *a geodesic sphere $G(r)$ of radius r and a tube $T(r)$ around totally geodesic $\mathbb{C}P^{n-1}$ with $0 < r < \pi/\sqrt{c}$,*

(A_2) *a tube $T_\ell(r)$ around totally geodesic $\mathbb{C}P^\ell (1 \leq \ell \leq n-2)$ with $0 < r < \pi/\sqrt{c}$,*

(B) *a tube* $R(r)$ *of radius* r *around a totally real totally geodesic* $\mathbb{R}P^n(c/4)$ *with* $0 < r < \pi/\big(2\sqrt{c}\big)$, *which is a tube of radius* $\pi/\big(2\sqrt{c}\big) - r$ *around complex hyperquadric* $\mathbb{C}Q^{n-1}$ *in another expression*;

(C) *a tube of radius* r *around the Segre embedding of* $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, *where* $0 < r < \pi/(2\sqrt{c})$ *and* n $(\geqq 5)$ *is odd,*

(D) *a tube of radius* r *around the Plücker embedding of a complex Grassmann* $\mathbb{C}G_{2,5}$, *which is the set of all* 2*-dimensional complex subspaces in* $\mathbb{C}^5$, *where* $0 < r < \pi/(2\sqrt{c})$ *and* $n = 9$,

(E) *a tube of radius* r *around a canonical embedding of a Hermitian symmetric space* $SO(10)/U(5)$, *where* $0 < r < \pi/(2\sqrt{c})$ *and* $n = 15$.

Here, a Segre embedding $\mathbb{C}P^k \times \mathbb{C}P^\ell \to \mathbb{C}P^{(k+1)(\ell+1)-1}$ is given by

$$\big([z_0, \ldots, z_k], [w_0, \ldots, w_\ell]\big) \mapsto [z_0w_0, z_0w_1, \ldots, z_iw_j, \ldots, z_kw_\ell]$$

with homogeneous coordinates, and a Plücker embedding $\mathbb{C}G_{2,5} \to \mathbb{C}P^9$ is given by $\alpha = Span(v, w) \mapsto [v \wedge w]$. These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D), (E), respectively. Gathering real hypersurfaces of types (A_1) and (A_2) together, we call them hypersurfaces of type (A). Their principal curvatures are as in Table 1.

	(A_1)	(A_2)	(B)	(C, D, E)
λ_1	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r\big)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r\big)$	—	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r\big)$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan\big(\frac{\sqrt{c}}{2}r\big)$	—	$-\frac{\sqrt{c}}{2}\tan\big(\frac{\sqrt{c}}{2}r\big)$
λ_3	—	—	$-\frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}r$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\big)$
λ_4	—	—	$\frac{\sqrt{c}}{2}\tan\frac{\sqrt{c}}{2}r$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\big)$
ν	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\tan(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$

Table 1. Principal curvatures of homogeneous Hopf hypersurfaces in $\mathbb{C}P^n$

We note that if we represent a real hypersurface of type (B) as a tube of radius r around complex quadric $\mathbb{C}Q^{n-1}$, we see that

$$\lambda_3 = \frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\Big),\ \ \lambda_4 = \frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\Big),\ \ \nu = \sqrt{c}\cot(\sqrt{c}\,r).$$

We also note that a geodesic sphere $G(r)$ and a tube $T\big(\frac{\pi}{\sqrt{c}} - r\big)$ in $\mathbb{C}P^n(c)$ are congruent to each other.

For a principal curvature λ_i, we set $V_{\lambda_i} = \big\{v \in T^0M \bigm| Av = \lambda_i v\big\}$. This is a subbundle of principal curvature vectors which are orthogonal to ξ and are associated with λ_i. We then have by Lemma 5.3 that

$$\phi(V_{\lambda_1}) = V_{\lambda_1},\ \phi(V_{\lambda_2}) = V_{\lambda_2},\ \phi(V_{\lambda_3}) = V_{\lambda_4},\ \phi(V_{\lambda_4}) = V_{\lambda_3}.$$

Here, when a real hypersurface does not have the principal curvature corresponding to λ_i in the table, we do not consider V_{λ_i} in the above relations. If we list the multiplicities of principal curvatures, they are as the following Table 2. We denote by $m(\lambda_i)$ and $m(\nu)$ the multiplicities of the principal curvatures λ_i and ν, respectively.

	(A_1)	(A_2)	(B)	(C)	(D)	(E)
$m(\lambda_1)$	$2n-2$	$2n-2\ell-2$	—	2	4	6
$m(\lambda_2)$	—	2ℓ	—	2	4	6
$m(\lambda_3)$	—	—	$n-1$	$n-3$	4	8
$m(\lambda_4)$	—	—	$n-1$	$n-3$	4	8
$m(\nu)$	1	1	1	1	1	1

TABLE 2. Multiplicities of principal curvatures of homogeneous Hopf hypersurfaces in $\mathbb{C}P^n$

For the sake of later use, we here give the inverse images of real hypersurfaces of types (A) and (B) in $\mathbb{C}P^n(4)$ with respect to a Hopf fibration $\varpi : S^{2n+1}(1)(\subset$

$\mathbb{C}^{n+1}) \to \mathbb{C}P^n(4)$. For a geodesic sphere $G(r)$ of radius r in $\mathbb{C}P^n(4)$, it is expressed as

$$\begin{aligned}\varpi^{-1}\big(G(r)\big) &= \big\{(z_0, ..., z_n) \in \mathbb{C}^{n+1} \bigm| |z_0| = \cos r,\ |z_1|^2 + ... + |z_n|^2 = \sin^2 r\big\}\\ &\equiv S^1(1/\cos^2 r) \times S^{2n-1}(1/\sin^2 r) \subset \mathbb{C} \times \mathbb{C}^n\end{aligned}$$

under some isometry of S^{2n+1}. This means that $\varpi^{-1}\big(G(r)\big)$ is isometric to the above set in the right-hand side. For a tube $T_\ell(r)$ around totally geodesic $\mathbb{C}P^\ell(4)$ with $1 \leq \ell \leq n-2$ in $\mathbb{C}P^n(4)$, it is expressed as

$$\begin{aligned}\varpi^{-1}\big(T_\ell(r)\big) &= \left\{(z_0, ..., z_n) \in \mathbb{C}^{n+1} \;\middle|\; \begin{array}{l}|z_0|^2 + ... + |z_\ell|^2 = \cos^2 r,\\ |z_{\ell+1}|^2 + ... + |z_n|^2 = \sin^2 r\end{array}\right\}\\ &\equiv S^{2\ell+1}(1/\cos^2 r) \times S^{2n-2\ell-1}(1/\sin^2 r) \subset \mathbb{C}^{\ell+1} \times \mathbb{C}^{n-\ell}\end{aligned}$$

under some isometry of S^{2n+1}. Thus we see that we may express a geodesic sphere $G(r)$ by $T_0(r)$ and that a tube $T(r) = T_{n-1}(r)$ around totally geodesic $\mathbb{C}P^{n-1}(4)$ is congrunet to $G((\pi/2) - r)$.

For a tube $R(r)$ $(0 < r < \pi/4)$ around totally geodesic $\mathbb{R}P^n(1)$ in $\mathbb{C}P^n(4)$, which is a real hypersurface of type (B), it is expressed as

$$\varpi^{-1}\big(R(r)\big) = \left\{(z_0, ..., z_n) \in \mathbb{C}^{n+1} \;\middle|\; \begin{array}{l}|z_0|^2 + \cdots + |z_n|^2 = 1,\\ |z_0^2 + \cdots + z_n^2| = \cos 2r\end{array}\right\}$$

under some isometry of S^{2n+1}.

5.3. **Standard hypersurfaces in a complex hyperbolic space.** For about homogeneous real hypersurfaces in a complex hyperbolic space, Berndt[20] gave a corresponding result as of M. Kimura.

Proposition 5.2 (Montiel[39], Berndt[20]). *In a complex hyperbolic space $\mathbb{C}H^n(c)$, every Hopf real hypersurface all of whose principal curvatures are constant is congruent to one of the following homogeneous hypersurfaces*:

(A_0) *a horosphere in $\mathbb{C}H^n$,*

($A_{1,0}$) *a geodesic sphere $G(r)$ of radius r*

($A_{1,1}$) *a tube $T(r)$ of radius r around totally geodesic $\mathbb{C}H^{n-1}$,* (A$_2$) *a tube* $T_\ell(r)$ *of radius* r *around totally geodesic* $\mathbb{C}H^\ell(1 \leq \ell \leq n-2)$ *in* $\mathbb{C}H^n$,

(B) *a tube* $R(r)$ *of radius* r *around a totally real totally geodesic* $\mathbb{R}H^n(-c/4)$ *with* $0 < r < \pi/(2\sqrt{|c|})$.

These real hypersurfaces are said to be of types (A$_0$), (A$_1$), (A$_1$), (A$_2$), (B), respectively. We should note that both geodesic spheres and tubes around totally geodesic $\mathbb{C}H^{n-1}$ are called hypersurfaces of type (A$_1$). Gathering real hypersurface of types (A$_0$), (A$_1$) and (A$_2$) together, we call them hypersurfaces of type (A). Their principal curvatures are as in the following Table 3.

	(A$_0$)	(A$_{1,0}$)	(A$_{1,1}$)	(A$_2$)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$
ν	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}\,r)$

TABLE 3. Principal curvatures of homogeneous Hopf hypersurfaces in $\mathbb{C}H^n$

By Lemma 5.3 we have the following. If M is a hypersurface of type (A), then subbundles of principal curvature vectors are invariant under the action of ϕ, that is, $\phi(V_{\lambda_1}) = V_{\lambda_1}$ and $\phi(V_{\lambda_2}) = V_{\lambda_2}$. If M is a hypersurface of type (B), then they satisfy $\phi(V_{\lambda_1}) = V_{\lambda_2}$ and $\phi(V_{\lambda_2}) = V_{\lambda_1}$. We list the multiplicities of principal curvatures.

	(A$_0$)	(A$_{1,0}$)	(A$_{1,1}$)	(A$_2$)	(B)
$m(\lambda_1)$	$2n-2$	$2n-2$	$2n-2$	$2n-2\ell-2$	$n-1$
$m(\lambda_2)$	—	—	—	2ℓ	$n-1$
$m(\nu)$	1	1	1	1	1

TABLE 4. Multiplicities of principal curvatures of homogeneous Hopf hypersurfaces in $\mathbb{C}H^n$

We here give the inverse images of real hypersurfaces of types (A) and (B) with respect to a canonical fibration $\varpi : H_1^{2n+1}\big(\subset \mathbb{C}_1^{n+1}\big) \to \mathbb{C}H^n(-4)$. For a geodesic sphere $G(r)$ of radius r in $\mathbb{C}H^n(-4)$, it is expressed as

$$\begin{aligned}\varpi^{-1}\big(G(r)\big) &= \big\{(z_0, ..., z_n) \in \mathbb{C}^{n+1} \bigm| |z_0| = \cosh r,\ |z_1|^2 + ... + |z_n|^2 = \sinh^2 r\big\}\\ &\equiv S^1(1/\cosh^2 r) \times S^{2n-1}(1/\sinh^2 r) \subset \mathbb{C} \times \mathbb{C}^n\end{aligned}$$

under some isometry of H_1^{2n+1}. For a tube $T(r)$ around totally geodesic $\mathbb{C}H^{n-1}(-4)$ in $\mathbb{C}H^n(-4)$, it is expressed as

$$\begin{aligned}\varpi^{-1}\big(T(r)\big) &= \left\{(z_0, ..., z_n) \in \mathbb{C}^{n+1} \,\middle|\, \begin{array}{c} -|z_0|^2 + |z_1|^2 + ... + |z_{n-1}|^2 = -\cosh^2 r, \\ |z_n| = \sinh r \end{array}\right\}\\ &\equiv H_1^{2n-1}(1/\cosh^2 r) \times S^1(1/\sinh^2 r) \subset \mathbb{C}^n \times \mathbb{C}\end{aligned}$$

under some isometry of H_1^{2n+1}. For a tube $T_\ell(r)$ around totally geodesic $\mathbb{C}H^\ell(-4)$ with $1 \leq \ell \leq n-2$ in $\mathbb{C}H^n(-4)$, it is expressed as

$$\begin{aligned}\varpi^{-1}\big(T_\ell(r)\big) &= \left\{(z_0, ..., z_n) \in \mathbb{C}^{n+1} \,\middle|\, \begin{array}{c} -|z_0|^2 + |z_1|^2 + ... + |z_\ell|^2 = -\cosh^2 r, \\ |z_{\ell+1}|^2 + ... + |z_n|^2 = \sinh^2 r \end{array}\right\}\\ &\equiv H_1^{2\ell+1}(1/\cosh^2 r) \times S^{2n-2\ell-1}(1/\sinh^2 r) \subset \mathbb{C}^{\ell+1} \times \mathbb{C}^{n-\ell}\end{aligned}$$

under some isometry of H_1^{2n+1}. For a horosphere HS in $\mathbb{C}H^n(-4)$, it is a bit different from these tubes and is expressed as

$$\varpi^{-1}\big(HS\big) = \left\{(z_0, ..., z_n) \in \mathbb{C}^{n+1} \,\middle|\, \begin{array}{c} -|z_0|^2 + |z_1|^2 + ... + |z_n|^2 = -1, \\ |z_0 - z_1| = 1 \end{array}\right\}$$

under some isometry of H_1^{2n+1}. For a tube $R(r)$ around totally geodesic $\mathbb{R}H^n(1)$ in $\mathbb{C}H^n(-4)$, it is expressed as

$$\varpi^{-1}\big(R(r)\big) = \left\{(z_0, ..., z_n) \in \mathbb{C}^{n+1} \,\middle|\, \begin{array}{c} -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = -1, \\ |-z_0^2 + z_1^2 + \cdots + z_n^2| = \cosh 2r \end{array}\right\}$$

under some isometry of H_1^{2n+1}.

We should note that there are non-Hopf homogeneous real hypersurfaces in $\mathbb{C}H^n(c)$. Following Berndt-Tamaru [22], a homogeneous real hypersurface in $\mathbb{C}H^n(c)$ is congruent to one of a Hopf real hypersurface of types (A) and (B) or hypersurfaces of the following:

(S) the minimal ruled real hypersurface W obtained by a horocyclic totally real circle (c.f. Adachi-Bao-Maeda [9]),

(W) a tube around the minimal ruled submanifold $W_{\psi}^{2n-\ell}$ for some ψ $(0 < \psi \leq \pi/2)$ and ℓ $(2 \leq \ell \leq n-1)$

(see [23] for more detail).

5.4. **Characterizations of hypersurfaces of type (A).** Since real hypersurfaces of type (A) are quite standard real hypersurfaces in nonflat complex space forms, many geometers studied their properties. We here summarize some of their characterizations which will be used in the following sections. We first give a characterization by the condition that shape operators and characteristic tensors are commutative (see [41], for example).

Lemma 5.4. *A real hypersurface M is of type (A) if and only if its shape operator A and its characteristic tensor field ϕ are simultaneously diagonalizable, that is, $A\phi = \phi A$.*

Proof. Suppose that M is a real hypersurface of type (A). Since the subbundles of principal curvature vectors orthogonal to ξ are invariant under the action of ϕ, if we take a principal curvature vector $v \in V_{\lambda}$, we then have $A\phi v = \lambda\phi v = \phi\lambda v = \phi A v$, This mean that $A\phi = \phi A$.

For about "if" part of this lemma, we only give an outline of proof. If $A\phi = \phi A$ holds, we have $\phi A\xi = A\phi\xi = 0$. Hence ξ is principal, which means that M is a Hopf hypersurface. We then have $(2\lambda - \nu)\lambda = \lambda\nu + (c/2)$ by Lemma 5.3. Thus, when $c > 0$, we have $\lambda = \frac{1}{2}(\nu + \sqrt{\nu^2 + c})$. As ν is locally constant, we find λ is also locally constant. Thus, Takagi's list shows that M is of type (A). When $c < 0$, we need to study the case that $\nu^2 = -c$ at some point. $\square$

On the other hand, it is know that real hypersurfaces of type (A) are characterized by a property on differentials of their shape operators (see [41], for example).

Lemma 5.5. *For a real hypersurface M in a nonflat complex space form $\mathbb{C}M^n(c)$, the following statements are mutually equivalent.*

(1) *The real hypersurface M is of type (A).*

(2) *The equality*

$$\langle(\nabla_X A)Y, Z\rangle = \frac{c}{4}\{-\eta(Y)\langle\phi X, Z\rangle - \eta(Z)\langle\phi X, Y\rangle\}$$

holds for arbitrary vector fields X, Y, Z on the real hypersurface M.

(3) *The shape operator A is cyclic parallel, that is for arbitrary vector fields X, Y, Z on the real hypersurface M the equality*

$$\langle(\nabla_X A)Y, Z\rangle + \langle(\nabla_Y A)Z, X\rangle + \langle(\nabla_Z A)X, Y\rangle = 0$$

holds.

5.5. **Sasakian space forms.** We here make mention of Sasakian space forms, which correspond to complex space forms. An odd dimensional smooth manifold admitting almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ is said to be *Sasakian* if its characteristic tensor ϕ satisfies

$$(\nabla_X \phi)Y = \langle X, Y\rangle\xi - \eta(Y)X$$

for all tangent vectors $X, Y \in TM$. We here note that we loosely use the terminology "Sasakian magnetic fields". Even on a real hypersurface which is not Sasakian we have Sasakian magnetic fields. For a Sasakian manifold M we consider its ϕ-section, a 2-plane of the tangent space T_pM at some point p which is spanned by tangent vectors v, $\phi v \in T_p^0 M$. We say the sectional curvature $\mathrm{Riem}(v, \phi v)$ to be its ϕ-sectional curvature. When M is complete simply connected Sasakian manifold of constant ϕ-sectional curvatures, we call it a *Sasakian space form.*

In his book [24], Blair give Sasakian space forms from the viewpoint of contact geometry. If we review them from the viewpoint of submanifold theory, they are obtained as real hypersurfaces in complex space forms. Sasakian space form of constant ϕ sectional curvature $c+1$ is isometric as manifolds with almost contact metric structures to the following (see Adachi-Kameda-Maeda [11]):

1) When $c > 0$, it is isometric to a geodesic sphere $G\big((2/\sqrt{c})\tan^{-1}(\sqrt{c}/2)\big)$ in $\mathbb{C}P^n(c)$;
2) When $c = 0$, it is isometric to a standard sphere $S^{2n-1}(1)$ in $\mathbb{C}^n$;
3) When $-3 < c < 0$, it is isometric to a geodesic sphere $G\big((2/\sqrt{|c|})\tan^{-1}(\sqrt{|c|}/2\big)$ in $\mathbb{C}H^n(c)$;
4) When $c = -4$, it is isometric to a horosphere HS in $\mathbb{C}H^n(-4)$;
5) When $c < -4$, it is isometric to a tube $T\big((2/\sqrt{|c|})\tan^{-1}(\sqrt{|c|}/2\big)$ around totally geodesic $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n(c)$.

Thus, our study on trajectories on real hypersurfaces of type (A) which will be given in the following sections also shows their behaviors on model spaces in contact geometry.

6. **Trajectories for Sasakian magnetic fields**

We shall now start our study on trajectories for Sasakian magnetic fields. Let M be a real hypersurface in a Kähler manifold $(\widetilde{M}, J, \langle\ ,\ \rangle)$. As we see in §3, the induced almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ defines a canonical closed form $\boldsymbol{F}_\phi$. For a Sasakian magnetic field $\boldsymbol{F}_\kappa = \kappa \boldsymbol{F}_\phi$ $(\kappa \in \mathbb{R})$, a smooth curve γ parameterized by its arclength is a trajectory if it satisfies $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma$. Being different from Kähler magnetic fields, as Sasakian magnetic fields are not uniform in general, trajectories for Sasakian magnetic fields are not so simple as curves.

6.1. **Structure torsions of trajectories.** As the subbundle $T^0M = \big\{v \in TM \bigm| \langle v, \xi\rangle = 0\big\}$ is invariant under the action of J if we regard it as a subbundle of $T\widetilde{M}\big|_M$, we may say that the direction of the characteristic vector ξ is a special direction. For a trajectory γ for a Sasakian magnetic field $\boldsymbol{F}_\kappa$ on M, we define its *structure torsion* ρ_γ by $\rho_\gamma = \langle \dot\gamma, \xi\rangle$. Generally, it is a function along γ. It is very important for investigating properties of trajectories for Sasakian magnetic fields. For example, as we have $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma$, we see its first geodesic curvature $k_1 = \|\nabla_{\dot\gamma}\dot\gamma\|$ is given as

$$k_1 = |\kappa|\,\|\phi\dot\gamma\| = |\kappa|\sqrt{1-\eta\big(\dot\gamma\big)^2} = |\kappa|\sqrt{1-\rho_\gamma^2}.$$

Lemma 6.1. *If the structure torsion of a trajectory γ for a Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a Hopf real hypersurface M is equal to ± 1 at some point $\gamma(t_0)$, then it is a geodesic of unit speed.*

Proof. We take an integral curve σ of a vector field ξ with $\sigma(0) = \gamma(t_0)$, which is a geodesic of unit speed by Lemma 5.1 because M is Hopf. Clearly this satisfies the equation $\nabla_{\dot\sigma}\dot\sigma = \kappa\phi\dot\sigma$ $(= 0)$ and initial condition $\dot\sigma(0) = \xi_{\gamma(t_0)}$. When $\rho_\gamma(t_0) = 1$, the curve γ satisfies $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma$ and $\dot\gamma(t_0) = \xi_{\gamma(t_0)}$. By the theorem on the uniqueness of solutions for ordinary linear differential equations, we find $\sigma(t) = \gamma(t+t_0)$. When $\rho_\gamma(t_0) = -1$, we consider a smooth curve σ_1 given by $\sigma_1(t) = \sigma(-t)$.

It is parameterized by its arclength. Since $\dot\sigma_1(t) = -\dot\sigma(-t)$, this curve satisfies $\nabla_{\dot\sigma_1}\dot\sigma_1(t) = \nabla_{\dot\sigma}\dot\sigma(-t) = 0 = \kappa\phi\dot\sigma_1(t)$ and $\dot\sigma_1(t_0) = -\xi_{\dot\sigma_1(t_0)}$, we find $\gamma(t) = \sigma_1(t+t_0)$. We hence get the conclusion. □

This lemma shows that if the structure torsion ρ_γ of a trajectory γ on a Hopf real hypersurface takes a value ± 1 at some point then it is a constant function: $\rho_\gamma \equiv \pm 1$. We next consider the constancy of structure torsion.

Lemma 6.2. *Let γ be a trajectory for a Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a real hypersurface M in a Kähler manifold $\widetilde{M}$. The derivative of its structure torsion is $\rho'_\gamma = (1/2)\big\langle(\phi A - A\phi)\dot\gamma, \dot\gamma\big\rangle$.*

Proof. By direct computation using (3.3) in Lemma 3.4, we have

$$\rho'_\gamma = \dot\gamma\langle\xi,\dot\gamma\rangle = \langle\nabla_{\dot\gamma}\dot\gamma,\xi\rangle + \langle\dot\gamma,\nabla_{\dot\gamma}\xi\rangle = \langle\kappa\phi\dot\gamma,\xi\rangle + \langle\dot\gamma,\phi A\dot\gamma\rangle = \langle\dot\gamma,\phi A\dot\gamma\rangle.$$

Since A is symmetric and ϕ is skew symmetric, we have $\langle\dot\gamma,\phi A\dot\gamma\rangle = -\langle A\phi\dot\gamma,\dot\gamma\rangle$, hence we see $\rho'_\gamma = \frac{1}{2}\big\langle(\phi A - A\phi)\dot\gamma,\dot\gamma\big\rangle$. □

Corollary 6.1. *The structure torsion ρ_γ of a trajectory for a Sasakian magnetic field is constant along γ if and only if $(A\phi - \phi A)\dot\gamma(t)$ is perpendicular to $\dot\gamma(t)$ at each point $\gamma(t)$. In particular, ρ_γ is constant along γ if $A\phi\dot\gamma(t) = \phi A\dot\gamma(t)$ holds for all t.*

6.2. **Circular trajectories.** Since circles are simple next to geodesics from the viewpoint of Frenet-Serre formula, we here consider conditions for trajectories for Sasakian magnetic fields to be circles.

Lemma 6.3. *Let γ be a trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ ($\kappa \neq 0$) on a real hypersurface M of a Kähler manifold $\widetilde{M}$.*

(1) *It is a geodesic if and only if $\rho_\gamma \equiv \pm 1$.*

(2) *It is a circle of positive geodesic curvature if and only if it satisfies both of the following conditions*:

i) $\rho'_\gamma = 0$ *and* $\rho_\gamma \neq \pm 1$,

ii) $-\kappa\rho_\gamma^2\dot\gamma + \rho_\gamma A\dot\gamma + \big(\kappa\rho_\gamma - \langle A\dot\gamma, \dot\gamma\rangle\big)\xi = 0$.

In this case, its geodesic curvature is $|\kappa|\sqrt{1-\rho_\gamma^2}$.

Proof. As we mentioned above, the first geodesic curvature of γ is $k_1 = |\kappa|\|\phi\dot\gamma\| = |\kappa|\sqrt{1-\rho_\gamma^2}$. Thus we see that γ is a geodesic if and only if $\rho_\gamma \equiv \pm 1$.

We next consider the case $\rho_\gamma \neq \pm 1$. As k_1 is constant if and only if ρ_γ is constant along γ, we calculate the derivative of the second vector field Y_2 in the Frenet frame of γ. In this case, we have $Y_2 = \mathrm{sgn}(\kappa)\,(\phi\dot\gamma)\big/\sqrt{1-\rho_\gamma^2}$, where $\mathrm{sgn}(\kappa)$ denotes the signature of κ. By (3.4) in Lemma 3.4, we get

$$\begin{aligned}\nabla_{\dot\gamma}(\phi\dot\gamma) &= (\nabla_{\dot\gamma}\phi)\dot\gamma + \phi(\nabla_{\dot\gamma}\dot\gamma)\\ &= \langle\dot\gamma, \xi\rangle A\dot\gamma - \langle A\dot\gamma, \dot\gamma\rangle\xi + \kappa\phi^2\dot\gamma\\ &= \rho_\gamma A\dot\gamma - \langle A\dot\gamma, \dot\gamma\rangle\xi - \kappa(\dot\gamma - \rho_\gamma\xi).\end{aligned}$$

Therefore we obtain that $\nabla_{\dot\gamma}Y_2 = -|\kappa|\sqrt{1-\rho_\gamma^2}\,\dot\gamma$ if and only if the condition 2-ii) holds. □

We should note that we do not suppose M is Hopf in Lemma 6.3. Therefore, we can not weaken the condition on structure torsion for geodesic trajectories into a condition at one point. We also note that we do not guarantee the "existence" of trajectories which are also geodesics or circles by this lemma. For example, if there is a trajectory γ for a non-trivial Sasakian magnetic field which is a geodesic, then $\xi_{\gamma(t)}$ is principal at each t. When a trajectory for a Sasakian magnetic field is also a circle of positive geodesic curvature, we shall say that it is a *circular* trajectory. By these lemmas we find that trajectories under Sasakian magnetic fields are not necessarily circles. But we here make mention that in the following sections we show on some real hypersurfaces in a complex space form there exist trajectories for Sasakian magnetic fields which are also circles of positive geodesic curvature.

7. **Circular trajectories on real hypersurfaces of type (A) in $\mathbb{C}P^n$**

Real hypersurfaces of type (A) in a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c ($c > 0$), which are geodesic spheres and tubes around totally geodesic $\mathbb{C}P^\ell$, are typical examples which have been studied in submanifold theory. In this section, we study trajectories for Sasakian magnetic fields on these real hypersurfaces. These hypersurfaces have quite a nice property in our study. By Lemma 5.4 these hypersurfaces are characterized as hypersurfaces whose shape operators and characteristic tensors are simultaneously diagonalizable. We therefore have the following by Corollary 6.1.

Corollary 7.1. *On a real hypersurface M of type (A) in a nonflat complex space form, the structure torsion ρ_γ of an arbitrary trajectory γ for an arbitrary Sasakian magnetic field $\boldsymbol{F}_\kappa$ is constant along γ.*

7.1. **Circular trajectories on geodesic spheres.**

We first consider trajectories on real hypersurfaces of type (A_1), that is geodesic spheres in $\mathbb{C}P^n(c)$. As we mentioned in §5, a geodesic sphere $G(r)$ of radius r in $\mathbb{C}P^n(c)$ has two principal curvatures $\lambda = (\sqrt{c}/2)\cot\big(\sqrt{c}\,r/2\big)$, $\nu = \sqrt{c}\cot\big(\sqrt{c}\,r\big)$. The characteristic vector field ξ satisfies $A\xi = \nu\xi$ and every tangent vector v orthogonal to ξ satisfies $Av = \lambda v$. We can hence rewrite the condition (2)-ii) in Lemma 6.3 by use of these principal curvatures. We here consider more generally and study the feature of all trajectories.

Proposition 7.1. *Let γ be a trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a geodesic sphere $G(r)$ in a complex projective space $\mathbb{C}P^n(c)$.*

(1) *It is a geodesic if and only if $\rho_\gamma = \pm 1$,*

(2) *It is a circle of positive geodesic curvature if and only if*

$$\kappa\rho_\gamma = (\sqrt{c}/2)\cot(\sqrt{c}r/2).$$

In this case, its geodesic curvature is $\sqrt{4\kappa^2 - c\cot^2(\sqrt{c}\,r/2)}/2$.

(3) *Otherwise, it is a helix of proper order* 3 *whose geodesic curvatures are* $k_1 = |\kappa|\sqrt{1-\rho_\gamma^2}$ *and* $k_2 = |\kappa\rho_\gamma - (\sqrt{c}/2)\cot(\sqrt{c}r/2)|$.

Proof. We compute the Frenet-Serre formula of γ. Since we have $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma$, as we see in the proof of Lemma 6.3, we find γ is a geodesic if and only if $\rho_\gamma = \pm 1$.

We next consider the case $\rho_\gamma \neq \pm 1$. We set $Y_2 = \mathrm{sgn}(\kappa)\phi\dot\gamma/\sqrt{1-\rho_\gamma^2}$. On $G(r)$, as it has only two principal curvatures, we have

$$A\dot\gamma = A\big(\rho_\gamma\xi + (\dot\gamma - \rho_\gamma\xi)\big) = \nu\rho_\gamma\xi + \lambda(\dot\gamma - \rho_\gamma\xi).$$

Therefore the second condition in Lemma 6.3 (2) turns to

$$\begin{aligned} 0 &= -\kappa\rho_\gamma^2\dot\gamma + \rho_\gamma A\dot\gamma + \big(\kappa\rho_\gamma - \langle A\dot\gamma, \dot\gamma\rangle\big)\xi \\ &= -\kappa\rho_\gamma^2\dot\gamma + \nu\rho_\gamma^2\xi + \lambda\rho_\gamma(\dot\gamma - \rho_\gamma\xi) + \kappa\rho_\gamma\xi - \big(\nu\rho_\gamma^2 + \lambda(1-\rho_\gamma^2)\big)\xi \\ &= (\kappa\rho_\gamma - \lambda)(\xi - \rho_\gamma\dot\gamma) \end{aligned}$$

Since $\rho_\gamma \neq \pm 1$, we have $\xi \neq \rho_\gamma\dot\gamma$. We hence find that γ is a circle of positive geodesic curvature if and only if $\kappa\rho_\gamma = \lambda$. In this case we can write the system of differential equation for circular trajectories as follows:

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma = \kappa\sqrt{1-\rho_\gamma^2}\,\dfrac{1}{\sqrt{1-\rho_\gamma^2}}\phi\dot\gamma, \\ \nabla_{\dot\gamma}\Big(\dfrac{1}{\sqrt{1-\rho_\gamma^2}}\phi\dot\gamma\Big) = -\kappa\sqrt{1-\rho_\gamma^2}\,\dot\gamma \end{cases}$$

Thus, the geodesic curvature of circular trajectory is $|\kappa|\sqrt{1-\rho_\gamma^2}$. Since we have $\lambda = (\sqrt{c}/2)\cot(\sqrt{c}r/2)$, We get the second assertion.

We now consider the case $\rho_\gamma \neq \pm 1$ and $\kappa\rho_\gamma \neq \lambda$. Our calculation in the proof of Lemma 6.3 and above calculation show that

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma, \\ \nabla_{\dot\gamma}\Big(\dfrac{1}{\sqrt{1-\rho_\gamma^2}}\phi\dot\gamma\Big) = -\kappa\sqrt{1-\rho_\gamma^2}\,\dot\gamma + \dfrac{\kappa\rho_\gamma - \lambda}{\sqrt{1-\rho_\gamma^2}}\,(\xi - \rho_\gamma\dot\gamma). \end{cases}$$

As we have $\nabla_X\xi = \phi AX$ by Lemma 3.4, we get

$$\nabla_{\dot\gamma}(\xi - \rho_\gamma\dot\gamma) = \phi A\dot\gamma - \rho_\gamma\kappa\phi\dot\gamma = (\lambda - \kappa\rho_\gamma)\phi\dot\gamma.$$

Therefore, we obtain

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma & = \kappa\phi\dot\gamma, \\ \nabla_{\dot\gamma}\Big(\dfrac{1}{\sqrt{1-\rho_\gamma^2}}\phi\dot\gamma\Big) & = -\kappa\sqrt{1-\rho_\gamma^2}\,\dot\gamma \ + \ \dfrac{\kappa\rho_\gamma-\lambda}{\sqrt{1-\rho_\gamma^2}}\,(\xi-\rho_\gamma\dot\gamma), \\ \nabla_{\dot\gamma}\Big(\dfrac{1}{\sqrt{1-\rho_\gamma^2}}(\xi-\rho_\gamma\dot\gamma)\Big) & = -\dfrac{\kappa\rho_\gamma-\lambda}{\sqrt{1-\rho_\gamma^2}}\,\phi\dot\gamma, \end{cases}$$

and find that γ is a helix of proper order 3 in this case. Since $\|\xi-\rho_\gamma\dot\gamma\| = \sqrt{1-\rho_\gamma^2}$, we have $Y_3 = \mathrm{sgn}\big(\kappa(\kappa\rho_\gamma-\lambda)\big)(\xi-\rho_\gamma\dot\gamma)/\sqrt{1-\rho_\gamma^2}$ and $k_2 = |\kappa\rho_\gamma-\lambda|$. This complete the proof. □

This proposition guarantees that there exist circular trajectories and "helical" trajectories for Sasakian magnetic fields on geodesic spheres in $\mathbb{C}P^n(c)$. Since structure torsion ρ_γ of a trajectory γ satisfies $|\rho_\gamma|\le 1$, we particularly have the following on circular trajectories.

Theorem 7.1. *Let $\boldsymbol{F}_\kappa$ be a non-trivial Sasakian magnetic field on a geodesic sphere $G(r)$ of radius r $(0<r<\pi/\sqrt{c})$ in $\mathbb{C}P^n(c)$.*

(1) *When $0<|\kappa|\le(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$, there are no circular trajectories for $\boldsymbol{F}_\kappa$.*

(2) *When $|\kappa|>(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$, a trajectory for $\boldsymbol{F}_\kappa$ is circular if and only if its structure torsion is $\rho_\gamma=(\sqrt{c}/2\kappa)\cot(\sqrt{c}\,r/2)$. In this case its geodesic curvature is $\sqrt{\kappa^2-(c/4)\cot^2(\sqrt{c}\,r/2)}$.*

In order to study the amount of circular trajectories on geodesic spheres, we here study their congruency.

Proposition 7.2 (Adachi[3])**.** *Trajectories γ_1 for a Sasakian magnetic field $\boldsymbol{F}_{\kappa_1}$ and γ_2 for $\boldsymbol{F}_{\kappa_2}$ on a geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$ are congruent to each other in strong sense if and only if one of the following conditions holds*:

i) $|\rho_{\gamma_1}|=|\rho_{\gamma_2}|=1$,

ii) $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$ *and* $|\kappa_1| = |\kappa_2|$,

iii) $0 < |\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$ *and* $\kappa_1\rho_{\gamma_1} = \kappa_2\rho_{\gamma_2}$.

In order to show this we here study isometries of geodesic spheres in $\mathbb{C}P^n$. Through an isometric immersion $\iota : G(r) \to \mathbb{C}P^n(c)$ we may consider that $TG(r)$ is a subset of $T\mathbb{C}P^n$.

Lemma 7.1. *Let $x, x' \in G(r)$ be arbitrary points on a geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$. Given unit tangent vectors $u \in \langle\xi_x\rangle^\perp \subset T_xG(r)$ and $u' \in \langle\xi_{x'}\rangle^\perp \subset T_{x'}G(r)$ which are orthogonal to ξ at x and x', there exist isometries $\tilde\varphi^+$, $\tilde\varphi^-$ of $\mathbb{C}P^n(c)$ satisfying the following conditions*:

i) $\tilde\varphi^+\big(G(r)\big) = \tilde\varphi^-\big(G(r)\big) = G(r)$,
(*i.e.* $G(r)$ *is invariant under the actions of* $\tilde\varphi^+$ *and* $\tilde\varphi^-$);

ii) $\tilde\varphi^+(x) = \tilde\varphi^-(x) = x'$;

iii) $d\tilde\varphi^+(u) = d\tilde\varphi^-(u) = u'$;

iv) $d\tilde\varphi^+ \circ J = J \circ d\tilde\varphi^+$ *and* $d\tilde\varphi^- \circ J = -J \circ d\tilde\varphi^-$,
in particular, $d\tilde\varphi^+(\xi_x) = \xi_{x'}$ *and* $d\tilde\varphi^-(\xi_x) = -\xi_{x'}$.

Proof. For the sake of simplicity we only consider the case $n = 2$ and $c = 4$. As we mentioned in §5.2, we may consider that

$$\varpi^{-1}\big(G(r)\big) = \big\{\hat z = (z_0, z_1, z_2) \in \mathbb{C}^3 \;\big|\; |z_0| = \cos r,\ |z_1|^2 + |z_2|^2 = \sin^2 r\big\}$$
$$= S^1 \times S^3 \subset \mathbb{C} \times \mathbb{C}^2.$$

We take an arbitrary point $\hat z = e^{\sqrt{-1}\delta}(\cos r, \alpha e^{\sqrt{-1}\theta}, \beta e^{\sqrt{-1}\psi}) \in \varpi^{-1}\big(G(r)\big)$, where $\alpha, \beta, \delta \in \mathbb{R}$ satisfy $\alpha^2 + \beta^2 = \sin^2 r$. The tangent space of $\widehat M = \varpi^{-1}\big(G(r)\big)$ at this point $\hat z$ is represented by

$$T_{\hat z}\widehat M = \big\{(\hat z, \hat v) \in \{\hat z\} \times \mathbb{C}^3 \;\big|\; \mathrm{Re}(z_0\bar v_0) = \mathrm{Re}(z_1\bar v_1 + z_2\bar v_2) = 0\big\}$$
$$= \Big\{\Big(z, e^{\sqrt{-1}\delta}\big(\sqrt{-1}a, \sqrt{-1}b\alpha e^{\sqrt{-1}\theta} - \beta\zeta e^{-\sqrt{-1}\psi}, \sqrt{-1}b\beta e^{\sqrt{-1}\psi} + \alpha\zeta e^{-\sqrt{-1}\theta}\big)\Big) \;\Big|\; a, b \in \mathbb{R}, \zeta \in \mathbb{C}\Big\}.$$

We denote by $\widehat{\mathcal{N}}_{\hat{z}} \in T_{\hat{z}}S^5$ the horizontal lift of the unit normal vector $\mathcal{N}_{\varpi(\hat{z})}$ of $G(r)$ in $\mathbb{C}P^2(4)$. Since $\widehat{\mathcal{N}}_{\hat{z}}$ is orthogonal to $T_{\hat{z}}\widehat{M}$, we find that it is represented as

$$\widehat{\mathcal{N}}_{\hat{z}} = \Big(\hat{z}, e^{\sqrt{-1}\delta}\big(-\sin r, \alpha \cot r\, e^{\sqrt{-1}\theta}, \beta \cot r\, e^{\sqrt{-1}\psi}\big)\Big).$$

More clearly, if $\hat{z} = (z_0, z_1, z_2)$ then we have $\widehat{\mathcal{N}}_{\hat{z}} = \big(\hat{z}, (-\tan r\, z_0, \cot r\, z_1, \cot r\, z_2)\big)$. We set $\hat{\xi}_{\hat{z}} = -J\widehat{\mathcal{N}}_{\hat{z}}$ with the complex structure J on $\mathbb{C}^3$. We hence have

$$\hat{\xi}_{\hat{z}} = \Big(\hat{z}, -\sqrt{-1}e^{\sqrt{-1}\delta}\big(-\sin r, \alpha \cot r\, e^{\sqrt{-1}\theta}, \beta \cot r\, e^{\sqrt{-1}\psi}\big)\Big).$$

We denote by $\langle \hat{\xi}_{\hat{z}} \rangle$ the real linear subspace of $T_z S^5$ spanned by $\hat{\xi}_{\hat{z}}$, and by $\langle \hat{\xi}_{\hat{z}} \rangle^{\perp}$ its orthogonal complement in $T_{\hat{z}}S^5$. The horizontal part $\langle \hat{\xi}_{\hat{z}} \rangle^{\perp} \cap \mathcal{H}_{\hat{z}}$ of $\langle \hat{\xi}_{\hat{z}} \rangle^{\perp}$ corresponds to $\langle \xi_{\varpi(\hat{z})} \rangle^{\perp}$, and is represented as

$$\begin{aligned}\langle \hat{\xi}_{\hat{z}} \rangle^{\perp} \cap \mathcal{H}_{\hat{z}} &= \big\{(\hat{z}, \hat{v}) \in \{\hat{z}\} \times \mathbb{C}^3 \;\big|\; v_0 = 0,\ z_1\overline{v}_1 + z_2\overline{v}_2 = 0\big\} \\ &= \Big\{\Big(\hat{z}, e^{\sqrt{-1}\delta}\big(0, -\beta\zeta e^{-\sqrt{-1}\psi}, \alpha\zeta e^{-\sqrt{-1}\theta}\big)\Big) \;\Big|\; \zeta \in \mathbb{C}\Big\}.\end{aligned}$$

In order to show the assertion, for arbitrary $\hat{z}, \hat{z}' \in \widehat{M}$ and unit tangent vectors $\hat{u} \in \langle \hat{\xi}_{\hat{z}} \rangle^{\perp} \cap \mathcal{H}_{\hat{z}}$, $\hat{u}' \in \langle \hat{\xi}_{\hat{z}'} \rangle^{\perp} \cap \mathcal{H}_{\hat{z}'}$, we construct isometries $\hat{\varphi}^+$, $\hat{\varphi}^-$ of S^5 satisfying the following conditions:

i) $\hat{\varphi}^+\big(\widehat{M}\big) = \hat{\varphi}^-\big(\widehat{M}\big) = \widehat{M}$,

ii) $\hat{\varphi}^+$ and $\hat{\varphi}^-$ are compatible with the S^1-action on S^5 given by $\hat{p} \mapsto e^{\sqrt{-1}\theta}\hat{p}$, that is, for arbitrary $\hat{p} \in \widehat{M}$ and $\theta \in [0, 2\pi)$ there is $\theta'_{\pm} \in [0, 2\pi)$ satisfying $e^{\sqrt{-1}\theta'_{\pm}}\hat{\varphi}^{\pm}(\hat{p}) = \hat{\varphi}^{\pm}\big(e^{\sqrt{-1}\theta}\hat{p}\big)$,

iii) $\hat{\varphi}^+(\hat{z}) = \hat{\varphi}^-(\hat{z}) = \hat{z}'$ and $d\hat{\varphi}^+(\hat{u}) = d\hat{\varphi}^-(\hat{u}) = \hat{u}'$,

iv) $d\hat{\varphi}^+ \circ \hat{\phi} = \hat{\phi} \circ d\hat{\varphi}^+$, $d\hat{\varphi}^- \circ \hat{\phi} = -\hat{\phi} \circ d\hat{\varphi}^-$, where $\hat{\phi}$ denotes the characteristic tensor on S^5 in $\mathbb{C}^3$,

v) $d\hat{\varphi}^{\pm}(\hat{\xi}_{\hat{z}}) = \pm\hat{\xi}_{\hat{z}'}$.

If we can construct them, then by the second condition we obtain isometries $\tilde{\varphi}^{\pm}$ of $\mathbb{C}P^n(4)$ satisfying $\tilde{\varphi}^{\pm} \circ \varpi = \varpi \circ \hat{\varphi}^{\pm}$. It is clear that these isometries satisfy the desired conditions.

We here take a point $\hat{z}_* = (\cos r, \sin r, 0) \in \widehat{M}$ and a unit tangent vector $\hat{u}_* = \big(\hat{z}_*, (0,0,1)\big) \in \langle \hat{\xi}_{\hat{z}_*}\rangle^{\perp} \cap \mathcal{H}_{\hat{z}_*}$. If we can construct isometries $\hat{\varphi}^{+}_{(\hat{z},\hat{u})}$ and $\hat{\varphi}^{-}_{(\hat{z},\hat{u})}$ of S^5 satisfying the above four conditions on $(\hat{z}_*, \hat{u}_*)$ and $(\hat{z}, \hat{u})$, then the isometries $\hat{\varphi}^{+}_{(\hat{z}',\hat{u}')} \circ (\hat{\varphi}^{+}_{(\hat{z},\hat{u})})^{-1}$ and $\hat{\varphi}^{-}_{(\hat{z}',\hat{u}')} \circ (\hat{\varphi}^{+}_{(\hat{z},\hat{u})})^{-1}$ satisfy the above 4 conditions on $(\hat{z}, \hat{u})$ and $(\hat{z}', \hat{u}')$. We therefore need to construct isometries for $(\hat{z}_*, \hat{u}_*)$ and $(\hat{z}, \hat{u})$ given by

$$\hat{z} = e^{\sqrt{-1}\delta}\big(\cos r, \alpha e^{\sqrt{-1}\theta}, \beta e^{\sqrt{-1}\psi}\big), \quad \hat{u} = \Big(\hat{z}, e^{\sqrt{-1}\delta}\big(0, -\beta\zeta e^{-\sqrt{-1}\psi}, \alpha\zeta e^{-\sqrt{-1}\theta}\big)\Big)$$

with $|\zeta| = 1$. Since the isometry group of $\widehat{M} = S^1 \times S^3 \subset \mathbb{C} \times \mathbb{C}^2$ is $O(2) \oplus O(4)$, we take a unitary matrix

$$U_+ = e^{\sqrt{-1}\delta}\begin{pmatrix} 1 & 0 & 0 \\ 0 & (\alpha/\sin r)e^{\sqrt{-1}\theta} & (-\beta\zeta/\sin r)e^{-\sqrt{-1}\psi} \\ 0 & (\beta/\sin r)e^{\sqrt{-1}\psi} & (\alpha/\sin r)e^{-\sqrt{-1}\theta} \end{pmatrix} \in U(1) \oplus U(2) \subset U(3).$$

We then find this matrix U_+ induces an isometry $\hat{\varphi}^{+}_{(\hat{z},\hat{u})}$ of S^5 which preserves $\widehat{M}$ and satisfies $\hat{\varphi}^{+}_{(\hat{z},\hat{u})}\big(e^{\sqrt{-1}\theta}\hat{p}\big) = e^{\sqrt{-1}\theta}\hat{\varphi}^{+}_{(\hat{z},\hat{u})}(\hat{p})$ for arbitrary $\theta \in [0, 2\pi)$ and $\hat{p} \in \widehat{M}$. As $U_+J = JU_+$ with the matrix $J = \sqrt{-1}E$, where E is the identity, it satisfies $d\hat{\varphi}^{+}_{(\hat{z},\hat{u})} \circ J = J \circ d\hat{\varphi}^{+}_{(\hat{z},\hat{u})}$. We also find that it satisfies $\hat{\varphi}^{+}_{(\hat{z},\hat{u})}(\hat{z}_*) = \hat{z}$, $d\hat{\varphi}^{+}_{(\hat{z},\hat{u})}(\hat{u}_*) = \hat{u}$ and $d\hat{\varphi}^{+}_{(\hat{z},\hat{u})}(\hat{\xi}_{\hat{z}_*}) = \hat{\xi}_{\hat{z}}$.

In order to define $\hat{\varphi}^{-}_{(\hat{z},\hat{u})}$ we consider a matrix

$$\varPsi = \begin{pmatrix} \epsilon & O & O \\ O & \epsilon & O \\ O & O & \epsilon \end{pmatrix} \in O(6) \quad \text{with} \quad \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(2).$$

This matrix induces a map $\mathbb{C}^3 \ni (w_0, w_1, w_2) \mapsto (\bar{w}_0, \bar{w}_1, \bar{w}_2) \in \mathbb{C}^3$. If we take a matrix $U_- = U_+\varPsi$, as $\varPsi J = -J\varPsi$, it induces an isometry $\hat{\varphi}^{-}_{(\hat{z},\hat{u})}$ of $\widehat{M}$. It is clear that it preserves $\widehat{M}$, satisfies $\hat{\varphi}^{+}_{(\hat{z},\hat{u})}\big(e^{\sqrt{-1}\theta}\hat{p}\big) = e^{-\sqrt{-1}\theta}\hat{\varphi}^{+}_{(\hat{z},\hat{u})}(\hat{p})$ for arbitrary $\theta \in [0, 2\pi)$ and $\hat{p} \in \widehat{M}$, and satisfies $d\hat{\varphi}^{+}_{(\hat{z},\hat{u})} \circ J = -J \circ d\hat{\varphi}^{+}_{(\hat{z},\hat{u})}$ and $\hat{\varphi}^{+}_{(\hat{z},\hat{u})}(\hat{z}_*) = \hat{z}$, $d\hat{\varphi}^{+}_{(\hat{z},\hat{u})}(\hat{u}_*) = \hat{u}$, $d\hat{\varphi}^{+}_{(\hat{z},\hat{u})}(\hat{\xi}_{\hat{z}_*}) = -\hat{\xi}_{\hat{z}}$. We hence get the conclusion for the case $n = 2$.

We now make mention of the case $n \geq 3$ breifly. As we may consider that

$$\begin{aligned} \varpi^{-1}\big(G(r)\big) &= \big\{\hat{z} = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \;\big|\; |z_0| = \cos r,\ |z_1|^2 + \cdots + |z_n|^2 = \sin^2 r\big\} \\ &= S^1 \times S^{2n-1} \subset \mathbb{C} \times \mathbb{C}^n \subset \mathbb{C}^{n+1}, \end{aligned}$$

at a point $\hat{z} \in \widehat{M} = \varpi^{-1}\big(G(r)\big)$ its tangent space is represented by

$$T_{\hat{z}}\widehat{M} = \big\{(\hat{z}, \hat{v}) \in \{\hat{z}\} \times \mathbb{C}^{n+1} \;\big|\; \mathrm{Re}(z_0\bar{v}_0) = \mathrm{Re}(z_1\bar{v}_1 + \cdots + z_n\bar{v}_n) = 0\big\}.$$

We can therefore find that the horizontal lift $\widehat{\mathcal{N}}_{\hat{z}}$ of the unit normal vector $\mathcal{N}_{\varpi(\hat{z})}$ of $G(r)$ in $\mathbb{C}P^n(4)$, which is a unit normal vector of $\widehat{M}$ in S^{2n+1}, is given as

$$\widehat{\mathcal{N}}_{\hat{z}} = \big(\hat{z}, (\tan r z_0, -\cot r z_1, \ldots, -\cot r z_n)\big).$$

If we put $\hat{\xi}_{\hat{z}} = -J\widehat{\mathcal{N}}_{\hat{z}}$ with the complex structure J on $\mathbb{C}^{n+1}$, which is a horizontal lift of $\xi_{\varpi(\hat{z})}$, it is given as

$$\hat{\xi}_{\hat{z}} = \Big(\hat{z}, \big(-\sqrt{-1}\tan r z_0, \sqrt{-1}\cot r z_1, \ldots, \sqrt{-1}\cot r z_n\big)\Big).$$

We hence obtain that

$$\langle\hat{\xi}_{\hat{z}}\rangle^{\perp} \cap \mathcal{H}_{\hat{z}} = \big\{(\hat{z}, \hat{v}) \in \{\hat{z}\} \times \mathbb{C}^{n+1} \;\big|\; v_0 = 0,\ z_1\bar{v}_1 + \cdots + z_n\bar{v}_n = 0\big\}.$$

We take a point $\hat{z}_* = (\cos r, \sin r, 0, \ldots, 0) \in \widehat{M}$ and a unit tangent vector $\hat{u}_* = \big(\hat{z}_*, (0, 0, 1, 0, \ldots, 0)\big) \in \langle\hat{\xi}_{\hat{z}_*}\rangle^{\perp} \cap \mathcal{H}_{\hat{z}_*}$. For an arbitrary point $\hat{z} \in \widehat{M}$ and an arbitrary unit tangent vector $\hat{u} \in \langle\hat{\xi}_{\hat{z}}\rangle^{\perp} \cap \mathcal{H}_{\hat{z}}$, we choose unit tangent vectors $\hat{u}^{(2)}, \ldots, \hat{u}^{(n)} \in \langle\hat{\xi}_{\hat{z}}\rangle^{\perp} \cap \mathcal{H}_{\hat{z}}$ so that $\{\hat{u}, \hat{u}^{(2)}, \ldots, \hat{u}^{(n)}\}$ forms a unitary orthonormal basis of $\langle\hat{\xi}_{\hat{z}}\rangle^{\perp} \cap \mathcal{H}_{\hat{z}}$, that is, a basis satisfying $(\!(\hat{u}^{(i)}, \hat{u}^{(j)})\!) = 0$ if $i \neq j$ with $\hat{u}^{(1)} = \hat{u}$. We define a unitary matrix U_+ by

$$U_+ = \begin{pmatrix} z_0/\cos r & 0 & 0 & 0 & \cdots & 0 \\ 0 & z_1/\sin r & u_1 & u_1^{(2)} & \cdots & u_1^{(n)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & z_n/\sin r & u_n & u_n^{(2)} & \cdots & u_n^{(n)} \end{pmatrix} \in U(1) \oplus U(n) \subset U(n+1).$$

As we have $\hat{\xi}_{\hat{z}_*} = \big(\hat{z}_*, (\sin r, -\cos r, 0, \ldots, 0)\big)$, it induces an isometry $\hat{\varphi}^{+}_{(\hat{z},\hat{u})}$ of S^{2n+1} which satisfies the following conditions:

i) $\hat{\varphi}^{+}_{(\hat{z},\hat{u})}\big(\widehat{M}\big) = \widehat{M}$,

ii) $\hat{\varphi}^{+}_{(\hat{z},\hat{u})}\big(e^{\sqrt{-1}\theta}\hat{p}\big) = e^{\sqrt{-1}\theta}\hat{\varphi}^{+}_{(\hat{z},\hat{u})}(\hat{p})$ for arbitrary $\theta \in [0, 2\pi)$ and $\hat{p} \in \widehat{M}$,

iii) $d\hat{\varphi}^{+}_{(\hat{z},\hat{u})}J = Jd\hat{\varphi}^{+}_{(\hat{z},\hat{u})}$, if we consider $TS^{2n+1} \subset T\mathbb{C}^{n+1}$,

iv) $\hat{\varphi}^{+}_{(\hat{z},\hat{u})}(\hat{z}_*) = \hat{z}$,

v) $d\hat{\varphi}^{+}_{(\hat{z},\hat{u})}(\hat{u}_*) = \hat{u},\ d\hat{\varphi}^{+}_{(\hat{z},\hat{u})}(\hat{\xi}_{\hat{z}_*}) = \hat{\xi}_{\hat{z}}$.

Along the same lines as for the case $n = 2$ we can get the conclusion. □

$$
\begin{array}{ccccccccc}
\mathbb{C}^{n+1} & \supset & S^{2n+1} & \supset & \widehat{M} & \xrightarrow{\hat{\varphi}} & \widehat{M} & \subset & S^{2n+1} \\
 & & \downarrow\varpi & & \downarrow\varpi & \circlearrowleft & \downarrow\varpi & & \downarrow \\
 & & \mathbb{C}P^n & \supset & G(r) & \xrightarrow{\varphi} & G(r) & \subset & \mathbb{C}P^n
\end{array}
$$

Remark 7.1. Every isometry φ of $G(r)$ in $\mathbb{C}P^n(c)$ is equivariant. That is, if we denote by $\iota : G(r) \to \mathbb{C}P^n(c)$ an isometric immersion, there is an isometry $\tilde{\varphi}$ of $\mathbb{C}P^n(c)$ satisfying $\tilde{\varphi} \circ \iota = \iota \circ \varphi$.

Proof. An isometry φ induces an isometry $\check{\varphi}$ of $\varpi^{-1}\big(G(r)\big)$ satisfying $\varphi \circ \varpi = \varpi \circ \check{\varphi}$. Since $\varpi^{-1}\big(G(r)\big) \cong S^1 \times S^{2n-1}$, whose isometry group is $O(2) \times O(2n)$, we find $\check{\varphi}$ extends to an isometry of $\mathbb{C}^{n+1}$. Since we have $d\varphi \circ \phi = \pm\phi \circ d\varphi$, we can see that it belongs to $U(1) \times U(n) \subset U(n+1)$. In particular, we have an isometry $\hat{\varphi}$ of S^{2n+1} which satisfies $\hat{\varphi}|_{\varpi^{-1}(G(r))} = \check{\varphi}$ and is compatible with the S^1-action of S^{2n+1}. Thus we obtain an isometry $\tilde{\varphi}$ of $\mathbb{C}P^n$ satisfying $\tilde{\varphi}|_{G(r)} = \varphi$. □

Proof of Proposition 7.2. "Only if" part. Suppose γ_1, γ_2 are congruent to each other. There is an isometry φ of $G(r)$ satisfying $\gamma_2 = \varphi \circ \gamma_1$. When φ satisfies $d\varphi \circ \phi = \phi \circ d\varphi$, that is, φ is a restriction of a holomorphic isometry of $\mathbb{C}P^n$, then we have

$$\kappa_2 \phi \dot{\gamma}_2 = \nabla_{\dot{\gamma}_2} \dot{\gamma}_2 = \nabla_{d\varphi(\dot{\gamma}_1)} d\varphi(\dot{\gamma}_1) = d\varphi\big(\nabla_{\dot{\gamma}_1}\dot{\gamma}_1\big) = d\varphi\big(\kappa_1 \phi \dot{\gamma}_1\big) = \kappa_1 \phi \dot{\gamma}_1,$$

hence $\kappa_1 = \kappa_2$. We also have

$$\rho_{\gamma_2} = \langle \xi_{\gamma_2}, \dot{\gamma}_2 \rangle = \langle \xi_{\varphi\circ\gamma_1}, d\varphi \circ \dot{\gamma}_1 \rangle = \langle d\varphi(\xi_{\gamma_1}), d\varphi(\dot{\gamma}_1) \rangle = \langle \xi_{\gamma_1}, \dot{\gamma}_1 \rangle = \rho_{\gamma_1}.$$

When φ satisfies $d\varphi \circ \phi = -\phi \circ d\varphi$, that is, φ is a restriction of an anti-holomorphic isometry of $\mathbb{C}P^n$, then we have

$$\kappa_2 \phi \dot{\gamma}_2 = \nabla_{\dot{\gamma}_2} \dot{\gamma}_2 = \nabla_{d\varphi(\dot{\gamma}_1)} d\varphi(\dot{\gamma}_1) = d\varphi\big(\nabla_{\dot{\gamma}_1} \dot{\gamma}_1\big) = d\varphi\big(\kappa_1 \phi \dot{\gamma}_1\big) = -\kappa_1 \phi \dot{\gamma}_1,$$

hence $\kappa_1 = -\kappa_2$. We also have

$$\rho_{\gamma_2} = \langle \xi_{\gamma_2}, \dot{\gamma}_2 \rangle = \langle \xi_{\varphi\circ\gamma_1}, d\varphi \circ \dot{\gamma}_1 \rangle = \langle -d\varphi(\xi_{\gamma_1}), d\varphi(\dot{\gamma}_1) \rangle = -\rho_{\gamma_1}.$$

Thus we find that one of three conditions holds.

"If" part. On the other hand, we suppose one of the three conditions holds.

1) If $\rho_{\gamma_1} = \rho_{\gamma_2} = \pm 1$, Lemma 7.1 shows that we have an isometry φ^+ of $G(r)$ satisfying $\varphi^+\big(\gamma_1(0)\big) = \gamma_2(0)$ and $d\varphi^+\big(\dot{\gamma}_1(0)\big) = \dot{\gamma}_2(0)$. If $\rho_{\gamma_1} = -\rho_{\gamma_2} = \pm 1$, we have an isometry φ^- of $G(r)$ satisfying $\varphi^-\big(\gamma_1(0)\big) = \gamma_2(0)$ and $d\varphi^-\big(\dot{\gamma}_1(0)\big) = -\dot{\gamma}_2(0)$. Since γ_1 and γ_2 are geodesics, we find that they are congruent to each other by the uniqueness of solutions of linear differential equations.

2),3) If $\rho_{\gamma_1} = \rho_{\gamma_2}$ and $\kappa_1 = \kappa_2$, by Lemma 7.1, we can take an isometry φ^+ of $G(r)$ which satisfies $\phi \circ d\varphi^+ = d\varphi^+ \circ \phi$ and

$$\varphi^+\big(\gamma_1(0)\big) = \gamma_2(0),\ d\varphi^+\big(\xi_{\gamma_1(0)}\big) = \xi_{\gamma_2(0)},\ d\varphi^+\big(\dot{\gamma}_1(0) - \rho_{\gamma_1}\xi_{\gamma_1(0)}\big) = \dot{\gamma}_2(0) - \rho_{\gamma_2}\xi_{\gamma_2(0)}.$$

We hence have

$$\begin{aligned} d\varphi^+\big(\dot{\gamma}_1(0)\big) &= d\varphi^+\big(\rho_{\gamma_1}\xi_{\gamma_1(0)} + \dot{\gamma}_1(0) - \rho_{\gamma_1}\xi_{\gamma_1(0)}\big) \\ &= \rho_{\gamma_1}\xi_{\gamma_2(0)} + \dot{\gamma}_2(0) - \rho_{\gamma_2}\xi_{\gamma_2(0)} = \dot{\gamma}_2(0). \end{aligned}$$

If $\rho_{\gamma_1} = -\rho_{\gamma_2}$ and $\kappa_1 = -\kappa_2$, by Lemma 7.1, we can take an isometry φ^+ of $G(r)$ which satisfies $\phi \circ d\varphi^+ = -d\varphi^+ \circ \phi$ and

$$\varphi^+\big(\gamma_1(0)\big) = \gamma_2(0),\ d\varphi^+\big(\xi_{\gamma_1(0)}\big) = -\xi_{\gamma_2(0)},\ d\varphi^+\big(\dot{\gamma}_1(0) - \rho_{\gamma_1}\xi_{\gamma_1(0)}\big) = \dot{\gamma}_2(0) - \rho_{\gamma_2}\xi_{\gamma_2(0)}.$$

We hence have

$$\begin{aligned} d\varphi^+\big(\dot{\gamma}_1(0)\big) &= d\varphi^+\big(\rho_{\gamma_1}\xi_{\gamma_1(0)} + \dot{\gamma}_1(0) - \rho_{\gamma_1}\xi_{\gamma_1(0)}\big) \\ &= -\rho_{\gamma_1}\xi_{\gamma_2(0)} + \dot{\gamma}_2(0) - \rho_{\gamma_2}\xi_{\gamma_2(0)} = \dot{\gamma}_2(0). \end{aligned}$$

With the aid of Lemma 3.6 we find they are congruent to each other in strong sense by the uniqueness of solutions of linear differential equations. □

Corollary 7.2. *Every trajectory for an arbitrary Sasakian magnetic field on a geodesic sphere in* $\mathbb{C}P^n(c)$ *is Killing.*

Proof. We take a trajectory γ. By Proposition 7.2, for each t there is an isometry φ_t satisfying $\varphi_t\big(\gamma(0)\big) = \gamma(t), d\varphi_t\big(\dot\gamma(0)\big) = \dot\gamma(t)$ and $d\varphi_t \circ \phi = \phi \circ d\varphi_t$. We find that γ is Killing. □

Corollary 7.3. *Circular trajectories for a given Sasakian magnetic field* $\boldsymbol{F}_\kappa$ *on a geodesic sphere in* $\mathbb{C}P^n(c)$ *are congruent to each other in strong sense.*

Corollary 7.4. *Geodesic trajectories for non-trival Sasakian magnetic fields on a geodesic sphere in* $\mathbb{C}P^n(c)$ *are congruent to each other in strong sense.*

7.2. **Circular trajectories on tubes around** $\mathbb{C}P^\ell$**.** Next we study trajectories for Sasakian magnetic fields on hypersurfaces of type (A_2) in $\mathbb{C}P^n(c)$. A real hypersurface M of type (A_2) in $\mathbb{C}P^n(c)$ is a tube around totally geodesic $\mathbb{C}P^\ell(c)$ ($1 \le \ell \le n-2$) and has three distinct principal curvatures. If its radius is r, that is $M = T_\ell(r)$, they are $\lambda = (\sqrt{c}/2)\cot\big(\sqrt{c}\,r/2\big)$, $\mu = -(\sqrt{c}/2)\tan\big(\sqrt{c}\,r/2\big)$, $\nu = \sqrt{c}\cot\big(\sqrt{c}\,r\big)$. The characteristic vector field ξ satisfies $A\xi = \nu\xi$ and the principal curvatures λ, μ are those for vectors orthogonal to ξ (see §5).

We denote by V_λ, V_μ the two subbundles of the holomorphic distribution T^0M consisted by principal curvature vectors associated with λ, μ, respectively. The tangent space of a real hypersurface $M = T_\ell(r)$ of type (A_2) hence splits as $TM = T^0M \oplus \mathbb{R}\xi = V_\lambda \oplus V_\mu \oplus \mathbb{R}\xi$. In order to classify trajectories on $T_\ell(r)$, we need another invariant for them. We denote the projections of the tangent bundle onto V_λ and V_μ by $\mathrm{Proj}_{V_\lambda} : TT_\ell(r) \to V_\lambda$ and $\mathrm{Proj}_{V_\mu} : TT_\ell(r) \to V_\mu$, respectively. We set

$\tau_\gamma = \|\mathrm{Proj}_{V_\lambda}\big(\dot\gamma\big)\|$ and call it the *principal torsion* of γ. Since we have

$$\|\mathrm{Proj}_{V_\lambda}\big(\dot\gamma(t)\big)\|^2 + \|\mathrm{Proj}_{V_\mu}\big(\dot\gamma(t)\big)\|^2 = 1-\rho_\gamma^2(t),$$

it satisfies $0 \leqq \tau_\gamma \leqq \sqrt{1-\rho_\gamma^2}$.

Lemma 7.2. *The principal torsion τ_γ for a trajectory γ for a Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a real hypersurface $T_\ell(r)$ $(1 \le \ell \le n-2)$ of type (A_2) in $\mathbb{C}P^n(c)$ is constant along γ.*

Proof. By Lemma 5.5 we have

$$\begin{aligned}
\nabla_{\dot\gamma}\langle A\dot\gamma,\dot\gamma\rangle &= \big\langle \big(\nabla_{\dot\gamma}A\big)\dot\gamma,\dot\gamma\big\rangle + \langle \kappa A\phi\dot\gamma,\dot\gamma\rangle + \langle A\dot\gamma,\kappa\phi\dot\gamma\rangle \\
&= -\frac{c}{4}\big\{\langle\phi\dot\gamma,\dot\gamma\rangle\langle\xi,\dot\gamma\rangle + \langle\dot\gamma,\xi\rangle\langle\phi\dot\gamma,\dot\gamma\rangle\big\} + \kappa\langle A\phi\dot\gamma,\dot\gamma\rangle - \kappa\langle\phi A\dot\gamma,\dot\gamma\rangle \\
&= -\frac{c}{2}\,\rho_\gamma\langle\phi\dot\gamma,\dot\gamma\rangle + \kappa\langle(A\phi-\phi A)\dot\gamma,\dot\gamma\rangle \\
&= 0,
\end{aligned}$$

hence $\langle A\dot\gamma,\dot\gamma\rangle$ is constant along γ. As we have $A\dot\gamma = \lambda\mathrm{Proj}_{V_\lambda}\big(\dot\gamma\big)+\mu\mathrm{Proj}_{V_\mu}\big(\dot\gamma\big)+\rho_\gamma\nu\xi$, we find

$$\langle A\dot\gamma,\dot\gamma\rangle = \nu\rho_\gamma^2 + \lambda\tau_\gamma^2 + \mu(1-\rho_\gamma^2-\tau_\gamma^2) = (\nu-\mu)\rho_\gamma^2 - \mu\rho_\gamma^2 + (\lambda-\mu)\tau_\gamma^2.$$

Since ρ_γ is constant along γ by Corollary 7.1, this shows that $(\lambda-\mu)\tau_\gamma^2$ is also constant along γ. As $\lambda \neq \mu$, we find that the principal torsion τ_γ is constant along γ. □

Structure torsions and principal torsions of trajectories determine the congruence classes of trajectories on hypersurfaces of type (A_2). Corresponding to Proposition 7.2, we have the following.

Proposition 7.3. *We consider a hypersurface $T_\ell(r)$ of type (A_2) in a complex projective space $\mathbb{C}P^n(c)$. Trajectories γ_1 for a Sasakian magnetic field $\boldsymbol{F}_{\kappa_1}$ and γ_2 for $\boldsymbol{F}_{\kappa_2}$ on $T_\ell(r)$ are congruent to each other in strong sense if and only if one of the following conditions holds*:

i) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| = 1$,

ii) $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$, $\tau_{\gamma_1} = \tau_{\gamma_2}$ *and* $|\kappa_1| = |\kappa_2|$,

iii) $0 < |\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$, $\tau_{\gamma_1} = \tau_{\gamma_2}$ *and* $\kappa_1\rho_{\gamma_1} = \kappa_2\rho_{\gamma_2}$.

We decompose the tangent space $T_xT_\ell(r)$ of a real hypersurface $T_\ell(r)$ of type (A_2) at x as $T_xT_\ell(r) = V_{\lambda,x} \oplus V_{\mu,x} \oplus \mathbb{R}\xi_x$, where $V_{\lambda,x}$ and $V_{\mu,x}$ are the subspaces of principal curvature vectors orthogonal to ξ_x which correspond to principal curvatures λ and μ, respectively. Through an isometric immersion $\iota : T_\ell(r) \to \mathbb{C}P^n(c)$ we consider $TT_\ell(r)$ as a subset of $T\mathbb{C}P^n(c)$.

Lemma 7.3. *Let* $x, x' \in T_\ell(r)$ *be arbitrary points on a hypersurface* $T_\ell(r)$ *of type* (A_2) *in* $\mathbb{C}P^n(c)$. *Given unit tangent vectors* $u \in V_{\lambda,x}$, $w \in V_{\mu,x}$ *and* $u' \in V_{\lambda,x'}$, $w' \in V_{\mu,x'}$, *there exist isometries* $\tilde\varphi^+$, $\tilde\varphi^-$ *of* $\mathbb{C}P^n(c)$ *satisfying the following conditions*:

i) $\tilde\varphi^+\big(T_\ell(r)\big) = \tilde\varphi^-\big(T_\ell(r)\big) = T_\ell(r)$,

(*i.e.* $T_\ell(r)$ *is invariant under the actions of* $\tilde\varphi^+$ *and* $\tilde\varphi^-$);

ii) $\tilde\varphi^+(x) = \tilde\varphi^-(x) = x'$;

iii) $d\tilde\varphi^+(u) = d\tilde\varphi^-(u) = u'$ *and* $d\tilde\varphi^+(w) = d\tilde\varphi^-(w) = w'$

iv) $d\tilde\varphi^+ \circ J = J \circ d\tilde\varphi^+$ *and* $d\tilde\varphi^- \circ J = -J \circ d\tilde\varphi^-$,

in particular, $d\tilde\varphi^+(\xi_x) = \xi_{x'}$ *and* $d\tilde\varphi^-(\xi_x) = -\xi_{x'}$.

Proof. For the sake of simplicity, we are enough to consider the case $n = 3$, $\ell = 1$ and $c = 4$. As we see in §5.2 we may consider that

$$\begin{aligned}\varpi^{-1}\big(T_1(r)\big) &= \big\{\hat z = (z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \;\big|\; |z_0|^2 + |z_1|^2 = \cos^2 r,\ |z_2|^2 + |z_3|^2 = \sin^2 r\big\} \\ &= S^3 \times S^3 \subset \mathbb{C}^2 \times \mathbb{C}^2.\end{aligned}$$

We take an arbitrary point $\hat z = (z_0, z_1, z_2, z_3) \in \varpi^{-1}\big(T_1(r)\big)$. The tangent space of $\widehat{M} = \varpi^{-1}\big(T_1(r)\big)$ at $\hat z$ is represented as

$$T_{\hat z}\widehat{M} = \left\{ (\hat z, \hat v) \in \{\hat z\} \times \mathbb{C}^4 \;\middle|\; \begin{array}{c} \mathrm{Re}(z_0\overline{v}_0 + z_1\overline{v}_1) = 0, \\ \mathrm{Re}(z_2\overline{v}_2 + z_3\overline{v}_3) = 0 \end{array} \right\}.$$

We denote by $\widehat{\mathcal{N}}_{\hat{z}} \in T_{\hat{z}}S^7$ the horizontal lift of the unit normal vector $\mathcal{N}_{\varpi(\hat{z})}$ of $T_1(r)$ in $\mathbb{C}P^3(4)$. Since $\widehat{\mathcal{N}}_{\hat{z}} \in T_{\hat{z}}S^7$ and is orthogonal to $T_{\hat{z}}\widehat{M}$, considering on each component we find that it is represented as

$$\widehat{\mathcal{N}}_{\hat{z}} = \big(\hat{z}, (\tan r\, z_0, \tan r\, z_1, -\cot r\, z_2, -\cot r\, z_3)\big).$$

We put $\hat{\xi}_{\hat{z}} = -J\widehat{\mathcal{N}}_{\hat{z}}$. We denote by $\langle\hat{\xi}_{\hat{z}}\rangle$ the real linear subspace of $T_{\hat{z}}S^7$ spanned by $\hat{\xi}_{\hat{z}}$, and by $\langle\hat{\xi}_{\hat{z}}\rangle^\perp$ its orthogonal complement in $T_{\hat{z}}S^7$. The horizontal part $\langle\hat{\xi}_{\hat{z}}\rangle^\perp \cap \mathcal{H}_{\hat{z}}$ of $\langle\hat{\xi}_{\hat{z}}\rangle^\perp$ corresponds to the complex vector space $\langle\xi_{\varpi(\hat{z})}\rangle^\perp$, and is represented as

$$\langle\hat{\xi}_{\hat{z}}\rangle^\perp \cap \mathcal{H}_{\hat{z}} = \big\{(\hat{z}, \hat{v}) \in \{\hat{z}\} \times \mathbb{C}^3 \;\big|\; z_0\overline{v}_0 + z_1\overline{v}_1 = 0,\ z_2\overline{v}_2 + z_3\overline{v}_3 = 0\big\}.$$

We should note that if we decompose $\mathcal{H}_{\hat{z}}$ as $\mathcal{H}_{\hat{z}} = \widehat{V}_{\lambda,\hat{z}} \oplus \widehat{V}_{\mu,\hat{z}} \oplus \mathbb{R}\hat{\xi}_{\hat{z}}$ corresponding to the decomposition of $T_{\varpi(\hat{z})}T_1(r)$ into subspaces of principal curvature vectors then we see

$$\widehat{V}_{\lambda,\hat{z}} = \big\{(\hat{z}, \hat{v}) \in \{\hat{z}\} \times \mathbb{C}^3 \;\big|\; z_2\overline{v}_2 + z_3\overline{v}_3 = 0,\ v_0 = v_1 = 0\big\},$$
$$\widehat{V}_{\mu,\hat{z}} = \big\{(\hat{z}, \hat{v}) \in \{\hat{z}\} \times \mathbb{C}^3 \;\big|\; z_0\overline{v}_0 + z_1\overline{v}_1 = 0,\ v_2 = v_3 = 0\big\}.$$

We take a point $\hat{z}_* = (\cos r, 0, \sin r, 0) \in \widehat{M}$ and unit tangent vectors $\hat{u}_* = \big(\hat{z}_*, (0,0,0,1)\big) \in \widehat{V}_{\lambda,\hat{z}_*}$, $\hat{w}_* = \big(\hat{z}, (0,1,0,0)\big) \in \widehat{V}_{\mu,\hat{z}_*}$. For an arbitrary $\hat{z} \in \widehat{M}$ and unit tangent vectors $\hat{u} = \big(\hat{z}, (0,0,u_2,u_3)\big) \in \widehat{V}_{\lambda,\hat{z}}$, $\hat{w} = \big(\hat{z}, (w_0,w_1,0,0)\big) \in \widehat{V}_{\mu,\hat{z}}$, which are expressed as

$$\hat{u} = \Big(\hat{z}, \Big(0, 0, \frac{\zeta_1\overline{z}_3}{\cos r}, -\frac{\zeta_1\overline{z}_2}{\cos r}\Big)\Big),\ \hat{w} = \Big(\hat{z}, \Big(\frac{\zeta_2\overline{z}_1}{\sin r}, -\frac{\zeta_2\overline{z}_0}{\sin r}, 0, 0\Big)\Big)$$

with some $\zeta_1, \zeta_2 \in \mathbb{C}$ satisfying $|\zeta_1| = |\zeta_2| = 1$, we define a unitary matrix

$$U_+ = \begin{pmatrix} z_0/\cos r & w_0 & 0 & 0 \\ z_1/\cos r & w_1 & 0 & 0 \\ 0 & 0 & z_2/\sin r & u_2 \\ 0 & 0 & z_3/\sin r & u_3 \end{pmatrix} \in U(2) \oplus U(2) \subset U(4).$$

This induces a linear transformation of $\mathbb{C}^4$ which preserves the Hermitian inner product $\langle\!\langle\ ,\ \rangle\!\rangle$, hence it induces an isometry $\hat{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}$ of S^7. It clearly satisfies $\hat{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}\big(\widehat{M}\big) = \widehat{M}$ and $\hat{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}\big(e^{\sqrt{-1}\theta}\hat{p}\big) = e^{\sqrt{-1}\theta}\hat{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}(\hat{p})$ for arbitrary $\theta \in [0, 2\pi)$

and $\hat{p} \in \widehat{M}$. Therefore $\hat{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}$ induces an isometry $\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}$ of $\mathbb{C}P^3(4)$ satisfying

$$\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})} \circ \varpi = \varpi \circ \hat{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}, \quad \tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}\big(\varpi(\hat{z}_*)\big) = \varpi(\hat{z}),$$
$$d\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}\big(d\varpi(\hat{u}_*)\big) = d\varpi(\hat{u}), \quad d\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}\big(d\varpi(\hat{w}_*)\big) = d\varpi(\hat{w}).$$

Since we have $U_+J = JU_+$ for the matrix $J = \sqrt{-1}E$, we find that $\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}$ is holomorphic, that is, $d\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})} \circ J = J \circ d\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}$. In particular, it satisfies $d\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}\big(\xi_{\varpi(\hat{z}_*)}\big) = \xi_{\varpi(\hat{z})}$.

We next consider a matrix

$$\Psi = \begin{pmatrix} \epsilon & O & O & O \\ O & \epsilon & O & O \\ O & O & \epsilon & O \\ O & O & O & \epsilon \end{pmatrix} \in O(8) \quad \text{with} \quad \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(2).$$

This matrix induces a map $\mathbb{C}^4 \ni (p_0, p_1, p_2, p_3) \mapsto (\bar{p}_0, \bar{p}_1, \bar{p}_2, \bar{p}_3) \in \mathbb{C}^4$. If we define a matrix U_- by $U_- = U_+\Psi$, it induces a linear transformation of $\mathbb{C}^4$ which preserves the Hermitian inner product. By the representation of $\widehat{M}$, we see it induces an isometry $\hat{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}$ of S^7 satisfying $\hat{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}\big(\widehat{M}\big) = \widehat{M}$. It is clear that it satisfies $\hat{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}\big(e^{\sqrt{-1}\theta}\hat{p}\big) = e^{-\sqrt{-1}\theta}\hat{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}(\hat{p})$ for arbitrary $\theta \in [0, 2\pi)$ and $\hat{p} \in \widehat{M}$. As we have $U_-J = JU_-$ for the matrix $J = \sqrt{-1}E$, we find that $\hat{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}$ induces an isometry $\tilde{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}$ of $\mathbb{C}P^3(4)$ satisfying

$$\tilde{\varphi}^-_{(\hat{z},\hat{u},\hat{w})} \circ \varpi = \varpi \circ \hat{\varphi}^-_{(z,u,w)}, \quad d\tilde{\varphi}^-_{(\hat{z},\hat{u},\hat{w})} \circ J = -J \circ d\tilde{\varphi}^-_{(\hat{z},\hat{u},\hat{w})},$$
$$\tilde{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}(\varpi(\hat{z}_*)) = \varpi(\hat{z}),$$
$$d\tilde{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}(d\varpi(\hat{u}_*)) = d\varpi(\hat{u}), \quad d\tilde{\varphi}^-_{((\hat{z},\hat{u},\hat{w}))}(d\varpi(\hat{w}_*)) = d\varpi(\hat{w}).$$

In particular, it satisfies $d\tilde{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}\big(\xi_{\varpi(\hat{z}_*)}\big) = -\xi_{\varpi(\hat{z})}$.

As we constructed desirable isometries for a fixed triplet $\big(\varpi(\hat{z}_*), d\varpi(\hat{u}_*), d\varpi(\hat{w}_*)\big)$ and an arbitrary triplet $\big(\varpi(\hat{z}), d\varpi(\hat{u}), d\varpi(\hat{w})\big)$, we can get our conclusion along the same lines as in the proof of Lemma 7.1. □

Remark 7.2. Every isometry of $T_\ell(r)$ in $\mathbb{C}P^n(c)$ is equivariant.

Proof of Proposition 7.3. "Only if" part. Suppose γ_1, γ_2 are congruent to each other. There is an isometry φ satisfying $\gamma_2 = \varphi \circ \gamma_1$. Since φ is equivariant, we see by Lemma 3.5, it preserves the decomposition of the holomorphic distribution $T^0T_\ell(r)$ into subbundles of principal curvature vectors. Hence we find $\tau_{\gamma_1} = \tau_{\gamma_2}$. When φ satisfies $d\varphi \circ \phi = \phi \circ d\varphi$, we find that $\kappa_1 = \kappa_2$, $\rho_{\gamma_1} = \rho_{\gamma_2}$, and when φ satisfies $d\varphi \circ \phi = -\phi \circ d\varphi$, we find that $\kappa_1 = -\kappa_2$, $\rho_{\gamma_1} = -\rho_{\gamma_2}$, by the same way as in the proof of Proposition 7.2.

"If" part. On the other hand, we suppose one of the three conditions holds.

1) If $\rho_{\gamma_1} = \rho_{\gamma_2} = \pm 1$, Lemma 7.3 shows that we have an isometry φ^+ of $G(r)$ satisfying $\varphi^+\big(\gamma_1(0)\big) = \gamma_2(0)$ and $d\varphi^+\big(\dot\gamma_1(0)\big) = \dot\gamma_2(0)$. If $\rho_{\gamma_1} = -\rho_{\gamma_2} = \pm 1$, we have an isometry φ^- of $G(r)$ satisfying $\varphi^-\big(\gamma_1(0)\big) = \gamma_2(0)$ and $d\varphi^-\big(\dot\gamma_1(0)\big) = -\dot\gamma_2(0)$. Since γ_1 and γ_2 are geodesics, we find that they are congruent to each other by the uniquness of solutions of linear differential equations.

2),3) If $\rho_{\gamma_1} = \rho_{\gamma_2}$ and $\kappa_1 = \kappa_2$, by Lemma 7.1, we can take an isometry φ^+ of $G(r)$ which satisfies $\phi \circ d\varphi^+ = d\varphi^+ \circ \phi$ and

$$\varphi^+\big(\gamma_1(0)\big) = \gamma_2(0),\ d\varphi^+\big(\xi_{\gamma_1(0)}\big) = \xi_{\gamma_2(0)},$$

$$d\varphi^+\big(\mathrm{Proj}_{V_\lambda}(\dot\gamma_1(0))\big) = \mathrm{Proj}_{V_\lambda}(\dot\gamma_2(0)),\ d\varphi^+\big(\mathrm{Proj}_{V_\mu}(\dot\gamma_1(0))\big) = \mathrm{Proj}_{V_\mu}(\dot\gamma_2(0)).$$

We hence have

$$\begin{aligned} d\varphi^+\big(\dot\gamma_1(0)\big) &= d\varphi^+\big(\mathrm{Proj}_{V_\lambda}(\dot\gamma_1(0)) + \mathrm{Proj}_{V_\mu}(\dot\gamma_1(0)) + \rho_{\gamma_1}\xi_{\gamma_1(0)}\big) \\ &= \mathrm{Proj}_{V_\lambda}(\dot\gamma_2(0)) + \mathrm{Proj}_{V_\mu}(\dot\gamma_2(0)) + \rho_{\gamma_1}\xi_{\gamma_2(0)} = \dot\gamma_2(0). \end{aligned}$$

If $\rho_{\gamma_1} = -\rho_{\gamma_2}$ and $\kappa_1 = -\kappa_2$, by Lemma 7.1, we can take an isometry φ^- of $G(r)$ which satisfies $\phi \circ d\varphi^- = -d\varphi^- \circ \phi$ and

$$\varphi^-\big(\gamma_1(0)\big) = \gamma_2(0),\ d\varphi^-\big(\xi_{\gamma_1(0)}\big) = -\xi_{\gamma_2(0)},$$

$$d\varphi^-\big(\mathrm{Proj}_{V_\lambda}(\dot\gamma_1(0))\big) = \mathrm{Proj}_{V_\lambda}(\dot\gamma_2(0)),\ d\varphi^-\big(\mathrm{Proj}_{V_\mu}(\dot\gamma_1(0))\big) = \mathrm{Proj}_{V_\mu}(\dot\gamma_2(0)).$$

We hence have

$$\begin{aligned} d\varphi^-\big(\dot\gamma_1(0)\big) &= d\varphi^-\big(\mathrm{Proj}_{V_\lambda}(\dot\gamma_1(0)) + \mathrm{Proj}_{V_\mu}(\dot\gamma_1(0)) + \rho_{\gamma_1}\xi_{\gamma_1(0)}\big) \\ &= \mathrm{Proj}_{V_\lambda}(\dot\gamma_2(0)) + \mathrm{Proj}_{V_\mu}(\dot\gamma_2(0)) - \rho_{\gamma_1}\xi_{\gamma_2(0)} = \dot\gamma_2(0). \end{aligned}$$

With the aid of Lemma 3.6 we find they are congruent to each other by the uniquness of solutions of linear differential equations. □

Corollary 7.5. *Every trajectory for an arbitrary Sasakian magnetic field on a hypersurface of type (A_2) in $\mathbb{C}P^n(c)$ is Killing.*

We are now in the position to investigate circular trajectories on hypersurfaces of type (A_2). We apply Lemma 6.3.

Proposition 7.4. *Let γ be a trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a hypersurface of type (A_2) in a complex projective space $\mathbb{C}P^n(c)$.*

(1) *It is a geodesic if and only if $\rho_\gamma = \pm 1$,*
(2) *It is a circle of positive geodesic curvature if and only if one of the following conditions holds*:
  i) $\tau_\gamma = 0$ *and* $\kappa\rho_\gamma = -(\sqrt{c}/2)\tan\big(\sqrt{c}\,r/2\big)$,
  ii) $\tau_\gamma = \sqrt{1-\rho_\gamma^2}$ *and* $\kappa\rho_\gamma = (\sqrt{c}/2)\cot\big(\sqrt{c}\,r/2\big)$,
  iii) $\rho_\gamma = 0$ *and* $\tau_\gamma = \sin\big(\sqrt{c}\,r/2\big)$.
  In these cases, the geodesic curvature of γ is $|\kappa|\sqrt{1-\rho_\gamma^2}$.

Proof. As structure torsions of trajectories are constant function by Corollary 7.1, we need to study the condition (2)-ii) in Lemma 6.3. Thus we study the case that $|\rho_\gamma| < 1$. As we calculated in the proof of Lemma 7.2, we have

$$A\dot\gamma = \lambda \mathrm{Proj}_{V_\lambda}\big(\dot\gamma\big) + \mu \mathrm{Proj}_{V_\mu}\big(\dot\gamma\big) + \rho_\gamma \nu\xi,$$
$$\langle A\dot\gamma, \dot\gamma\rangle = (\nu-\mu)\rho_\gamma^2 - \mu\rho_\gamma^2 + (\lambda-\mu)\tau_\gamma^2.$$

We therefore find that the condition (2)-ii) in Lemma 6.3 turns to

$$\begin{aligned} &\rho_\gamma(\lambda-\kappa\rho_\gamma)\,\mathrm{Proj}_{V_\lambda}(\dot\gamma) + \rho_\gamma(\mu-\kappa\rho_\gamma)\,\mathrm{Proj}_{V_\mu}(\dot\gamma) \\ &\qquad + \{\kappa\rho_\gamma - \kappa\rho_\gamma^3 - \lambda\tau_\gamma^2 - \mu(1-\rho_\gamma^2-\tau_\gamma^2)\}\xi = 0. \end{aligned} \tag{7.1}$$

We first consider the case of $\mathrm{Proj}_{V_\lambda}(\dot\gamma)=0$, which is the case that $\tau_\gamma=0$. In this case, the above equality (7.1) turns to

$$(\kappa\rho_\gamma-\mu)\big\{-\rho_\gamma\mathrm{Proj}_{V_\mu}(\dot\gamma)+(1-\rho_\gamma^2)\xi\big\}=0.$$

As we have $|\rho_\gamma|<1$, this shows $\kappa\rho_\gamma=\mu$.

Next we consider the case of $\mathrm{Proj}_{V_\mu}(\dot\gamma)=0$, which is the case that $\tau_\gamma=\sqrt{1-\rho_\gamma^2}$. The equality (7.1) turns to

$$(\kappa\rho_\gamma-\lambda)\big\{-\rho_\gamma\mathrm{Proj}_{V_\lambda}(\dot\gamma)+(1-\rho_\gamma^2)\xi\big\}=0.$$

Hence we have $\kappa\rho_\gamma=\lambda$.

Finally we consider the case of $0<\tau_\gamma<\sqrt{1-\rho_\gamma^2}$. In this case, as $\mathrm{Proj}_{V_\lambda}(\dot\gamma)$, $\mathrm{Proj}_{V_\mu}(\dot\gamma)$ and ξ are linearly independent, we get

$$\begin{cases}\rho_\gamma(\lambda-\kappa\rho_\gamma)=0,\\ \rho_\gamma(\mu-\kappa\rho_\gamma)=0,\\ \kappa\rho_\gamma-\kappa\rho_\gamma^3-\lambda\tau_\gamma^2-\mu(1-\rho_\gamma^2-\tau_\gamma^2)=0,\end{cases}$$

Since $\lambda\neq\mu$, we get $\rho_\gamma=0$ from the first two equalities. Hence we get $\lambda\tau_\gamma^2+\mu(1-\tau_\gamma^2)=0$. We here substitute principal curvatures to this condition. We have

$$0=\cot\Big(\frac{\sqrt{c}\,r}{2}\Big)\tau_\gamma^2-\tan\Big(\frac{\sqrt{c}\,r}{2}\Big)(1-\tau_\gamma^2)=\frac{\tau_\gamma^2}{\sin\big(\frac{\sqrt{c}\,r}{2}\big)\cos\big(\frac{\sqrt{c}\,r}{2}\big)}-\tan\Big(\frac{\sqrt{c}\,r}{2}\Big),$$

which shows $\tau_\gamma^2=\sin^2\big(\sqrt{c}\,r/2\big)$. As $\tau_\gamma\geq 0$, we get the conclusion. □

Corollary 7.6. *For a given Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a hypersurface of type (A_2) in $\mathbb{C}P^n(c)$, we have at most three congruence classes of circular trajectories in strong sense.*

Corollary 7.7. *Geodesic trajectories for non-trivial Sasakian magnetic fields on a hypersurface of type (A_2) in $\mathbb{C}P^n(c)$ are congruent to each other in strong sense.*

As we have $\cot\big(\sqrt{c}\,r/2\big)>\tan\big(\sqrt{c}\,r/2\big)$ when $0<r<\pi/(2\sqrt{c})$ and $\cot\big(\sqrt{c}\,r/2\big)<\tan\big(\sqrt{c}\,r/2\big)$ when $\pi/(2\sqrt{c})<r<\pi/\sqrt{c}$, the situation changes whether the radius r

of tube is less than, equal to or greater than $\pi/(2\sqrt{c})$. Since the geodesic curvature of a circular trajectory γ for $\boldsymbol{F}_\kappa$ is $|\kappa|\sqrt{1-\rho_\gamma^2}$, we obtain the following results.

Theorem 7.2. *Let $\boldsymbol{F}_\kappa$ be a non-trivial Sasakian magnetic field on a tube $T_\ell(r)$ of radius r $(0 < r < \pi/(2\sqrt{c}))$ around totally geodesic $\mathbb{C}P^\ell(c)$ $(1 \le \ell \le n-2)$ in $\mathbb{C}P^n(c)$.*

(1) *A trajectory γ for $\boldsymbol{F}_\kappa$ with $\rho_\gamma = 0$ and $\tau_\gamma = \sin(\sqrt{c}\,r/2)$ is circular. In this case its geodesic curvature is $|\kappa|$.*

(2) *When $0 < |\kappa| \le (\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$, there are no circular trajectories for $\boldsymbol{F}_\kappa$ other than those in (1).*

(3) *When $(\sqrt{c}/2)\tan(\sqrt{c}\,r/2) < |\kappa| \le (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$, a trajectory for $\boldsymbol{F}_\kappa$ is circular if and only if it satisfies either the condition in (1) or $\rho_\gamma = -(\sqrt{c}/(2\kappa))\tan(\sqrt{c}\,r/2)$ and $\tau_\gamma = 0$. In the latter case its geodesic curvature is $\sqrt{\kappa^2 - (c/4)\tan^2(\sqrt{c}\,r/2)}$.*

(4) *When $|\kappa| > (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$, a trajectory for $\boldsymbol{F}_\kappa$ is circular if and only if it satisfies one of the following;*
   i) *the condition in (1),*
   ii) *$\rho_\gamma = -(\sqrt{c}/(2\kappa))\tan(\sqrt{c}\,r/2)$ and $\tau_\gamma = 0$,*
   iii) *$\rho_\gamma = (\sqrt{c}/(2\kappa))\cot(\sqrt{c}\,r/2)$ and $\tau_\gamma = \sqrt{1-\rho_\gamma^2}$.*

   In the second case, its geodesic curvature is $\sqrt{\kappa^2 - (c/4)\tan^2(\sqrt{c}\,r/2)}$, and in the third case, its geodesic curvature is $\sqrt{\kappa^2 - (c/4)\cot^2(\sqrt{c}\,r/2)}$.

Theorem 7.3. *Let $\boldsymbol{F}_\kappa$ be a non-trivial Sasakian magnetic field on a tube $T_\ell(\pi/2\sqrt{c})$ $(1 \le \ell \le n-2)$ in $\mathbb{C}P^n(c)$.*

(1) *A trajectory γ for $\boldsymbol{F}_\kappa$ with $\rho_\gamma = 0$ and $\tau_\gamma = \sqrt{2}/2$ is circular. In this case its geodesic curvature is $|\kappa|$.*

(2) *When $0 < |\kappa| \le \sqrt{c}/2$, there are no circular trajectories fot $\boldsymbol{F}_\kappa$ other than those in (1).*

(3) *When* $|\kappa| > \sqrt{c}/2$, *a trajectory for* $\boldsymbol{F}_\kappa$ *is circular if and only if it satisfies one of the following*;

i) *the condition in* (1),

ii) $\rho_\gamma = -\sqrt{c}/(2\kappa)$ *and* $\tau_\gamma = 0$,

iii) $\rho_\gamma = \sqrt{c}/(2\kappa)$ *and* $\tau_\gamma = \sqrt{1-\rho_\gamma^2}$.

In the second and the third cases its geodesic curvature is $\sqrt{\kappa^2 - (c/4)}$.

Theorem 7.4. *Let* $\boldsymbol{F}_\kappa$ *be a non-trivial Sasakian magnetic field on a tube* $T_\ell(r)$ *of radius* r $(\pi/(2\sqrt{c}) < r < \pi/\sqrt{c})$ *around totally geodesic* $\mathbb{C}P^\ell(c)$ $(1 \leq \ell \leq n-2)$ *in* $\mathbb{C}P^n(c)$.

(1) *A trajectory* γ *for* $\boldsymbol{F}_\kappa$ *with* $\rho_\gamma = 0$ *and* $\tau_\gamma = \sin(\sqrt{c}\,r/2)$ *is circular. In this case its geodesic curvature is* $|\kappa|$.

(2) *When* $0 < |\kappa| \leq (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$, *there are no circular trajectories for* $\boldsymbol{F}_\kappa$ *other than those in* (1).

(3) *When* $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) < |\kappa| \leq (\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$, *a trajectory for* $\boldsymbol{F}_\kappa$ *is circular if and only if it satisfies either the condition in* (1) *or* $\rho_\gamma = (\sqrt{c}/2\kappa)\cot(\sqrt{c}\,r/2)$ *and* $\tau_\gamma = \sqrt{1-\rho_\gamma^2}$. *In the latter case its geodesic curvature is* $\sqrt{\kappa^2 - (c/4)\cot^2(\sqrt{c}\,r/2)}$.

(4) *When* $|\kappa| > (\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$, *a trajectory for* $\boldsymbol{F}_\kappa$ *is circular if and only if it satisfies one of the following*;

i) *the condition in* (1),

ii) $\rho_\gamma = -(\sqrt{c}/2\kappa)\tan(\sqrt{c}\,r/2)$ *and* $\tau_\gamma = 0$,

iii) $\rho_\gamma = (\sqrt{c}/2\kappa)\cot(\sqrt{c}\,r/2)$ *and* $\tau_\gamma = \sqrt{1-\rho_\gamma^2}$.

In the second case, its geodesic curvature is $\sqrt{\kappa^2 - (c/4)\tan^2(\sqrt{c}\,r/2)}$, *and in the third case, its geodesic curvature is* $\sqrt{\kappa^2 - (c/4)\cot^2(\sqrt{c}\,r/2)}$.

With these theorems we can refine Corollary 7.6 by considering strengths of Sasakian magnetic fields.

By Proposition 7.3, in order to study features of trajectories we only need to consider $T_1(r)$ in $\mathbb{C}P^3(c)$, which is real 5 dimensional. Therefore we find that trajectories are at most of proper order 5 as Frenet curves. For more detail, we will discuss in the forthcoming paper.

8. Extrinsic shapes of circular trajectories on geodesic spheres in $\mathbb{C}P^n$

When we study curves on a submanifold M in an ambient space $\widetilde{M}$, it is one of basic ways to study these curves by looking them from $\widetilde{M}$. Let $\iota : M \to \widetilde{M}$ be an isometric immersion. For a curve γ on M we call the curve $\tilde{\gamma} = \iota \circ \gamma$ its *extrinsic shape*. In order to simplify the notations we sometimes denote $\tilde{\gamma}$ also by γ.

In this section we study extrinsic shapes of circular trajectories on geodesic spheres in a complex projective space $\mathbb{C}P^n(c)$. Since it is known that every isometry φ of a geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$ is equivariant, that is there is an isometry $\tilde{\varphi}$ of $\mathbb{C}P^n(c)$ satisfying $\tilde{\varphi} \circ \iota = \iota \circ \varphi$ with an isometric immersion $\iota : G(r) \to \mathbb{C}P^n(c)$ (Remark 7.1), and since every trajectory for a Sasakian magnetic field is Killing (Corollary 7.2), we see the extrinsic shape of an arbitrary trajectory is also Killing. We here study more detail on their extrinsic shapes of trajectories. To simplify our calculation we first consider the case $c = 4$.

Proposition 8.1. *Let $G(r)$ be a geodesic sphere of radius r $(0 < r < \pi/2)$ in $\mathbb{C}P^n(4)$.*

1. *When $\pi/4 < r < \pi/2$, the extrinsic shape of circular $\boldsymbol{F}_{\pm 1}$-trajectory is a circle of geodesic curvature $k_1 = \sqrt{1-\cot^2 r}$ and of complex torsion $\tau_{12} = \mp\sqrt{1-\cot^2 r}$.*
2. *Otherwise, the extrinsic shape of circular $\boldsymbol{F}_\kappa$-trajectory is an essential Killing helix of proper order 4 which satisfies the condition (II) in Lemma 4.9. Its geodesic curvatures are given as*

$$k_1 = \frac{1}{\kappa^2}\sqrt{\kappa^6 + (1-2\kappa^2)\cot^2 r}, \quad k_2 = \frac{|\kappa^2-1|\cot r\sqrt{\kappa^2-\cot^2 r}}{\kappa^2\sqrt{\kappa^6 + (1-2\kappa^2)\cot^2 r}},$$
$$k_3 = \frac{\kappa^2-\cot^2 r}{\sqrt{\kappa^6 + (1-2\kappa^2)\cot^2 r}}.$$

Proof. Since a circular $\boldsymbol{F}_\kappa$-trajectory γ satisfies $\kappa\rho_\gamma = \cot r$ and $\rho_\gamma \neq \pm 1$, we see

$$A\dot\gamma = (\cot r)\dot\gamma - (\rho_\gamma \tan r)\xi = \kappa\rho_\gamma\dot\gamma - \kappa^{-1}\xi,$$
$$\langle A\dot\gamma, \dot\gamma\rangle = \cot r - \rho_\gamma^2 \tan r = \rho_\gamma(\kappa - \kappa^{-1}).$$

Hence we have

$$\begin{aligned}\widetilde\nabla_{\dot\gamma}\dot\gamma &= \kappa\phi\dot\gamma + \langle A\dot\gamma, \dot\gamma\rangle\mathcal{N} = \kappa(J\dot\gamma - \rho_\gamma\mathcal{N}) + \rho_\gamma(\kappa - \kappa^{-1})\mathcal{N} \\ &= \kappa J\dot\gamma - \rho_\gamma\kappa^{-1}\mathcal{N},\end{aligned}$$

with a unit normal vector field $\mathcal{N}$ on $G(r)$ in $\mathbb{C}P^n(4)$. Thus we obtain $\widetilde\nabla_{\dot\gamma}\dot\gamma = k_1 Y_2$ with

$$\begin{aligned}k_1 &= \sqrt{\kappa^2(1-\rho_\gamma^2) + \rho_\gamma^2(\kappa-\kappa^{-1})^2} = \sqrt{\kappa^2 - 2\rho_\gamma^2 + \kappa^{-2}\rho_\gamma^2}\ \ (\neq 0), \\ Y_2 &= \frac{1}{k_1}\Big(\kappa J\dot\gamma - \rho_\gamma\kappa^{-1}\mathcal{N}\Big).\end{aligned}$$

Continuing calculation we have

$$\begin{aligned}\widetilde\nabla_{\dot\gamma}\Big(\kappa J\dot\gamma - \rho_\gamma\kappa^{-1}\mathcal{N}\Big) &= -\kappa^2\dot\gamma + \rho_\gamma\kappa^{-1}A\dot\gamma + \rho_\gamma\xi \\ &= -(\kappa^2 - 2\rho_\gamma^2 + \kappa^{-2}\rho_\gamma^2)\dot\gamma + \rho_\gamma(\kappa^{-2} - 1)(\rho_\gamma\dot\gamma - \xi).\end{aligned}$$

When $\kappa = \pm 1$, which is the case that $\pi/4 < r < \pi/2$ and $\rho_\gamma = \pm\cot r$, we have $k_1\widetilde\nabla_{\dot\gamma}\dot\gamma = -(1 - \rho_\gamma^2)\dot\gamma$, hence find that the extrinsic shape of γ is a circle of positive geodesic curvature $\sqrt{1 - \cot^2 r}$. Otherwise we have $k_1\widetilde\nabla_{\dot\gamma}Y_2 = -k_1^2\dot\gamma + k_1k_2Y_3$ with

$$k_2 = k_1^{-1}|\rho_\gamma(\kappa^{-2} - 1)|\sqrt{1 - \rho_\gamma^2}, \quad Y_3 = \frac{\operatorname{sgn}\big(\rho_\gamma(\kappa^{-2} - 1)\big)}{\sqrt{1 - \rho_\gamma^2}}(\rho_\gamma\dot\gamma - \xi),$$

where, as usual, $\operatorname{sgn}(\alpha)$ denotes the signature of a number α. By (3.2) we have

$$\begin{aligned}\widetilde\nabla_{\dot\gamma}(\rho_\gamma\dot\gamma - \xi) &= \rho_\gamma\widetilde\nabla_{\dot\gamma}\dot\gamma + \widetilde\nabla_{\dot\gamma}(J\mathcal{N}) = \rho_\gamma\big(\kappa J\dot\gamma - \rho_\gamma\kappa^{-1}\mathcal{N}\big) - JA\dot\gamma \\ &= \rho_\gamma\big(\kappa J\dot\gamma - \rho_\gamma\kappa^{-1}\mathcal{N}\big) - J\big(\kappa\rho_\gamma\dot\gamma - \kappa^{-1}\xi\big) = (1 - \rho_\gamma^2)\kappa^{-1}\mathcal{N}.\end{aligned}$$

Separating the component which is parallel to Y_2 we have

$$\widetilde\nabla_{\dot\gamma}(\rho_\gamma\dot\gamma - \xi) = \operatorname{sgn}\big(\rho_\gamma(\kappa^{-2} - 1)\big)\sqrt{1 - \rho_\gamma^2}\,(-k_2Y_2 + k_2Y_2) + (1 - \rho_\gamma^2)\kappa^{-1}\mathcal{N}$$

$$
\begin{aligned}
&= -\frac{\rho_\gamma(\kappa^{-2}-1)(1-\rho_\gamma^2)}{k_1}\cdot\frac{1}{k_1}(\kappa J\dot\gamma-\rho_\gamma\kappa^{-1}\mathcal{N})\\
&\quad+\frac{\rho_\gamma\kappa^{-1}(1-\kappa^2)(1-\rho_\gamma^2)}{k_1^2}(\phi\dot\gamma+\rho_\gamma\mathcal{N})\\
&\quad+\frac{\kappa^{-1}(1-\rho_\gamma^2)}{k_1^2}\{\rho_\gamma^2(1-\kappa^{-2})+k_1^2\}\mathcal{N}\\
&= -\frac{\rho_\gamma(\kappa^{-2}-1)(1-\rho_\gamma^2)}{k_1}\cdot\frac{1}{k_1}(\kappa J\dot\gamma-\rho_\gamma\kappa^{-1}\mathcal{N})\\
&\quad+\frac{1-\rho_\gamma^2}{k_1^2}\{-\rho_\gamma(\kappa-\kappa^{-1})\phi\dot\gamma+\kappa(1-\rho_\gamma^2)\mathcal{N}\},
\end{aligned}
$$

hence we obtain $\widetilde\nabla_{\dot\gamma}Y_3=-k_2Y_2+k_3Y_4$ with

$$
k_3=k_1^{-1}(1-\rho_\gamma^2)\ (>0),\quad Y_4=\frac{\mathrm{sgn}\big(\rho_\gamma(\kappa^{-2}-1)\big)}{k_1\sqrt{1-\rho_\gamma^2}}\{-\rho_\gamma(\kappa-\kappa^{-1})\phi\dot\gamma+\kappa(1-\rho_\gamma^2)\mathcal{N}\}.
$$

Since we see

$$
\begin{aligned}
&\widetilde\nabla_{\dot\gamma}\{-\rho_\gamma(\kappa-\kappa^{-1})\phi\dot\gamma+\kappa(1-\rho_\gamma^2)\mathcal{N}\}\\
&\qquad=\widetilde\nabla_{\dot\gamma}\{-\rho_\gamma(\kappa-\kappa^{-1})J\dot\gamma+(\kappa-\rho_\gamma^2\kappa^{-1})\mathcal{N}\}\\
&\qquad=-\rho_\gamma(\kappa-\kappa^{-1})J\big(\kappa J\dot\gamma-\rho_\gamma\kappa^{-1}\mathcal{N}\big)-(\kappa-\rho_\gamma^2\kappa^{-1})A\dot\gamma\\
&\qquad=\rho_\gamma(\kappa^2-1)\dot\gamma-\rho_\gamma^2(1-\kappa^{-2})\xi-(\kappa-\rho_\gamma^2\kappa^{-1})\big(\kappa\rho_\gamma\dot\gamma-\kappa^{-1}\xi\big)\\
&\qquad=-(1-\rho_\gamma^2)(\rho_\gamma\dot\gamma-\xi),
\end{aligned}
$$

the extrinsic shape is a helix of proper order 4. In view of the Frenet frame $\{\dot\gamma,Y_2,Y_3,Y_4\}$ of the extrinsic shape, as they are formed by $\dot\gamma,J\dot\gamma,\mathcal{N},J\mathcal{N}=-\xi$, we find that it lies on some totally geodesic $\mathbb{C}P^2$. Therefore we see it is essential. Moreover, as we have

$$
\begin{aligned}
k_1-k_3&=\frac{\kappa^6+(1-2\kappa^2)\cot^2 r-\kappa^2(\kappa^2-\cot^2 r)}{\kappa^2\sqrt{\kappa^6+(1-2\kappa^2)\cot^2 r}}=\frac{(\kappa^2-1)(\kappa^4-\cot^2 r)}{\kappa^2\sqrt{\kappa^6+(1-2\kappa^2)\cot^2 r}}\\
&=\frac{(\kappa^2-1)(\kappa^2-\rho_\gamma^2)}{|\kappa|\sqrt{\kappa^4-2\kappa^2\rho_\gamma^2+\rho_\gamma^2}},\\
k_2^2+(k_1-k_3)^2&=\frac{(\kappa^2-1)^2(1-\rho_\gamma^2)\rho_\gamma^2}{\kappa^2(\kappa^4-2\kappa^2\rho_\gamma^2+\rho_\gamma^2)}+\frac{(\kappa^2-1)^2(\kappa^2-\rho_\gamma^2)^2}{\kappa^2(\kappa^4-2\kappa^2\rho_\gamma^2+\rho_\gamma^2)^2}=(\kappa-\kappa^{-1})^2,
\end{aligned}
$$

we see

$$\frac{k_1 - k_3}{\sqrt{k_2^2+(k_1-k_3)^2}} = \frac{\kappa(\kappa-\kappa^{-1})(\kappa^2-\rho_\gamma^2)}{|\kappa-\kappa^{-1}|\,|\kappa|\sqrt{\kappa^4-2\kappa^2\rho_\gamma^2+\rho_\gamma^2}} = \frac{\mathrm{sgn}(\kappa^2-1)(\kappa^2-\rho_\gamma^2)}{\sqrt{\kappa^4-2\kappa^2\rho_\gamma^2+\rho_\gamma^2}}.$$

We hence find

$$\begin{aligned}\tau_{12} &= \langle\dot\gamma, JY_2\rangle = \frac{1}{k_1}\langle\dot\gamma, J\big(\kappa J\dot\gamma - \rho_\gamma\kappa^{-1}\mathcal{N}\big)\rangle = \frac{1}{k_1}\langle\dot\gamma, -\kappa\dot\gamma + \rho_\gamma\kappa^{-1}\xi\rangle \\ &= \frac{\mathrm{sgn}(\kappa)\cdot(\rho_\gamma^2-\kappa^2)}{\sqrt{\kappa^4-2\kappa^2\rho_\gamma^2+\rho_\gamma^2}} = \frac{-\mathrm{sgn}(\kappa-\kappa^{-1})\cdot(k_1-k_3)}{\sqrt{k_2^2+(k_1-k_3)^2}}, \\ \tau_{34} &= \langle Y_3, JY_4\rangle = \frac{1}{k_1(1-\rho_\gamma^2)}\big\langle\rho_\gamma\dot\gamma-\xi, \rho_\gamma(\kappa-\kappa^{-1})\dot\gamma - (\kappa-\rho_\gamma^2\kappa^{-1})\xi\big\rangle \\ &= \frac{|\kappa|(\kappa^2-\rho_\gamma^2)}{\kappa\sqrt{\kappa^4-2\kappa^2\rho_\gamma^2+\rho_\gamma^2}} = \frac{\mathrm{sgn}(\kappa)\cdot(\kappa^2-\rho_\gamma^2)}{\sqrt{\kappa^4-2\kappa^2\rho_\gamma^2+\rho_\gamma^2}} = \frac{\mathrm{sgn}(\kappa-\kappa^{-1})\cdot(k_1-k_3)}{\sqrt{k_2^2+(k_1-k_3)^2}}.\end{aligned}$$

Therefore the extrinsic shape satisfies the condition (II) in Lemma 4.9 when it is of proper order 4. □

Remark 8.1. When $\pi/4 < r < \pi/2$, by the above proof we find that the extrinsic shape of a circular $\boldsymbol{F}_{\pm\sqrt{\cot r}}$-trajectory has the complex torsions $\tau_{12} = \tau_{34} = 0$, because $\kappa\rho_\gamma = \cot r$. Hence we can conclude that it is a moderate Killing helix. That is, an essential Killing helix whose complex torsions satisfy $\tau_{12} = \tau_{13} = \tau_{24} = \tau_{34} = 0$ and $\tau_{23} = -\tau_{14} = 1$.

In order to study trajectories on geodesic spheres in $\mathbb{C}P^n(c)$, it is useful to make use of homothetical changes of metrics. Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold. Another Riemannian metric $\langle\ ,\ \rangle'$ is said to be *homothetic* to $\langle\ ,\ \rangle$ if there is a positive α satisfying $\langle\ ,\ \rangle' = \alpha^2\langle\ ,\ \rangle$, and is said to be conformal to $\langle\ ,\ \rangle$ if there is a positive function $\alpha : M \to \mathbb{R}$ satisfying $\langle\ ,\ \rangle'_p = \alpha(p)^2\langle\ ,\ \rangle_p$ at each point $p \in M$. In the rest of this section, we use the symbol "$\,'\,$" to show that the object corresponds to the metric $\langle\ ,\ \rangle'$. For example, we denote by ∇' the Riemannian connection associated with $\langle\ ,\ \rangle'$

By using Christoffel's symbols, we obtain the following by direct computation.

Lemma 8.1. *Suppose* $\langle\ ,\ \rangle'$ *is conformal to* $\langle\ ,\ \rangle$ *(i.e.* $\langle\ ,\ \rangle' = \alpha^2\langle\ ,\ \rangle$ *with some positive function* α*). If we put* $\beta = \log\alpha$*, we have*

$$\nabla'_X Y = \nabla_X Y + (X\beta)Y - \langle X, Y\rangle\nabla\beta.$$

In particular, if $\langle\ ,\ \rangle'$ *is homothetic to* $\langle\ ,\ \rangle$*, we have* $\nabla'_X Y = \nabla_X Y$*.*

Lemma 8.2. *Suppose two Riemannian metrics* $\langle\ ,\ \rangle$ *and* $\langle\ ,\ \rangle'$ *on a manifold* M *are homothetic* ($\langle\ ,\ \rangle' = \alpha^2\langle\ ,\ \rangle$)*.*

(1) *For a smooth curve* $\gamma : I \to M$*, its lengths satisfy* $\mathrm{length}'(\gamma) = \alpha\,\mathrm{length}(\gamma)$*. In particular, distance functions satisfy* $d'_M = \alpha\, d_M$*.*

(2) *For an arbitrary* 2*-plane* $\varDelta$ *in* TM*, its sectional curvatures satisfy* $\mathrm{Riem}'(\varDelta) = \alpha^{-2}\,\mathrm{Riem}(\varDelta)$*.*

Proof. (1) By the definition we have

$$\mathrm{length}'(\gamma) = \int_I \left\| \frac{d\gamma}{dt}(t) \right\|' dt = \int_I \alpha \left\| \frac{d\gamma}{dt}(t) \right\| dt = \alpha\,\mathrm{length}(\gamma).$$

(2) We take an orthonormal vectors $\{v, w\}$ with respect to $\langle\ ,\ \rangle$ which span $\varDelta$. Then the vectors $\{\alpha^{-1}v, \alpha^{-1}w\}$ are orthonormal with respect to $\langle\ ,\ \rangle'$. Since curvature tonsors R and R' with respect to $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ satisfy $R' = R$ by Lemma 8.1, we have

$$\begin{aligned}\mathrm{Riem}'(\varDelta) &= \langle R'(\alpha^{-1}v, \alpha^{-1}w)\alpha^{-1}w, \alpha^{-1}v\rangle' = \alpha^2\langle R(\alpha^{-1}v, \alpha^{-1}w)\alpha^{-1}w, \alpha^{-1}v\rangle \\ &= \alpha^{-2}\langle R(v, w)w, v\rangle = \alpha^{-2}\mathrm{Riem}(\varDelta)\end{aligned}$$

by multilinearity of curvature tensors. □

Lemma 8.3. *Suppose two Riemannian metrics* $\langle\ ,\ \rangle$ *and* $\langle\ ,\ \rangle'$ *on a manifold* M *are homothetic* ($\langle\ ,\ \rangle' = \alpha^2\langle\ ,\ \rangle$)*. If* γ *is a helix of proper order* d *with respect to* $\langle\ ,\ \rangle$*, then the curve* σ *given by* $\sigma(s) = \gamma(s/\alpha)$ *is also a helix of proper order* d *with respect to* $\langle\ ,\ \rangle'$*. Their geodesic curvatures satisfy* $k'_j = k_j/\alpha$ *for* $j = 1, \ldots, d-1$*.*

When M *is a Kähler manifold, their complex torsions satisfy* $\tau'_{ij} = \tau_{ij}$ *for* $1 \le i < j \le d$*.*

Proof. As we have $\dfrac{d\sigma}{ds} = \dfrac{dt}{ds}\dfrac{d\gamma}{dt}$, where $t = s/\alpha$, we see

$$\Big\|\frac{d\sigma}{ds}\Big\|' = \alpha\Big\|\frac{d\sigma}{ds}\Big\| = \alpha\,\Big\|\frac{1}{\alpha}\,\frac{d\gamma}{dt}\Big\| = \Big\|\frac{d\gamma}{dt}\Big\| = 1,$$

hence find that σ is parameterized by its arclength with respect to $\langle\ ,\ \rangle'$. Let $\{Y_1 = \dot\gamma, Y_2, \dots, Y_d\}$ be the Frenet frame of γ. Then $\{\alpha^{-1}Y_1, \alpha^{-1}Y_2, \dots, \alpha^{-1}Y_d\}$ are orthonomal with respect to $\langle\ ,\ \rangle'$, where $\alpha^{-1}Y_1$ means $\dfrac{d\sigma}{ds}$. By Lemma 8.1 we obtain

$$\begin{aligned}\nabla'_{\alpha^{-1}Y_1}\big(\alpha^{-1}Y_j\big) &= \alpha^{-2}\nabla'_{\dot\gamma}Y_j = \alpha^{-2}\nabla_{\dot\gamma}Y_j = \alpha^{-2}\big(-k_{j-1}Y_{j-1} + k_jY_{j+1}\big)\\ &= -(k_{j-1}/\alpha)\alpha^{-1}Y_{j-1} + (k_j/\alpha)\alpha^{-1}Y_{j+1}.\end{aligned}$$

We hence get $k'_j = k_j/\alpha$.

When M is a Kähler manifold, we have

$$\tau'_{ij} = \langle Y'_i, Y'_j\rangle' = \alpha^2\langle\alpha^{-1}Y_i, \alpha^{-1}Y_j\rangle = \langle Y_i, Y_j\rangle = \tau_{ij}.$$

This complete the proof. □

Lemma 8.4. *Let M be a hypersurface of $\widetilde{M}$. Suppose two Riemannian metrics $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ on $\widetilde{M}$ are homothetic. If we consider induced metrics on M, shape operators A_M and A'_M with respect to them satisfy $A'_M = \alpha^{-1}A_M$. In particular, principal curvatures with respect to $\langle\ ,\ \rangle'$ are α^{-1}-times of principal curvatures with respect to $\langle\ ,\ \rangle$*

Proof. By Lemma 8.1 we find that their second fundamental forms coincide:

$$\sigma'_M(X,Y) = \widetilde{\nabla}'_XY - \nabla'_XY = \widetilde{\nabla}_XY - \nabla_XY = \sigma_M(X,Y)$$

holds for arbitrary vector fields X, Y on M. We take a unit normal $\mathcal{N}$ of M with respect to $\langle\ ,\ \rangle$. Then $\alpha^{-1}\mathcal{N}$ is a unit normal with respect to $\langle\ ,\ \rangle'$. We hence have

$$\langle A'_Mv, w\rangle' = \langle\sigma'_M(v,w), \alpha^{-1}\mathcal{N}\rangle' = \alpha\langle\sigma_M(v,w), \mathcal{N}\rangle = \alpha\langle A_Mv, w\rangle = \langle\alpha^{-1}A_Mv, w\rangle'$$

for arbitrary $v, w \in T_pM$ at each point $p \in M$. We hence get $A'_M = \alpha^{-1}A_M$. □

We now study on geodesic spheres in $\mathbb{C}P^n(c)$. Let γ be a trajectory for a Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$. On $\mathbb{C}P^n(c)$ we consider a new metric $\langle\ ,\ \rangle' = (c/4)\langle\ ,\ \rangle$, which is homothetic to the original metric. By Lemma 8.2, holomorphic sectional curvatures are 4 and radius of the geodesic sphere is $\sqrt{c}\,r/2$ with respect to the new metric. If we define a curve σ by $\sigma(s) = \gamma\big(2t/\sqrt{c}\big)$, it is a trajectory for a Sasakian magnetic field $\boldsymbol{F}'_{2\kappa/\sqrt{c}}$ with respect to the new metric by Lemmas 3.2 and 8.1. If we put $\mathcal{N}' = (2/\sqrt{c})\mathcal{N}$, it is a unit normal with respect to the new metric. Thus we find

$$\rho_\sigma = \Big\langle \frac{d\sigma}{ds}, -J\mathcal{N}'\Big\rangle' = \frac{c}{4}\Big\langle \frac{2}{\sqrt{c}}\frac{d\gamma}{dt}, -\frac{2}{\sqrt{c}}J\mathcal{N}\Big\rangle = \Big\langle \frac{d\gamma}{dt}, -J\mathcal{N}\Big\rangle = \rho_\gamma.$$

Since geodesic curvatures k_j of the extrinsic shape of γ satisfy $k_j = (\sqrt{c}/2)k'_j$ with the geodesic curvatures k'_j of the extrinsic shape of σ by Lemma 8.3, and since we have properties of the extrinsic shape of σ by Proposition 8.1, we obtain the following.

Proposition 8.2. *Let $G(r)$ be a geodesic sphere of radius r $(0 < r < \pi/\sqrt{c})$ in $\mathbb{C}P^n(c)$.*

(1) *When $\pi/(2\sqrt{c}) < r < \pi/\sqrt{c}$, the extrinsic shape of a circular $\boldsymbol{F}_{\pm\sqrt{c}/2}$-trajectory is a circle of geodesic curvature $k_1 = \sqrt{c\big\{1-\cot^2(\sqrt{c}\,r/2)\big\}}/2$ and of complex torsion $\tau_{12} = \mp\sqrt{1-\cot^2(\sqrt{c}\,r/2)}$.*

(2) *Otherwise, the extrinsic shape of a circular $\boldsymbol{F}_\kappa$-trajectory is an essential Killing helix of proper order 4 which satisfies the condition (II) in Lemma 4.9. Its geodesic curvatures are given as*

$$k_1 = \frac{1}{8\kappa^2}\sqrt{64\kappa^6 + c^2(c-8\kappa^2)\cot^2(\sqrt{c}\,r/2)},$$
$$k_2 = \frac{c|4\kappa^2 - c|\cot(\sqrt{c}\,r/2)\sqrt{4c\kappa^2 - c^2\cot^2(\sqrt{c}\,r/2)}}{8\kappa^2\sqrt{64\kappa^6 + c^2(c-8\kappa^2)\cot^2(\sqrt{c}\,r/2)}},$$
$$k_3 = \frac{c(4\kappa^2 - c\cot^2(\sqrt{c}\,r/2))}{2\sqrt{64\kappa^6 + c^2(c-8\kappa^2)\cot^2(\sqrt{c}\,r/2)}}.$$

Proof. We here make clear our computation. We take a trajectory γ for a Sasakian magnetic field $\boldsymbol{F}_\kappa$ on $G(r)$ in $\mathbb{C}P^n(c)$. We consider a new metric $\langle\ ,\ \rangle' = (c/4)\langle\ ,\ \rangle$, which is homothetic to the original one. If we consider a curve σ given by $\sigma(s) = \gamma\big(2t/\sqrt{c}\big)$, it is a trajectory for a Sasakian magnetic field $\boldsymbol{F}'_{2\kappa/\sqrt{c}}$ with respect to the new metric. With respect to this new metric the radius of the geodesic sphere $G(r)$ is $r' = \sqrt{c}\,r/2$. That is, it can be seen as $G'(\sqrt{c}\,r/2)$ in $\mathbb{C}P^n(4)$.

When $\kappa' = 2\kappa/\sqrt{c} = \pm 1$, that is when $\kappa = \pm\sqrt{c}/2$, the extrinsic shape of the trajectory σ is a circle of geodesic curvature $k_1' = \sqrt{1-\cot^2(\sqrt{c}\,r/2)}$ with respect to $\langle\ ,\ \rangle'$ by Proposition 8.1. We therefore find that the extrinsic shape of γ is a circle of geodesic curvature $k_1 = (\sqrt{c}/2)k_1' = \sqrt{c\{1-\cot^2(\sqrt{c}\,r/2)\}}/2$.

When $\kappa' = 2\kappa/\sqrt{c} \neq \pm 1$, the extrinsic shape of σ is an essential Killing helix of proper order 4 with geodesic curvatures

$$\begin{aligned}
k_1' &= \kappa'^{-2}\sqrt{\kappa'^6 + (1-2\kappa'^2)\cot^2(\sqrt{c}\,r/2)},\\
k_2' &= \frac{|\kappa'^2-1|\cot(\sqrt{c}\,r/2)\sqrt{\kappa'^2-\cot^2(\sqrt{c}\,r/2)}}{\kappa'^2\sqrt{\kappa'^6 + (1-2\kappa'^2)\cot^2(\sqrt{c}\,r/2)}},\\
k_3' &= \frac{\kappa'^2-\cot^2(\sqrt{c}\,r/2)}{\sqrt{\kappa'^6 + (1-2\kappa'^2)\cot^2(\sqrt{c}\,r/2)}}
\end{aligned}$$

with respect to the new metric. Therefore we find that the extrinsic shape of γ is an essential Killing helix of proper order 4 of geodesic curvatures

$$\begin{aligned}
k_1 &= \frac{\sqrt{c}}{2}k_1' = \frac{\sqrt{c}}{2}\,\frac{c}{4\kappa^2}\sqrt{\frac{64\kappa^6}{c^3} + \Big(1-\frac{8\kappa^2}{c}\Big)\cot^2(\sqrt{c}\,r/2)},\\
k_2 &= \frac{\sqrt{c}}{2}k_2' = \frac{\sqrt{c}}{2}\,\frac{\Big|\frac{4\kappa^2}{c}-1\Big|\cot(\sqrt{c}\,r/2)\sqrt{\frac{4\kappa^2}{c}-\cot^2(\sqrt{c}\,r/2)}}{\frac{4\kappa^2}{c}\sqrt{\frac{64\kappa^6}{c^3} + \Big(1-\frac{8\kappa^2}{c}\Big)\cot^2(\sqrt{c}\,r/2)}},\\
k_3 &= \frac{\sqrt{c}}{2}k_3' = \frac{\sqrt{c}}{2}\,\frac{\frac{4\kappa^2}{c}-\cot^2(\sqrt{c}\,r/2)}{\sqrt{\frac{64\kappa^6}{c^3} + \Big(1-\frac{8\kappa^2}{c}\Big)\cot^2(\sqrt{c}\,r/2)}}.
\end{aligned}$$

Since complex torsions stay invariant under the homothetic change of metrics, we get the conclusion □

Table 5. Homothetic change of metrics and correspondence of geometric datas

	$(\mathbb{C}P^n,\ \langle\ ,\ \rangle)$	$\longrightarrow$	$(\mathbb{C}P^n,\ \langle\ ,\ \rangle')$
holomorphic sectional curvature	c		4
a geodesic sphere	$G(r)$	$=$	$G'(\sqrt{c}\,r/2)$
trajectory	γ for $\boldsymbol{F}_\kappa$		σ for $\boldsymbol{F'}_{2\kappa/\sqrt{c}}$
geometric datas	k_j		$k_j' = (2/\sqrt{c})k_j$
	$\mathrm{length}(\gamma)$		$\mathrm{length}'(\sigma) = (\sqrt{c}/2)\mathrm{length}(\gamma)$

For about extrinsic shapes of trajectories for Sasakian magnetic fields on a real hypersurface of type (A_2) in $\mathbb{C}P^n(c)$, we find that they are Killing helices of order at most 6 by Corollary 7.5 and by the fact that every isometry of a real hypersurface of type (A_2) is equivariant. But for more detail we will discuss in the forthcoming paper. We here only note that we have a corresponding result on geodesics on hypersurfaces of type (A_2) in $\mathbb{C}P^n$ given by Adachi[4].

9. **Length of circular trajectories on geodesic spheres in $\mathbb{C}P^n$**

In this section we study whether circular trajectories on a geodesic sphere in $\mathbb{C}P^n(c)$ are closed or not. A smooth curve γ parameterized by its arclength on a Riemannian manifold is said to be *closed* it there is a positive constant t_c satisfying $\gamma(t) = \gamma(t+t_c)$ for all t. The minimum positive t_c with this property is called the *length* of γ and is denoted by length(γ). When a smooth curve is not closed we say it is *open* and set length$(\gamma) = \infty$. Our goal in this section is to show the following.

Theorem 9.1. *Let $G(r)$ be a geodesic sphere of radius r $(0<r<\pi/\sqrt{c})$ in a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c. A circular trajectory γ for a Sasakian magnetic field $\boldsymbol{F}_\kappa$ on $G(r)$ satisfies the following*:

(1) *When $\pi/2\sqrt{c} < r < \pi/\sqrt{c}$ and $\kappa^2 = c\bigl\{3\sqrt{2\{\cot^2(\sqrt{c}\,r/2)+1\}}-4\bigr\}/8$, it is closed and its length is $4\pi\sqrt{(1/c)\sin(\sqrt{c}\,r/2)\bigl\{3\sqrt{2}-4\sin(\sqrt{c}\,r/2)\bigr\}}$.*

(2) *Otherwise, γ is closed if and only if*

$$\frac{(2\kappa^2+c)\bigl|32\kappa^4+32c\kappa^2-c^2\bigl(9\cot^2(\sqrt{c}\,r/2)+1\bigr)\bigr|}{\bigl\{16\kappa^4+16c\kappa^2-c^2\bigl(3\cot^2(\sqrt{c}\,r/2)-1\bigr)\bigr\}^{3/2}} = \frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}}$$

holds with some relatively prime positive integers p, q satisfying $p > q$. In this case its length is given as

$$4\pi\delta(p,q)|\kappa|\sqrt{(3p^2+q^2)/\bigl\{16\kappa^4+16c\kappa^2-c^2\bigl(3\cot^2(\sqrt{c}\,r/2)-1\bigr)\bigr\}},$$

where $\delta(p,q) = 1$ when pq is odd and $\delta(p,q) = 2$ when pq is even.

9.1. **Relation of connections.** In order to study curves on $\mathbb{C}P^n(4)$, it is one of natural ways to make use of a Hopf fibration $\varpi : S^{2n+1}(1) \to \mathbb{C}P^n(4)$. As $S^{2n+1}(1)$ is contained in $\mathbb{C}^{n+1}$, it connects the geometry of complex projective spaces with the geometry of complex Euclidean spaces. Let $\widehat{\mathcal{N}}$ denote the outward unit normal of $S^{2n+1}(1)$ in $\mathbb{C}^{n+1}$. We denote by $\widehat{\nabla}$ and $\overline{\nabla}$ the Riemannian connections on $S^{2n+1}(1)$ and $\mathbb{C}^{n+1}$, respectively. We also denote by $\widetilde{\nabla}$ the Riemannian connection on $\mathbb{C}P^n(4)$.

Lemma 9.1. *Let* X, $Y \in \mathcal{X}(\mathbb{C}P^n(4))$ *be vector fields on* $\mathbb{C}P^n(4)$. *If we regard them as horizontal vector fields on* $S^{2n+1}(1)$, *we have the following*:

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y - \langle X, Y\rangle \widehat{\mathcal{N}} + \langle X, JY\rangle J\widehat{\mathcal{N}}, \tag{9.1}$$

where $\widehat{\mathcal{N}}$ *denotes the outward unit normal vector field on* S^{2n+1} *in* $\mathbb{C}^{n+1}$.

Proof. We first consider the relationship between the Riemannian connections $\overline{\nabla}$ on $\mathbb{C}^{n+1}$ and $\widehat{\nabla}$ on $S^{2n+1}(1)$. Since $S^{2n+1}(1)$ is a real hypersurface, its shape operator is quite simple. They are related as $\widehat{\nabla}_{\widehat{X}}\widehat{Y} = \overline{\nabla}_{\widehat{X}}\widehat{Y} - \langle \overline{\nabla}_{\widehat{X}}\widehat{Y}, \widehat{\mathcal{N}}\rangle\widehat{\mathcal{N}}$ for vector fields $\widehat{X}$, $\widehat{Y}$ on $S^{2n+1}(1)$, As we have $\widehat{N}_{\hat{z}} = (\hat{z}, \hat{z}) \in \{\hat{z}\} \times \mathbb{C}^{n+1}$ at $\hat{z} \in S^{2n+1}$, we have

$$\langle \overline{\nabla}_{\widehat{X}}\widehat{Y}, \widehat{\mathcal{N}}\rangle = \overline{\nabla}_{\widehat{X}}\langle \widehat{Y}, \widehat{\mathcal{N}}\rangle - \langle \widehat{Y}, \overline{\nabla}_{\widehat{X}}\widehat{\mathcal{N}}\rangle = -\langle \widehat{Y}, \widehat{X}\rangle,$$

we find

$$\overline{\nabla}_{\widehat{X}}\widehat{Y} = \widehat{\nabla}_{\widehat{X}}\widehat{Y} - \langle \widehat{X}, \widehat{Y}\rangle\widehat{\mathcal{N}}. \tag{9.2}$$

Next, we study the relationship between the connections $\widehat{\nabla}$ on $S^{2n+1}(1)$ and $\widetilde{\nabla}$ on $\mathbb{C}P^n(4)$. As we see in §2.2, the vertical part of the tangent space $T_{\hat{z}}S^{2n+1}$ at $\hat{z}$ is generated by $J\widehat{N}_{\hat{z}} = (\hat{z}, \sqrt{-1}\hat{z})$. Hence they are related as $\widetilde{\nabla}_X Y = \widehat{\nabla}_X Y - \langle \widehat{\nabla}_X Y, J\widehat{\mathcal{N}}\rangle J\widehat{\mathcal{N}}$ for vector fields X, Y on $\mathbb{C}P^n(4)$, which are regarded as horizontal vector fields on $S^{2n+1}(1)$, because the horizontal lift of $\widetilde{\nabla}_X Y$ does not have vertical component. Since we have

$$\begin{aligned}\langle \widehat{\nabla}_X Y, J\widehat{\mathcal{N}}\rangle &= \widehat{\nabla}_X\langle Y, J\widehat{\mathcal{N}}\rangle - \langle Y, \widehat{\nabla}_X(J\widehat{\mathcal{N}})\rangle = -\big\langle Y, \overline{\nabla}_X(J\widehat{\mathcal{N}}) + \langle X, J\widehat{\mathcal{N}}\rangle\widehat{\mathcal{N}}\big\rangle \\ &= -\langle Y, J\overline{\nabla}_X\widehat{\mathcal{N}}\rangle = -\langle Y, JX\rangle = \langle X, JY\rangle,\end{aligned}$$

we obtain

$$\widehat{\nabla}_X Y = \widetilde{\nabla}_X Y + \langle X, JY\rangle J\widehat{\mathcal{N}}. \tag{9.3}$$

Combining these relations (9.2) and (9.3) we get the conclusion. □

As a consequence of this relationship between connections, we have an explicit expression of geodesics on $\mathbb{C}P^n(4)$.

Corollary 9.1. *On $\mathbb{C}P^n(4)$, a geodesic γ with $\gamma(0) = \varpi(\hat{z})$ and $\dot{\gamma}(0) = d\varpi\big((\hat{z}, \hat{u})\big)$ is given as $\gamma(t) = \varpi\big(\hat{z}\cos t + \hat{u}\sin t\big)$. Here, we take $(\hat{z}, \hat{u})$ horizontal.*

Proof. Let $\hat{\gamma}$ be a horizontal lift of γ. Regarding this curve as a curve in $\mathbb{C}^{n+1}$, we find it satisfies $\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = -\widehat{\mathcal{N}}_{\hat{\gamma}}$ by Lemma 9.1. On S^{2n+1} in $\mathbb{C}^{n+1}$, we can identify the position vector with outward unit normal at that point. Therefore this differential equation can be written as $\hat{\gamma}'' + \hat{\gamma} = 0$. Solving this equation, we get the conclusion. □

9.2. **Review of circles on $\mathbb{C}P^n(4)$.** In order to show Theorem 9.1 we here recall the result on lengths of circles on $\mathbb{C}P^n(4)$ due to T. Adachi and S. Maeda.

Lemma 9.2 (Adachi-Maeda[10]). *Let σ be a circle of geodesic curvature $1/\sqrt{2}$ and of complex torsion τ $(0 \le |\tau| < 1)$ on $\mathbb{C}P^n(4)$. We denote by a_τ, b_τ, c_τ $(a_\tau < b_\tau < c_\tau)$ the three distinct real solutions of the cubic equation*

$$\Theta^3 - \frac{3}{2}\Theta + \frac{\tau}{\sqrt{2}} = 0. \tag{9.4}$$

*If one of the ratios $b_\tau/a_\tau, c_\tau/b_\tau, a_\tau/c_\tau$ is rational, hence equivalently if all of these ratios are rational, then the curve σ is closed with length $2\pi\times$*L.C.M.$\big\{(b_\tau - a_\tau)^{-1}, (c_\tau - b_\tau)^{-1}\big\}$. *Here,* L.C.M.$(\alpha, \beta)$ *for positive α, β denotes the least common multiple of α, β, which is the minimum of the set $\{m\alpha \mid m = 1, 2, \ldots\} \cap \{m\beta \mid m = 1, 2, \ldots\}$.*

Proof. As σ satisfies

$$\widetilde{\nabla}_{\dot{\sigma}}\dot{\sigma} = \frac{1}{\sqrt{2}}Y, \;\; \widetilde{\nabla}_{\dot{\sigma}}Y = -\frac{1}{\sqrt{2}}\dot{\sigma},$$

we have by (9.1) that its horizontal lift $\hat{\sigma}$ satisfies

$$\overline{\nabla}_{\dot{\hat{\sigma}}}\dot{\hat{\sigma}} = \frac{1}{\sqrt{2}}Y - \widehat{\mathcal{N}}, \;\; \overline{\nabla}_{\dot{\hat{\sigma}}}Y = -\frac{1}{\sqrt{2}}\dot{\hat{\sigma}} + \tau J\widehat{\mathcal{N}}.$$

Therefore we find that it satisfies

$$\overline{\nabla}_{\dot{\hat{\sigma}}}\overline{\nabla}_{\dot{\hat{\sigma}}}\dot{\hat{\sigma}} = \frac{1}{\sqrt{2}}\overline{\nabla}_{\dot{\hat{\sigma}}}Y - \overline{\nabla}_{\dot{\hat{\sigma}}}\widehat{\mathcal{N}} = -\frac{1}{2}\dot{\hat{\sigma}} + \frac{\tau}{\sqrt{2}}J\widehat{\mathcal{N}} - \dot{\hat{\sigma}},$$

which is equivalent to

$$\hat{\sigma}''' + \frac{3}{2}\hat{\sigma}' - \sqrt{-1}\frac{\tau}{\sqrt{2}}\hat{\sigma} = 0$$

as a curve in $\mathbb{C}^{n+1}$. Its characteristic equation is

$$\Lambda^3 + \frac{3}{2}\Lambda - \sqrt{\frac{1}{2}}\,\tau = 0.$$

In order to realize it, if we put $\Theta = -\sqrt{-1}\Lambda$, we then obtain (9.4).

Since $|\tau| < 1$, we find that this cubic equation (9.4) have three distinct solutions by considering the differential of its left hand side with respect to Θ. By use of these solutions $a_\tau,\ b_\tau,\ c_\tau\ (a_\tau < b_\tau < c_\tau)$ we have $\hat{\sigma}(t) = A_\tau e^{\sqrt{-1}a_\tau t} + B_\tau e^{\sqrt{-1}b_\tau t} + C_\tau e^{\sqrt{-1}c_\tau t}$. with some $A_\tau, B_\tau, C_\tau \in \mathbb{C}^{n+1}$ defined by $\hat{\sigma}(0), \dot{\hat{\sigma}}(0)$ and horizontal lift of $Y(0)$. Since $\tau \neq \pm 1$, and since $\dot{\hat{\sigma}}(0)$ and horizontal lift of $Y(0)$ are orthogonal to both $\hat{\sigma}(0)$ and $J\hat{\sigma}(0)$, we find that they are linearly independent over $\mathbb{C}$. As σ is closed if and only if

$$A_\tau e^{\sqrt{-1}a_\tau(t+t_0)} + B_\tau e^{\sqrt{-1}b_\tau(t+t_0)} + C_\tau e^{\sqrt{-1}c_\tau(t+t_0)}$$
$$= e^{\sqrt{-1}\theta(t)}\left\{A_\tau e^{\sqrt{-1}a_\tau t} + B_\tau e^{\sqrt{-1}b_\tau t} + C_\tau e^{\sqrt{-1}c_\tau t}\right\}$$

with some $\theta(t) \in \mathbb{R}$, linearly independency of $\{A_\tau, B_\tau, C_\tau\}$ shows that it is equivalent to

$$\begin{cases} e^{\sqrt{-1}a_\tau(t+t_0)} = e^{\sqrt{-1}(a_\tau t+\theta(t))}, \\ e^{\sqrt{-1}b_\tau(t+t_0)} = e^{\sqrt{-1}(b_\tau t+\theta(t))}, \\ e^{\sqrt{-1}c_\tau(t+t_0)} = e^{\sqrt{-1}(c_\tau t+\theta(t))}, \end{cases} \quad \text{hence to} \quad \begin{cases} e^{\sqrt{-1}a_\tau t_0} = e^{\sqrt{-1}\theta(t)}, \\ e^{\sqrt{-1}b_\tau t_0} = e^{\sqrt{-1}\theta(t)}, \\ e^{\sqrt{-1}c_\tau t_0} = e^{\sqrt{-1}\theta(t)}, \end{cases}$$

Clearly, it is equivalent to $e^{\sqrt{-1}(b_\tau - a_\tau)t_0} = e^{\sqrt{-1}(c_\tau - a_\tau)t_0} = 1$. Thus we obtain that σ is closed if and only if $(b_\tau - a_\tau)t_0 \in 2\pi\mathbb{Z}$ and $(c_\tau - a_\tau)t_0 \in 2\pi\mathbb{Z}$, which is equivalent to the condition that $(c_\tau - a_\tau)/(b_\tau - a_\tau)$ is rational. Since the coefficient of the term Θ^2 is zero, we have $a_\tau + b_\tau + c_\tau = 0$, hence we see

$$\frac{c_\tau - a_\tau}{b_\tau - a_\tau} = -1 - \frac{3a_\tau}{b_\tau - a_\tau} = -1 - \frac{3}{(b_\tau/a_\tau) - 1}.$$

Thus $(c_\tau - a_\tau)/(b_\tau - a_\tau)$ is rational if and only if b_τ/a_τ is rational. Again as we have $a_\tau + b_\tau + c_\tau = 0$, we find one of $b_\tau/a_\tau,\ c_\tau/b_\tau,\ a_\tau/c_\tau$ is rational if and only if all of them are rational. Since the length of closed σ satisfies $(b_\tau - a_\tau)\mathrm{length}(\sigma) \in 2\pi\mathbb{Z}$ and $(c_\tau - a_\tau)\mathrm{length}(\sigma) \in 2\pi\mathbb{Z}$, we get the conclusion. □

This Lemma 9.2 gives an arithmetic condition on circles to be closed. On the other hand, we can fortunately obtain all circles of geodesic curvature $1/\sqrt{2}$ geometrically. We consider a parallel embedding $S^1 \times S^{n-1}/\!\sim\, \to \mathbb{C}P^n(4)$ defined by

$$\big(e^{\sqrt{-1}\theta}, a_1, \ldots, a_n\big) \mapsto \varpi \begin{pmatrix} \big(e^{-2\sqrt{-1}\theta/3} + 2a_1 e^{\sqrt{-1}\theta/3}\big)/3 \\ \sqrt{2}\big(e^{-2\sqrt{-1}\theta/3} - a_1 e^{\sqrt{-1}\theta/3}\big)/3 \\ \sqrt{-1}(2/\sqrt{6})\, a_2 e^{\sqrt{-1}\theta/3} \\ \vdots \\ \sqrt{-1}(2/\sqrt{6})\, a_m e^{\sqrt{-1}\theta/3} \end{pmatrix}$$

with a Hopf fibration $\varpi : S^{2n+1}(1) \to \mathbb{C}P^n(4)$. Here $\sim$ denotes the equivalence relation obtained by identifying two points $(e^{\sqrt{-1}\theta}, a_1, \ldots, a_n)$ and $(-e^{\sqrt{-1}\theta}, -a_1, \ldots, -a_n)$ on $S^1 \times S^{n-1}$. The metric on $S^1 \times S^{n-1}/\!\sim$ is induced by the metric

$$\langle (u_1, v_1), (u_2, v_2)\rangle = \frac{2}{9}\langle u_1, u_2\rangle_{S^1} + \frac{2}{3}\langle v_1, v_2\rangle_{S^{n-1}}$$

for $(u_1, v_1), (u_2, v_2) \in T_{(p_1,p_2)}(S^1 \times S^{n-1}) \cong T_{p_1}S^1 \times T_{p_2}S^{n-1}$ defined with standard metrics on S^1 and S^{n-1}. As was shown by Naitoh[40], the second fundamental form $\sigma_{S^1\times S^{n-1}}$ of this embedding is expressed as

$$\begin{cases} \sigma_{S^1\times S^{n-1}}(u,u) = -(1/\sqrt{2})Ju, \\ \sigma_{S^1\times S^{n-1}}(u,v) = (1/\sqrt{2})Jv, \\ \sigma_{S^1\times S^{n-1}}(v,v) = (1/\sqrt{2})Ju \end{cases}$$

for an arbitrary unit tangent vector $v \in TS^{n-1}$ and the normalized vector u of $\partial/\partial\theta$. geodesics on $S^1 \times S^{n-1}/\!\sim$ are mapped to circles of geodesic curvature $1/\sqrt{2}$ on $\mathbb{C}P^n(4)$. By the definition of the metric on $S^1 \times S^{n-1}/\!\sim$ we have the following:

Proposition 9.1 (Adachi-Maeda[10], Adachi-Maeda-Udagawa[13]). *A circle of geodesic curvature $1/\sqrt{2}$ and of complex torsion τ on $\mathbb{C}P^n(4)$ has the following properties*:

(1) *If $\tau = \pm 1$, it is closed of length $2\sqrt{2}\pi/3$.*

(2) *If* $\tau = 0$, *it is closed of length* $2\sqrt{6}\pi/3$.

(3) *If* $0 < |\tau| < 1$, *it is closed if and only if* $\tau = \pm q(9p^2 - q^2)(3p^2 + q^2)^{-3/2}$ *holds with some relatively prime positive integers* p, q $(p > q)$. *In this case, its length is given as* $\pi\delta(p,q)\sqrt{2(3p^2+q^2)}/3$, *where* $\delta(p,q) = 1$ *when* pq *is odd and* $\delta(p,q) = 2$ *when* pq *is even.*

9.3. **Length of closed circular trajectories on $\mathbb{C}P^n$.** We are now in the position to show Theorem 9.1. For the sake of simplicity we first consider the case $c = 4$.

Lemma 9.3. *Let* γ *be a circular trajectory* γ *on a geodesic sphere* $G(r)$ *in* $\mathbb{C}P^n(4)$. *Let* a_κ, b_κ, c_κ $\ (a_\kappa < b_\kappa < c_\kappa)$ *be three distinct solutions of the cubic equation*

$$\Theta^3 - (\kappa - \kappa^{-1})\Theta^2 - (2 - \rho_\gamma^2)\Theta - (1 - \rho_\gamma^2)\kappa^{-1} = 0 \tag{9.5}$$

Then, γ *is closed if and only if there exists a constant* d_κ *satisfying that all of the ratios*

$$(a_\kappa - d_\kappa)/(b_\kappa - d_\kappa),\ (b_\kappa - d_\kappa)/(c_\kappa - d_\kappa),\ (c_\kappa - d_\kappa)/(a_\kappa - d_\kappa)$$

are rational. In this case, its length is $2\pi \times \mathrm{L.C.M.}\big\{(b_\kappa - a_\kappa)^{-1}, (c_\kappa - b_\kappa)^{-1}\big\}$.

Proof. By Proposition 8.1 the extrinsic shape of γ is either a circle or an essentially Killing helix of proper order 4. We take a horizontal lift $\hat{\gamma}$ of the extrinsic shape of γ with respect to a Hopf fibration ϖ and regard it as a curve in $\mathbb{C}^{n+1}$.

We first consider the latter case that the extrinsic shape of γ is an essentially Killing helix of proper order 4. In this case it is determined by the differential equations

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = k_1 Y_2, \quad \widetilde{\nabla}_{\dot\gamma} Y_2 = -k_1\dot\gamma + k_2 Y_3,$$

and the equations $\widetilde{\nabla}_{\dot\gamma} Y_3 = -k_2 Y_2 + k_3 Y_4$, $\widetilde{\nabla}_{\dot\gamma} Y_4 = -k_3 Y_3$ are auxiliary (see Lemma 4.9 and also see Adachi[6]). We hence consider the first two equations. By use of

(9.1) and Lemma 4.9 and Proposition 8.1, we find that its horizontal lift $\hat{\gamma}$ satisfies

$$\begin{cases} \overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = k_1 Y_2 - \widehat{\mathcal{N}}, \\ \overline{\nabla}_{\dot{\hat{\gamma}}} Y_2 = -k_1\dot{\hat{\gamma}} + k_2 Y_3 + \tau_{12} J\widehat{\mathcal{N}} \\ \qquad\quad = -k_3\dot{\hat{\gamma}} + \mathrm{sgn}(\kappa - \kappa^{-1})\sqrt{k_2^2 + (k_1 - k_3)^2}\, JY_2 + \tau_{12} J\widehat{\mathcal{N}}. \end{cases}$$

Since we have

$$k_1 = \frac{1}{|\kappa|}\sqrt{\kappa^4 - 2\kappa^2\rho_\gamma^2 + \rho_\gamma^2}, \qquad k_3 = (1-\rho_\gamma^2)/k_1,$$
$$\mathrm{sgn}(\kappa - \kappa^{-1})\sqrt{k_2^2 + (k_1 - k_3)^2} = \kappa - \kappa^{-1}, \quad \tau_{12} = \frac{{\rho_\gamma}^2 - \kappa^2}{\kappa k_1},$$

by the proof of Proposition 8.1, we obtain

$$\begin{aligned} \overline{\nabla}_{\dot{\hat{\gamma}}}\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} &= k_1 \overline{\nabla}_{\dot{\hat{\gamma}}} Y_2 - \dot{\hat{\gamma}} \\ &= -(k_1k_3 + 1)\dot{\hat{\gamma}} + k_1(\kappa - \kappa^{-1}) JY_2 + \kappa^{-1}(\rho_\gamma^2 - \kappa^2) J\widehat{\mathcal{N}} \\ &= -(2 - \rho_\gamma^2)\dot{\hat{\gamma}} + (\kappa - \kappa^{-1}) J\big(\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} + \mathcal{N}\big) + \kappa^{-1}(\rho_\gamma^2 - \kappa^2) J\widehat{\mathcal{N}} \\ &= -(2 - \rho_\gamma^2)\dot{\hat{\gamma}} + (\kappa - \kappa^{-1}) J\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} + \kappa^{-1}(\rho_\gamma^2 - 1) J\widehat{\mathcal{N}}. \end{aligned}$$

Thus we obtain that it satisfies the following differential equation

$$\hat{\gamma}''' - \sqrt{-1}(\kappa - \kappa^{-1})\hat{\gamma}'' + (2 - \rho_\gamma^2)\hat{\gamma}' + \sqrt{-1}(1 - \rho_\gamma^2)\kappa^{-1}\hat{\gamma} = 0. \tag{9.6}$$

Next we consider the case that the extrinsic shape of γ is a circle. In this case $\kappa = \pm 1$. If we consider its horizontal lift $\hat{\gamma}$, then it satisfies

$$\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = k_1 Y_2 - \widehat{\mathcal{N}}, \qquad \overline{\nabla}_{\dot{\hat{\gamma}}} Y_2 = -k_1\dot{\hat{\gamma}} + \tau_{12} J\widehat{\mathcal{N}}$$

with $k_1 = \sqrt{1 - \cot^2 r}$, $\tau_{12} = \mp\sqrt{1 - \cot^2 r}$. We therefore obtain

$$\overline{\nabla}_{\dot{\hat{\gamma}}}\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = k_1\overline{\nabla}_{\dot{\hat{\gamma}}} Y_2 - \dot{\hat{\gamma}} = -(k_1^2 + 1)\dot{\hat{\gamma}} + k_1\tau_{12} J\widehat{\mathcal{N}} = -(2 - \cot^2 r)\dot{\hat{\gamma}} \mp (1 - \cot^2 r) J\widehat{\mathcal{N}}.$$

Since we have $\kappa = \pm 1$, hence have $\rho_\gamma^2 = \cot^2 r$, we find it also satisfies (9.6).

We here consider the characteristic equation of the linear differential equation (9.6) of constant coefficients, which is given by

$$\Lambda^3 - \sqrt{-1}(\kappa - \kappa^{-1})\Lambda^2 + (2 - \rho_\gamma^2)\Lambda + \sqrt{-1}(1 - \rho_\gamma^2)\kappa^{-1} = 0.$$

In order to realize it, we put $\Theta = -\sqrt{-1}\Lambda$, then obtain (9.5). Since $\hat{\gamma}$ lies on $S^{2n+1}(1)$, the cubic equation (9.5) should have three distinct real solutions, otherwise $\hat{\gamma}$ is unbounded. By use of these solutions a_κ, b_κ, c_κ we find that a horizontal lift $\hat{\gamma}$ on S^{2n+1} in $\mathbb{C}^{n+1}$ is of the form

$$\hat{\gamma}(t) = A_\kappa e^{\sqrt{-1}a_\kappa t} + B_\kappa e^{\sqrt{-1}b_\kappa t} + C_\kappa e^{\sqrt{-1}c_\kappa t}$$

with some $A_\kappa, B_\kappa, C_\kappa \in \mathbb{C}^{n+1}$. These $A_\kappa, B_\kappa, C_\kappa$ are determined by $\hat{\gamma}(0), \dot{\hat{\gamma}}(0)$ and $Y_2(0)$. Here, we consider $Y_2(0)$ as its horizontal lift. The complex torsion satisfies $\tau_{12} \neq \pm 1$ because $\bigl(\kappa^4 - 2\kappa^2\rho_\gamma^2 + \rho_\gamma^2\bigr) - (\kappa^2 - \rho_\gamma^2)^2 = \rho_\gamma^2(1-\rho_\gamma^2) \neq 0$. Here we used the property that $0 < |\rho_\gamma| < 1$ for a circular trajectory γ. Since $\dot{\hat{\gamma}}(0)$ and $Y_2(0)$ are orthogonal to both $\hat{\gamma}(0)$ and $J\hat{\gamma}(0)$, with the aid of $\tau_{12} \neq \pm 1$, we find $A_\kappa, B_\kappa, C_\kappa$ are linearly independent over $\mathbb{C}$. Hence by the same argument as in the proof of Lemma 9.2, we find that γ is closed if and only if there exists a constant d_κ satisfying that all of the ratios

$$(a_\kappa - d_\kappa)/(b_\kappa - d_\kappa),\ (b_\kappa - d_\kappa)/(c_\kappa - d_\kappa),\ (c_\kappa - d_\kappa)/(a_\kappa - d_\kappa)$$

are rational, and that its length in this case is $2\pi\times\text{L.C.M.}\bigl\{(b_\kappa - a_\kappa)^{-1}, (c_\kappa - b_\kappa)^{-1}\bigr\}$. $\square$

We shall transplant the result on circles on $\mathbb{C}P^n(4)$ to our trajectories. To do this we modify the cubic equation (9.5) by changing the variable. First we make a parallel translation by putting $\Theta_1 = \Theta - \frac{1}{3}(\kappa - \kappa^{-1})$. We then find that (9.5) turns to

$$\Theta_1^3 - \frac{1}{3}(\kappa^2 + \kappa^{-2} + 4 - 3\rho_\gamma^2)\Theta_1 - \frac{1}{27}\bigl(2\kappa^3 + 12\kappa - 9\kappa\rho_\gamma^2 + 15\kappa^{-1} - 18\kappa^{-1}\rho_\gamma^2 - 2\kappa^{-3}\bigr) = 0.$$

Next we make the coefficient of degree one of this cubic equation to be 3/2 by putting $\vartheta = \bigl(3/\sqrt{2(\kappa^2+\kappa^{-2}+4-3\rho_\gamma^2)}\bigr)\Theta_1$. We should note that $\kappa^2+\kappa^{-2}+4-3\rho_\gamma^2 > 0$ because $|\rho_\gamma| < 1$. Thus by putting

$$\vartheta = \bigl(3\Theta - \kappa + \kappa^{-1}\bigr)/\sqrt{2(\kappa^2+\kappa^{-2}+4-3\rho_\gamma^2)},$$

we find the equation (9.5) turns to

$$\vartheta^3 - \frac{3}{2}\vartheta - \frac{\operatorname{sgn}(\kappa)\cdot(\kappa^2+2)\{2\kappa^4+(8-9\rho_\gamma^2)\kappa^2-1\}}{2\sqrt{2}(\kappa^4+4\kappa^2-3\rho_\gamma^2\kappa^2+1)^{3/2}} = 0. \tag{9.7}$$

We set

$$\tau(\kappa;r) = -\frac{\operatorname{sgn}(\kappa)\cdot(\kappa^2+2)\{2\kappa^4+(8-9\rho_\gamma^2)\kappa^2-1\}}{2(\kappa^4+4\kappa^2-3\rho_\gamma^2\kappa^2+1)^{3/2}}.$$

Here, as we have

$$\begin{aligned}
&4(\kappa^4+4\kappa^2-3\rho_\gamma^2\kappa^2+1)^3-(\kappa^2+2)^2\{2\kappa^4+(8-9\rho_\gamma^2)\kappa^2-1\}^2 \\
&\qquad = 27\kappa^8+18(16+7\rho_\gamma^2-3\rho_\gamma^6)\kappa^6+4\{7+54\rho_\gamma^2(1-\rho_\gamma^2)\}\kappa^4+108(1-\rho_\gamma^2)>0,
\end{aligned}$$

we obtain $|\tau(\kappa;r)|<1$. This guarantees that the equality (9.5) has three distinct real solutions directly.

We now compare (9.7) and (9.4). By use of the solutions a_τ, b_τ, c_τ ($a_\tau<b_\tau<c_\tau$) for (9.4) with $\tau=\tau(\kappa;r)$, the change of variables of Θ to ϑ shows that

$$\begin{aligned}
a_\kappa &= \big(a_\tau\sqrt{2(\kappa^2+\kappa^{-2}+4-3\rho_\gamma^2)}+\kappa-\kappa^{-1}\big)/3,\\
b_\kappa &= \big(b_\tau\sqrt{2(\kappa^2+\kappa^{-2}+4-3\rho_\gamma^2)}+\kappa-\kappa^{-1}\big)/3,\\
c_\kappa &= \big(c_\tau\sqrt{2(\kappa^2+\kappa^{-2}+4-3\rho_\gamma^2)}+\kappa-\kappa^{-1}\big)/3.
\end{aligned}$$

Thus by putting $d_\kappa=(\kappa-\kappa^{-1})/3$ we find

$$\frac{a_\kappa-d_\kappa}{b_\kappa-d_\kappa}=\frac{a_\tau}{b_\tau},\quad \frac{b_\kappa-d_\kappa}{c_\kappa-d_\kappa}=\frac{b_\tau}{c_\tau},\quad \frac{c_\kappa-d_\kappa}{a_\kappa-d_\kappa}=\frac{c_\tau}{a_\tau}.$$

Therefore, we find that γ is closed if and only if a circle σ of geodesic curvature $1/\sqrt{2}$ and of complex torsion $\tau(\kappa;r)$ on $\mathbb{C}P^n(4)$ is closed. Moreover, in this case, we obtain

$$\begin{aligned}
\operatorname{length}(\gamma) &= 2\pi\times\text{L.C.M.}\big\{(b_\kappa-a_\kappa)^{-1},(c_\kappa-b_\kappa)^{-1}\big\}\\
&= 2\pi\times\text{L.C.M.}\big\{(b_\tau-a_\tau)^{-1},(c_\tau-b_\tau)^{-1}\big\}\times 3\{2(\kappa^2+\kappa^{-2}+4-3\rho_\gamma^2)\}^{-1/2}\\
&= 3\operatorname{length}(\sigma)\big/\sqrt{2(\kappa^2+\kappa^{-2}+4-3\rho_\gamma^2)}.
\end{aligned}$$

First we consider the case that $\tau(\kappa;r)=\pm q(9p^2-q^2)(3p^2+q^2)^{-3/2}$. By Proposition 9.1 we find that this circular trajectory γ is closed and that its length is given by

$$\begin{aligned}&\frac{1}{3}\pi\delta(p,q)\sqrt{2(3p^2+q^2)}\times\frac{3}{\sqrt{2\{\kappa^2+(1-3\cot^2 r)\kappa^{-2}+4\}}}\\&=\pi\delta(p,q)\sqrt{\frac{3p^2+q^2}{\kappa^2+(1-3\cot^2 r)\kappa^{-2}+4}}.\end{aligned}$$

We next consider the case corresponding to the case of $\tau=0$. Clearly we have $\tau(\kappa;r)=0$ if and only if $2\kappa^4+8\kappa^2-9\cot^2 r-1=0$, because $\kappa\rho_\gamma=\cot r$. The solution of this equation $2\kappa^4+8\kappa^2-9\cot^2 r-1=0$ is

$$\kappa^2=\big(3\sqrt{2(\cot^2 r+1)}-4\big)/2=\big(3\sqrt{2}(\sin r)^{-1}-4\big)/2.$$

(We note that $3\sqrt{2}/4>1$.) Since the condition $\kappa\rho_\gamma=\cot r$ shows $|\kappa|>\cot r$, we have to check whether this occurs. As we see $\big(3\sqrt{2}(\sin r)^{-1}-4\big)/2>\cot^2 r$ if and only if $(2\cot^2 r+1)(\cot^2 r-1)<0$, we find that there is κ satisfying $\tau(\kappa;r)=0$ when $\pi/4<r<\pi/2$. Thus we have $\tau(\kappa;r)=0$ if and only if $\pi/4<r<\pi/2$ and $\kappa^2=\big(3\sqrt{2}(\sin r)^{-1}-4\big)/2$.

In this case γ is closed and is of length

$$\begin{aligned}&\frac{2\sqrt{6}}{3}\pi\times\frac{3}{\sqrt{2\{\kappa^2+(1-3\cot^2 r)\kappa^{-2}+4\}}}=\frac{2\sqrt{6}\pi|\kappa|}{\sqrt{9\cot^2 r+1+2(1-3\cot^2 r)}}\\&=\frac{2\sqrt{2}\pi}{\cot^2 r+1}\times\sqrt{\big(3\sqrt{2}(\sin r)^{-1}-4\big)/2}=2\pi\sqrt{\sin r(3\sqrt{2}-4\sin r)}.\end{aligned}$$

Therefore we have proved the following.

Theorem 9.2. *Let γ be a circular $\boldsymbol{F}_\kappa$-trajectory on a geodesic sphere $G(r)$ of radius r $(0<r<\pi/2)$ in $\mathbb{C}P^n(4)$.*

(1) *When $\pi/4<r<\pi/2$ and $\kappa^2=\big(3\sqrt{2(\cot^2 r+1)}-4\big)/2$, it is closed and its length is $2\pi\sqrt{\sin r\big(3\sqrt{2}-4\sin r\big)}$.*

(2) *Otherwise, γ is closed if and only if*

$$\frac{(\kappa^2+2)\big|2\kappa^4+8\kappa^2-9\cot^2 r-1\big|}{2(\kappa^4+4\kappa^2-3\cot^2 r+1)^{3/2}}=\frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}}$$

holds with some relatively prime positive integers p, q satisfying $p > q$. In this case, its length is given as $\pi\delta(p,q)|\kappa|\sqrt{(3p^2+q^2)/(\kappa^4+4\kappa^2-3\cot^2 r+1)}$, where $\delta(p,q)=1$ when pq is odd and $\delta(p,q)=2$ when pq is even.

In order to study trajectories on geodesic spheres in a general complex projective space $\mathbb{C}P^n(c)$, we make use of homothetic changes of metrics.

Proof of Theorem 9.1. Let γ be a circular trajectory for a Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$. As was mentioned in §8, on $\mathbb{C}P^n(c)$ we consider a new metric $\langle\ ,\ \rangle' = (c/4)\langle\ ,\ \rangle$. Then holomorphic sectional curvatures are 4 and the radius of the geodesic sphere is $\sqrt{c}\,r/2$ with respect to the new metric. If we define a curve σ by $\sigma(s) = \gamma\big(2t/\sqrt{c}\big)$, it is a circular trajectory for a Sasakian magnetic field ${\boldsymbol{F}'}_{2\kappa/\sqrt{c}}$ with respect to the new metric.

When $\kappa' = 2\kappa/\sqrt{c} = \Big(3\sqrt{2\big(\cot^2(\sqrt{c}\,r/2)+1\big)}-4\Big)\Big/2$, the trajectory σ is closed and its length is $2\pi\sqrt{\sin(\sqrt{c}\,r/2)\big(3\sqrt{2}-4\sin(\sqrt{c}\,r/2)\big)}$. Since we have $\mathrm{length}'(\sigma) = (\sqrt{c}/2)\mathrm{length}(\gamma)$, we get the first assertion.

When κ' satisfies

$$\frac{(\kappa'^2+2)\big|2\kappa'^4+8\kappa'^2-9\cot^2(\sqrt{c}\,r/2)-1\big|}{2(\kappa'^4+4\kappa'^2-3\cot^2(\sqrt{c}\,r/2)+1)^{3/2}} = \frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}},$$

that is, when κ satisfies

$$\begin{aligned}\frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}} &= \frac{\big(\frac{4\kappa^2}{c}+2\big)\big|\frac{32\kappa^4}{c^2}+\frac{32\kappa^2}{c}-9\cot^2(\sqrt{c}\,r/2)-1\big|}{2\big(\frac{16\kappa^4}{c^2}+\frac{16\kappa^2}{c}-3\cot^2(\sqrt{c}\,r/2)+1\big)^{3/2}} \\ &= \frac{(2\kappa^2+c)\big|32\kappa^4+32c\kappa^2-9c^2\cot^2(\sqrt{c}\,r/2)-c^2\big|}{\big(16\kappa^4+16c\kappa^2-3c^2\cot^2(\sqrt{c}\,r/2)+c^2\big)^{3/2}},\end{aligned}$$

the trajectory σ is closed. Hence γ is closed and its length is

$$\begin{aligned}\mathrm{length}(\gamma) &= \frac{2}{\sqrt{c}}\,\mathrm{length}'(\sigma) = \frac{2}{\sqrt{c}}\,\pi\delta(p,q)|\kappa'|\sqrt{\frac{3p^2+q^2}{\kappa'^4+4\kappa'^2-3\cot^2(\sqrt{c}\,r/2)+1}}\\ &= \frac{2}{\sqrt{c}}\,\pi\delta(p,q)\frac{2|\kappa|}{\sqrt{c}}\sqrt{\frac{3p^2+q^2}{\frac{16\kappa^4}{c^2}+\frac{16\kappa^2}{c}-3\cot^2(\sqrt{c}\,r/2)+1}}\\ &= 4\pi\delta(p,q)|\kappa|\sqrt{\frac{3p^2+q^2}{16\kappa^4+16c\kappa^2-3c^2\cot^2(\sqrt{c}\,r/2)+c^2}}.\end{aligned}$$

This complete the proof. □

10. **Hadamard manifolds**

Let M be a topological space and $p_0 \in M$ be a fixed point. A continuous map $\gamma : [0,1] \to M$ is called a closed curve or a loop with base point p_0 if it satisfies $\gamma(0) = \gamma(1) = p_0$. We denote by $\mathcal{C}(M;p_0)$ the set of all loops on M with base point p_0. For $\gamma \in \mathcal{C}(M;p_0)$ we have $\gamma^{-1} \in \mathcal{C}(M;p_0)$ which is given by $\gamma^{-1}(t) = \gamma(1-t)$. For two loops $\gamma_1, \gamma_2 \in \mathcal{C}(M;p_0)$ we define $\gamma_1 \cdot \gamma_2 \in \mathcal{C}(M;p_0)$ by

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le 1/2, \\ \gamma_2(2t-1) & \text{if } 1/2 < t \le 1. \end{cases}$$

We say $\gamma_1 \in \mathcal{C}(M;p_0)$ is homotopic to $\gamma_2 \in \mathcal{C}(M;p_0)$ if there is a continuous map $\mathit{\Gamma} : [0,1] \times [0,1] \to M$ satisfying $\mathit{\Gamma}(t,0) = \gamma_1(t)$ and $\mathit{\Gamma}(t,1) = \gamma_2(t)$ for all $t \in [0,1]$. For $\gamma \in \mathcal{C}(M;p_0)$, considering a continuous map $\mathit{\Gamma} : [0,1] \times [0,1] \to M$ given by $\mathit{\Gamma}(t,s) = \gamma(t)$, we see γ is homotopic to itself. If γ_1 is homotopic to γ_2 by a continuous map $\mathit{\Gamma}$, by a continuous map $\mathit{\Upsilon} : [0,1] \times [0,1] \to M$ given by $\mathit{\Upsilon}(t,s) = \mathit{\Gamma}(t, 1-s)$ we find γ_2 is homotopic to γ_1. If γ_1 is homotopic to γ_2 by a continuous map $\mathit{\Gamma}_1$ and if γ_2 is homotopic to γ_3 by a continuous map $\mathit{\Gamma}_2$, then considering a continuous map $\mathit{\Gamma} : [0,1] \times [0,1] \to M$ given by

$$\mathit{\Gamma}(t,s) = \begin{cases} \mathit{\Gamma}_1(t, 2s) & \text{if } 0 \le s \le 1/2, \\ \mathit{\Gamma}_2(t, 2s-1) & \text{if } 1/2 < s \le 1, \end{cases}$$

we find γ_1 is homotopic to γ_3. Thus we see the homotpoic property gives an equivalence relation. The quotient space of $\mathcal{C}(M;p_0)$ under this equivalence relation is called the fundamental group of M and is denoted by $\pi_1(M, p_0)$. When M is arcwise connected, this group does not depend on the choice of p_0. We call M *simply connected* if $\pi_1(M)$ consists only of the identity, which means that every loop is homotopic to a constant curve $\gamma_0 : [0,1] \to M$ with $\gamma_0(t) = p_0$.

A complete, simply connected Reimannian manifold of nonpositive curvature is said to be a *Hadamard manifold*. We here breifly make mention of variations of geodesics. Given a geodesic γ on a Riemannian manifold M, we call a smooth map $\alpha : \mathbb{R} \times (-\epsilon, \epsilon) \to M$ a variation of geodesics if $\alpha(\cdot, s)$ is a geodesic for each $s \in (-\epsilon, \epsilon)$. A vector field $\partial\alpha/\partial s(t, 0)$ along γ is called a Jacobi field. A Jacobi field Y along γ is hence a vector field along γ satisfying $\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y + R(Y, \dot\gamma)\dot\gamma = 0$. Given a point $p \in M$ we define an exponential map $\exp_p : T_pM \to M$ by

$$\exp_p(v) = \begin{cases} \gamma_{v/\|v\|}(\|v\|), & \text{if } v \neq 0_p, \\ p, & \text{if } v = 0_p, \end{cases}$$

where γ_u denotes the geodesic satisfying $\dot\gamma_u(0) = u$. By Rauch's comparison theorem on Jacobi fields (see Cheeger-Ebin[27] or Sakai[43], for example), every exponential map on a Hadamard manifold does not have singular points, hence is bijective. Therefore a Hadamard manifold is diffeomorphic to a Euclidean space $\mathbb{R}^m$ if its dimension is m. Since a complex hyperbolic space $\mathbb{C}H^n(c)$ is a typical example of Hadamard manifolds, we here make mention of some fundamental notations and some basic results on Hadamard manifolds (see Eberlein-O'Neil[29] and Eberlein[28], for more detail).

10.1. **Ideal boundary.** Let $\widetilde{M}$ be a Hadamard manifold of dimension m (≥ 2). Two geodesic half lines $\gamma_1, \gamma_2 : [0, \infty) \to \widetilde{M}$ of unit speed on $\widetilde{M}$ are said to be *asymptotic* if the distance $d\big(\gamma_1(t), \gamma_2(t)\big)$ between $\gamma_1(t)$ and $\gamma_2(t)$ is a bounded function with respect to t. As we usually consider geodesics are defined on $\mathbb{R}$ on a complete Riemannian manifold, we here use a terminology "geodesic half lines" to make clear that they are only defined on a half line $[0, \infty)$. It is clear that asymptotic relation is an equivalence relation on the set of all geodesic half lines of unit speed on $\widetilde{M}$. We call an equivalence class of asymptotic geodesic half lines on $\widetilde{M}$ a *point at infinity* for $\widetilde{M}$. We denote by $\widetilde{M}(\infty)$ the set of all points at infinity for $\widetilde{M}$. For a geodesic half line γ, the equivalence class containing γ is denoted by $\gamma(\infty)$. When γ is a geodesic of unit speed, we can consider two geodesic half lines $t \mapsto \gamma(t)$ and $t \mapsto \gamma(-t)$, hence denote by $\gamma(\infty)$ and $\gamma(-\infty)$ the points at infinity corresponding to them.

For example, a Euclidean space $\mathbb{R}^m$ is a Hadamard manifold. Two geodesic half lines, which are nothing but usual half lines, are asymptotic if and only if they are parallel. Therefore, the set $\mathbb{R}^m(\infty)$ of points at infinity is bijective to S^{m-1}. The same property holds in general. For an arbitrary point p on a Hadamard manifold $\widetilde{M}$, for given a geodesic half line γ on $\widetilde{M}$ there is a unique geodesic half line σ of unit speed with $\sigma(0) = p$ and which is asymptotic to γ (see Eberlein-O'Neil[29]). Therefore, the set $\widetilde{M}(\infty)$ is bijective to the unit tangent space $U_p\widetilde{M} \simeq S^{m-1}$.

We now induce a topology on $\overline{\widetilde{M}} = \widetilde{M} \cup \widetilde{M}(\infty)$. On $\widetilde{M}$ we induce the original topology of $\widetilde{M}$. This means that the restricted topology on the subset $\widetilde{M}$ of $\overline{\widetilde{M}}$ coincide with the original topology. We are hence enough to induce a topology on $\widetilde{M}(\infty)$. Let $p_0 \in \widetilde{M}$ be an arbitrary point. We define the angle at p_0 in the following way.

i) For given a point $q \in \widetilde{M}$, we take a geodesic segment of unit speed γ_{p_0q} satisfying $\gamma_{p_0q}(0) = p_0$ and $\gamma_{p_0q}\big(d(p_0, q)\big) = q$.
ii) For given a point $z \in \widetilde{M}(\infty)$, we take a geodesic half line of unit speed γ_{p_0z} satisfying $\gamma_{p_0z}(0) = p_0$ and $\gamma_{p_0z}(\infty) = z$.

For given two points $p, q \in \overline{\widetilde{M}} = \widetilde{M} \cup \widetilde{M}(\infty)$, we set $\angle_{p_0}(p, q)$ the angle between two vectors $\dot{\gamma}_{p_0p}(0)$ and $\dot{\gamma}_{p_0q}(0)$. We take an arbitrary point $z \in \widetilde{M}(\infty)$ and arbitrary positive numbers ϵ, r, and put

$$V_{p_0}(z; \epsilon, r) = \big\{q \in \overline{\widetilde{M}} \;\big|\; \angle_{p_0}(z, q) < \epsilon\big\} \setminus \big\{q \in \widetilde{M} \;\big|\; d(p_0, q) \le r\big\}.$$

Considering $\widetilde{M}(\infty) \simeq U_{p_0}\widetilde{M}$, we can induce a neighborhood basis of z by $\big\{V_{p_0}(z; \epsilon, r) \;\big|\; \epsilon, r\big\}$. This topology on $\overline{\widetilde{M}}$ is called the *cone topology.* With this topology $\overline{\widetilde{M}}$ is homeomorphic to a closed unit ball $B^m = \big\{x \in \mathbb{R}^m \;\big|\; \|x\| \le 1\big\}$, hence is compact, and $\widetilde{M}$ is a dense open subset in $\overline{\widetilde{M}}$. We call $\widetilde{M}(\infty)$ with cone topology the *ideal boundary* of $\widetilde{M}$ and is also denoted by $\partial\widetilde{M}$.

10.2. **Ball model of a complex hyperbolic space.** We define a complex hyperbolic space $\mathbb{C}H^n$ in §2 as a quatient space of an anti-de Sitter space H_1^{2n+1} under the S^1 action. We here give another representation, which is called a *ball model* of a complex hyperbolic space. Let $\boldsymbol{D}^n$ denote a unit open ball in $\mathbb{C}^n$, that is

$$\boldsymbol{D}^n = \bigl\{w = (w_1, w_2, \dots, w_n) \in \mathbb{C}^n \bigm| |w_1|+\cdots+|w_n| < 1\bigr\}.$$

We identify a point $\varpi(z) \in \mathbb{C}H^n,\ z = (z_0, z_1, \dots, z_n) \in H_1^{2n+1} \subset \mathbb{C}_1^{n+1}$ with the point $\bigl(z_1/z_0, \dots, z_n/z_0\bigr) \in \boldsymbol{D}^n$. Since $-|z_0|^2+|z_1|^2+\cdots+|z_n| = -1$, we see $z_0 \neq 0$ and $|z_1/z_0|^2+\cdots+|z_n/z_0|^2 = 1 - \frac{1}{|z_0|^2} < 1$. On this model, images of geodesics are circle-arcs which meet orthogonal to the topological boundary $\partial\boldsymbol{D}^n$. Therefore, the ideal boundary with respect to the cone topology coincides with the topological boundary.

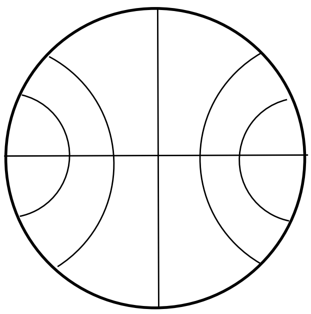


FIGURE 1. Geodesics on a ball model of $\mathbb{C}H^n$

11. **Circular trajectories on horospheres in $\mathbb{C}H^n$**

We devote this section and the next three sections to study trajectories on standard real hypersurfaces in a complex hyperbolic space. In this section we study on a horosphere *HS* in $\mathbb{C}H^n(c)$, which is called a real hypersurface of type (A_0). A horosphere *HS* has two distinct principal curvatures $\lambda = \sqrt{|c|}/2$ and $\nu = \sqrt{|c|}$. Its characteristic vector field ξ satisfies $A\xi = \nu\xi$ and every tangent vector v orthogonal to ξ satisfies $Av = \lambda v$ with the shape operator A. In particular, the shape operator and the characteristic tensor satisfy $A\phi = \phi A$.

11.1. **Trajectories on horospheres.** We first study trajectories from the viewpoint of Frenet-Serre formula. By Corollary 7.1, we know that every trajectory on *HS* has constant structure torsion. By the same proof as of Proposition 7.1, that is, by substituting $\lambda = \sqrt{|c|}/2$ in that proof, we have the following.

Proposition 11.1. *Let γ be a trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a horosphere HS in a complex hyperbolic space $\mathbb{C}H^n(c)$.*

(1) *It is a geodesic if and only if $\rho_\gamma = \pm 1$,*
(2) *It is a circle of positive geodesic curvature if and only if $\kappa\rho_\gamma = \sqrt{|c|}/2$. In this case, its geodesic curvature is $\sqrt{4\kappa^2 + c}/2$.*
(3) *Otherwise, it is a helix of proper order 3 whose geodesic curvatures are $|\kappa|\sqrt{1-\rho_\gamma^2}$ and $|2\kappa\rho_\gamma - \sqrt{c}|/2$.*

If we restrict ourselves to circular trajectories, as $|\rho_\gamma| < 1$, we have the following result.

Theorem 11.1. *Let $\boldsymbol{F}_\kappa$ be a non-trivial Sasakian magnetic field on a horosphere HS in $\mathbb{C}H^n(c)$.*

(1) *When $0 < |\kappa| \le \sqrt{|c|}/2$, there are no circular trajectories for $\boldsymbol{F}_\kappa$.*

(2) *When* $|\kappa| > \sqrt{|c|}/2$, *a trajectory for* $\boldsymbol{F}_\kappa$ *is circular if and only if* $\rho_\gamma = \sqrt{|c|}/(2\kappa)$. *In this case its geodesic curvature is* $\sqrt{\kappa^2 + (c/4)}$.

Congruency conditions on circular trajectories on a horosphere are given in the same way as in Proposition 7.2.

Proposition 11.2 (Adachi[3]). *Trajectories* γ_1 *for a Sasakian magnetic field* $\boldsymbol{F}_{\kappa_1}$ *and* γ_2 *for* $\boldsymbol{F}_{\kappa_2}$ *on a horosphere HS in* $\mathbb{C}H^n(c)$ *are congruent to each other in strong sense if and only if one of the following conditions holds*:

i) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| = 1$,

ii) $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$ *and* $|\kappa_1| = |\kappa_2|$,

iii) $0 < |\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$ *and* $\kappa_1\rho_{\gamma_1} = \kappa_2\rho_{\gamma_2}$.

In order to show this proposition we need to construct some isometries of horospheres. Once we show the following Lemma, then we can prove the above Proposition by just the same way as of Proposition 7.2. Through an isometric immersion $\iota : HS \to \mathbb{C}H^n(c)$ we consider THS as a subset of $T\mathbb{C}H^n(c)$.

Lemma 11.1. *Let* $x, x' \in HS$ *be arbitrary points on a horosphere HS in* $\mathbb{C}H^n(c)$. *Given unit tangent vectors* $u \in \langle\xi_x\rangle^\perp \subset T_xHS$ *and* $u' \in \langle\xi_{x'}\rangle^\perp \subset T_{x'}HS$ *which are orthogonal to* ξ *at* x *and* x', *there exist isometries* $\tilde{\varphi}^+$, $\tilde{\varphi}^-$ *of* $\mathbb{C}H^n(c)$ *satisfying the following conditions*:

i) $\tilde{\varphi}^+\big(HS\big) = \tilde{\varphi}^-\big(HS\big) = HS$,

(*i.e. HS is invariant under the actions of* $\tilde{\varphi}^+$ *and* $\tilde{\varphi}^-$);

ii) $\tilde{\varphi}^+(x) = \tilde{\varphi}^-(x) = x'$;

iii) $d\tilde{\varphi}^+(u) = d\tilde{\varphi}^-(u) = u'$,

iv) $d\tilde{\varphi}^+ \circ J = J \circ d\tilde{\varphi}^+$ *and* $d\tilde{\varphi}^- \circ J = -J \circ d\tilde{\varphi}^-$,

in particular, $d\tilde{\varphi}^+(\xi_x) = \xi_{x'}$ *and* $d\tilde{\varphi}^-(\xi_x) = -\xi_{x'}$.

Proof. For the sake of simplicity, we only treat the case $n = 2$ and $c = -4$. As we see in §5.3 we may consider that

$$\varpi^{-1}(HS) = \big\{(z_0, z_1, z_2) \in \mathbb{C}^3 \ \big|\ -|z_0|^2+|z_1|^2+|z_2|^2 = -1,\ |z_0-z_1| = 1\big\}$$

through a canonical fibration $\varpi : H_1^5 \to \mathbb{C}H^2$. We take an arbitrary point $\hat{z} = (z_0, z_1, z_2) \in \varpi^{-1}(HS)$. The tangent space of $\widehat{M} = \varpi^{-1}(HS)$ at $\hat{z}$ is represented as

$$T_{\hat{z}}\widehat{M} = \left\{(\hat{z}, \hat{v}) \in \{\hat{z}\}\times\mathbb{C}^3 \ \middle|\ \begin{array}{l}\mathrm{Re}\big(-z_0\overline{v}_0+z_1\overline{v}_1+z_2\overline{v}_2\big) = 0,\\ \mathrm{Re}\big((z_0-z_1)(\overline{v}_0-\overline{v}_1)\big) = 0\end{array}\right\}.$$

We denote by $\widehat{\mathcal{N}}_{\hat{z}} \in T_{\hat{z}}H_1^5$ the horizontal lift of the unit normal vector $\mathcal{N}_{\varpi(\hat{z})}$ of HS in $\mathbb{C}H^2(-4)$. Since $\widehat{\mathcal{N}}_{\hat{z}}$ is orthogonal to $T_{\hat{z}}\widehat{M}$, we find it is represented as

$$\widehat{\mathcal{N}}_{\hat{z}} = \big(\hat{z}, (-z_1, z_0-2z_1, -z_2)\big).$$

In fact, for $(\hat{z}, \hat{v}) \in T_{\hat{z}}\widehat{M}$ we have

$$\mathrm{Re}\big(z_1\overline{v}_0+(z_0-2z_1)\overline{v}_1-z_2\overline{v}_2\big) = \mathrm{Re}\big(-(z_0-z_1)(\overline{v}_0-\overline{v}_1)+z_0\overline{v}_0-z_1\overline{v}_1-z_2\overline{v}_2\big) = 0,$$

hence it is orthogonal to $\widehat{\mathcal{N}}_{\hat{z}}$. We put $\hat{\xi}_{\hat{z}} = -J\widehat{\mathcal{N}}_{\hat{z}} = \big(\hat{z}, \sqrt{-1}(z_1, -z_0+2z_1, z_2)\big)$. We denote by $\langle\hat{\xi}_{\hat{z}}\rangle$ the real linear subspace of $T_{\hat{z}}H_1^5$ spanned by $\hat{\xi}_{\hat{z}}$, and by $\langle\hat{\xi}_{\hat{z}}\rangle^{\perp}$ its orthogonal complement in $T_{\hat{z}}H_1^5$. The horizontal part $\langle\hat{\xi}_{\hat{z}}\rangle^{\perp}\cap\mathcal{H}_{\hat{z}}$ of $\langle\hat{\xi}_{\hat{z}}\rangle^{\perp}$ corresponds to $\langle\xi_{\varpi(\hat{z})}\rangle^{\perp}$, and is represented as

$$\begin{aligned}\langle\hat{\xi}_{\hat{z}}\rangle^{\perp}\cap\mathcal{H}_{\hat{z}} &= \left\{(\hat{z}, \hat{v}) \in \{\hat{z}\}\times\mathbb{C}^3 \ \middle|\ \begin{array}{l}-z_0\overline{v}_0+z_1\overline{v}_1+z_2\overline{v}_2 = 0,\\ (z_0-z_1)(\overline{v}_0-\overline{v}_1) = 0\end{array}\right\}\\ &= \left\{\left(\hat{z}, \Big(\frac{\alpha\overline{z}_2}{\overline{z}_0-\overline{z}_1}, \frac{\alpha\overline{z}_2}{\overline{z}_0-\overline{z}_1}, \alpha\Big)\right) \ \middle|\ \alpha\in\mathbb{C}\right\}.\end{aligned}$$

We take a point $\hat{z}_* = (1,0,0) \in \widehat{M}$ and a unit tangent vector $\hat{u}_* = \big(\hat{z}_*, (0,0,1)\big) \in \langle\hat{\xi}_{\hat{z}_*}\rangle^{\perp}\cap\mathcal{H}_{\hat{z}_*}$. For $\hat{z}\in\widehat{M}$ and $\alpha\in\mathbb{C}$ with $|\alpha|^2 = |z_0-z_1|^2/(|z_0-z_1|^2+|z_2|^2)$, we put

$$\hat{u} = \big(\hat{z}, (\alpha\overline{z}_2/(\overline{z}_0-\overline{z}_1), \alpha\overline{z}_2/(\overline{z}_0-\overline{z}_1), \alpha)\big) \in U_z\widehat{M},$$

and consider a matrix

$$U_+ = \begin{pmatrix} z_0 & -z_1 & \alpha\overline{z}_2/(\overline{z}_0-\overline{z}_1)\\ z_1 & z_0-2z_1 & \alpha\overline{z}_2/(\overline{z}_0-\overline{z}_1)\\ z_2 & -z_2 & \alpha\end{pmatrix} \in U(2,1),$$

This induces a linear transformation of $\mathbb{C}^3_1$ which preserves the Hermitian product $\langle\!\langle\ ,\ \rangle\!\rangle$, hence it induces an isometry $\hat\varphi^+_{(\hat z,\hat u)}$ of H^5_1. If we take an arbitrary $\hat p = (p_0, p_1, p_2) \in \widehat{M}$, the point

$$\begin{aligned}\hat\varphi^+_{(\hat z,\hat u)}(\hat z) &= (p_0', p_1', p_2')\\ &= \big(z_0p_0 - z_1p_1 + u_1p_2, z_1p_0 - (z_0 - 2z_1)p_1 + u_1p_2, z_2(p_0 - p_1) + \alpha p_2\big)\end{aligned}$$

satisfies

$$|p_0' - p_1'| = \big|(z_0 - z_1)(p_0 - p_1)\big| = |p_0 - p_1| = 1.$$

Therefore we see $\hat\varphi^+_{(\hat z,\hat u)}\big(\widehat{M}\big) = \widehat{M}$. It is clear that $\hat\varphi^+_{(\hat z,\hat u)}(e^{\sqrt{-1}\theta}\hat p) = e^{\sqrt{-1}\theta}\hat\varphi^+_{(\hat z,\hat u)}(\hat p)$ for arbitrary $\theta \in [0, 2\pi)$ and $\hat p \in \widehat{M}$. Since it is needless to say that $U_+J = JU_+$ for the matrix $J = \sqrt{-1}E$, we find that $\hat\varphi^+_{(\hat z,\hat u)}$ induces an isomtry $\tilde\varphi^+_{(\hat z,\hat u)}$ of $\mathbb{C}H^2(-4)$ satisfying

$$\begin{aligned}&\tilde\varphi^+_{(\hat z,\hat u)} \circ \varpi = \varpi \circ \hat\varphi^+_{(\hat z,\hat u)}, \quad d\tilde\varphi^+_{(\hat z,\hat u)} \circ J = J \circ d\tilde\varphi^+_{(\hat z,\hat u)},\\ &\tilde\varphi^+_{((\hat z,\hat u))}(\varpi(\hat z_*)) = \varpi(\hat z), \quad d\tilde\varphi^+_{(\hat z,\hat u)}\big(d\varpi(\hat u_*)\big) = d\varpi(\hat u).\end{aligned}$$

In particular, it satisfies $d\tilde\varphi^+_{(\hat z,\hat u)}\big(\xi_{\hat z_*}\big) = \xi_{\hat z}$.

We next consider a matrix

$$\Psi = \begin{pmatrix}\epsilon & O & O\\ O & \epsilon & O\\ O & O & \epsilon\end{pmatrix} \in O(6) \quad \text{with} \quad \epsilon = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \in O(2).$$

This matrix induces a map $\mathbb{C}^3 \ni (p_0, p_1, p_2) \mapsto (\bar p_0, \bar p_1, \bar p_2) \in \mathbb{C}^3$. If we define a matrix U_- by $U_- = U_+\Psi$, it induces a linear transformation of $\mathbb{C}^3_1$ which preserves the Hermitian product. By the representation of $\widehat{M}$, we see it induces an isometry $\hat\varphi^-_{(\hat z,\hat u)}$ of H^5_1 satisfying $\hat\varphi^-_{(\hat z,\hat u)}\big(\widehat{M}\big) = \widehat{M}$. It is clear that it satisfies $\hat\varphi^-_{(\hat z,\hat u)}\big(e^{\sqrt{-1}\theta}\hat p\big) = e^{-\sqrt{-1}\theta}\hat\varphi^-_{(\hat z,\hat u)}(\hat p)$ for arbitrary $\theta \in [0, 2\pi)$ and $\hat p \in \widehat{M}$. As we have $U_-J = -JU_-$ for the matrix $J = \sqrt{-1}E$, we find that $\hat\varphi^-_{(\hat z,\hat u)}$ induces an isometry $\tilde\varphi^-_{(\hat z,\hat u)}$ of $\mathbb{C}H^n(-4)$ satisfying

$$\begin{aligned}&\tilde\varphi^-_{(\hat z,\hat u)} \circ \varpi = \varpi \circ \hat\varphi^-_{(z,u)}, \quad d\tilde\varphi^-_{(\hat z,\hat u)} \circ J = -J \circ d\tilde\varphi^+_{(\hat z,\hat u)},\\ &\tilde\varphi^-_{(\hat z,\hat u)}(\varpi(\hat z_*)) = \varpi(\hat z), \quad d\tilde\varphi^-_{(\hat z,\hat u)}\big(d\varpi(\hat u_*)\big) = d\varpi(\hat u).\end{aligned}$$

In particular, it satisfies $d\tilde\varphi^-_{(\hat z,\hat u)}\big(\xi_{\hat z_*}\big) = -\xi_{\hat z}$.

As we constructed desirable isometries for a fixed pair $\big(\varpi(\hat{z}_*), d\varpi(\hat{u}_*)\big)$ and an arbitrary pair $\big(\varpi(\hat{z}), d\varpi(\hat{u})\big)$, we can get our conclusion. □

Corollary 11.1. *Every circular trajectory on HS in* $\mathbb{C}H^n(c)$ *is Killing.*

Corollary 11.2. *Circular trajectories for a given Sasakian magnetic field* $\boldsymbol{F}_\kappa$ *on a horosphere in* $\mathbb{C}H^n(c)$ *are congruent to each other in strong sense.*

Corollary 11.3. *Geodesic trajectories for non-trivial Sasakian magnetic fields on a horosphere in* $\mathbb{C}H^n(c)$ *are congruent to each other in strong sense.*

11.2. **Extrinsic shapes of circular trajectories on** *HS***.** We next study extrinsic shapes of circular trajectories on a horosphere *HS* in $\mathbb{C}H^n(c)$.

Proposition 11.3. *The extrinsic shape of a circular trajectory for a non-trivial Sasakian magnetic field* $\boldsymbol{F}_\kappa$ *on HS in* $\mathbb{C}H^n(-4)$ *is an essential Killing helix of proper order* 4 *and satisfies the condition* (*I*) *in Lemma 4.9. Its geodesic curvatures are*

$$k_1 = \frac{1}{\kappa^2}\sqrt{\kappa^6+2\kappa^2+1},\ k_2 = \frac{(\kappa^2+1)\sqrt{\kappa^2-1}}{\kappa^2\sqrt{\kappa^6+2\kappa^2+1}},\ k_3 = \frac{\kappa^2-1}{\sqrt{\kappa^6+2\kappa^2+1}},$$

and its complex torsions satisfy

$$\tau_{12} = \tau_{34} = -\frac{\kappa+\kappa^{-3}}{k_1} = \frac{-\mathrm{sgn}(\kappa)\cdot(k_1+k_3)}{\sqrt{k_2^2+(k_1+k_3)^2}}, \quad \tau_{13} = \tau_{24} = 0,$$

$$\tau_{23} = \tau_{14} = -\frac{\mathrm{sgn}(\kappa)\sqrt{1-\kappa^{-2}}}{\kappa^2 k_1} = \frac{-\mathrm{sgn}(\kappa)\cdot k_2}{\sqrt{k_2^2+(k_1+k_3)^2}},$$

Proof. Since a circular $\boldsymbol{F}_\kappa$-trajectory γ satisfies $\kappa\rho_\gamma = 1$, we have

$$A\dot\gamma = \dot\gamma + \rho_\gamma\xi = \dot\gamma + \kappa^{-1}\xi,$$

$$\langle A\dot\gamma, \dot\gamma\rangle = 1 + \rho_\gamma^2 = 1 + \kappa^{-2}.$$

By use of Gauss formula (3.1), we have

$$\widetilde\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma + \langle A\dot\gamma, \dot\gamma\rangle\mathcal{N} = \kappa\phi\dot\gamma + (1+\kappa^{-2})\mathcal{N} = \kappa J\dot\gamma + \kappa^{-2}\mathcal{N}$$

with a unit normal $\mathcal{N}$ on HS in $\mathbb{C}H^n(-c)$. If we set

$$k_1 = \|\kappa J\dot{\gamma} + \kappa^{-2}\mathcal{N}\| = \sqrt{\kappa^2+2\rho_\gamma\kappa^{-1}+\kappa^{-4}} = \sqrt{\kappa^2+2\kappa^{-2}+\kappa^{-4}}\ (>0),$$
$$Y_2 = \big(\kappa J\dot{\gamma} + \kappa^{-2}\mathcal{N}\big)/k_1,$$

we have $\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = k_1 Y_2$. Differentiating Y_2 we have

$$\widetilde{\nabla}_{\dot{\gamma}}(\kappa J\dot{\gamma} + \kappa^{-2}\mathcal{N}) = \kappa J\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} - \kappa^{-2}A\dot{\gamma} = -(\kappa^2+\kappa^{-2})\dot{\gamma} - \kappa^{-1}(1+\kappa^{-2})\xi$$
$$= -(\kappa^2+2\kappa^{-2}+\kappa^{-4})\dot{\gamma} - \kappa^{-1}(1+\kappa^{-2})(\xi - \kappa^{-1}\dot{\gamma}).$$

By setting

$$k_2 = |\kappa^{-1}|(1+\kappa^{-2})\sqrt{1-\kappa^{-2}}/k_1\ (>0), \quad Y_3 = \mathrm{sgn}(\kappa)(\kappa^{-1}\dot{\gamma} - \xi)\big/\sqrt{1-\kappa^{-2}},$$

we see $\widetilde{\nabla}_{\dot{\gamma}}Y_2 = -k_1\dot{\gamma} + k_2 Y_3$. Continuing calculation we have

$$\widetilde{\nabla}_{\dot{\gamma}}(\kappa^{-1}\dot{\gamma} - \xi) = J\dot{\gamma} + \kappa^{-3}\mathcal{N} - JA\dot{\gamma} = \kappa^{-1}(\kappa^{-2}-1)\mathcal{N}$$
$$= -\mathrm{sgn}(\kappa)\sqrt{1-\kappa^{-2}}\,k_2Y_2 + \kappa^{-1}(\kappa^{-2}-1)\mathcal{N} + \mathrm{sgn}(\kappa)\sqrt{1-\kappa^{-2}}\,k_2Y_2$$
$$= -\mathrm{sgn}(\kappa)\sqrt{1-\kappa^{-2}}k_2Y_2 + \kappa^{-1}(\kappa^{-2}-1)\mathcal{N}$$
$$+ k_1^{-2}\kappa^{-1}(1+\kappa^{-2})(1-\kappa^{-2})\big(\kappa J\dot{\gamma} + \kappa^{-2}\mathcal{N}\big)$$
$$= -\mathrm{sgn}(\kappa)k_2Y_2 + k_1^{-2}(1-\kappa^{-2})\big\{(1+\kappa^{-2})J\dot{\gamma} - (\kappa+\kappa^{-3})\mathcal{N}\big\}$$
$$= -\mathrm{sgn}(\kappa)k_2Y_2 + k_1^{-2}(1-\kappa^{-2})\big\{(1+\kappa^{-2})\phi\dot{\gamma} - (\kappa-\kappa^{-1})\mathcal{N}\big\}.$$

We therefore set

$$k_3 = k_1^{-2}\sqrt{1-\kappa^{-2}}\|(1+\kappa^{-2})\phi\dot{\gamma} - (\kappa-\kappa^{-1})\mathcal{N}\| = k_1^{-1}(1-\kappa^{-2})\ (>0),$$
$$Y_4 = \mathrm{sgn}(\kappa)k_1^{-1}(1-\kappa^{-2})^{-1/2}\big\{(1+\kappa^{-2})\phi\dot{\gamma} - (\kappa-\kappa^{-1})\mathcal{N}\big\}.$$

We then have $\widetilde{\nabla}_{\dot{\gamma}}Y_3 = -k_2Y_2 + k_3Y_4$. Moreover we have

$$\widetilde{\nabla}_{\dot{\gamma}}\big\{(1+\kappa^{-2})\phi\dot{\gamma} - (\kappa-\kappa^{-1})\mathcal{N}\big\} = \widetilde{\nabla}_{\dot{\gamma}}\big\{(1+\kappa^{-2})J\dot{\gamma} - (\kappa+\kappa^{-3})\mathcal{N}\big\}$$
$$= (1+\kappa^{-2})J\big(\kappa J\dot{\gamma} + \kappa^{-2}\mathcal{N}\big) + (\kappa+\kappa^{-3})A\dot{\gamma}$$
$$= -(\kappa+\kappa^{-1})\dot{\gamma} - \kappa^{-2}(1+\kappa^{-2})\xi + (\kappa+\kappa^{-3})(\dot{\gamma} + \kappa^{-1}\xi)$$
$$= -(1-\kappa^{-2})(\kappa^{-1}\dot{\gamma} - \xi).$$

We hence get $\widetilde{\nabla}_{\dot{\gamma}}Y_4 = -k_3Y_3$, and find that the extrinsic shape of γ is a helix of proper order 4. In view of the Frenet frame $\{\dot{\gamma}, Y_2, Y_3, Y_4\}$ of the extrinsic shape, as they are formed by $\dot\gamma, J\dot\gamma, \mathcal{N}, J\mathcal{N} = -\xi$, we find that it lies on some totally geodesic $\mathbb{C}H^2$. Therefore we see it is essential.

Next we compute its complex torsions and study their relations to geodesic curvatures. We have

$$k_1 + k_3 = \sqrt{\kappa^2+2\kappa^{-2}+\kappa^{-4}} + \frac{1-\kappa^{-2}}{\sqrt{\kappa^2+2\kappa^{-2}+\kappa^{-4}}} = \frac{(\kappa^2+1)(1+\kappa^{-4})}{\sqrt{\kappa^2+2\kappa^{-2}+\kappa^{-4}}},$$

$$k_2^2 + (k_1+k_3)^2 = \frac{\kappa^{-2}(1+\kappa^{-2})^2(1-\kappa^{-2}) + (\kappa^2+1)^2(1+\kappa^{-4})^2}{\kappa^2+2\kappa^{-2}+\kappa^{-4}} = (\kappa+\kappa^{-1})^2.$$

We also have that complex torsions are as follows:

$$\tau_{12} = \frac{1}{k_1}\langle\dot\gamma, -\kappa\dot\gamma - \kappa^{-2}\xi\rangle = -\frac{\kappa+\kappa^{-3}}{k_1} = \frac{-\mathrm{sgn}(\kappa)\cdot(\kappa_1+\kappa_3)}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}},$$

$$\tau_{34} = \frac{1}{k_1(1-\kappa^{-2})}\big\langle\kappa^{-1}\dot\gamma - \xi, -(1+\kappa^{-2})\dot\gamma + (\kappa+\kappa^{-3})\xi\big\rangle = -\frac{\kappa+\kappa^{-3}}{k_1} = \tau_{12},$$

$$\begin{aligned}\tau_{23} &= \frac{\mathrm{sgn}(\kappa)}{k_1\sqrt{1-\kappa^{-2}}}\big\langle\kappa J\dot\gamma + \kappa^{-2}\mathcal{N}, \kappa^{-1}J\dot\gamma - \mathcal{N}\big\rangle \\ &= -\frac{\mathrm{sgn}(\kappa)\sqrt{1-\kappa^{-2}}}{\kappa^2 k_1} = \frac{-\mathrm{sgn}(\kappa)\cdot k_2}{\sqrt{k_2^2+(k_1+k_3)^2}},\end{aligned}$$

$$\tau_{14} = \frac{\mathrm{sgn}(\kappa)}{k_1(1-\kappa^{-2})}\big\langle\dot\gamma, -(1+\kappa^{-2})\dot\gamma + (\kappa+\kappa^{-3})\xi\big\rangle. = -\frac{\mathrm{sgn}(\kappa)\sqrt{1-\kappa^{-2}}}{\kappa^2 k_1} = \tau_{23}.$$

This completes the proof. □

Based on Proposition 11.3, by making use of homothetic changes of metrics, we can obtain properties of extrinsic shapes of circular trajectories on a horospheres in a general complex hyperbolic space $\mathbb{C}H^n(c)$.

Proposition 11.4. *The extrinsic shape of a circular trajectory for a non-trivial Sasakian magnetic field* $\boldsymbol{F}_\kappa$ *on HS in* $\mathbb{C}H^n(c)$ *is an essential Killing helix of proper order* 4 *and satisfies the condition* (*I*) *in Lemma 4.9. Its geodesic curvatures are*

$$k_1 = \frac{\sqrt{64\kappa^6+8c^2\kappa^2-c^3}}{8\kappa^2},\ k_2 = \frac{|c|^{3/2}(4\kappa^2-c)\sqrt{4\kappa^2+c}}{8\kappa^2\sqrt{64\kappa^6+8c^2\kappa^2-c^3}},\ k_3 = \frac{|c|(4\kappa^2+c)}{2\sqrt{64\kappa^6+8c^2\kappa^2-c^3}}.$$

Proof. We consider a new metric $\langle\ ,\ \rangle' = (|c|/4)\langle\ ,\ \rangle$ on $\mathbb{C}H^n$, which is homothetic to the original metric. Then it has constant holomorphic sectional curvatures -4

with respect to this metric. Though the levels of Busemann functions change, *HS* is also a horosphere with respect to this new metric.

We take a trajectory γ for a Sasakian magnetic field $\boldsymbol{F}_\kappa$ and consider a curve σ given by $\sigma(s) = \gamma\big(2s/\sqrt{|c|}\big)$, which is a trajectory for a Sasakian magnetic field $\boldsymbol{F}'_{2\kappa/\sqrt{|c|}}$ with respect to the new metric. By Proposition 11.3, its extrinsic shape in $\mathbb{C}H^n(-4)$ is an essential Killing helix of proper order 4 with geodesic curvatures

$$k_1' = \frac{1}{\kappa'^2}\sqrt{\kappa'^6+2\kappa'^2+1},\ k_2' = \frac{(\kappa'^2+1)\sqrt{\kappa'^2-1}}{\kappa'^2\sqrt{\kappa'^6+2\kappa'^2+1}},\ k_3' = \frac{\kappa'^2-1}{\sqrt{\kappa'^6+2\kappa'^2+1}},$$

where $\kappa' = 2\kappa/\sqrt{|c|}$. Therefore, by Lemma 8.3, the extrinsic shape of γ in $\mathbb{C}H^n(c)$ is an essential Killing helix of proper order 4 with geodesic curvatures

$$k_1 = \frac{\sqrt{|c|}}{2}\,k_1' = \frac{\sqrt{|c|}}{2}\,\frac{|c|}{4\kappa^2}\sqrt{\frac{64\kappa^6}{|c|^3}+\frac{8\kappa^2}{|c|}+1} = \frac{1}{8\kappa^2}\sqrt{64\kappa^6+8|c|^2\kappa^2+|c|^3},$$

$$k_2 = \frac{\sqrt{|c|}}{2}\,k_2' = \frac{\sqrt{|c|}}{2}\,\frac{\big(\frac{4\kappa^2}{|c|}+1\big)\sqrt{\frac{4\kappa^2}{|c|}-1}}{\frac{4\kappa^2}{|c|}\sqrt{\frac{64\kappa^6}{|c|^3}+\frac{8\kappa^2}{|c|}+1}} = \frac{|c|^{3/2}(4\kappa^2+|c|)\sqrt{4\kappa^2-|c|}}{8\kappa^2\sqrt{64\kappa^6+8|c|^2\kappa^2+|c|^3}},$$

$$k_3 = \frac{\sqrt{|c|}}{2}\,k_3' = \frac{\sqrt{|c|}}{2}\,\frac{\frac{4\kappa^2}{|c|}-1}{\sqrt{\frac{64\kappa^6}{|c|^3}+\frac{8\kappa^2}{|c|}+1}} = \frac{|c|(4\kappa^2-|c|)}{2\sqrt{64\kappa^6+8|c|^2\kappa^2+|c|^3}}.$$

Since complex torsions of helices are invariant under homothetic changes of metrics, we get the conclusion. □

11.3. **Relation of connections on $\mathbb{C}H^n$ and $\mathbb{C}^{n+1}$.** As we studied circular trajectories on geodesic spheres in $\mathbb{C}P^n$ through Hopf fibrations, we study circular trajectories on horospheres through the fibration $\varpi : H_1^{2n+1} \to \mathbb{C}H^n(-4)$ of anti-de Sitter space H_1^{2n+1}. Let $\widehat{\mathcal{N}}$ denote a normal vector field on H_1^{2n+1} in $\mathbb{C}_1^{n+1}$ with $\langle\!\langle \widehat{\mathcal{N}}, \widehat{\mathcal{N}} \rangle\!\rangle = -1$. Here, like in §2, we denote by $\mathbb{C}_1^{n+1}$ a complex Euclidean space $\mathbb{C}^{n+1}$ admitting a Hermitian form $\langle\!\langle\ ,\ \rangle\!\rangle$. We denote by $\widehat{\nabla}$ and $\overline{\nabla}$ the connections associated with $\langle\ ,\ \rangle = \mathrm{Re}\langle\!\langle\ ,\ \rangle\!\rangle$ on H_1^{2n+1} and $\mathbb{C}_1^{n+1}$, respectively. We also denote by $\widetilde{\nabla}$ the Riemmanian connection on $\mathbb{C}H^n(-4)$.

Lemma 11.2. *Let X, $Y \in \mathcal{X}(\mathbb{C}H^n)$ be vector fields on $\mathbb{C}H^n(-4)$. If we regard them as horizontal vector fields on H_1^{2n+1} through the fibration ϖ, then they satisfy*

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + \langle X, Y\rangle \widehat{\mathcal{N}} - \langle X, JY\rangle J\widehat{\mathcal{N}}. \tag{11.1}$$

Proof. The connections $\overline{\nabla}$ on $\mathbb{C}_1^{n+1}$ and $\widehat{\nabla}$ on H_1^{2n+1} are related as

$$\widehat{\nabla}_X Y = \overline{\nabla}_X Y - \frac{\langle \widehat{\nabla}_X Y, \widehat{\mathcal{N}}\rangle}{\langle \widehat{\mathcal{N}}, \widehat{\mathcal{N}}\rangle}\widehat{\mathcal{N}} = \overline{\nabla}_X Y + \langle \widehat{\nabla}_X Y, \widehat{\mathcal{N}}\rangle \widehat{\mathcal{N}}.$$

Since Y is orthogonal to $\widehat{\mathcal{N}}$, we have

$$\langle \widehat{\nabla}_X Y, \widehat{\mathcal{N}}\rangle = \widehat{\nabla}_X \langle Y, \widehat{\mathcal{N}}\rangle - \langle Y, \widehat{\nabla}_X \widehat{\mathcal{N}}\rangle = -\langle Y, X\rangle,$$

hence obtain

$$\overline{\nabla}_X Y = \widehat{\nabla}_X Y + \langle X, Y\rangle \widehat{\mathcal{N}}. \tag{11.2}$$

Next we study the relationship between the connections $\widehat{\nabla}$ on H_1^{2n+1} and $\widetilde{\nabla}$ on $\mathbb{C}H^n(c)$. Since we need to delete the vertical component, they are related as

$$\widetilde{\nabla}_X Y = \widehat{\nabla}_X Y - \frac{\langle \widehat{\nabla}_X Y, J\widehat{\mathcal{N}}\rangle}{\langle J\widehat{\mathcal{N}}, J\widehat{\mathcal{N}}\rangle} J\widehat{\mathcal{N}} = \widehat{\nabla}_X Y + \langle \widehat{\nabla}_X Y, J\widehat{\mathcal{N}}\rangle J\widehat{\mathcal{N}}.$$

Since Y is horizontal, by use of (11.2) we have

$$\begin{aligned}\langle \widehat{\nabla}_X Y, J\widehat{\mathcal{N}}\rangle &= \widehat{\nabla}_X \langle Y, J\widehat{\mathcal{N}}\rangle - \langle Y, \widehat{\nabla}_X (J\widehat{\mathcal{N}})\rangle = -\big\langle Y, \overline{\nabla}_X (J\widehat{\mathcal{N}}) - \langle X, J\widehat{\mathcal{N}}\rangle \widehat{\mathcal{N}}\big\rangle \\ &= -\langle Y, J\overline{\nabla}_X \widehat{\mathcal{N}}\rangle = -\langle Y, JX\rangle = \langle X, JY\rangle.\end{aligned}$$

Thus we obtain

$$\widehat{\nabla}_X Y = \widetilde{\nabla}_X Y - \langle X, JY\rangle J\widehat{\mathcal{N}}. \tag{11.3}$$

Combining (11.2) and (11.3), we get the conclusion. □

11.4. **Behaviors of circular trajectories on *HS*.** We now study behaviors of trajectories on a horosphere in $\mathbb{C}H^n(c)$. A smooth curve $\sigma : \mathbb{R} \to \mathbb{C}H^n(c)$ parameterized by its arc-length on $\mathbb{C}H^n$ is said to be unbounded in both directions if both of the sets $\sigma\big([0,\infty)\big)$, $\sigma\big((-\infty, 0]\big)$ are unbounded. For a smooth curve σ which is unbounded in both directions, we set $\sigma(\infty) = \lim_{t\to\infty} \sigma(t)$, $\sigma(-\infty) = \lim_{t\to-\infty} \sigma(t) \in \partial\mathbb{C}H^n$

if they exist and call them *points at infinity*. Since σ is parameterized by its arclength, if $\sigma(\infty)$ or $\sigma(-\infty)$ exists, then it lies on the ideal boundary $\partial\mathbb{C}H^n$. When σ is a curve on a real hypersurface in $\mathbb{C}H^n(c)$, considering its extrinsic shape, we also employ these terminologies to this curve.

Lemma 11.3. *Let γ be a circular trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a horosphere HS in $\mathbb{C}H^n(-4)$. A horizontal lift $\hat{\gamma}$ of the extrinsic shape of γ satisfies*

$$\hat{\gamma}''' - \sqrt{-1}(\kappa+\kappa^{-1})\hat{\gamma}'' - (2-\kappa^{-2})\hat{\gamma}' + \sqrt{-1}\kappa^{-1}(1-\kappa^{-2})\hat{\gamma} = 0 \tag{11.4}$$

as a curve in $\mathbb{C}^{n+1}_1$.

Proof. By Proposition 11.3, the extrinsic shape of γ is an essential Killing helix of proper order 4. In this case, it is determined by the differential equations $\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = k_1Y_2$, $\widetilde{\nabla}_{\dot{\gamma}}Y_2 = -k_1\dot{\gamma} + k_2Y_3$. By use of (11.1) and Lemma 4.9, we find that a horizontal lift $\hat{\gamma}$ of the extrinsic shape of γ satisfies

$$\begin{cases} \overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = k_1Y_2 + \widehat{\mathcal{N}}, \\ \overline{\nabla}_{\dot{\hat{\gamma}}}Y_2 = -k_1\dot{\hat{\gamma}} + k_2Y_3 - \tau_{12}J\widehat{\mathcal{N}} = k_3\dot{\hat{\gamma}} + \mathrm{sgn}(\kappa)\sqrt{k_2^2 + (k_1+k_3)^2}\,JY_2 - \tau_{12}J\widehat{\mathcal{N}}. \end{cases}$$

We therefore have

$$\begin{aligned} \overline{\nabla}_{\dot{\hat{\gamma}}}\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} &= k_1\overline{\nabla}_{\dot{\hat{\gamma}}}Y_2 + \dot{\hat{\gamma}} \\ &= (1 + k_1k_3)\dot{\hat{\gamma}} + (\kappa+\kappa^{-1})k_1JY_2 + (\kappa+\kappa^{-3})J\widehat{\mathcal{N}} \\ &= (2-\kappa^{-2})\dot{\hat{\gamma}} + (\kappa+\kappa^{-1})J\big(\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} - \widehat{\mathcal{N}}\big) + (\kappa+\kappa^{-3})J\widehat{\mathcal{N}} \\ &= (2-\kappa^{-2})\dot{\hat{\gamma}} + (\kappa+\kappa^{-1})J\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} - \kappa^{-1}(1-\kappa^{-2})J\widehat{\mathcal{N}}, \end{aligned}$$

and get the conclusion. □

Theorem 11.2. *Every circular trajectory γ on a horosphere HS in $\mathbb{C}H^n(c)$ is unbounded in both directions. In particular, it has a single point at infinity; $\gamma(\infty) = \gamma(-\infty)$.*

Proof. We are enough to consider the case $c = -4$. By Lemma 11.3, we have a horizontal lift of the extrinsic shape of γ satisfies the equation (11.4). Its characteristic equation

$$\Lambda^3 - \sqrt{-1}(\kappa+\kappa^{-1})\Lambda^2 - (2-\kappa^{-2})\Lambda + \sqrt{-1}\kappa^{-1}(1-\kappa^{-2}) = 0$$

has a double solution $\sqrt{-1}/\kappa$ and a solution $\sqrt{-1}(\kappa^2-1)/\kappa$. Therefore $\hat{\gamma}$ is of the form

$$\hat{\gamma}(t) = (A + tB)e^{\sqrt{-1}t/\kappa} + Ce^{\sqrt{-1}(\kappa^2-1)t/\kappa}$$

with some $A, B, C \in \mathbb{C}^{n+1}$. This shows that $\hat{\gamma}$ is unbounded in $\mathbb{C}_1^{n+1}$, hence γ is unbounded. As *HS* has only one point at infinity we have $\gamma(\infty) = \gamma(-\infty)$. We here show this directly. If we describe the extrinsic shape of γ on the ball model $\boldsymbol{D}^n$ we have

$$\gamma(t) = \left(\frac{(A_1 + tB_1)e^{\sqrt{-1}t/\kappa} + C_1e^{\sqrt{-1}(\kappa^2-1)t/\kappa}}{(A_0 + tB_0)e^{\sqrt{-1}t/\kappa} + C_0e^{\sqrt{-1}(\kappa^2-1)t/\kappa}},\right.$$
$$\left. \ldots\ldots, \frac{(A_n + tB_n)e^{\sqrt{-1}t/\kappa} + C_ne^{\sqrt{-1}(\kappa^2-1)t/\kappa}}{(A_0 + tB_0)e^{\sqrt{-1}t/\kappa} + C_0e^{\sqrt{-1}(\kappa^2-1)t/\kappa}}\right),$$

where $A = (a_0, \ldots, A_n),\ B = (B_0, \ldots, B_n),\ C = (C_0, \ldots, C_n) \in \mathbb{C}_1^{n+1}$. Therefore we find $\lim_{t\to\infty}\gamma(t) = \big(B_1/B_0, \ldots, B_n/B_0\big) = \lim_{t\to-\infty}\gamma(t) \in S^{2n-1} \in \partial\boldsymbol{D}^n$. $\square$

12. **Circular trajectories on geodesic spheres in $\mathbb{C}H^n$**

In this section we study circular trajectories on geodesic spheres in a complex hyperbolic space $\mathbb{C}H^n(c)$. A geodesic sphere also have two principal curvatures $\lambda = (\sqrt{|c|}/2)\coth\big(\sqrt{|c|}\,r/2\big)$ and $\nu = \sqrt{|c|}\coth\big(\sqrt{|c|}\,r\big)$. Its characteristic vector field ξ satisfies $A\xi = \nu\xi$ and every tangent vector v orthogonal to ξ satisfies $Av = \lambda v$ with the shape operator A. In particular, the shape operator and the characteristic tensor satisfy $A\phi = \phi A$.

12.1. **Trajectories on geodesic spheres in $\mathbb{C}H^n$.** We first study trajectories from the viewpoint of Frenet-Serre formula. By Corollary 7.1, we know that every trajectory on a geodesic sphere $G(r)$ has constant structure torsion. By the same proof as of Proposition 7.1, that is, by substituting $\lambda = (\sqrt{|c|}/2)\coth\big(\sqrt{|c|}\,r/2\big)$, we have the following.

Proposition 12.1. *Let γ be a trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_k$ on a geodesic sphere $G(r)$ in a complex hyperbolic space $\mathbb{C}H^n(c)$.*

(1) *It is a geodesic if and only if $\rho_\gamma = \pm 1$,*

(2) *It is a circle of positive geodesic curvature if and only if*
$$\kappa\rho_\gamma = (\sqrt{|c|}/2)\coth\big(\sqrt{|c|}\,r/2\big).$$
In this case, its geodesic curvature is $\sqrt{4\kappa^2 + c\coth^2\big(\sqrt{|c|}\,r/2\big)}\,/2$.

(3) *Otherwise, it is a helix of proper order 3 whose geodesic curvatures are $|\kappa|\sqrt{1-\rho_\gamma^2}$ and $\big|2\kappa\rho_\gamma - \sqrt{|c|}\big|\coth\big(\sqrt{|c|}\,r/2\big)/2$.*

If we restrict ourselves to circular trajectories, as $|\rho_\gamma| < 1$, we have the following result.

Theorem 12.1. *Let $\boldsymbol{F}_\kappa$ be a non-trivial Sasakian magnetic field on a geodesic sphere $G(r)$ of radius r in $\mathbb{C}H^n(c)$.*

(1) *When* $0 < |\kappa| \le (\sqrt{|c|}/2)\coth(\sqrt{|c|}r/2)$, *there are no circular trajectories for* $\boldsymbol{F}_\kappa$.

(2) *When* $|\kappa| > (\sqrt{|c|}/2)\coth(\sqrt{|c|}r/2)$, *a trajectory for* $\boldsymbol{F}_\kappa$ *is circular if and only if* $\rho_\gamma = (\sqrt{|c|}/(2\kappa))\coth(\sqrt{|c|}r/2)$. *In this case its geodesic curvature is* $\sqrt{\kappa^2 + (c/4)\coth^2(\sqrt{|c|}r/2)}$.

Congruency conditions on circular trajectories on a geodesic sphere are given in the same way as in Proposition 7.2.

Proposition 12.2 (Adachi[3]). *Trajectories* γ_1 *for a Sasakian magnetic field* $\boldsymbol{F}_{\kappa_1}$ *and* γ_2 *for* $\boldsymbol{F}_{\kappa_2}$ *on a geodesic sphere* $G(r)$ *in* $\mathbb{C}H^n(c)$ *are congruent to each other in strong sense if and only if one of the following conditions holds*:

i) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| = 1$,

ii) $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$ *and* $|\kappa_1| = |\kappa_2|$,

iii) $0 < |\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$ *and* $\kappa_1\rho_{\gamma_1} = \kappa_2\rho_{\gamma_2}$.

In order to show this we here study isometries of a geodesic sphere $G(r)$ in $\mathbb{C}H^n$. Once we show the following Lemma, then we can prove the above Proposition by just the same way as of Proposition 7.2. Through an isometric immersion $\iota : G(r) \to \mathbb{C}H^n(c)$ we may consider that $TG(r)$ is a subset of $T\mathbb{C}H^n$.

Lemma 12.1. *Let* $x, x' \in G(r)$ *be arbitrary points on a geodesic sphere* $G(r)$ *in* $\mathbb{C}H^n(c)$. *Given unit tangent vectors* $u \in \langle\xi_x\rangle^\perp \subset T_xG(r)$ *and* $u' \in \langle\xi_{x'}\rangle^\perp \subset T_{x'}G(r)$ *which are orthogonal to* ξ *at* x *and* x', *there exist isometries* $\tilde\varphi^+$, $\tilde\varphi^-$ *of* $\mathbb{C}H^n(c)$ *satisfying the following conditions*:

i) $\tilde\varphi^+\big(G(r)\big) = \tilde\varphi^-\big(G(r)\big) = G(r)$,
(*i.e.* $G(r)$ *is invariant under the actions of* $\tilde\varphi^+$ *and* $\tilde\varphi^-$);

ii) $\tilde\varphi^+(x) = \tilde\varphi^-(x) = x'$;

iii) $d\tilde\varphi^+(u) = d\tilde\varphi^-(u) = u'$,

iv) $d\tilde{\varphi}^{+}\circ J = J\circ d\tilde{\varphi}^{+}$ *and* $d\tilde{\varphi}^{-}\circ J = -J\circ d\tilde{\varphi}^{-}$,

in particular, $d\tilde{\varphi}^{+}(\xi_x) = \xi_{x'}$ *and* $d\tilde{\varphi}^{-}(\xi_x) = -\xi_{x'}$.

Proof. For the sake of simplicity we only consider the case $n = 2$ and $c = -4$. As we mentioned in §5.3, we may consider that

$$\begin{aligned}\varpi^{-1}\big(G(r)\big) &= \big\{z = (z_0, z_1, z_2) \in \mathbb{C}^3 \;\big|\; |z_0| = \cosh r,\ |z_1|^2 + |z_2|^2 = \sinh^2 r\big\}\\ &= S^1(1/\cosh^2 r) \times S^3(1/\sinh^2 r) \subset \mathbb{C}\times\mathbb{C}^2\end{aligned}$$

through a Hopf fibration $\varpi : H_1^5 \to \mathbb{C}H^2$. We take an arbitrary point $\hat{z} = (z_0, z_1, z_2) \in \varpi^{-1}\big(G(r)\big)$. The tangent space of $\widehat{M} = \varpi^{-1}\big(G(r)\big)$ at this point $\hat{z}$ is represented by

$$T_{\hat{z}}\widehat{M} = \big\{(\hat{z}, \hat{v}) \in \{\hat{z}\}\times\mathbb{C}^3 \;\big|\; \mathrm{Re}(z_0\bar{v}_0) = \mathrm{Re}(z_1\bar{v}_1 + z_2\bar{v}_2) = 0\big\}.$$

We denote by $\widehat{\mathcal{N}}_{\hat{z}} \in T_{\hat{z}}H_1^5$ the horizontal lift of the unit normal vector $\mathcal{N}_{\varpi(\hat{z})}$ of $G(r)$ in $\mathbb{C}H^2(-4)$. Since $\widehat{\mathcal{N}}_{\hat{z}}$ is orthogonal to $T_{\hat{z}}\widehat{M}$, we find that it is represented as

$$\widehat{\mathcal{N}}_{\hat{z}} = \big(\hat{z}, (-\tanh r z_0, \coth r z_1, \coth r z_2)\big).$$

We set $\hat{\xi}_{\hat{z}} = -J\widehat{\mathcal{N}}_{\hat{z}}$ with the complex structure J on $\mathbb{C}_1^3$. We hence have

$$\hat{\xi}_{\hat{z}} = \big(\hat{z}, -\sqrt{-1}(-\tanh r z_0, \coth r z_1, \coth r z_2)\big).$$

We denote by $\langle\hat{\xi}_{\hat{z}}\rangle$ the real linear subspace of $T_zH_1^5$ spanned by $\hat{\xi}_{\hat{z}}$, and by $\langle\hat{\xi}_{\hat{z}}\rangle^{\perp}$ its orthogonal complement in $T_{\hat{z}}{H_1}^5$. The horizontal part $\langle\hat{\xi}_{\hat{z}}\rangle^{\perp}\cap\mathcal{H}_{\hat{z}}$ of $\langle\hat{\xi}_{\hat{z}}\rangle^{\perp}$ corresponds to $\langle\xi_{\varpi(\hat{z})}\rangle^{\perp}$, and is represented as

$$\langle\hat{\xi}_{\hat{z}}\rangle^{\perp}\cap\mathcal{H}_{\hat{z}} = \big\{(\hat{z}, \hat{v}) \in \{\hat{z}\}\times\mathbb{C}^3 \;\big|\; v_0 = 0,\ z_1\bar{v}_1 + z_2\bar{v}_2 = 0\big\}.$$

We here take a point $\hat{z}_* = (\cosh r, \sinh r, 0) \in \widehat{M}$ and a unit tangent vector $\hat{u}_* = \big(\hat{z}_*, (0,0,1)\big) \in \langle\hat{\xi}_{\hat{z}_*}\rangle^{\perp}\cap\mathcal{H}_{\hat{z}_*}$. At this point we see $\hat{\xi}_{\hat{z}_*} = \big(\hat{z}_*, \sqrt{-1}(\sinh r, -\cosh r, 0)\big)$. For an arbitrary point $\hat{z} = \big(z_0, z_1, z_2\big) \in \widehat{M}$ and an arbitrary unit tangent vector $\hat{u} = \Big(\hat{z}, \big(0, u_1, u_2\big)\Big) \in \langle\hat{\xi}_{\hat{z}}\rangle^{\perp}\cap\mathcal{H}_{\hat{z}}$, which is expressed as

$$\hat{u} = \left(\hat{z}, \left(0, \frac{\zeta\overline{z}_2}{\sinh r}, -\frac{\zeta\overline{z}_1}{\sinh r}\right)\right)$$

with some $\zeta \in \mathbb{C}$ satisfying $|\zeta| = 1$, we take a unitary matrix

$$U_+ = \begin{pmatrix} z_0/\cosh r & 0 & 0 \\ 0 & z_1/\sinh r & u_1 \\ 0 & z_2/\sinh r & u_2 \end{pmatrix} \in U(0,1) \oplus U(2) \subset U(2,1).$$

This induces a linear transformation on $\mathbb{C}^3$ which preserves the Hermitian product $\langle\!\langle\ \ \rangle\!\rangle$, hence it induces an isometry $\hat{\varphi}^+_{(\hat{z},\hat{u})}$ of H^5_1. It is clear that it preserves $\widehat{M}$ and satisfies $\hat{\varphi}^+_{(\hat{z},\hat{u})}\big(e^{\sqrt{-1}\theta}\hat{p}\big) = e^{\sqrt{-1}\theta}\hat{\varphi}^+_{(\hat{z},\hat{u})}(\hat{p})$ for arbitrary $\theta \in [0, 2\pi)$ and $\hat{p} \in \widehat{M}$. Therefore, we find that $\hat{\varphi}^+_{(\hat{z},\hat{u})}$ induces an isometry $\tilde{\varphi}^+_{(\hat{z},\hat{u})}$ of $\mathbb{C}H^n(-4)$ satisfying $\tilde{\varphi}^+_{(\hat{z},\hat{u})} \circ \varpi = \varpi \circ \hat{\varphi}^+_{(\hat{z},\hat{u})}$ and $\tilde{\varphi}^+_{(\hat{z},\hat{u})}\big(\varpi(\hat{z}_*)\big) = \varpi(\hat{z})$, $d\tilde{\varphi}^+_{(\hat{z},\hat{u})}\big(d\varpi(\hat{u}_*)\big) = d\varpi(\hat{u})$, $d\tilde{\varphi}^+_{(\hat{z},\hat{u})}(\xi_{\hat{z}_*}) = \xi_{\hat{z}}$. As $U_+J = JU_+$ with the matrix $J = \sqrt{-1}E$, where E is the identity, we find that $\tilde{\varphi}^+_{(\hat{z},\hat{u})}$ satisfies $d\tilde{\varphi}^+_{(\hat{z},\hat{u})} \circ J = J \circ d\tilde{\varphi}^+_{(\hat{z},\hat{u})}$.

Next we consider a matrix

$$\Psi = \begin{pmatrix} \epsilon & O & O \\ O & \epsilon & O \\ O & O & \epsilon \end{pmatrix} \in O(6) \quad \text{with} \quad \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(2).$$

This matrix induces a map $\mathbb{C}^3 \ni (w_0, w_1, w_2) \mapsto (\bar{w}_0, \bar{w}_1, \bar{w}_2) \in \mathbb{C}^3$. If we take a matrix $U_- = U_+\Psi$, as $\Psi J = -J\Psi$, it induces an isometry $\hat{\varphi}^-_{(\hat{z},\hat{u})}$ of H^5_1 which preserves $\widehat{M}$. It is clear that it satisfies $\hat{\varphi}^-_{(\hat{z},\hat{u})}\big(e^{\sqrt{-1}\theta}\hat{p}\big) = e^{-\sqrt{-1}\theta}\hat{\varphi}^-_{(\hat{z},\hat{u})}(\hat{p})$ for arbitrary $\theta \in [0, 2\pi)$ and $\hat{p} \in \widehat{M}$. As we have $U_-J = -JU_-$ for the matrix $J = \sqrt{-1}E$, we find that $\hat{\varphi}^-_{(\hat{z},\hat{u})}$ induces an isometry $\tilde{\varphi}^-_{(\hat{z},\hat{u})}$ of $\mathbb{C}H^n(-4)$ satisfying

$$\tilde{\varphi}^-_{(\hat{z},\hat{u})} \circ \varpi = \varpi \circ \hat{\varphi}^-_{(\hat{z},\hat{u})}, \quad d\hat{\varphi}^-_{(\hat{z},\hat{u})} \circ J = -J \circ d\hat{\varphi}^-_{(\hat{z},\hat{u})}.$$
$$\tilde{\varphi}^-_{(\hat{z},\hat{u})}\big(\varpi(\hat{z}_*)\big) = \varpi(\hat{z}), \quad d\hat{\varphi}^-_{(\hat{z},\hat{u})}\big(d\varpi(\hat{u}_*)\big) = d\varpi(\hat{u}).$$

In particular, it satisfies $d\hat{\varphi}^-_{(\hat{z},\hat{u})}(\xi_{\hat{z}_*}) = -\xi_{\hat{z}}$.

We constructed desireble isometries for a fix pair $\big(\varpi(\hat{z}_*), d\varpi(\hat{u}_*)\big)$ and an arbitrary pair $\big(\varpi(\hat{z}), d\varpi(\hat{u})\big)$. For arbitrary pairs (z, u), (z', u'), we consider isometries $\tilde{\varphi}^+_{(\hat{z}',\hat{u}')} \circ \big(\tilde{\varphi}^+_{(\hat{z},\hat{u})}\big)^{-1}$, $\tilde{\varphi}^-_{(\hat{z}',\hat{u}')} \circ \big(\tilde{\varphi}^+_{(\hat{z},\hat{u})}\big)^{-1}$, where $\varpi(\hat{z}) = z$, $\varpi(\hat{u}) = u$, $\varpi(\hat{z}') = z'$, $\varpi(\hat{u}') = u'$. We then get the conclusion by these isometries. $\square$

$$\begin{array}{ccccccccc}
\mathbb{C}^{n+1} & \supset & H_1^{2n+1} & \supset & \widehat{M} & \xrightarrow{\hat{\varphi}} & \widehat{M} & \subset & H_1^{2n+1} \\
 & & \downarrow \varpi & & \downarrow \varpi & \circlearrowright & \downarrow \varpi & & \downarrow \\
 & & \mathbb{C}H^n & \supset & G(r) & \xrightarrow{\tilde{\varphi}} & G(r) & \subset & \mathbb{C}H^n
\end{array}$$

Remark 12.1. Every isometry φ of $G(r)$ in $\mathbb{C}H^n(c)$ is equivariant. That is, if we denote by $\iota : G(r) \to \mathbb{C}H^n(c)$ an isometric immersion, there is an isometry $\tilde{\varphi}$ of $\mathbb{C}H^n(c)$ satisfying $\tilde{\varphi} \circ \iota = \iota \circ \varphi$.

Corollary 12.1. *Every circular trajectory on $G(r)$ in $\mathbb{C}H^n(c)$ is Killing.*

Corollary 12.2. *Circular trajectories for a given Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a geodesic sphere in $\mathbb{C}H^n(c)$ are congruent to each other in strong sense.*

Corollary 12.3. *Geodesic trajectories for non-trivial Sasakian magnetic fields on a geodesic sphere in $\mathbb{C}H^n(c)$ are congruent to each other in strong sense.*

12.2. **Extrinsic shapes of circular trajectories on $G(r)$ in $\mathbb{C}H^n$.** We next study extrinsic shapes of circular trajectories on a geodesic sphere $G(r)$ in $\mathbb{C}H^n(c)$.

Proposition 12.3. *The extrinsic shape of a circular trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a geodesic sphere $G(r)$ in $\mathbb{C}H^n(-4)$ is an essential Killing helix of proper order 4 and satisfies the condition (I) of Lemma 4.9. Its geodesic curvatures are*

$$k_1 = \frac{1}{\kappa^2}\sqrt{\kappa^6+(2\kappa^2+1)\coth^2 r}, \quad k_2 = \frac{(\kappa^2+1)\coth r\sqrt{\kappa^2-\coth^2 r}}{\kappa^2\sqrt{\kappa^6+(2\kappa^2+1)\coth^2 r}},$$

$$k_3 = \frac{\kappa^2-\coth^2 r}{\sqrt{\kappa^6+(2\kappa^2+1)\coth^2 r}}.$$

and its complex torsions satisfy

$$\tau_{12} = \tau_{34} = -\frac{\kappa + \kappa^{-3}\coth^2 r}{k_1} = \frac{-\mathrm{sgn}(\kappa)\cdot(k_1+k_3)}{\sqrt{k_2^2 + (k_1+k_3)^2}}, \quad \tau_{13} = \tau_{24} = 0,$$
$$\tau_{23} = \tau_{14} = -\frac{\coth r\sqrt{\kappa^2 - \coth^2 r}}{\kappa^3 k_1} = \frac{-\mathrm{sgn}(\kappa)\cdot k_2}{\sqrt{k_2^2 + (k_1+k_3)^2}}.$$

Proof. Since $c = -4$, we have $\lambda = \coth r$, $\nu = 2\coth 2r$, hence find $\nu - \lambda = \tanh r$. We calculate geodesic curvatures by just the same way as in the proof of Proposition 8.1. By use of the circular condition $\kappa\rho_\gamma = \coth r$, we have

$$A\dot\gamma = A\big(\rho_\gamma\xi + (\dot\gamma - \rho_\gamma\xi)\big) = \lambda\dot\gamma + (\nu-\lambda)\rho_\gamma\xi = \coth r\dot\gamma + \kappa^{-1}\xi,$$
$$\langle A\dot\gamma, \dot\gamma\rangle = \lambda + (\nu-\lambda)\rho_\gamma^2 = (1+\kappa^{-2})\coth r.$$

By use of Gauss formula (3.1), we obtain

$$\widetilde\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma + \langle A\dot\gamma, \dot\gamma\rangle\mathcal{N} = \kappa J\dot\gamma + \kappa^{-2}\coth r\,\mathcal{N}.$$

If we put

$$k_1 = \|\kappa J\dot\gamma + \kappa^{-2}\coth r\,\mathcal{N}\| = \sqrt{\kappa^2 + 2\kappa^{-1}\rho_\gamma\coth r + \kappa^{-4}\coth^2 r}$$
$$= \sqrt{\kappa^2 + (2\kappa^{-2} + \kappa^{-4})\coth^2 r},$$
$$Y_2 = (\kappa J\dot\gamma + \kappa^{-2}\coth r\,\mathcal{N})/k_1,$$

we have $\widetilde\nabla_{\dot\gamma}\dot\gamma = k_1 Y_2$. We differentiate Y_2. By (3.2) we have

$$\widetilde\nabla_{\dot\gamma}\big(\kappa J\dot\gamma + \kappa^{-2}\coth r\,\mathcal{N}\big) = \kappa J\big(\kappa J\dot\gamma + \kappa^{-2}\coth r\,\mathcal{N}\big) - \kappa^{-2}\coth r A\dot\gamma$$
$$= -(\kappa^2 + \kappa^{-2}\coth^2 r)\dot\gamma - (\kappa^{-1} + \kappa^{-3})\coth r\,\xi$$
$$= -k_1^2\dot\gamma + (\kappa^{-2} + \kappa^{-4})\coth^2 r\dot\gamma - (\kappa^{-1} + \kappa^{-3})\coth r\,\xi$$
$$= -k_1^2\dot\gamma + \kappa^{-4}(\kappa^2+1)\coth r(\coth r\dot\gamma - \kappa\xi).$$

Since $|\kappa| > \coth r$, by putting

$$k_2 = {k_1}^{-1}(\kappa^2+1)\kappa^{-4}\coth r\sqrt{\kappa^2 - \coth^2 r},$$
$$Y_3 = (\coth r\dot\gamma - \kappa\xi)\big/\sqrt{\kappa^2 - \coth^2 r},$$

we have $\widetilde\nabla_{\dot\gamma}Y_2 = -k_1\dot\gamma + k_2 Y_3$. Continuing calculations we have

$$\widetilde\nabla_{\dot\gamma}(\coth r\dot\gamma - \kappa\xi) = \kappa\coth r J\dot\gamma + \kappa^{-2}\coth^2 r\,\mathcal{N} + \kappa J\widetilde\nabla_{\dot\gamma}\mathcal{N} = (\kappa^{-2}\coth^2 r - 1)\mathcal{N}.$$

We therefore find that

$$\begin{aligned}
\widetilde{\nabla}_{\dot\gamma} Y_3 &= -k_2Y_2 + k_2Y_2 - \kappa^{-2}\sqrt{\kappa^2-\coth^2 r}\,\mathcal{N} \\
&= -k_2Y_2 + \frac{(\kappa^2+1)\coth r\sqrt{\kappa^2-\coth^2 r}}{k_1^2\kappa^4}\big\{\kappa\phi\dot\gamma + (\kappa^{-2}+1)\coth r\,\mathcal{N}\big\} \\
&\qquad - \kappa^{-2}\sqrt{\kappa^2-\coth^2 r}\,\mathcal{N} \\
&= -k_2Y_2 + \frac{(\kappa^2+1)\coth r\sqrt{\kappa^2-\coth^2 r}}{k_1^2\kappa^3}\,\phi\dot\gamma \\
&\qquad + \frac{\sqrt{\kappa^2-\coth^2 r}}{k_1^2\kappa^2}\big\{(\kappa^{-2}+1)^2\coth^2 r - \kappa^2 - (2\kappa^{-2}+\kappa^{-4})\coth^2 r\big\}\mathcal{N} \\
&= -k_2Y_2 + \frac{\sqrt{\kappa^2-\coth^2 r}}{k_1^2\kappa^2}\big\{(\kappa+\kappa^{-1})\coth r\phi\dot\gamma - (\kappa^2-\coth^2 r)\mathcal{N}\big\}.
\end{aligned}$$

As we have

$$\|(\kappa+\kappa^{-1})\coth r\phi\dot\gamma - (\kappa^2-\coth^2 r)\mathcal{N}\|^2 = (\kappa^2-\coth^2 r)k_1^2,$$

by putting

$$\begin{aligned}
k_3 &= {k_1}^{-1}(1-\kappa^{-2}\coth^2 r), \\
Y_4 &= \frac{1}{k_1\sqrt{\kappa^2-\coth^2 r}}\{(\kappa+\kappa^{-1})\coth r\,\phi\dot\gamma - (\kappa^2-\coth^2 r)\mathcal{N}\},
\end{aligned}$$

we see $\widetilde{\nabla}_{\dot\gamma} Y_3 = -k_2Y_2 + k_3Y_4$. Moreover, we have

$$\begin{aligned}
&\widetilde{\nabla}_{\dot\gamma}\{(\kappa+\kappa^{-1})\coth r\,\phi\dot\gamma - (\kappa^2-\coth^2 r)\mathcal{N}\} \\
&\quad = (\kappa+\kappa^{-1})\coth r\,J(\kappa J\dot\gamma + \kappa^{-2}\coth r\,\mathcal{N}) + (\kappa^2+\kappa^{-2}\coth^2 r)(\coth r\dot\gamma + \kappa^{-1}\xi) \\
&\quad = -(1-\kappa^{-2}\coth^2 r)(\coth r\dot\gamma - \kappa\xi),
\end{aligned}$$

hence have $\widetilde{\nabla}_{\dot\gamma} Y_4 = -k_3Y_3$. We therefore find that the extrinsic shape of γ is a helix of proper order 4.

Next we compute complex torsions of the extrinsic shape of γ. We first note that

$$\begin{aligned}
k_1 + k_3 &= k_1^{-1}(\kappa^2+1)(1+\kappa^{-4}\coth^2 r), \\
k_2^2 + (k_1+k_3)^2 &= (\kappa+\kappa^{-1})^2.
\end{aligned}$$

By direct computation we get

$$\begin{aligned}
\tau_{12} &= \frac{1}{k_1}\langle\dot\gamma, J(\kappa J\dot\gamma+\kappa^{-2}\coth r\mathcal{N})\rangle = -\frac{\kappa+\kappa^{-3}\coth^2 r}{k_1} = \frac{-\mathrm{sgn}(\kappa)\cdot(k_1+k_3)}{\sqrt{k_2^2+(k_1+k_3)^2}},\\
\tau_{34} &= \frac{1}{k_1(\kappa^2-\coth^2 r)}\big\langle\cosh r\,\dot\gamma-\kappa\xi, -(\kappa+\kappa^{-1})\coth r\,\dot\gamma+(\kappa^2+\kappa^{-2}\coth^2 r)\xi\big\rangle\\
&= -\frac{\kappa+\kappa^{-3}\coth^2 r}{k_1},\\
\tau_{23} &= \frac{1}{k_1\sqrt{\kappa^2-\coth^2 r}}\big\langle\kappa J\dot\gamma+\kappa^{-2}\coth r\mathcal{N}, J(\coth r\dot\gamma-\kappa\xi)\big\rangle\\
&= -\frac{\coth r\sqrt{\kappa^2-\coth^2 r}}{\kappa^3 k_1} = \frac{-\mathrm{sgn}(\kappa)\cdot k_2}{\sqrt{k_2^2+(k_1+k_3)^2}},\\
\tau_{14} &= \frac{1}{k_1\sqrt{\kappa^2-\coth^2 r}}\big\langle\dot\gamma, -(\kappa+\kappa^{-1})\coth r\,\dot\gamma+(\kappa^2+\kappa^{-2}\coth^2 r)\xi\big\rangle\\
&= -\frac{\coth r\sqrt{\kappa^2-\coth^2 r}}{\kappa^3 k_1}.
\end{aligned}$$

Thus, we obtain the conclusion. □

We here rewrite the above result into a result on extrinsic shapes of circular trajectories on geodesic spheres in a complex hyperbolic space $\mathbb{C}H^n(c)$.

Proposition 12.4. *The extrinsic shape of a circular trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a geodesic sphere $G(r)$ in $\mathbb{C}H^n(c)$ is an essential Killing helix of proper order 4 and satisfies the condition (I) of Lemma 4.9. Its geodesic curvatures are*

$$\begin{aligned}
k_1 &= \frac{1}{8\kappa^2}\sqrt{64\kappa^6+c^2(8\kappa^2-c)\coth^2(\sqrt{|c|}\,r/2)},\\
k_2 &= \frac{|c|^{3/2}(4\kappa^2-c)\coth(\sqrt{|c|}\,r/2)\sqrt{4\kappa^2+c\coth^2(\sqrt{|c|}\,r/2)}}{8\kappa^2\sqrt{64\kappa^6+c^2(8\kappa^2-c)\coth^2(\sqrt{|c|}\,r/2)}},\\
k_3 &= \frac{|c|(4\kappa^2+c\coth^2(\sqrt{|c|}\,r/2))}{2\sqrt{64\kappa^6+c^2(8\kappa^2-c)\coth^2(\sqrt{|c|}\,r/2)}}.
\end{aligned}$$

Proof. As usual, we consider a new metric $\langle\ ,\ \rangle' = (|c|/4)\langle\ ,\ \rangle$ on $\mathbb{C}H^n$. Then it has constant holomorphic sectional curvatures -4, and the radius of the geodesic sphere turns to $r' = \sqrt{|c|}\,r/2$. If we define a smooth curve σ by $\sigma(s) = \gamma(2s/\sqrt{|c|})$, then it is a circular trajectory for a Sasakian magnetic field $\mathbf{F}'_{2\kappa/\sqrt{|c|}}$ with respect to the new metric. By Proposition 12.3 we see the extrinsic shape of σ is an essential Killing helix of proper order 4 with geodesic curvatures

$$k'_1 = \frac{1}{\kappa'^2}\sqrt{\kappa'^6+(2\kappa'^2+1)\coth^2 r'},\quad k'_2 = \frac{(\kappa'^2+1)\coth r'\sqrt{\kappa'^2-\coth^2 r'}}{\kappa'^2\sqrt{\kappa'^6+(2\kappa'^2+1)\coth^2 r'}},$$
$$k'_3 = \frac{\kappa'^2-\coth^2 r'}{\sqrt{\kappa'^6+(2\kappa'^2+1)\coth^2 r'}},$$

where $\kappa' = 2\kappa/\sqrt{|c|}$. Therefore, by Lemma 8.3, the extrinsic shape of γ in $\mathbb{C}H^n(c)$ is an essential Killing helix of proper order 4 with geodesic curvatures

$$\begin{aligned}
k_1 &= \frac{\sqrt{|c|}}{2}\,k'_1 = \frac{\sqrt{|c|}}{2}\,\frac{|c|}{4\kappa^2}\sqrt{\frac{64\kappa^6}{|c|^3}+\Big(\frac{8\kappa^2}{|c|}+1\Big)\coth^2(\sqrt{|c|}\,r/2)}\\
&= \frac{1}{8\kappa^2}\sqrt{64\kappa^6+(8|c|^2\kappa^2+|c|^3)\coth^2(\sqrt{|c|}\,r/2)},\\
k_2 &= \frac{\sqrt{|c|}}{2}\,k'_2 = \frac{\sqrt{|c|}}{2}\,\frac{\big(\frac{4\kappa^2}{|c|}+1\big)\coth(\sqrt{|c|}\,r/2)\sqrt{\frac{4\kappa^2}{|c|}-\coth^2(\sqrt{|c|}\,r/2)}}{\frac{4\kappa^2}{|c|}\sqrt{\frac{64\kappa^6}{|c|^3}+\big(\frac{8\kappa^2}{|c|}+1\big)\coth^2(\sqrt{|c|}\,r/2)}}\\
&= \frac{|c|^{3/2}(4\kappa^2+|c|)\coth(\sqrt{|c|}\,r/2)\sqrt{4\kappa^2-|c|\coth^2(\sqrt{|c|}\,r/2)}}{8\kappa^2\sqrt{64\kappa^6+(8|c|^2\kappa^2+|c|^3)\coth^2(\sqrt{|c|}\,r/2)}},\\
k_3 &= \frac{\sqrt{|c|}}{2}\,k'_3 = \frac{\sqrt{|c|}}{2}\,\frac{\frac{4\kappa^2}{|c|}-\coth^2(\sqrt{|c|}\,r/2)}{\sqrt{\frac{64\kappa^6}{|c|^3}+\big(\frac{8\kappa^2}{|c|}+1\big)\coth^2(\sqrt{|c|}\,r/2)}}\\
&= \frac{|c|\big(4\kappa^2-|c|\coth^2(\sqrt{|c|}\,r/2)\big)}{2\sqrt{64\kappa^6+(8|c|^2\kappa^2+|c|^3)\coth^2(\sqrt{|c|}\,r/2)}}.
\end{aligned}$$

Since complex torsions of helices are invariant under homothetic changes of metrics, we get the conclusion. □

12.3. **Lengths of circular trajectories on $G(r)$ in $\mathbb{C}H^n$.** We now study properties of circular trajectories on $G(r)$ in $\mathbb{C}H^n(c)$. Since geodesic spheres are compact, it is clear that extrinsic shapes of trajectories for Sasakian magnetic fields are bounded curves in $\mathbb{C}H^n(c)$. We therefore interested in whether circular trajectories are closed or not. For this study we first consider horizontal lifts of extrinsic shapes of circular trajectories with respect to the canonical fibration $\varpi : H_1^{2n+1} \to \mathbb{C}H^n$. We can get the following by just the same way as of the proof of Lemma 11.3 by use of Lemma 4.9 and the equality (11.1).

Lemma 12.2. *Let γ be a circular trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a geodesic sphere $G(r)$ in $\mathbb{C}H^n(-4)$. A horizontal lift $\hat\gamma$ of the extrinsic shape of γ satisfies*

$$(12.1)\quad \hat\gamma''' - \sqrt{-1}(\kappa+\kappa^{-1})\hat\gamma'' - (2-\kappa^{-2}\coth^2 r)\hat\gamma' + \sqrt{-1}(\kappa^{-1}-\kappa^{-3}\coth^2 r)\hat\gamma = 0.$$

as a curve in $\mathbb{C}_1^{n+1}$.

Proof. By Proposition 12.3, the extrinsic shape of γ is an essential Killing helix of proper order 4. In this case, it is determined by the differential equations

$$\widetilde\nabla_{\dot\gamma}\dot\gamma = k_1 Y_2,\ \widetilde\nabla_{\dot\gamma} Y_2 = -k_1\dot\gamma + k_2 Y_3.$$

By use of the relationship (11.1) of connections and Lemma 4.9, we find that a horizontal lift of the extrinsic shape of γ satisfies

$$\begin{cases} \overline\nabla_{\dot{\hat\gamma}}\dot{\hat\gamma} = k_1 Y_2 + \widehat{\mathcal{N}}, \\ \overline\nabla_{\dot{\hat\gamma}} Y_2 = -k_1\dot{\hat\gamma} + k_2 Y_3 - \tau_{12} J\widehat{\mathcal{N}} = k_3\dot{\hat\gamma} + \mathrm{sgn}(\kappa)\sqrt{k_2^2 + (k_1+k_3)^2}\, JY_2 - \tau_{12} J\widehat{\mathcal{N}}, \end{cases}$$

where

$$k_3 = {k_1}^{-1}(1-\kappa^{-2}\coth^2 r), \quad \sqrt{k_2^2 + (k_1+k_3)^2} = |\kappa+\kappa^{-1}|,$$
$$\tau_{12} = -k_1^{-1}(\kappa+\kappa^{-3}\coth^2 r).$$

We therefore have

$$
\begin{aligned}
\overline{\nabla}_{\dot{\hat{\gamma}}}\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} &= k_1\overline{\nabla}_{\dot{\hat{\gamma}}}Y_2 + \dot{\hat{\gamma}} \\
&= (1+k_1k_3)\dot{\hat{\gamma}} + (\kappa+\kappa^{-1})k_1JY_2 + (\kappa+\kappa^{-3}\coth^2 r)J\widehat{\mathcal{N}} \\
&= (2-\kappa^{-2}\coth^2 r)\dot{\hat{\gamma}} + (\kappa+\kappa^{-1})J\big(\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} - \widehat{\mathcal{N}}\big) + (\kappa+\kappa^{-3}\coth^2 r)J\widehat{\mathcal{N}} \\
&= (2-\kappa^{-2}\coth^2)\dot{\hat{\gamma}} + (\kappa+\kappa^{-1})J\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} - \kappa^{-1}(1-\kappa^{-2}\coth^2 r)J\widehat{\mathcal{N}},
\end{aligned}
$$

and get the conclusion. □

We consider the characteristic equation of the linear differential equation (12.1) of constant coefficents, which is given by

$$\Lambda^3 - \sqrt{-1}(\kappa+\kappa^{-1})\Lambda^2 - (2-\kappa^{-2}\coth^2 r)\Lambda + \sqrt{-1}(\kappa^{-1} - \kappa^{-3}\coth^2 r) = 0.$$

Since the extrinsic shape of a trajectory γ on $G(r)$ is bounded, this cubic equation should have three distinct pure imaginary solutions. We shall check this by direct computation later (in the proof of Theorem 12.2). If we put $\Theta = -\sqrt{-1}\Lambda$ to realize it, then along the same lines as in the proof of Lemma 9.2 we obtain the following.

Corollary 12.4. *Let γ be a circular trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a geodesic sphere $G(r)$ in $\mathbb{C}H^n(-4)$. Let a_κ, b_κ, c_κ $(a_\kappa < b_\kappa < c_\kappa)$ be three distinct solutions of the cubic equation*

$$\Theta^3 - (\kappa+\kappa^{-1})\Theta^2 + (2-\kappa^{-2}\coth^2 r)\Theta - (\kappa^{-1} - \kappa^{-3}\coth^2 r) = 0. \tag{12.2}$$

Then, γ is closed if and only if there exists a constant d_κ satisfying that all of the ratios

$$(a_\kappa-d_\kappa)/(b_\kappa-d_\kappa),\ (b_\kappa-d_\kappa)/(c_\kappa-d_\kappa),\ (c_\kappa-d_\kappa)/(a_\kappa-d_\kappa)$$

are rational. In this case, its length is $2\pi\times\mathrm{L.C.M.}\big\{(b_\kappa-a_\kappa)^{-1}, (c_\kappa-b_\kappa)^{-1}\big\}$.

We are now in the position to make use of geometric properties of circles on $\mathbb{C}P^n(4)$ for getting conditions that circular trajectories on $\mathbb{C}H^n$ are closed.

Theorem 12.2. *Let γ be a circular trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a geodesic sphere $G(r)$ of radius r in $\mathbb{C}H^n(-4)$.*

(1) *When $r>\log\bigl(\sqrt{2}+1\bigr)$ and $\kappa=\pm\sqrt{2}$, it is closed and its length is $2\sqrt{2}\pi\sinh r$.*

(2) *Otherwise, it is closed if and only if*

$$\frac{\bigl|\kappa^2-2\bigr|(2\kappa^4-8\kappa^2+9\coth^2 r-1)}{2(\kappa^4-4\kappa^2+3\coth^2 r+1)^{3/2}}=\frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}}$$

holds with some relatively prime positive integers p,q satisfying $p>q$. In this case its length is given as $\pi\delta(p,q)|\kappa|\sqrt{(3p^2+q^2)/(\kappa^4-4\kappa^2+3\coth^2 r+1)}$, where $\delta(p,q)=1$ when pq is odd and $\delta(p,q)=2$ when pq is even.

Proof. In order to transplant the result on circles on $\mathbb{C}P^n(4)$ to our trajectories, we modify the cubic equation (12.2). First we make a parallel translation by putting $\varTheta_1=\varTheta-\frac{1}{3}(\kappa+\kappa^{-1})$. We then find that (12.2) turns to

$$\begin{aligned}\varTheta_1^3&-\frac{1}{3}\bigl\{\kappa^2-4+\kappa^{-2}(3\coth^2 r+1)\bigr\}\varTheta_1\\&-\frac{1}{27}\bigl\{2\kappa^3-12\kappa+3\kappa^{-1}(3\coth^2 r+5)-2\kappa^{-3}(9\coth^2 r-1)\bigr\}=0.\end{aligned}$$

As we have $\kappa^2-4+\kappa^{-2}(3\coth^2 r+1)=(\kappa-2\kappa^{-1})^2+3\kappa^{-2}(\coth^2 r-1)>0$, we make the coefficient of degree one of this cubic equation to be $3/2$ by putting

$$\begin{aligned}\vartheta&=\Bigl(3/\sqrt{2\{\kappa^2-4+\kappa^{-2}(3\coth^2 r+1)\}}\Bigr)\varTheta_1\\&=\bigl(3\varTheta-\kappa-\kappa^{-1}\bigr)/\sqrt{2\{\kappa^2-4+\kappa^{-2}(3\coth^2 r+1)\}}.\end{aligned}$$

We find the equation (12.2) turns to

$$\vartheta^3-\frac{3}{2}\vartheta-\frac{\mathrm{sgn}(\kappa)\cdot(\kappa^2-2)(2\kappa^4-8\kappa^2+9\coth^2 r-1)}{2\sqrt{2}(\kappa^4-4\kappa^2+3\coth^2 r+1)^{3/2}}=0. \tag{12.3}$$

We set

$$\tau_G(\kappa;r)=-\frac{\mathrm{sgn}(\kappa)\cdot(\kappa^2-2)(2\kappa^4-8\kappa^2+9\coth^2 r-1)}{2(\kappa^4-4\kappa^2+3\coth^2 r+1)^{3/2}}.$$

Here, as we have

$$\begin{aligned}4(\kappa^4-4\kappa^2+3\coth^2 r+1)^3 - (\kappa^2-2)^2(2\kappa^4-8\kappa^2+9\coth^2 r-1)^2\\ = 27(\coth^2 r-1)^2(\kappa^4-4\kappa^2+4\coth^2 r)\\ = 27(\coth^2 r-1)^2\big\{(\kappa^2-2)^2+4(\coth^2 r-1)\big\} > 0,\end{aligned}$$

we obtain $|\tau_G(\kappa; r)| < 1$. This guarantees that the equality (12.2) has three distinct real solutions directly.

We now compare (12.3) and (9.4). With the solutions a_τ, b_τ, c_τ ($a_\tau < b_\tau < c_\tau$) for (9.4) with $\tau = \tau_G(\kappa; r)$, the change of variables of Θ to ϑ shows that the solutions a_κ, b_κ, c_κ of (12.2) satisfy

$$\begin{aligned}a_\kappa &= \big(a_\tau\sqrt{2\{\kappa^2-4+\kappa^{-2}(3\coth^2 r+1)\}} + \kappa+\kappa^{-1}\big)/3,\\ b_\kappa &= \big(b_\tau\sqrt{2\{\kappa^2-4+\kappa^{-2}(3\coth^2 r+1)\}} + \kappa+\kappa^{-1}\big)/3,\\ c_\kappa &= \big(c_\tau\sqrt{2\{\kappa^2-4+\kappa^{-2}(3\coth^2 r+1)\}} + \kappa+\kappa^{-1}\big)/3.\end{aligned}$$

Thus by putting $d_\kappa = (\kappa+\kappa^{-1})/3$ we find

$$\frac{a_\kappa-d_\kappa}{b_\kappa-d_\kappa} = \frac{a_\tau}{b_\tau}, \quad \frac{b_\kappa-d_\kappa}{c_\kappa-d_\kappa} = \frac{b_\tau}{c_\tau}, \quad \frac{c_\kappa-d_\kappa}{a_\kappa-d_\kappa} = \frac{c_\tau}{a_\tau}.$$

Therefore, we find that γ is closed if and only if a circle σ of geodesic curvature $1/\sqrt{2}$ and of complex torsion $\tau(\kappa; r)$ on $\mathbb{C}P^n(4)$ is closed. Moreover, in this case, we obtain

$$\begin{aligned}\mathrm{length}(\gamma) &= 2\pi\times\mathrm{L.C.M.}\big\{(b_\kappa-a_\kappa)^{-1}, (c_\kappa-b_\kappa)^{-1}\big\}\\ &= 2\pi\times\mathrm{L.C.M.}\big\{(b_\tau-a_\tau)^{-1}, (c_\tau-b_\tau)^{-1}\big\}\\ &\qquad\times\big\{2\{\kappa^2-4+\kappa^{-2}(3\coth^2 r+1)\}\big\}^{-1/2}\\ &= 3\,\mathrm{length}(\sigma)\big/\sqrt{2\{\kappa^2-4+\kappa^{-2}(3\coth^2 r+1)\}}.\end{aligned}$$

First we consider the case that $\tau_G(\kappa; r) = \pm q(9p^2-q^2)(3p^2+q^2)^{-3/2}$. By Proposition 9.1 we find that this circular trajectory γ is closed and that its length is given by

$$\frac{1}{3}\pi\delta(p,q)\sqrt{2(3p^2+q^2)}\times\frac{3}{\sqrt{2\{\kappa^2-4+\kappa^{-2}(3\coth^2 r+1)\}}}$$
$$=\pi\delta(p,q)\sqrt{\frac{3p^2+q^2}{\kappa^2-4+\kappa^{-2}(3\coth^2 r+1)}}.$$

We next consider the case corresponding to the case of $\tau=0$. Since

$$2\kappa^4-8\kappa^2+9\coth^2 r-1=2(\kappa^2-2)^2+9(\coth^2 r-1)>0,$$

we have $\tau_G(\kappa;r)=0$ if and only if $\kappa=\pm\sqrt{2}$. Since we need $|\kappa|>\coth r$ by the circular condition, we have $\coth r<\sqrt{2}$, which is equivalent to $\cosh^2 r>2$. We therefore have $r>\log(\sqrt{2}+1)$. In this case, we find that γ is closed and its length is

$$\frac{2\sqrt{6}\pi}{3}\times\frac{3}{\sqrt{3(\coth^2 r-1)}}=2\sqrt{2}\pi\sinh r.$$

This complete the proof. □

We now study circular trajectories on geodesic spheres in $\mathbb{C}H^n(c)$ by make use of the homothetic change of metrics given in Proposition 12.4.

Theorem 12.3. *Let γ be a circular trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a geodesic sphere $G(r)$ of radius r in $\mathbb{C}H^n(c)$.*

(1) *When $r>2/\sqrt{|c|}\log\big(\sqrt{2}+1\big)$ and $\kappa=\pm\sqrt{|c|/2}$, it is closed and its length is $4\sqrt{2/|c|}\pi\sinh r$.*

(2) *Otherwise, it is closed if and only if*

$$\frac{\big|2\kappa^2+c\big|\{32\kappa^4+32c\kappa^2+c^2\big(9\coth^2(\sqrt{|c|}\,r/2)-1\big)\}}{\{16\kappa^4+16c\kappa^2+c\big(3\coth^2(\sqrt{|c|}\,r/2)+1\big)\}^{3/2}}=\frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}}$$

holds with some relatively prime positive integers p,q satisfying $p>q$. In this case its length is given as

$$4\pi\delta(p,q)|\kappa|\sqrt{\frac{3p^2+q^2}{16\kappa^4+16c\kappa^2+c^2\big(3\coth^2(\sqrt{|c|}\,r/2)+1\big)}},$$

where $\delta(p,q)=1$ when pq is odd and $\delta(p,q)=2$ when pq is even.

Proof. We consider a new metric on $\mathbb{C}H^n$ given by $\langle\ ,\ \rangle' = |c|/4\langle\ ,\ \rangle$. Then it has constant holomorphic sectional curvatures -4, and the radius of the geodesic sphere turns to $r' = \sqrt{|c|}\,r/2$. If we define a smooth curve σ by $\sigma(s) = \gamma(2s/\sqrt{|c|})$, then it is a circular trajectory for a Sasakian magnetic field $\mathbf{F}'_{2\kappa/\sqrt{|c|}}$ with respect to the new metric.

When $r' > \log(\sqrt{2}+1)$ and $\kappa' = \pm\sqrt{2}$, that is, when $r > (2/\sqrt{|c|})\log(\sqrt{2}+1)$ and $\kappa = \pm\sqrt{|c|/2}$, the trajectory σ is closed and $\mathrm{length}'(\sigma) = 2\sqrt{2}\pi\sinh r'$. We hence find that γ is closed and

$$\mathrm{length}(\gamma) = \frac{2}{\sqrt{|c|}} \times \mathrm{length}'(\sigma) = 4\sqrt{\frac{2}{|c|}}\,\pi\sinh(\sqrt{|c|}\,r/2)$$

(see Table 5 in §8).

In other case, the trajectory σ is closed if and only if

$$\frac{\bigl|\kappa'^2-2\bigr|(2\kappa'^4-8\kappa'^2+9\coth^2 r'-1)}{2(\kappa'^4-4\kappa'^2+3\coth^2 r'+1)^{3/2}} = \frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}}$$

holds with some relatively prime positive integers p, q satisfying $p > q$, and its length is given as

$$\mathrm{length}'(\sigma) = \pi\delta(p,q)|\kappa'|\sqrt{(3p^2+q^2)/(\kappa'^4-4\kappa'^2+3\coth^2 r'+1)}.$$

Substituting $\kappa' = 2\kappa/\sqrt{|c|}$ and $r' = \sqrt{|c|}\,r/2$, we find that γ is closed if and only if

$$\begin{aligned}\frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}} &= \frac{\bigl|\frac{4\kappa^2}{|c|}-2\bigr|\bigl(\frac{32\kappa^4}{|c|^2}-\frac{32\kappa^2}{|c|}+9\coth^2(\sqrt{|c|}\,r/2)-1\bigr)}{2\bigl(\frac{16\kappa^4}{|c|^2}-\frac{16\kappa^2}{|c|}+3\coth^2(\sqrt{|c|}\,r/2)+1\bigr)^{3/2}} \\ &= \frac{\bigl|2\kappa^2-|c|\bigr|\{32\kappa^4-32|c|\kappa^2+|c|^2\bigl(9\coth^2(\sqrt{|c|}\,r/2)-1\bigr)\}}{\{16\kappa^4-16|c|\kappa^2+|c|^2\bigl(3\coth^2(\sqrt{|c|}\,r/2)+1\bigr)\}^{3/2}},\end{aligned}$$

and its length is given by

$$\begin{aligned}\mathrm{length}(\gamma) &= \frac{2}{\sqrt{|c|}} \times \mathrm{length}'(\sigma) \\ &= \frac{2\pi}{\sqrt{|c|}}\delta(p,q)\frac{2|\kappa|}{\sqrt{|c|}}\sqrt{\frac{3p^2+q^2}{\frac{16\kappa^4}{|c|^2}-\frac{16\kappa^2}{|c|}+3\coth^2(\sqrt{|c|}\,r/2)+1}}\end{aligned}$$

$$= 4\pi\delta(p,q)|\kappa|\sqrt{\frac{3p^2+q^2}{16\kappa^4-16|c|\kappa^2+|c|^2\big(3\coth^2(\sqrt{|c|}\,r/2)+1\big)}}.$$

This complete the proof. □

13. **Circular trajectories on tubes around totally geodesic $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n$**

In this section we study circular trajectories on tubes around totally geodesic $\mathbb{C}H^{n-1}(c)$ in a complex hyperbolic space $\mathbb{C}H^n(c)$. A tube $T(r)$ around totally geodesic $\mathbb{C}H^{n-1}$ also have two principal curvatures $\lambda = (\sqrt{|c|}/2)\tanh\bigl(\sqrt{|c|}\,r/2\bigr)$ and $\nu = \sqrt{|c|}\coth\bigl(\sqrt{|c|}\,r\bigr)$. Its characteristic vector field ξ satisfies $A\xi = \nu\xi$ and every vector v orthogonal to ξ satisfies $Av = \lambda v$ with the shape operator A. In particular, the shape operator and the characteristic tensor satisfy $A\phi = \phi A$.

13.1. **Trajectories on tubes around $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n$.**

We first study trajectories from the viewpoint of Frenet-Serre formula. By Corollary 7.1, we know that every trajectory on a tube $T(r)$ has constant structure torsion. By the same proof as of Proposition 7.1, we have the following.

Proposition 13.1. *Let γ be a trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a tube $T(r)$ of radius r around totally geodesic $\mathbb{C}H^{n-1}(c)$ in a complex hyperbolic space $\mathbb{C}H^n(c)$.*

1. *It is a geodesic if and only if $\rho_\gamma = \pm 1$.*
2. *It is a circle of positive geodesic curvature if and only if*
   $$\kappa\rho_\gamma = (\sqrt{|c|}/2)\tanh\bigl(\sqrt{|c|}\,r/2\bigr).$$
   In this case, its geodesic curvature is $\sqrt{4\kappa^2 + c\tanh^2\bigl(\sqrt{|c|}\,r/2\bigr)}\,/2$.
3. *Otherwise, it is a helix of proper order 3 whose geodesic curvatures are $|\kappa|\sqrt{1-\rho_\gamma^2}$ and $\bigl|2\kappa\rho_\gamma - \sqrt{|c|}\bigr|\tanh\bigl(\sqrt{|c|}\,r/2\bigr)/2$.*

If we restrict ourselves to circular trajectories, as $|\rho_\gamma| < 1$, we have the following result.

Theorem 13.1. *Let $\boldsymbol{F}_\kappa$ be a non-trivial Sasakian magnetic field on a tube $T(r)$ of radius r around $\mathbb{C}H^{n-1}(c)$ in $\mathbb{C}H^n(c)$.*

(1) *When* $0 < |\kappa| \le (\sqrt{|c|}/2)\tanh(\sqrt{|c|}r/2)$, *there are no circular trajectories for* $\boldsymbol{F}_\kappa$.

(2) *When* $|\kappa| > (\sqrt{|c|}/2)\tanh(\sqrt{|c|}r/2)$, *a trajectory for* $\boldsymbol{F}_\kappa$ *is circular if and only if* $\rho_\gamma = (\sqrt{|c|}/(2\kappa))\tanh(\sqrt{|c|}r/2)$. *In this case its geodesic curvature is* $\sqrt{\kappa^2 + (c/4)\tanh^2(\sqrt{|c|}r/2)}$.

Congruency conditions on circular trajectories on a tube around totally geodesic complex hypersurface are given in the same way as in Proposition 7.2.

Proposition 13.2 (Adachi[3]). *Trajectories* γ_1 *for a Sasakian magnetic field* $\boldsymbol{F}_{\kappa_1}$ *and* γ_2 *for* $\boldsymbol{F}_{\kappa_2}$ *on a tube* $T(r)$ *around totally geodesic* $\mathbb{C}H^{n-1}(c)$ *in* $\mathbb{C}H^n(c)$ *are congruent to each other in strong sense if and only if one of the following conditions holds*:

i) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| = 1$,

ii) $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$ *and* $|\kappa_1| = |\kappa_2|$,

iii) $0 < |\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$ *and* $\kappa_1\rho_{\gamma_1} = \kappa_2\rho_{\gamma_2}$.

In order to show this proposition, by the same argument as in the proof of Proposition 7.2, we only need the following.

Lemma 13.1. *Let* $x, x' \in T(r)$ *be arbitrary points on a tube* $T(r)$ *in* $\mathbb{C}H^n(c)$. *Given unit tangent vectors* $u \in \langle\xi_x\rangle^\perp \subset T_xT(r)$ *and* $u' \in \langle\xi_{x'}\rangle^\perp \subset T_{x'}T(r)$ *which are orthogonal to* ξ *at* x *and* x', *there exist isometries* $\tilde\varphi^+$, $\tilde\varphi^-$ *of* $\mathbb{C}H^n(c)$ *satisfying the following conditions*:

i) $\tilde\varphi^+\big(T(r)\big) = \tilde\varphi^-\big(T(r)\big) = T(r)$,
(*i.e.* $T(r)$ *is invariant under the actions of* $\tilde\varphi^+$ *and* $\tilde\varphi^-$);

ii) $\tilde\varphi^+(x) = \tilde\varphi^-(x) = x'$;

iii) $d\tilde\varphi^+(u) = d\tilde\varphi^-(u) = u'$,

iv) $d\tilde{\varphi}^+ \circ J = J \circ d\tilde{\varphi}^+$ *and* $d\tilde{\varphi}^- \circ J = -J \circ d\tilde{\varphi}^-$,

in particular, $d\tilde{\varphi}^+(\xi_x) = \xi_{x'}$ *and* $d\tilde{\varphi}^-(\xi_x) = -\xi_{x'}$.

Proof. For the sake of simplicity, we only treat the case $n = 2$ and $c = -4$. As we see in §5.3 we may consider that

$$\varpi^{-1}\big(T(r)\big) = \big\{(z_0, z_1, z_2) \in \mathbb{C}^3 \bigm| -|z_0|^2 + |z_1|^2 = -\cosh^2 r,\ |z_2| = \sinh r\big\}$$
$$= H_1^3 \times S^1 \subset \mathbb{C}^2 \times \mathbb{C}$$

through a canonical fibration $\varpi : H_1^5 \to \mathbb{C}H^2$. We take an arbitrary point $\hat{z} = (z_0, z_1, z_2) \in \varpi^{-1}\big(T(r)\big)$. The tangent space of $\widehat{M} = \varpi^{-1}\big(T(r)\big)$ at $\hat{z}$ is represented as

$$T_{\hat{z}}\widehat{M} = \big\{(\hat{z}, \hat{v}) \in \{\hat{z}\} \times \mathbb{C}^3 \bigm| \mathrm{Re}\big(-z_0\overline{v}_0 + z_1\overline{v}_1\big) = \mathrm{Re}(z_2\overline{v}_2) = 0\big\}.$$

We denote by $\widehat{\mathcal{N}}_{\hat{z}} \in T_{\hat{z}}H_1^5$ the horizontal lift of the unit normal vector $\mathcal{N}_{\varpi(\hat{z})}$ of $T(r)$ in $\mathbb{C}H^2(-4)$. Since $\widehat{\mathcal{N}}_{\hat{z}}$ is orthogonal to $T_{\hat{z}}\widehat{M}$, we find it is represented as

$$\mathcal{N}_{\hat{z}} = \big(\hat{z}, (-\tanh r\, z_0, -\tanh r\, z_1, -\coth r\, z_2)\big).$$

By putting $\hat{\xi}_{\hat{z}} = -J\mathcal{N}_{\hat{z}} = \big(\hat{z}, \sqrt{-1}(\tanh r\, z_0, \tanh r\, z_1, \coth r\, z_2)\big)$, we denote by $\langle\hat{\xi}_{\hat{z}}\rangle$ the real linear subspace of $T_{\hat{z}}H_1^5$ spanned by $\hat{\xi}_{\hat{z}}$, and by $\langle\hat{\xi}_{\hat{z}}\rangle^\perp$ its orthogonal complement in $T_{\hat{z}}H_1^5$. The horizontal part $\langle\hat{\xi}_{\hat{z}}\rangle^\perp \cap \mathcal{H}_{\hat{z}}$ of $\langle\hat{\xi}_{\hat{z}}\rangle^\perp$ corresponds to $\langle\xi_{\varpi(\hat{z})}\rangle^\perp$, and is represented as

$$\langle\hat{\xi}_{\hat{z}}\rangle^\perp \cap \mathcal{H}_{\hat{z}} = \big\{(\hat{z}, \hat{v}) \in \{\hat{z}\} \times \mathbb{C}^3 \bigm| -z_0\overline{v}_0 + z_1\overline{v}_1 = 0,\ v_2 = 0\big\}.$$

We take a point $\hat{z}_* = (\cosh r, 0, \sinh r) \in \widehat{M}$ and a unit tangent vector $\hat{u}_* = \big(\hat{z}_*, (0, 1, 0)\big) \in \langle\hat{\xi}_{\hat{z}_*}\rangle^\perp \cap \mathcal{H}_{\hat{z}_*}$. For an arbitrary point $\hat{z} \in \widehat{M}$ and an arbitrary unit tangent vector $\hat{u} \in \langle\hat{\xi}_{\hat{z}}\rangle^\perp \cap \mathcal{H}_{\hat{z}}$, which is expressed as $\hat{u} = \big(\hat{z}, (\zeta\overline{z}_1/\cosh r, \zeta\overline{z}_0/\cosh r, 0)\big)$ with some $\zeta \in \mathbb{C}$ satisfying $|\zeta| = 1$, we consider a matrix

$$U_+ = \begin{pmatrix} z_0/\cosh r & u_0 & 0 \\ z_1/\cosh r & u_1 & 0 \\ 0 & 0 & z_2/\sinh r \end{pmatrix} \in U(1,1) \oplus U(1).$$

This induces a linear transformation of $\mathbb{C}_1^3$ which preserves the Hermitian product $\langle\!\langle\ ,\ \rangle\!\rangle$, hence it induces an isometry $\hat\varphi^+_{(\hat z,\hat u)}$ of H_1^5. It clearly satisfies $\hat\varphi^+_{(\hat z,\hat u)}\big(\widehat{M}\big)=\widehat{M}$ and $\hat\varphi^+_{(\hat z,\hat u)}\big(e^{\sqrt{-1}\theta}\hat p\big)=e^{\sqrt{-1}\theta}\hat\varphi^+_{(\hat z,\hat u)}(\hat p)$ for arbitrary $\theta\in[0,2\pi)$ and $\hat p\in\widehat{M}$. Therefore, we find that $\hat\varphi^+_{(\hat z,\hat u)}$ induces an isometry $\tilde\varphi^+_{(\hat z,\hat u)}$ of $\mathbb{C}H^n(-4)$ satisfying

$$\tilde\varphi^+_{(\hat z,\hat u)}\circ\varpi=\varpi\circ\hat\varphi^+_{(\hat z,\hat u)},\quad \tilde\varphi^+_{(\hat z,\hat u)}\big(\varpi(\hat z_*)\big)=\varpi(\hat z),\quad d\tilde\varphi^+_{(\hat z,\hat u)}\big(d\varpi(\hat u_*)\big)=d\varpi(\hat u).$$

Since we have $U_+J=JU_+$ for the matrix $J=\sqrt{-1}E$, we find that $\tilde\varphi^+_{(\hat z,\hat u)}$ is holomorphic, that is, $d\tilde\varphi^+_{(\hat z,\hat u)}J=Jd\tilde\varphi^+_{(\hat z,\hat u)}$. In particular, it satisfies $d\tilde\varphi^+_{(\hat z,\hat u)}\big(\xi_{\varpi(\hat z_*)}\big)=\xi_{\varpi(\hat z)}$.

We next consider a matrix

$$\varPsi=\begin{pmatrix}\epsilon & O & O\\ O & \epsilon & O\\ O & O & \epsilon\end{pmatrix}\in O(6)\quad\text{with}\quad \epsilon=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\in O(2).$$

If we define a matrix U_- by $U_-=U_+\varPsi$, it induces a linear transformation of $\mathbb{C}_1^3$ which preserves the Hermitian product, hence it induces an isometry $\hat\varphi^-_{(\hat z,\hat u)}$ of H_1^5 which preserves $\widehat{M}$. It is clear that it satisfies $\hat\varphi^-_{(\hat z,\hat u)}\big(e^{\sqrt{-1}\theta}\hat p\big)=e^{-\sqrt{-1}\theta}\hat\varphi^-_{(\hat z,\hat u)}(\hat p)$ for arbitrary $\theta\in[0,2\pi)$ and $\hat p\in\widehat{M}$. As we have $U_-J=-JU_-$ for the matrix $J=\sqrt{-1}E$, we find that $\hat\varphi^-_{(\hat z,\hat u)}$ induces an isometry $\tilde\varphi^-_{(\hat z,\hat u)}$ of $\mathbb{C}H^n(-4)$ satisfying

$$\tilde\varphi^-_{(\hat z,\hat u)}\circ\varpi=\varpi\circ\hat\varphi^-_{(z,u)},\quad d\tilde\varphi^-_{(\hat z,\hat u)}J=-Jd\tilde\varphi^+_{(\hat z,\hat u)},$$

$$\tilde\varphi^-_{(\hat z,\hat u)}(\varpi(\hat z_*))=\varpi(\hat z),\quad d\tilde\varphi^-_{(\hat z,\hat u)}(d\varpi(\hat u_*))=d\varpi(\hat u).$$

In particular, it satisfies $d\tilde\varphi^-_{(\hat z,\hat u)}\big(\xi_{\varpi(\hat z_*)}\big)=-\xi_{\varpi(\hat z)}$.

As we constructed desireble isometries for a fixed pair $\big(\varpi(\hat z_*),d\varpi(\hat u_*)\big)$ and an arbitrary pair $\big(\varpi(\hat z),d\varpi(\hat u)\big)$, we can get our conclusion. □

Remark 13.1. Every isometry of $T(r)$ in $\mathbb{C}H^n(c)$ is equivariant.

Corollary 13.1. *Every circular trajectory on a tube $T(r)$ around $\mathbb{C}H^{n-1}(c)$ in $\mathbb{C}H^n(c)$ is Killing.*

Corollary 13.2. *Circular trajectories for a given Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a tube $T(r)$ in $\mathbb{C}H^n(c)$ are congruent to each other in strong sense.*

Corollary 13.3. *Geodesic trajectories for non-trivial Sasakian magnetic fields on a tube $T(r)$ in $\mathbb{C}H^n(c)$ are congruent to each other in strong sense.*

13.2. **Extrinsic shapes of circular trajectories on $T(r)$ in $\mathbb{C}H^n$.** We next study extrinsic shapes of circular trajectories on a tube $T(r)$ in $\mathbb{C}H^n(c)$. We first treat the case $c = -4$.

Proposition 13.3. *The extrinsic shape of a circular trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a tube $T(r)$ around $\mathbb{C}H^{n-1}(-4)$ in $\mathbb{C}H^n(-4)$ is an essential Killing helix of proper order 4 and satisfies the condition (I) of Lemma 4.9. Its geodesic curvatures are*

$$k_1 = \frac{1}{\kappa^2}\sqrt{\kappa^6 + (1+2\kappa^2)\tanh^2 r}, \quad k_2 = \frac{(\kappa^2+1)\tanh r\sqrt{\kappa^2 - \tanh^2 r}}{\kappa^2\sqrt{\kappa^6+(1+2\kappa^2)\tanh^2 r}},$$

$$k_3 = \frac{\kappa^2 - \tanh^2 r}{\sqrt{\kappa^6+(1+2\kappa^2)\tanh^2 r}},$$

and its complex torsions satisfy

$$\tau_{12} = \tau_{34} = -\frac{\kappa + \kappa^{-3}\tanh^2 r}{k_1} = \frac{-\mathrm{sgn}(\kappa)\cdot(k_1+k_3)}{\sqrt{k_2^2 + (k_1+k_3)^2}}, \quad \tau_{13} = \tau_{24} = 0,$$

$$\tau_{23} = \tau_{14} = -\frac{\tanh r\sqrt{\kappa^2 - \tanh^2 r}}{\kappa^3 k_1} = \frac{-\mathrm{sgn}(\kappa)\cdot k_2}{\sqrt{k_2^2 + (k_1+k_3)^2}}.$$

Proof. Since $c = -4$, we have $\lambda = \tanh r$, $\nu = 2\coth 2r$, hence find $\nu - \lambda = \coth r$. By use of the circular condition $\kappa\rho_\gamma = \tanh r$, we have

$$A\dot\gamma = A\big(\rho_\gamma\xi + (\dot\gamma - \rho_\gamma\xi)\big) = \lambda\dot\gamma + (\nu-\lambda)\rho_\gamma\xi = \tanh r\,\dot\gamma + \kappa^{-1}\xi,$$

$$\langle A\dot\gamma, \dot\gamma\rangle = \lambda + (\nu-\lambda)\rho_\gamma^2 = (1+\kappa^{-2})\tanh r.$$

Thus, our calculation on Frenet-Serre formula of the extrinsic shape of a circular trajectory γ goes through by just the same way as in the proof of Proposition 8.1 if we change $\coth r$ to $\tanh r$ in that proof. Its geodesic curvatures and its Frenet frame are given as

$$k_1 = \sqrt{\kappa^2 + (2\kappa^{-2} + \kappa^{-4})\tanh^2 r}, \quad k_2 = {k_1}^{-1}(\kappa^2+1)\kappa^{-4}\tanh r\sqrt{\kappa^2 - \tanh^2 r},$$
$$k_3 = {k_1}^{-1}(1-\kappa^{-2}\tanh^2 r),$$

and

$$\left\{\begin{array}{l} \dot\gamma, \quad Y_2 = (\kappa J\dot\gamma + \kappa^{-2}\tanh r\,\mathcal{N})/k_1, \quad Y_3 = (\tanh r\dot\gamma - \kappa\xi)\big/\sqrt{\kappa^2-\tanh^2 r}, \\ Y_4 = \dfrac{1}{k_1\sqrt{\kappa^2-\tanh^2 r}}\{(\kappa+\kappa^{-1})\tanh r\,\phi\dot\gamma - (\kappa^2 - \tanh^2 r)\mathcal{N}\} \end{array}\right\}.$$

As we have

$$k_1 + k_3 = k_1^{-1}(\kappa^2+1)(1+\kappa^{-4}\tanh^2 r), \quad k_2^2 + (k_1+k_3)^2 = (\kappa+\kappa^{-1})^2,$$

we obtain the conclusion. □

We here rewrite the above result into a result on extrinsic shapes of circular trajectories on a tube $T(r)$ around a complex hypersurface in a complex hyperbolic space $\mathbb{C}H^n(c)$ by use of a homothetic change of metrics. By the proof of Proposition 13.3, we only need to change $\coth(\sqrt{|c|}\,r/2)$ to $\tanh(\sqrt{|c|}\,r/2)$ in the result of Proposition 12.4.

Proposition 13.4. *The extrinsic shape of a circular trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a tube $T(r)$ around $\mathbb{C}H^{n-1}(c)$ in $\mathbb{C}H^n(c)$ is an essential Killing helix of proper order 4 and satisfies the condition (I) of Lemma 4.9. Its geodesic curvatures are*

$$k_1 = \frac{1}{8\kappa^2}\sqrt{64\kappa^6 + c^2(8\kappa^2-c)\tanh^2(\sqrt{|c|}\,r/2)},$$
$$k_2 = \frac{|c|^{3/2}(4\kappa^2-c)\tanh(\sqrt{|c|}\,r/2)\sqrt{4\kappa^2+c\tanh^2(\sqrt{|c|}\,r/2)}}{8\kappa^2\sqrt{64\kappa^6+c^2(8\kappa^2-c)\tanh^2(\sqrt{|c|}\,r/2)}},$$
$$k_3 = \frac{|c|\{4\kappa^2+c\tanh^2(\sqrt{|c|}\,r/2)\}}{2\sqrt{64\kappa^6+c^2(8\kappa^2-c)\tanh^2(\sqrt{|c|}\,r/2)}}.$$

13.3. **Behaviors of circular trajectories on $T(r)$ in $\mathbb{C}H^n$.** We now study properties of circular trajectories on $T(r)$ in $\mathbb{C}H^n(c)$. Since $T(r)$ is not compact, we have to consider whether trajectories are bounded or not at first. As we do not make use of principal curvatures essentially in the proof of Lemma 12.2, it also works in this case.

Lemma 13.2. *Let γ be a circular trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a tube $T(r)$ in $\mathbb{C}H^n(-4)$. A horizontal lift $\hat{\gamma}$ of the extrinsic shape of γ satisfies*

$$\text{(13.1)}\quad \hat{\gamma}''' - \sqrt{-1}(\kappa+\kappa^{-1})\hat{\gamma}'' - (2-\kappa^{-2}\tanh^2 r)\hat{\gamma}' + \sqrt{-1}\kappa^{-1}(1-\kappa^{-2}\tanh^2 r)\hat{\gamma} = 0$$

as a curve in $\mathbb{C}^{n+1}_1$.

Proof. Since the extrinsic shape of γ is an essential Killing helix of proper order 4 by Proposition 13.3, we find $\hat{\gamma}$ satisfies

$$\begin{cases} \overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = k_1 Y_2 + \widehat{\mathcal{N}}, \\ \overline{\nabla}_{\dot{\hat{\gamma}}} Y_2 = k_3\dot{\hat{\gamma}} + \mathrm{sgn}(\kappa)\sqrt{k_2^2 + (k_1+k_3)^2}\, J Y_2 - \tau_{12} J\widehat{\mathcal{N}}, \end{cases}$$

with

$$k_3 = k_1{}^{-1}(1-\kappa^{-2}\tanh^2 r), \quad \sqrt{k_2^2 + (k_1+k_3)^2} = |\kappa+\kappa^{-1}|,$$
$$\tau_{12} = -k_1^{-1}(\kappa+\kappa^{-3}\tanh^2 r),$$

by Lemma 4.9. We hence obtain

$$\begin{aligned} \overline{\nabla}_{\dot{\hat{\gamma}}}\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} &= k_1\overline{\nabla}_{\dot{\hat{\gamma}}} Y_2 + \dot{\hat{\gamma}} \\ &= (1 + k_1 k_3)\dot{\hat{\gamma}} + (\kappa+\kappa^{-1})k_1 J Y_2 + (\kappa+\kappa^{-3}\tanh^2 r) J\widehat{\mathcal{N}} \\ &= (2-\kappa^{-2}\tanh^2 r)\dot{\hat{\gamma}} + (\kappa+\kappa^{-1}) J\big(\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} - \widehat{\mathcal{N}}\big) + (\kappa+\kappa^{-3}\tanh^2 r) J\widehat{\mathcal{N}}, \end{aligned}$$

and get the conclusion. □

We now consider the characteristic equation of the differential equation for a horizontal lift $\hat{\gamma}$ of the extrinsic shape of a circular trajectory γ on $T(r)$ in $\mathbb{C}H^n(-4)$.

It is given as

$$(13.2)\quad \Lambda^3 - \sqrt{-1}(\kappa+\kappa^{-1})\Lambda^2 - (2-\kappa^{-2}\tanh^2 r)\Lambda + \sqrt{-1}(\kappa^{-1}-\kappa^{-3}\tanh^2 r) = 0.$$

We study its solutions. We first realize this equation by putting $\Theta = -\sqrt{-1}\Lambda$. We have

$$(13.3)\qquad \Theta^3 - (\kappa+\kappa^{-1})\Theta^2 + (2-\kappa^{-2}\tanh^2 r)\Theta - (\kappa^{-1}-\kappa^{-3}\tanh^2 r) = 0.$$

If we put $\Theta_1 = \Theta - (\kappa+\kappa^{-1})/3$, we find this cubic equation turns to

$$(13.4)\qquad \begin{aligned} &\Theta_1^3 - \frac{1}{3}\big\{\kappa^2-4+(1+3\tanh^2 r)\kappa^{-2}\big\}\Theta_1 \\ &-\frac{1}{27}\big\{2\kappa^3-12\kappa+3(5+3\tanh^2 r)\kappa^{-1}+2(1-9\tanh^2 r)\kappa^{-3}\big\} = 0. \end{aligned}$$

We set $\zeta(\kappa;r) = \kappa^2-4+(1+3\tanh^2 r)\kappa^{-2}$, which is the coefficient of Θ_1 of the left hand side of the equation (13.4). When $\zeta(\kappa;r) = 0$, which is the case

$$\kappa^2 = 2 \pm \sqrt{3(1-\tanh^2 r)} = 2 \pm (\sqrt{3}/\cosh r),$$

we find that the coefficient of order zero of the left hand side does not vanish. Hence, the equation (13.4) has only one real solution. Similarly, when $\zeta(\kappa;r) < 0$, which is the case $2-\sqrt{3}(\cosh r)^{-1} \le \kappa^2 \le 2+\sqrt{3}(\cosh r)^{-1}$, the left hand side of (13.4) is monotone increasing with respect to Θ_1. We hence find that the equation (13.4) also has only one real solution in this case. When $\zeta(\kappa;r) > 0$, by putting $\vartheta = 3\Theta_1/\sqrt{2\zeta(\kappa;r)}$ we see (13.4) turns to

$$(13.5)\qquad \vartheta^3 - (3/2)\vartheta + \tau_T(\kappa;r)/\sqrt{2} = 0,$$

where

$$\tau_T(\kappa;r) = -\mathrm{sgn}(\kappa)\frac{(\kappa^2-2)(2\kappa^4-8\kappa^2+9\tanh^2 r-1)}{2\big(\kappa^4-4\kappa^2+3\tanh^2 r+1\big)^{3/2}}.$$

Thus we see the equation (13.5) has three distinct real solutions if and only if $|\tau_T(\kappa;r)| < 1$, hence so does the equation (13.4). As we have

$$\begin{aligned} &4(\kappa^4-4\kappa^2+3\tanh^2 r+1)^3 - (\kappa^2-2)^2(2\kappa^4-8\kappa^2+9\tanh^2 r-1)^2 \\ &\quad = 27(\tanh^2 r-1)^2(\kappa^4-4\kappa^2+4\tanh^2 r), \end{aligned}$$

we obtain that $|\tau_T(\kappa;r)|<1$ if and only if

$$\kappa^2<2\{1-(\cosh r)^{-1}\} \quad \text{or} \quad \kappa^2>2\{1+(\cosh r)^{-1}\}.$$

When $\tau_T(\kappa;r)=\pm 1$, the equation (13.4) has a double real solution and a single real solution. When $|\tau_T(\kappa;r)|>1$, it has only one real solution.

Under the above consideration on solutions of the characteristic equation (13.2), we study conditions for circular trajectories to be bounded.

Theorem 13.2. *On a tube $T(r)$ of radius r around totally geodesic $\mathbb{C}H^{n-1}(-4)$ in $\mathbb{C}H^n(-4)$ the behavior of a circular trajectory γ for $\boldsymbol{F}_\kappa$ is as follows;*

(1) *If κ satisfies $2\big\{1-(\cosh r)^{-1}\big\}<\kappa^2<2\big\{1+(\cosh r)^{-1}\big\}$, it is unbounded in both directions and has two distinct points at infinity.*

(2) *When $\kappa^2=2\big\{1\pm(\cosh r)^{-1}\big\}$, it is also unbounded in both directions but has a single point at infinity.*

(3) *If κ satisfies either $\tanh^2 r<\kappa^2<2\big\{1-(\cosh r)^{-1}\big\}$ or $\kappa^2>2\big\{1+(\cosh r)^{-1}\big\}$, then it is bounded.*

Proof. We first consider the case $\zeta(\kappa,r)\le 0$ for κ with $|\kappa|>\tanh r$. Such case occurs when $2-\sqrt{3}(\cosh r)^{-1}\le\kappa^2\le 2+\sqrt{3}(\cosh r)^{-1}$. In this case, as the characteristic equation (13.2) has one pure imaginary solution and two distinct solutions which are not pure imaginary, we find that γ is unbounded in both directions. More precisely, the solutions of (13.4) are of the form $-2\alpha_\kappa, \alpha_\kappa\pm\sqrt{-1}\beta_\kappa$ with real numbers $\alpha_\kappa,\beta_\kappa$ satisfying $3(3\alpha_\kappa^2-\beta_\kappa^2)=\zeta(\kappa;r)$, $\alpha_\kappa\neq 0$ and $\beta_\kappa\neq 0$. We hence find that a horizontal lift $\hat\gamma$ of the extrinsic shape of γ is of the form

$$\hat\gamma(t)=Ae^{\sqrt{-1}\{-2\alpha_\kappa+(\kappa+\kappa^{-1})/3\}t}+(Be^{\beta_\kappa t}+Ce^{-\beta_\kappa t})e^{\sqrt{-1}\{\alpha_\kappa+(\kappa+\kappa^{-1})/3\}t}$$

with $\mathbb{C}$-linearly independent $A,B,C\in\mathbb{C}^{n+1}$. Hence, rewriting this expression on the ball model $\boldsymbol{D}^n$ of a complex hyperbolic space, we obtain that γ has two distinct points at infinity, which are $\big(\frac{B_1}{B_0},\dots,\frac{B_n}{B_0}\big)$, $\big(\frac{C_1}{C_0},\dots,\frac{C_n}{C_0}\big)\in\overline{\boldsymbol{D}^n}$ if we consider in $\overline{\boldsymbol{D}^n}$.

We next study the case $\zeta(\kappa, r) > 0$ and $|\tau_T(\kappa; r)| < 1$ for κ with $|\kappa| > \tanh r$. Such a case occurs if $\tanh^2 r < \kappa^2 < 2\big\{1-(\cosh r)^{-1}\big\}$ or $\kappa^2 > 2\big\{1+(\cosh r)^{-1}\big\}$. In this case, the equation (13.2) has 3 distinct pure imaginary solutions. We hence find that γ is bounded.

We consider the case $\zeta(\kappa, r) > 0$ and $|\tau_T(\kappa; r)| > 1$ for κ with $|\kappa| > \tanh r$. Such a case occurs if one of the following holds:

i) $2\big\{1-(\cosh r)^{-1}\big\} < \kappa^2 < 2-\sqrt{3}(\cosh r)^{-1}$,

ii) $2+\sqrt{3}(\cosh r)^{-1} < \kappa^2 < 2\big\{1+(\cosh r)^{-1}\big\}$.

The solutions of (13.5) are of the form $-2\alpha_\kappa, \alpha_\kappa \pm \sqrt{-1}\beta_\kappa$ with real numbers $\alpha_\kappa, \beta_\kappa$ satisfying $2(3\alpha_\kappa^2 - \beta_\kappa^2) = 3$ and $2\sqrt{2}\alpha_\kappa(\alpha_\kappa^2+\beta_\kappa^2) = \tau_T(\kappa; r)$. In particular, we have $\beta_\kappa \neq 0$. Thus we find that a horizontal lift $\hat{\gamma}$ of the extrinsic shape of γ is of the form

$$\begin{aligned}\hat{\gamma}(t) = {} & Ae^{\sqrt{-1}\{-2\sqrt{2\zeta(\kappa;r)}\,\alpha_\kappa + (\kappa+\kappa^{-1})\}t/3} \\ & + \big(Be^{\sqrt{2\zeta(\kappa;r)}\,\beta_\kappa t/3} + Ce^{-\sqrt{2\zeta(\kappa;r)}\,\beta_\kappa t/3}\big)e^{\sqrt{-1}\{\sqrt{2\zeta(\kappa;r)}\,\alpha_\kappa + (\kappa+\kappa^{-1})\}t/3}\end{aligned}$$

with $\mathbb{C}$-linearly independent $A, B, C \in \mathbb{C}^{n+1}$. This shows that γ is unbounded and has two distinct points at infinity in this case.

We finally consider the case $\tau_T(\kappa; r) = \pm 1$. This case occurs when $\kappa^2 = 2\big\{1 \pm (\cosh r)^{-1}\big\}$. Here, the double signs for $\tau_T(\kappa; r)$ and for κ^2 are independent. Since the equation (13.5) has a double solution $\pm 1/\sqrt{2}$ and a simple solution $\mp\sqrt{2}$ in this case, where the double signs correspond to the double sign for $\tau_T(\kappa; r)$, we find that a horizontal lift $\hat{\gamma}$ of the extrinsic shape of γ is of the form

$$\hat{\gamma}(t) = Ae^{\sqrt{-1}\{\mp 2\sqrt{\zeta(\kappa;r)} + (\kappa+\kappa^{-1})\}t/3} + (B + Ct)e^{\sqrt{-1}\{\pm\sqrt{\zeta(\kappa;r)} + (\kappa+\kappa^{-1})\}t/3}$$

with $\mathbb{C}$-linearly independent $A, B, C \in \mathbb{C}^{n+1}$. Thus we find γ is unbounded and has a single point at infinity, which is expressed as $\big(\frac{C_1}{C_0}, \ldots, \frac{C_n}{C_0}\big) \in \overline{\boldsymbol{D}^n}$, in this case. We hence get the conclusion. $\square$

Since trajectories are defined by their initial velocity vectors, it is clear that unbounded trajectories are open. We hence study whether bounded circular trajectories are closed or not on tubes around totally geodesic $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n$. We shall show the following by just the same way as we studied circular trajectories on geodesic spheres in $\mathbb{C}H^n$.

Theorem 13.3. *Let γ be a bounded circular trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a tube $T(r)$ of radius r around $\mathbb{C}H^{n-1}(-4)$ in $\mathbb{C}H^n(-4)$.*

(1) *When $r \leq \log(\sqrt{2}+1)$ and $\kappa^2 = \{4+3\sqrt{2}(\cosh r)^{-1}\}/2$, it is closed of length $2\pi\sqrt{\cosh r\bigl(4\cosh r+3\sqrt{2}\bigr)}$.*

(2) *When $r > \log(\sqrt{2}+1)$ and $\kappa^2 = \{4\pm3\sqrt{2}(\cosh r)^{-1}\}/2$, it is closed of length $2\pi\sqrt{\cosh r\bigl(4\cosh r\pm3\sqrt{2}\bigr)}$, where double signs take the same signatures.*

(3) *If κ satisfies either $\tanh^2 r < \kappa^2 < 2\bigl\{1-(\cosh r)^{-1}\bigr\}$ or $\kappa^2 > 2\bigl\{1+(\cosh r)^{-1}\bigr\}$ and is not in the cases of (1) and (2), it is closed if and only if*

$$\frac{|\kappa^2-2|\,\bigl|2\kappa^4-8\kappa^2+9\tanh^2 r-1\bigr|}{2(\kappa^4-4\kappa^2+3\tanh^2 r+1)^{3/2}} = \frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}}$$

holds with some relatively prime positive integers p, q satisfying $p > q$. In this case its length is given as $\pi\delta(p,q)|\kappa|\sqrt{(3p^2+q^2)/(\kappa^4-4\kappa^2+3\tanh^2 r+1)}$, where $\delta(p,q) = 1$ when pq is odd and $\delta(p,q) = 2$ when pq is even.

Proof. By Theorem 13.2, we need to consider the case that three conditions $\zeta(\kappa; r) > 0$, $|\tau_T(\kappa; r)| < 1$ and $|\kappa| > \tanh r$ hold. We compare (13.5) with the characteristic equation (9.4) for circles on $\mathbb{C}P^n(4)$ of geodesic curvature $1/\sqrt{2}$ and complex torsion $\tau = \tau_T(\kappa; r)$.

First we consider the case $\tau_T(\kappa; r) = 0$. By the definition of $\tau_T(\kappa; r)$, we see this occurs if $\kappa^2 = 2$ or $\kappa^2 = \{4 \pm 3\sqrt{2}(\cosh r)^{-1}\}/2$. But as the conditions $\zeta(\kappa; r) > 0$ and $|\tau_T(\kappa; r)| < 1$ show that $\kappa^2 < 2\{1-(\cosh r)^{-1}\}$ or $\kappa^2 > 2\{1+(\cosh r)^{-1}\}$, we need not consider the case $\kappa^2 = 2$. We also have to consider the condition $\kappa^2 > \tanh^2 r$.

It is clear that both

$$\{4+3\sqrt{2}(\cosh r)^{-1}\}/2 > \tanh^2 r \quad \text{and} \quad \{4+3\sqrt{2}(\cosh r)^{-1}\}/2 > 2\{1+(\cosh r)^{-1}\}$$

hold. On the other hand, though $\{4-3\sqrt{2}(\cosh r)^{-1}\}/2 < 2\{1-(\cosh r)^{-1}\}$ clearly holds, by direct computation we find that $\{4-3\sqrt{2}(\cosh r)^{-1}\}/2 > \tanh^2 r$ holds if and only if $2\tanh^2 r > 1$, which is equivalent to $r > \log(\sqrt{2}+1)$. Thus we see that bounded circular trajectories satisfy $\tau_T(\kappa; r) = 0$ if and only if one of the following holds:

i) $\kappa^2 = \{4+3\sqrt{2}(\cosh r)^{-1}\}/2$,

ii) $r > \log(\sqrt{2}+1)$ and $\kappa^2 = \{4-3\sqrt{2}(\cosh r)^{-1}\}/2$ (> 0).

In this case we have $\zeta(\kappa; r) = 3/\{\cosh r(4\cosh r \pm 3\sqrt{2})\}$. We now transplant the properties of circles on $\mathbb{C}P^n(4)$ to circular trajectories. Since the solutions of (9.4) with $\tau = 0$ are $\pm\sqrt{6}/2, 0$, we find the solutions of (13.3) are

$$a_\kappa = -c_\kappa = \frac{1}{3}\big((\sqrt{6}/2)\sqrt{2\zeta(\kappa; r)} + \kappa + \kappa^{-1}\big),\ b_\kappa = 0.$$

Thus, we find that γ is closed and its length is

$$\text{length}(\gamma) = \frac{2\sqrt{6}\pi}{3} \times \frac{3}{\sqrt{2\zeta(\kappa; r)}} = 2\pi\sqrt{\cosh r(4\cosh r \pm 3\sqrt{2})}.$$

This shows the first and the second assertions.

We next consider the case $0 < |\tau_T(\kappa; r)| < 1$ under the assumption that $\zeta(\kappa; r) > 0$ and $\kappa^2 > \tanh^2 r$. Such a case occurs if and only if one of the following holds:

i) $\tanh^2 r < \kappa^2 < 2\big\{1-(\cosh r)^{-1}\big\}$ and $\kappa^2 \neq \{4-3\sqrt{2}(\cosh r)^{-1}\}/2$,

ii) $\kappa^2 > 2\big\{1+(\cosh r)^{-1}\big\}$ and $\kappa^2 \neq \{4+3\sqrt{2}(\cosh r)^{-1}\}/2$.

In this case, by use of the solutions $a_\tau, b_\tau.c_\tau$ of (9.4) with $\tau = \tau_T(\kappa; r)$, we see the solutions $a_\kappa, b_\kappa, c_\kappa$ of (13.3) are given as

$$a_\kappa = \big(a_\tau\sqrt{2\zeta(\kappa; r)}+\kappa+\kappa^{-1}\big)/3, \quad b_\kappa = \big(b_\tau\sqrt{2\zeta(\kappa; r)}+\kappa+\kappa^{-1}\big)/3,$$
$$c_\kappa = \big(c_\tau\sqrt{2\zeta(\kappa; r)}+\kappa+\kappa^{-1}\big)/3.$$

Hence, γ is closed if and only if $\tau_T(\kappa;r) = \pm q(9p^2-q^2)(3p^2+q^2)^{-3/2}$ with some relativly prime positive integers p, q satisfying $p > q$, and its length is

$$\begin{aligned}\text{length}(\gamma) &= \frac{1}{3}\pi\delta(p,q)\sqrt{2(3p^2+q^2)} \times \frac{3}{\sqrt{2\zeta(\kappa;r)}} \\ &= \pi\delta(p,q)|\kappa|\sqrt{\frac{3p^2+q^2}{\kappa^4-4\kappa^2+3\tanh^2 r+1}}.\end{aligned}$$

This complete the proof. □

We now study circular trajectories on tubes in a general complex hyperbolic space by use of a homothetic change of metrics.

Theorem 13.4. *Let γ be a circular trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a tube $T(r)$ of radius r around totally geodesic $\mathbb{C}H^{n-1}(c)$ in $\mathbb{C}H^n(c)$.*

(1) *If κ satisfies*

$$(|c|/2)\bigl\{1-\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\}<\kappa^2<(|c|/2)\bigl\{1+\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\},$$

it is unbounded in both directions and has two distinct points at infinity.

(2) *When $\kappa^2 = (|c|/2)\bigl\{1 \pm \bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\}$, it is also unbounded in both directions but has a single point at infinity.*

(3) *If κ satisfies one of the following conditions*

i) $(|c|/4)\tanh^2(\sqrt{|c|}\,r/2) < \kappa^2 < (|c|/2)\bigl\{1-\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\}$,

ii) $\kappa^2 > (|c|/2)\bigl\{1+\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\}$,

then it is bounded and satisfies the following.

1) *When*

$$r \le (2/\sqrt{|c|})\log(\sqrt{2}+1) \quad \textit{and} \quad \kappa^2 = |c|\{4+3\sqrt{2}\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\}/8,$$

it is closed of length

$$4\pi\sqrt{\cosh(\sqrt{|c|}\,r/2)\bigl(4\cosh(\sqrt{|c|}\,r/2)+3\sqrt{2}\bigr)/|c|}\,.$$

2) *When*

$$r > (2/\sqrt{|c|})\log(\sqrt{2}+1) \quad \textit{and} \quad \kappa^2 = |c|\{4\pm3\sqrt{2}\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\}/8,$$

it is closed of length

$$4\pi\sqrt{\cosh(\sqrt{|c|}\,r/2)\bigl(4\cosh(\sqrt{|c|}\,r/2)\pm 3\sqrt{2}\bigr)/c}\,,$$

where double signs take the same signatures.

3) *If* κ *satisfies one of the following conditions*

i) $(|c|/4)\tanh^2(\sqrt{c}\,r/2) < \kappa^2 < (|c|/2)\bigl\{1-\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\}$,

ii) $\kappa^2 > (|c|/2)\bigl\{1+\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\}$

and is not in the cases of 3−1) *and* 3−2)*, it is closed if and only if*

$$\frac{|2\kappa^2+c|\,\bigl|32\kappa^4+32c\kappa^2+c^2\bigl(9\tanh^2(\sqrt{|c|}\,r/2)-1\bigr)\bigr|}{\bigl\{16\kappa^4+16c\kappa^2+c^2\bigl(3\tanh^2(\sqrt{|c|}\,r/2)+1\bigr)\bigr\}^{3/2}} = \frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}}$$

holds with some relatively prime positive integers p, q *satisfying* $p > q$. *In this case its length is given as*

$$4\pi\delta(p,q)|\kappa|\sqrt{\frac{3p^2+q^2}{16\kappa^4+16c\kappa^2+c^2\bigl(3\tanh^2(\sqrt{|c|}\,r/2)+1\bigr)}},$$

where $\delta(p,q)=1$ *when* pq *is odd and* $\delta(p,q)=2$ *when* pq *is even.*

Proof. We consider a new metric on $\mathbb{C}H^n$ given by $\langle\ ,\ \rangle' = (|c|/4)\langle\ ,\ \rangle$. Then it has constant holomorphic sectional curvatures -4, and the radius of the tube around complex hypersurface turns to $r' = \sqrt{|c|}\,r/2$. If we define a smooth curve σ by $\sigma(s) = \gamma(2s/\sqrt{|c|})$, then it is a circular trajectory for a Sasakian magnetic field $\mathbf{F}'_{\kappa'}$ $(\kappa' = 2\kappa/\sqrt{|c|})$ with respect to the new metric.

First we consider whether γ is bounded or not. Since σ is unbounded with two distinct points at infinity in the case $2\bigl\{1-(\cosh r')^{-1}\bigr\}<\kappa'^2<2\bigl\{1+(\cosh r')^{-1}\bigr\}$, we find γ is unbounded and has distinct points at infinity if and only if

$$\frac{c}{2}\bigl\{1-\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\} < \kappa^2 < \frac{c}{2}\bigl\{1+\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\}.$$

Similarly we find γ is unbounded in both directions and has a single point at infinity if and only if $\kappa'^2 = 2\bigl\{1\pm(\cosh r')^{-1}\bigr\}$, which is quivalent to $\kappa^2 = (c/2)\bigl\{1\pm\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\}$.

Next we consider the case that γ is bounded. This occurs if and only if one of the following holds:

i) $\kappa^2 > (c/2)\bigl\{1+\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\}$,

ii) $(c/4)\tanh^2 r < \kappa^2 < (c/2)\bigl\{1-\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\}$.

When $r' \leq \log(\sqrt{2}+1)$ and $\kappa'^2 = \{4+3\sqrt{2}(\cosh r')^{-1}\}/2$, that is, when $r \leq (2/\sqrt{|c|})\log(\sqrt{2}+1)$ and $\kappa^2 = |c|\bigl\{4+3\sqrt{2}\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\}/8$, the trajectory σ is closed and $\mathrm{length}'(\sigma) = 2\pi\sqrt{\cosh r'\bigl(4\cosh r'+3\sqrt{2}\bigr)}$. We hence find that γ is closed and

$$\mathrm{length}(\gamma) = \frac{2}{\sqrt{|c|}}\mathrm{length}'(\sigma) = 4\pi\sqrt{\cosh(\sqrt{|c|}\,r/2)\bigl(4\cosh(\sqrt{|c|}\,r/2)+3\sqrt{2}\bigr)/|c|}$$

(see Table 5 in §8).

When $r' > \log(\sqrt{2}+1)$ and $\kappa'^2 = \{4\pm3\sqrt{2}(\cosh r')^{-1}\}/2$, that is, when $r > (2/\sqrt{|c|})\log(\sqrt{2}+1)$ and $\kappa^2 = |c|\bigl\{4\pm3\sqrt{2}\bigl(\cosh(\sqrt{|c|}\,r/2)\bigr)^{-1}\bigr\}/8$, the trajectory σ is closed and $\mathrm{length}'(\sigma) = 2\pi\sqrt{\cosh r'\bigl(4\cosh r'\pm3\sqrt{2}\bigr)}$. We hence find that γ is closed and

$$\mathrm{length}(\gamma) = 4\pi\sqrt{\cosh(\sqrt{|c|}\,r/2)\bigl(4\cosh(\sqrt{|c|}\,r/2)\pm3\sqrt{2}\bigr)|c|}\,.$$

In other case, the trajectory σ is closed if and only if

$$\frac{\bigl|\kappa'^2-2\bigr|\,|2\kappa'^4-8\kappa'^2+9\tanh^2 r'-1|}{2(\kappa'^4-4\kappa'^2+3\tanh^2 r'+1)^{3/2}} = \frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}}$$

holds with some relatively prime positive integers p, q satisfying $p > q$, and its length is given as

$$\mathrm{length}'(\sigma) = \pi\delta(p,q)|\kappa'|\sqrt{(3p^2+q^2)/(\kappa'^4-4\kappa'^2+3\tanh^2 r'+1)}.$$

Substituting $\kappa' = 2\kappa/\sqrt{|c|}$ and $r' = \sqrt{|c|}\,r/2$, we find that γ is closed if and only if

$$\begin{aligned}\frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}} &= \frac{\bigl|\frac{4\kappa^2}{|c|}-2\bigr|\,\bigl|\frac{32\kappa^4}{|c|^2}-\frac{32\kappa^2}{|c|}+9\tanh^2(\sqrt{|c|}\,r/2)-1\bigr|}{2\bigl(\frac{16\kappa^4}{|c|^2}-\frac{16\kappa^2}{|c|}+3\tanh^2(\sqrt{|c|}\,r/2)+1\bigr)^{3/2}}\\ &= \frac{\bigl|2\kappa^2-|c|\bigr|\,\bigl|\{32\kappa^4-32|c|\kappa^2+|c|^2(9\tanh^2(\sqrt{|c|}\,r/2)-1)\}}{\{16\kappa^4-16|c|\kappa^2+|c|^2(3\tanh^2(\sqrt{|c|}\,r/2)+1)\}^{3/2}},\end{aligned}$$

and its length is given by

$$
\begin{aligned}
\text{length}(\gamma) &= (2/\sqrt{|c|}) \times \text{length}'(\sigma) \\
&= \frac{2\pi}{\sqrt{|c|}}\delta(p,q)\frac{2|\kappa|}{\sqrt{|c|}}\sqrt{\frac{3p^2+q^2}{\frac{16\kappa^4}{|c|^2}-\frac{16\kappa^2}{|c|}+3\tanh^2(\sqrt{|c|}\,r/2)+1}} \\
&= 4\pi\delta(p,q)|\kappa|\sqrt{\frac{3p^2+q^2}{16\kappa^4-16|c|\kappa^2+|c|^2(3\tanh^2(\sqrt{|c|}\,r/2)+1)}}.
\end{aligned}
$$

This complete the proof. □

14. **Circular trajectories on real hypersurfaces of type** (A_2) **in** $\mathbb{C}H^n$

In this section we study circular trajectories on a tube $T_\ell(r)$ of radius r around totally geodesic $\mathbb{C}H^\ell$ $(1 \le \ell \le n-2)$ in a complex hyperbolic space $\mathbb{C}H^n(c)$. A tube $T_\ell(r)$ in $\mathbb{C}H^n(c)$ also have three principal curvatures

$$\lambda = (\sqrt{|c|}/2)\coth\bigl(\sqrt{|c|}\,r/2\bigr),\ \mu = (\sqrt{|c|}/2)\tanh\bigl(\sqrt{|c|}\,r/2\bigr),\ \nu = \sqrt{|c|}\coth\bigl(\sqrt{|c|}\,r\bigr).$$

Its characteristic vector field ξ satisfies $A\xi = \nu\xi$ and λ, μ are principal curvatures for vectors orthogonal to ξ. On this real hypersurface, the shape operator and the characteristic tensor also satisfy $A\phi = \phi A$.

Proposition 14.1. *Let γ be a trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a hypersurface of type (A_2) in a nonflat complex hyperbolic space $\mathbb{C}H^n(c)$.*

(1) *It is a geodesic if and only if $\rho_\gamma = \pm 1$.*

(2) *It is a circle of positive geodesic curvature if and only if one of the following condition holds*:

i) $\tau_\gamma = 0$ *and* $\kappa\rho_\gamma = (\sqrt{|c|}/2)\tanh\bigl(\sqrt{|c|}\,r/2\bigr)$,

ii) $\tau_\gamma = \sqrt{1-\rho_\gamma^2}$ *and* $\kappa\rho_\gamma = (\sqrt{|c|}/2)\coth\bigl(\sqrt{|c|}\,r/2\bigr)$.

In these cases, the geodesic curvature of γ is $|\kappa|\sqrt{1-\rho_\gamma^2}$.

Proof. Just like the proof of Proposition 7.4, we consider the condition (2)-ii) in Lemma 6.3 by decomposing $\dot\gamma$ into principal curvature vectors. We denote by $T^0M = V_\lambda \oplus V_\mu$ the decomposition into subbundles of principal curvature vectors. We then have γ is circular if and only if the following holds:

$$\rho_\gamma(\lambda - \kappa\rho_\gamma)\,\mathrm{Proj}_{V_\lambda}(\dot\gamma) + \rho_\gamma(\mu - \kappa\rho_\gamma)\,\mathrm{Proj}_{V_\mu}(\dot\gamma) + \{\kappa\rho_\gamma - \kappa\rho_\gamma^3 - \lambda\tau_\gamma^2 - \mu(1-\rho_\gamma^2-\tau_\gamma^2)\}\xi = 0,$$

where $\mathrm{Proj}_{V_\lambda} : TM \to V_\lambda$ and $\mathrm{Proj}_{V_\mu} : TM \to V_\mu$ denote the projections. As was shown in the proof of Proposition 7.4, we see this holds if and only if one of the following conditions holds

i) $\tau_\gamma = 0$ and $\kappa\rho_\gamma = \mu$,

ii) $\tau_\gamma = \sqrt{1-\rho_\gamma^2}$ and $\kappa\rho_\gamma = \lambda$,

iii) $\rho_\gamma = 0$ and $\lambda\tau_\gamma^2 + \mu(1-\tau_\gamma^2) = 0$.

Being different from the case of tubes in $\mathbb{C}P^n$, the third case does not occur. As a matter of fact, by substituting principal curvatures we have

$$\lambda\tau_\gamma^2 + \mu(1-\tau_\gamma^2) = \frac{\sqrt{|c|}\,\tau_\gamma^2}{\sinh\bigl(\sqrt{|c|}\,r\bigr)} + \frac{\sqrt{|c|}}{2}\tanh\frac{\sqrt{|c|}\,r}{2} > 0.$$

We hence get the conclusion. □

Since $|\rho_\gamma| < 1$ and $\coth\bigl(\sqrt{|c|}\,r/2\bigr) > \tanh\bigl(\sqrt{|c|}\,r/2\bigr)$, we have the following.

Theorem 14.1. *We consider a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a tube $T_\ell(r)$ of radius r around $\mathbb{C}H^\ell(c)$ $(1 \le \ell \le n-2)$ in $\mathbb{C}H^n(c)$.*

(1) *When $0 < |\kappa| \le \bigl(\sqrt{|c|}/2\bigr)\tanh\bigl(\sqrt{|c|}r/2\bigr)$, there are no circular trajectories for $\boldsymbol{F}_\kappa$.*

(2) *When $\bigl(\sqrt{|c|}/2\bigr)\tanh\bigl(\sqrt{|c|}\,r/2\bigr) < |\kappa| \le \bigl(\sqrt{|c|}/2\bigr)\coth\bigl(\sqrt{|c|}\,r/2\bigr)$, a trajectory γ for $\boldsymbol{F}_\kappa$ is circular if and only if it satisfies the condition*

$\rho_\gamma = \bigl(\sqrt{|c|}/(2\kappa)\bigr)\tanh(\sqrt{|c|}r/2\bigr)$ *and* $\tau_\gamma = 0$.

(3) *When $|\kappa| > \bigl(\sqrt{|c|}/2\bigr)\coth(\sqrt{|c|}r/2\bigr)$, a $\boldsymbol{F}_\kappa$-trajectory γ is circular if and only if it satisfies one of the following*:

i) $\rho_\gamma = \bigl(\sqrt{|c|}/2\kappa\bigr)\tanh\bigl(\sqrt{|c|}\,r/2\bigr)$ *and* $\tau_\gamma = 0$,

ii) $\rho_\gamma = \bigl(\sqrt{|c|}/2\kappa\bigr)\coth\bigl(\sqrt{|c|}\,r/2\bigr)$ *and* $\tau_\gamma = \sqrt{1-\rho_\gamma^2}$.

Before we close this section we here make mention of congruence theorem on trajectories on hypersurfaces of type (A_2) in $\mathbb{C}H^n$.

Proposition 14.2. *We consider a hypersurface $T_\ell(r)$ of type (A_2) in a complex hyperbolic space $\mathbb{C}H^n(c)$. Trajectories γ_1 for a Sasakian magnetic field $\boldsymbol{F}_{\kappa_1}$ and γ_2 for $\boldsymbol{F}_{\kappa_2}$ on $T_\ell(r)$ are congruent to each other in strong sense if and only if one of the following conditions holds*:

i) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| = 1$,

ii) $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$, $\tau_{\gamma_1} = \tau_{\gamma_2}$ *and* $|\kappa_1| = |\kappa_2|$,

iii) $0 < |\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$, $\tau_{\gamma_1} = \tau_{\gamma_2}$ *and* $\kappa_1\rho_{\gamma_1} = \kappa_2\rho_{\gamma_2}$.

We can show this proposition by just the same way as of Proposition 7.3. We decompose the tangent space $T_xT_\ell(r)$ of a real hypersurface $T_\ell(r)$ of type (A_2) at x as $T_xT_\ell(r) = V_{\lambda,x} \oplus V_{\mu,x} \oplus \mathbb{R}\xi_x$, where $V_{\lambda,x}$ and $V_{\mu,x}$ are the subspaces of principal curvature vectors orthogonal to ξ_x which correspond to principal curvatures λ and μ, respectively. Through an isometric immersion $\iota : T_\ell(r) \to \mathbb{C}H^n(c)$ we consider $TT_\ell(r)$ as a subset of $T\mathbb{C}H^n(c)$. We can prove the following Lemma by just the same way as of Lemma 7.3.

Lemma 14.1. *Let $x, x' \in T_\ell(r)$ be arbitrary points on a hypersurface $T_\ell(r)$ of type (A_2) in $\mathbb{C}H^n(c)$. Given unit tangent vectors $u \in V_{\lambda,x}$, $w \in V_{\mu,x}$ and $u' \in V_{\lambda,x'}$, $w' \in V_{\mu,x'}$, there exist isometries $\tilde{\varphi}^+$, $\tilde{\varphi}^-$ of $\mathbb{C}H^n(c)$ satisfying the following conditions*:

i) $\tilde{\varphi}^+\big(T_\ell(r)\big) = \tilde{\varphi}^-\big(T_\ell(r)\big) = T_\ell(r)$,
   (*i.e.* $T_\ell(r)$ *is invariant under the actions of* $\tilde{\varphi}^+$ *and* $\tilde{\varphi}^-$);

ii) $\tilde{\varphi}^+(x) = \tilde{\varphi}^-(x) = x'$;

iii) $d\tilde{\varphi}^+(u) = d\tilde{\varphi}^-(u) = u'$ *and* $d\tilde{\varphi}^+(w) = d\tilde{\varphi}^-(w) = w'$

iv) $d\tilde{\varphi}^+ \circ J = J \circ d\tilde{\varphi}^+$ *and* $d\tilde{\varphi}^- \circ J = -J \circ d\tilde{\varphi}^-$,
   in particular, $d\tilde{\varphi}^+(\xi_x) = \xi_{x'}$ *and* $d\tilde{\varphi}^-(\xi_x) = -\xi_{x'}$.

Proof. For the sake of simplicity, we are enough to consider the case $n = 3$, $\ell = 1$ and $c = -4$. As we see in §5.3 we may consider that

$$\begin{aligned}\varpi^{-1}\big(T_1(r)\big) &= \left\{ (z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \;\middle|\; \begin{array}{c} -|z_0|^2 + |z_1|^2 = -\cosh^2 r, \\ |z_2|^2 + |z_3|^2 = \sinh^2 r \end{array} \right\} \\ &= H_1^3 \times S^3 \subset \mathbb{C}^2 \times \mathbb{C}^2.\end{aligned}$$

We take an arbitrary point $\hat z = (z_0, z_1, z_2, z_3) \in \varpi^{-1}\big(T_1(r)\big)$. The tangent space $\widehat{M} = \varpi^{-1}\big(T_1(r)\big)$ at $\hat z$ is represented as

$$T_{\hat z}\widehat{M} = \left\{ (\hat z, \hat v) \in \{\hat z\} \times \mathbb{C}^4 \;\middle|\; \begin{array}{c} \mathrm{Re}(-z_0\overline{v}_0 + z_1\overline{v}_1) = 0, \\ \mathrm{Re}(z_2\overline{v}_2 + z_3\overline{v}_3) = 0 \end{array} \right\}.$$

We denote by $\widehat{\mathcal{N}}_{\hat z} \in T_{\hat z}H_1^7$ the horizontal lift of the unit normal vector $\mathcal{N}_{\varpi(\hat z)}$ of $T_1(r)$ in $\mathbb{C}H^3(-4)$. Since $\widehat{\mathcal{N}}_{\hat z} \in T_{\hat z}H_1^7$ and is orthogonal to $T_{\hat z}\widehat{M}$, considering on each component, we find that it is represented as

$$\widehat{\mathcal{N}}_{\hat z} = \big(\hat z, (-\tanh r\, z_0, -\tanh r\, z_1, -\coth r\, z_2, -\coth r\, z_3)\big).$$

By putting $\hat\xi_{\hat z} = -J\widehat{\mathcal{N}}_{\hat z}$, we denote by $\langle \hat\xi_{\hat z}\rangle$ the real linear subspace of $T_{\hat z}H_1^7$ spanned by $\hat\xi_{\hat z}$, and by $\langle \hat\xi_{\hat z}\rangle^\perp$ its orthogonal complement in $T_{\hat z}H_1^7$. The horizontal part $\langle \hat\xi_{\hat z}\rangle^\perp \cap \mathcal{H}_{\hat z}$ of $\langle \hat\xi_{\hat z}\rangle^\perp$ corresponds to the complex vector space $\langle \xi_{\varpi(\hat z)}\rangle^\perp$, and is represented as

$$\langle \hat\xi_{\hat z}\rangle^\perp \cap \mathcal{H}_{\hat z} = \big\{ (\hat z, \hat v) \in \{\hat z\} \times \mathbb{C}^3 \;\big|\; -z_0\overline{v}_0 + z_1\overline{v}_1 = 0,\ z_2\overline{v}_2 + z_3\overline{v}_3 = 0 \big\}.$$

We should note that if we decompose $\mathcal{H}_{\hat z}$ as $\mathcal{H}_{\hat z} = \widehat{V}_{\lambda,\hat z} \oplus \widehat{V}_{\mu,\hat z} \oplus \mathbb{R}\hat\xi_{\hat z}$ corresponding to the decomposition of $T_{\varpi(\hat z)}T_1(r)$ into subspaces of principal curvature vectors then we see

$$\widehat{V}_{\lambda,\hat z} = \big\{ (\hat z, \hat v) \in \{\hat z\} \times \mathbb{C}^3 \;\big|\; z_2\overline{v}_2 + z_3\overline{v}_3 = 0,\ v_0 = v_1 = 0 \big\},$$
$$\widehat{V}_{\mu,\hat z} = \big\{ (\hat z, \hat v) \in \{\hat z\} \times \mathbb{C}^3 \;\big|\; -z_0\overline{v}_0 + z_1\overline{v}_1 = 0,\ v_2 = v_3 = 0 \big\}.$$

We take a point $\hat z_* = (\cosh r, 0, \sinh r, 0) \in \widehat{M}$ and unit tangent vectors $\hat u_* = \big(\hat z_*, (0,0,0,1)\big) \in \widehat{V}_{\lambda,\hat z_*}$, $\hat w_* = \big(\hat z, (0,1,0,0)\big) \in \widehat{V}_{\mu,\hat z_*}$. For an arbitrary $\hat z \in \widehat{M}$ and unit tangent vectors $\hat u = \big(\hat z, (0,0,u_2,u_3)\big) \in \widehat{V}_{\lambda,\hat z}$, $\hat w = \big(\hat z, (w_0,w_1,0,0)\big) \in \widehat{V}_{\mu,\hat z}$, which are expressed as

$$\hat u = \left(\hat z, \left(0, 0, \frac{\zeta_1\overline{z}_3}{\cosh r}, \frac{\zeta_1\overline{z}_2}{\cosh r}\right)\right),\ \hat w = \left(\hat z, \left(\frac{\zeta_2\overline{z}_1}{\sinh r}, -\frac{\zeta_2\overline{z}_0}{\sinh r}, 0, 0\right)\right)$$

with some $\zeta_1, \zeta_2 \in \mathbb{C}$ satisfying $|\zeta_1| = |\zeta_2| = 1$, we define a unitary matrix

$$U_+ = \begin{pmatrix} z_0/\cosh r & w_0 & 0 & 0 \\ z_1/\cosh r & w_1 & 0 & 0 \\ 0 & 0 & z_2/\sinh r & u_2 \\ 0 & 0 & z_3/\sinh r & u_3 \end{pmatrix} \in U(1,1) \oplus U(2) \subset U(3,1).$$

This induces a linear transformation of $\mathbb{C}_1^4$ which preserves the Hermitian product $\langle\!\langle\ ,\ \rangle\!\rangle$, hence it induces an isometry $\hat{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}$ of H_1^7. Clearly, it satisfies $\hat{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}\big(\widehat{M}\big) = \widehat{M}$ and $\hat{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}\big(e^{\sqrt{-1}\theta}\hat{p}\big) = e^{\sqrt{-1}\theta}\hat{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}(\hat{p})$ for arbitrary $\theta \in [0, 2\pi)$ and $\hat{p} \in \widehat{M}$. Therefore $\hat{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}$ induces an isometry $\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}$ of $\mathbb{C}H^3(-4)$ satisfying

$$\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})} \circ \varpi = \varpi \circ \hat{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}, \quad \tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}\big(\varpi(\hat{z}_*)\big) = \varpi(\hat{z}),$$
$$d\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}\big(d\varpi(\hat{u}_*)\big) = d\varpi(\hat{u}), \quad d\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}\big(d\varpi(\hat{w}_*)\big) = d\varpi(\hat{w}).$$

Since we have $U_+J = JU_+$ for the matrix $J = \sqrt{-1}E$, we find that $\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}$ is holomorphic, that is, $d\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}J = Jd\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}$. In particular, it satisfies $d\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})}\big(\xi_{\varpi(\hat{z}_*)}\big) = \xi_{\varpi(\hat{z})}$.

We next consider a matrix

$$\Psi = \begin{pmatrix} \epsilon & O & O & O \\ O & \epsilon & O & O \\ O & O & \epsilon & O \\ O & O & O & \epsilon \end{pmatrix} \in O(8) \quad \text{with} \quad \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(2).$$

This matrix induces a map $\mathbb{C}_1^4 \ni (p_0, p_1, p_2, p_3) \mapsto (\bar{p}_0, \bar{p}_1, \bar{p}_2, \bar{p}_3) \in \mathbb{C}_1^4$. If we define a matrix U_- by $U_- = U_+\Psi$, it induces a linear transformation of $\mathbb{C}_1^4$ which preserves the Hermitian product. By the representation of $\widehat{M}$, we see it induces an isometry $\hat{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}$ of H_1^7 satisfying $\hat{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}\big(\widehat{M}\big) = \widehat{M}$. It is clear that it satisfies $\hat{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}\big(e^{\sqrt{-1}\theta}\hat{p}\big) = e^{-\sqrt{-1}\theta}\hat{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}(\hat{p})$ for arbitrary $\theta \in [0, 2\pi)$ and $\hat{p} \in \widehat{M}$. As we have $U_-J = JU_-$ for the matrix $J = \sqrt{-1}E$, we find that $\hat{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}$ induces an isometry $\tilde{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}$ of $\mathbb{C}H^3(-4)$ satisfying

$$\tilde{\varphi}^-_{(\hat{z},\hat{u},\hat{w})} \circ \varpi = \varpi \circ \hat{\varphi}^-_{(z,u,w)}, \quad d\tilde{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}J = -Jd\tilde{\varphi}^+_{(\hat{z},\hat{u},\hat{w})},$$
$$\tilde{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}(\varpi(\hat{z}_*)) = \varpi(\hat{z}),$$
$$d\tilde{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}(d\varpi(\hat{u}_*)) = d\varpi(\hat{u}), \quad d\tilde{\varphi}^-_{((\hat{z},\hat{u},\hat{w}))}(d\varpi(\hat{w}_*)) = d\varpi(\hat{w}).$$

In particular, it satisfies $d\tilde{\varphi}^-_{(\hat{z},\hat{u},\hat{w})}\big(\xi_{\varpi(\hat{z}_*)}\big) = -\xi_{\varpi(\hat{z})}$.

As we constructed desirable isometries for a fixed triplet $\big(\varpi(\hat{z}_*), d\varpi(\hat{u}_*), d\varpi(\hat{w}_*)\big)$ and an arbitrary triplet $\big(\varpi(\hat{z}), d\varpi(\hat{u}), d\varpi(\hat{w})\big)$, we can get our conclusion along the same lines as in Lemma 7.1. □

Remark 14.1. Every isometry of $T_\ell(r)$ in $\mathbb{C}H^n(c)$ is equivariant.

Corollary 14.1. *Every circular trajectory on a real hypersurface $T_\ell(r)$ in $\mathbb{C}H^n(c)$ is Killing.*

By Proposition 14.1 we have the following.

Corollary 14.2. *For a given Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a real hypersurface $T_\ell(r)$ of type (A_2) in $\mathbb{C}H^n(c)$, we have at most two congruence classes of circular trajectories in strong sense.*

If we take into account Theorem 14.1, we can refine the above by considering strengths of Sasakian magnetic fields.

Corollary 14.3. *Geodesic trajectories for non-trivial Sasakian magnetic fields on a real hypersurface $T_\ell(r)$ of type (A_2) in $\mathbb{C}H^n(c)$ are congruent to each other in strong sense.*

For about extrinsic shapes of trajectories for Sasakian magnetic fields on a real hypersurface of type (A_2) in $\mathbb{C}H^n(c)$, we find that they are Killing helices of order at most 6 by Corollary 13.1 and by the fact that every isometry of a real hypersurface of type (A_2) is equivariant. But for more detail we will discuss in the forthcoming paper. For geodesics on this hypersurface we have a corresponding result in [4].

15. **Trajectories on geodesic spheres in a complex Euclidean space**

In ths section we study trajectories on some real hypersurfaces in a complex Euclidean space $\mathbb{C}^n$. In $\mathbb{C}^n$, typical examples of real hypersurfaces are a standard sphere $S^{2n-1}(c)$ of radius $1/\sqrt{c}$ and a Euclidean space $\mathbb{R}^{2n-1}$. Since $\mathbb{R}^{2n-1}$ is a horosphere in a Hadamard manifold $\mathbb{C}^n$, it is interesting to compare properties of trajectories on these hypersurfaces and those on standard real hypersurfaces in $\mathbb{C}H^n(c)$.

15.1. **Trajectories on Euclidean hypersurfaces.** First we consider a condition for trajectories on Euclidean hyperplanes to be circles. Without loss of generality, we can represent a Euclidean hyperplane $\mathbb{R}^{2n-1}$ in $\mathbb{C}^n$ by $\{(x_1+\sqrt{-1}y_1,\dots,x_{n-1}+\sqrt{-1}y_{n-1},x_n)\}$. In this case, we have $\mathcal{N}=(0,\dots,0,\sqrt{-1})$, hence find that $\xi=(0,\dots,0,1)$ is a parallel vector field on $\mathbb{R}^{2n-1}$.

Let γ be a trajectory for a Sasakian magnetic field $\boldsymbol{F}_\kappa$ on $\mathbb{R}^{2n-1}$. Its structure torsion ρ_γ is constant along γ because

$$\rho'_\gamma=\dot\gamma\langle\dot\gamma,\xi\rangle=\langle\kappa\phi\dot\gamma,\xi\rangle=0.$$

In order to study the behavior of γ, we denote as $\gamma=(\gamma_\sharp,\gamma_*)$, where $\gamma_*:\mathbb{R}\to\mathbb{R}$ is the last component and $\gamma_\sharp:\mathbb{R}\to\mathbb{R}^{2n-2}\cong\mathbb{C}^{n-1}$. Then the equation $\nabla_{\dot\gamma}\dot\gamma=\kappa\phi\dot\gamma$ (i.e. $\gamma''=\kappa\phi\gamma'$) is equivalent to the system of equations

$$\begin{cases}\gamma''_\sharp=\kappa J\gamma'_\sharp,\\ \gamma''_*=0.\end{cases}$$

Solving this system of differential equations, we find

$$\gamma(t)=\gamma(0)+\Big(\frac{1}{\kappa}\gamma'_\sharp(0)\{\sin\kappa t+\sqrt{-1}(1-\cos\kappa t)\},\rho_\gamma t\Big).$$

Thus we obtain the following.

Proposition 15.1. *On a Euclidean real hypersurface $\mathbb{R}^{2n-1}$ in a complex Euclidean space $\mathbb{C}^n$, a trajectory γ for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ is as follows.*

(1) *It is a geodesic if and only if $\rho_\gamma=\pm1$.*

(2) *It is circular if and only if its structure torsion satisfies* $\rho_\gamma = 0$. *In this case, its geodesic curvature is* $|\kappa|$ *and it is closed of length* $2\pi/|\kappa|$.

(3) *Otherwise, it is a helix of proper order* 3 *whose geodesic curvatures are* $k_1 = |\kappa|\sqrt{1-{\rho_\gamma}^2}$ *and* $k_2 = |\kappa\rho_\gamma|$. *In particular it is unbounded in both directions.*

Proof. We shall write down the Frenet formula for γ by the expression of components. As its velocity vector is expressed as $\gamma' = (\gamma'_\sharp, \gamma'_*)$, its structure torsion is given by

$$\rho_\gamma = \big\langle (\gamma'_\sharp, \gamma'_*), (0,1)\big\rangle = \gamma'_*.$$

When $\rho_\gamma \neq 0, \pm 1$, by direct computation we have

$$\gamma'' = (\kappa J\gamma'_\sharp, 0) = |\kappa|\sqrt{1-\rho_\gamma^2}\Big(\frac{\mathrm{sgn}(\kappa)}{\sqrt{1-\rho_\gamma^2}}J\gamma'_\sharp, 0\Big),$$

$$\begin{aligned}\frac{\mathrm{sgn}(\kappa)}{\sqrt{1-\rho_\gamma^2}}\big(J\gamma'_\sharp, 0\big)' &= \frac{\mathrm{sgn}(\kappa)}{\sqrt{1-\rho_\gamma^2}}\big(-\kappa\gamma'_\sharp, 0\big) \\ &= -|\kappa|\sqrt{1-\rho_\gamma^2}\gamma' + \frac{|\kappa|}{\sqrt{1-\rho_\gamma^2}}\big(-\rho_\gamma^2\gamma'_\sharp, (1-\rho_\gamma^2)\gamma'_*\big) \\ &= -|\kappa|\sqrt{1-\rho_\gamma^2}\gamma' + |\kappa\rho_\gamma|\Big(-\frac{|\rho_\gamma|}{\sqrt{1-\rho_\gamma^2}}\gamma'_\sharp, \frac{\sqrt{1-\rho_\gamma^2}}{|\rho_\gamma|}\gamma'_*\Big),\end{aligned}$$

$$\Big(-\frac{|\rho_\gamma|}{\sqrt{1-\rho_\gamma^2}}\gamma'_\sharp, \frac{\sqrt{1-\rho_\gamma^2}}{|\rho_\gamma|}\gamma'_*\Big)' = -\frac{\kappa|\rho_\gamma|}{\sqrt{1-\rho_\gamma^2}}\big(J\gamma'_\sharp, 0\big) = -|\kappa\rho_\gamma|\Big(\frac{\mathrm{sgn}(\kappa)}{\sqrt{1-\rho_\gamma^2}}J\gamma'_\sharp, 0\Big).$$

Hence it is a helix of proper order 3. When $\rho_\gamma = \pm 1$, as $\gamma = (0, \gamma_*)$, we see $\gamma'' = 0$, hence is a geodesic. When $\rho_\gamma = 0$, as we have $\gamma_* \equiv 0$ and $\gamma = (\gamma_\sharp, 0)$, we see $\gamma''' = -\kappa^2\gamma'$, hence is a circle of geodesic curvature $|\kappa|$.

On a Euclidean space $\mathbb{R}^{2n-1}$, it is known that every circle of positive geodesic curvature k is closed of length $2\pi/k$ and that all helix of proper order 3, which are also called ordinary helices, are unbounded homogeneous curve. We hence get the conclusion. □

We note that as a hyperplane $\mathbb{R}^{2n-1}$ is totally geodesic in $\mathbb{C}^n$ the extrinsic shapes of trajectories on $\mathbb{R}^{2n-1}$ in $\mathbb{C}^n$ is the same as in Proposition 15.1.

15.2. **Trajectories on standard spheres.** We next consider hyperspheres in $\mathbb{C}^n$. A standard sphere $S^{2n-1}(c)$ in $\mathbb{C}^n$ of constant sectional curvature c, which is also called a hypersphere and is expressed as a sphere of radius $1/\sqrt{c}$;

$$S^{2n-1}(c) = \big\{(z_1, \dots, z_n) \in \mathbb{C}^n \bigm| |z_1|^2 + \cdots |z_n|^2 = 1/c\big\}.$$

We take its inward unit normal $\mathcal{N}$. That is, at a point $z \in S^{2n-1}(c)$ the vector is given as $\mathcal{N}_z = -\sqrt{c}z$. We denote by ∇ and $\overline{\nabla}$ the Riemannian connections on $S^{2n-1}(c)$ and $\mathbb{C}^n$, respectively. We then have

$$\text{(15.1)}\quad \nabla_X Y = \overline{\nabla}_X Y - \langle \overline{\nabla}_X Y, \mathcal{N}\rangle \mathcal{N} = \overline{\nabla}_X Y + \langle Y, \overline{\nabla}_X \mathcal{N}\rangle \mathcal{N} = \overline{\nabla}_X Y - \sqrt{c}\langle X, Y\rangle \mathcal{N}$$

for all vector fields $X, Y \in \mathcal{X}\big(S^{2n-1}(c)\big)$. Therefore we have $\langle AX, Y\rangle = \sqrt{c}\langle X, Y\rangle$, hence have $Av = \sqrt{c}v$ for all $v \in TS^{2n-1}(c)$ and find that it is totally umbilic. In particular, we see the characteristic tensor satisfies $A\phi = \phi A$, and find that the structure torsion of each trajectory γ for a Sasakian magnetic field $\boldsymbol{F}_\kappa$ on this standard sphere is constant along γ. For properties of trajectories on spheres in the sense of Frenet-Serre formula we have the following.

Proposition 15.2. *Let $\boldsymbol{F}_\kappa$ be a non-trivial Sasakian magnetic field on a standard sphere $S^{2n-1}(c)$ in $\mathbb{C}^n$.*

(1) *A trajectory γ is a geodesic if and only if $\rho_\gamma = \pm 1$.*
(2) *When $|\kappa| \leq \sqrt{c}$, there are no circular trajectories for $\boldsymbol{F}_\kappa$.*
(3) *When $|\kappa| > \sqrt{c}$, a trajectory γ for $\boldsymbol{F}_\kappa$ is circular if and only if $\rho_\gamma = \sqrt{c}/\kappa$. In this case its geodesic curvature is $\sqrt{\kappa^2 - c}$, hence it is closed of length $2\pi/|\kappa|$.*
(4) *If a trajectory γ for $\boldsymbol{F}_\kappa$ is not a circle, that is $\rho_\gamma \neq \pm 1, \sqrt{c}/\kappa$, then it is a helix of proper order 3 whose geodesic curvatures are $|\kappa|\sqrt{1-\rho_\gamma^2}$ and $|\kappa\rho_\gamma - \sqrt{c}|$.*

Proof. We do the same calculation as in Lemma 6.3 and Proposition 7.1. When $\rho_\gamma = \pm 1$, as we have $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma = 0$, it is a geodesic. When $\rho_\gamma \neq \pm 1$, we set

$Y_2 = \big(\mathrm{sgn}(\kappa)/\sqrt{1-\rho_\gamma^2}\big)\phi\dot\gamma$. Since $A\dot\gamma = \sqrt{c}\dot\gamma$, we have

$$\begin{aligned}\nabla_{\dot\gamma}(\phi\dot\gamma) &= \rho_\gamma A\dot\gamma - \langle A\dot\gamma, \dot\gamma\rangle\xi - \kappa\big(\dot\gamma - \rho_\gamma\xi\big)\\ &= -\kappa(1-\rho_\gamma^2)\dot\gamma + (\kappa\rho_\gamma - \sqrt{c})(\xi - \rho_\gamma\dot\gamma).\end{aligned}$$

Thus, we find γ is a circle if and only if $\kappa\rho_\gamma - \sqrt{c} = 0$. Since $|\rho_\gamma| < 1$ in this case, this equality does not hold when $|\kappa| \le \sqrt{c}$. When $\kappa\rho_\gamma \ne \sqrt{c}$, we have

$$\nabla_{\dot\gamma}(\xi - \rho_\gamma\dot\gamma) = \phi A\dot\gamma - \rho_\gamma\kappa\phi\dot\gamma = (\sqrt{c} - \kappa\rho_\gamma)\phi\dot\gamma.$$

Therefore we find it is a helix of proper order 3 in this case.

On a standard sphere $S^{2n-1}(c)$, it is known that every circle of geodesic curvature k is closed of length $2\pi/\sqrt{k^2+c}$. We hence get the conclusion. □

We here compare Propositions 15.1, 15.2 and Theorems 9.1, 12.3, 13.4 on circular trajectories on geodesic spheres and tubes around totally geodesic hyperplanes, which are real hypersurfaces of type (A_1), in a nonflat $\mathbb{C}M^n(c)$. On hyperplanes and hyperspheres in $\mathbb{C}^n$ every circular trajectory is closed. On the contrary, on real hypersurfaces of type (A_1) in a nonflat $\mathbb{C}M^n(c)$, there are infinitely many open circular trajectories and infinitly many closed circular trajectories. This is quite different feature of circular trajectories.

We also point out that lengths of circular trajectories on standard spheres do not depend on sectional curvatures of spheres. In order to make clear this point, we study the extrinsic shapes of circular trajectories.

Proposition 15.3. *Let γ be a circular trajectory for $\boldsymbol{F}_\kappa$ on $S^{2n-1}(c)$. Then its extrinsic shape in $\mathbb{C}^n$ is a circle of geodesic curvature $|\kappa|$.*

Proof. We take the inword unit normal $\mathcal{N}$ on S^{2n-1} in $\mathbb{C}^n$. Since $\overline{\nabla}_X Y = \nabla_X Y + \sqrt{c}\langle X, Y\rangle\mathcal{N}$ by (15.1), we find the following:

$$\begin{cases}\overline{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + \sqrt{c}\mathcal{N} = \kappa\phi\dot\gamma + \sqrt{c}\mathcal{N},\\ \overline{\nabla}_{\dot\gamma}\big(\kappa\phi\dot\gamma + \sqrt{c}\mathcal{N}\big) = -\kappa^2(1-\rho_\gamma^2)\dot\gamma + \sqrt{c}A\dot\gamma = -\{\kappa^2(1-\rho_\gamma^2) + c\}\dot\gamma.\end{cases}$$

As we have

$$\|\kappa\phi\dot{\gamma} + \sqrt{c}\mathcal{N}\|^2 = \sqrt{\kappa^2(1-\rho_\gamma^2)+c} = \kappa^2$$

by the circular condition $\kappa\rho_\gamma = \sqrt{c}$, we get the conclusion. $\square$

16. Trajectories which are also curves of order 2 on real hypersurfaces of type (B) in $\mathbb{C}H^n$

In previous sections we studied circular trajectories on real hypersurfaces of type (A). We are hence interested in investigating whether there exist circular trajectories on homogeneous Hopf real hypersurfaces other than of type (A). As we see in Lemma 6.3, if a trajectory γ for a Sasakian magnetic field is a circle of positive geodesic curvature, we know that its structure torsion ρ_γ has to be constant. As we have $\rho'_\gamma = \frac{1}{2}\langle(\phi A - A\phi)\dot\gamma, \dot\gamma\rangle$ by Lemma 6.2, we need to study the behavior of ϕA–$A\phi$. For real hypersurfaces of type (A), as their shape operators and characteristic tensors are simultaneously diagonalizable (recall Lemma 5.4), we obtained that structure torsions of trajectories are always constant. But such properties do not hold for other standard real hypersurfaces. Though there is a result on differentials of shape operators for real hypersurfaces of type (B) given by Ki-Kim-Nakagawa [33], the author could not follow them. We therefore study structure torsions of trajectories from other point of view.

Our original interest lies on the existence of trajectories which are "simple" as curves. In this sense, circles are quite good objects. But as Sasakian magnetic fields are not uniform, the condition that geodesic curvatures are constant seems too hard on hypersurfaces which do not have so high symmetries. Therefore, we weaken the condition on geodesic curvatures, and consider the case that geodesic curvatures are functions. Since plane curves are sometimes treated in submanifold theory to study shapes of submanifolds, employing the notion of curves of order 2, which includes both of the notions of Frenet curve of order 2 and plane curves, we study trajectories which are also curves of order 2 in this section and next two sections.

16.1. **Trajectories which are curves of order 2 on hypersurfaces of type (B).** We first study on real hypersurfaces of type (B) in a nonflat complex space form $\mathbb{C}M^n(c)$. For a real hypersurface M of type (B), the holomorphic distribution $T^0M = \{v \in TM \mid \langle V,\ \xi\rangle = 0\}$ of its tangent bundle TM splits into two subbundles of principal curvature vectors which are orthogonal to ξ as $T^0M = V_\lambda \oplus V_\mu$. Here V_λ and V_μ correspond to the principal curvatures

$$\lambda = \begin{cases} -(\sqrt{c}/2)\cot\bigl(\sqrt{c}\, r/2\bigr), & \text{if } M \subset \mathbb{C}P^n(c), \\ (\sqrt{|c|}/2)\coth\bigl(\sqrt{|c|}\, r/2\bigr), & \text{if } M \subset \mathbb{C}H^n(c), \end{cases}$$

$$\mu = \begin{cases} (\sqrt{c}/2)\tan\bigl(\sqrt{c}\, r/2\bigr), & \text{if } M \subset \mathbb{C}P^n(c), \\ (\sqrt{|c|}/2)\tanh\bigl(\sqrt{|c|}\, r/2\bigr), & \text{if } M \subset \mathbb{C}H^n(c), \end{cases}$$

respectively. We note that the principal curvature of ξ is

$$\nu = \begin{cases} \sqrt{c}\tan\sqrt{c}r, & \text{if } M \subset \mathbb{C}P^n(c), \\ \sqrt{|c|}\tanh\sqrt{|c|}r, & \text{if } M \subset \mathbb{C}H^n(c). \end{cases}$$

The characteristic tensor ϕ acts on $TM = V_\lambda \oplus V_\mu \oplus \mathbb{R}\xi$ as $\phi(V_\lambda) = V_\mu$ and $\phi(V_\mu) = V_\lambda$.

Proposition 16.1. *Let γ be a trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a real hypersurface M of type (B) in a nonflat $\mathbb{C}M^n(c)$. Suppose γ is also a curve of order 2 and $|\rho_\gamma| < 1$. If we decompose its velocity vector as $\dot\gamma = X_\gamma + Y_\gamma + \rho_\gamma\xi \in V_\lambda \oplus V_\mu \oplus \mathbb{R}\xi$, we have*

$$\begin{cases} \rho_\gamma(1-\rho_\gamma^2)(\kappa\rho_\gamma - \lambda)X_\gamma = \rho_\gamma(\lambda-\mu)\langle\phi X_\gamma, Y_\gamma\rangle\phi Y_\gamma, \\ \rho_\gamma(1-\rho_\gamma^2)(\kappa\rho_\gamma - \mu)Y_\gamma = \rho_\gamma(\lambda-\mu)\langle\phi X_\gamma, Y_\gamma\rangle\phi X_\gamma, \\ \kappa\rho_\gamma(1-\rho_\gamma^2) = \lambda\|X_\gamma\|^2 + \mu\|Y_\gamma\|^2, \end{cases} \tag{16.1}$$

and

$$\|X_\gamma\|^2 = \frac{(1-\rho_\gamma^2)(\kappa\rho_\gamma - \mu)}{\lambda-\mu}, \quad \|Y_\gamma\|^2 = \frac{(1-\rho_\gamma^2)(\lambda - \kappa\rho_\gamma)}{\lambda-\mu}. \tag{16.2}$$

In particular, ρ_γ satisfies $\min\{\lambda,\mu\} \le \kappa\rho_\gamma \le \max\{\lambda,\mu\}$.

Moreover, ρ_γ does not vanishes on a real hypersurface M in a complex hyperbolic space $\mathbb{C}H^n(c)$.

Proof. Since γ is a curve of order 2, it satisfies

$$\|\nabla_{\dot\gamma}\dot\gamma\|^2\bigl(\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \|\nabla_{\dot\gamma}\dot\gamma\|^2\dot\gamma\bigr) = \langle\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma\rangle\nabla_{\dot\gamma}\dot\gamma$$

by definition. We here take account of the condition that γ is a trajectory. By direct computation we obtain

$$\begin{aligned}\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma &= \kappa\nabla_{\dot\gamma}\big(\phi\dot\gamma\big) = \kappa\big\{(\nabla_{\dot\gamma}\phi)\dot\gamma + \phi\nabla_{\dot\gamma}\dot\gamma\big\}\\ &= \kappa\big\{\rho_\gamma A\dot\gamma - \langle A\dot\gamma, \dot\gamma\rangle\xi + \kappa\phi^2\dot\gamma\big\}.\end{aligned}$$

By use of the decomposition $\dot\gamma$ as $\dot\gamma = X_\gamma + Y_\gamma + \rho_\gamma\xi \in V_\lambda \oplus V_\mu \oplus \mathbb{R}\xi$, we find

$$\nabla_{\dot\gamma}\dot\gamma = \kappa\big(\phi Y_\gamma + \phi X_\gamma\big),$$

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = \kappa\big\{(\rho_\gamma\lambda - \kappa)X_\gamma + (\rho_\gamma\mu - \kappa)Y_\gamma - \big(\lambda\|X_\gamma\|^2 + \mu\|Y_\gamma\|^2\big)\xi\big\}.$$

Since $\phi(X_\gamma) \in V_\mu,\ \phi(Y_\gamma) \in V_\lambda$, by substituting these into the above equality, we obtain

$$\begin{aligned}\kappa^3(1-\rho_\gamma^2)\big\{&\rho_\gamma(\lambda - \kappa\rho_\gamma)X_\gamma + \rho_\gamma(\mu - \kappa\rho_\gamma)Y_\gamma\\ &+ \big(\kappa\rho_\gamma(1-\rho_\gamma^2) - \lambda\|X_\gamma\|^2 - \mu\|Y_\gamma\|^2\big)\xi\big\}\\ &= \kappa^3\rho_\gamma(\mu - \lambda)\langle\phi X_\gamma, Y_\gamma\rangle(\phi Y_\gamma + \phi X_\gamma)\end{aligned}$$

Comparing each components, we find that a trajectory γ is a curve of order 2 if and only if the equalities (16.1) hold.

Next we consider norms of X_γ and Y_γ. If we suppose $\rho_\gamma(t_0) = 0$ at some t_0, the third equation in (16.1) and the definition of ρ_γ show that

$$\begin{cases}\lambda\|X_\gamma(t_0)\|^2 + \mu\|Y_\gamma(t_0)\|^2 = 0,\\ \|X_\gamma(t_0)\|^2 + \|Y_\gamma(t_0)\|^2 = 1.\end{cases}$$

When M is a real hypersurface in $\mathbb{C}H^n(c)$, as $\lambda > \mu > 0$, we do not have solutions, hence find that ρ_γ never vanishes. When M is a real hypersurface in $\mathbb{C}P^n(c)$, as $\lambda < 0 < \mu$, we see

$$\|X_\gamma(t_0)\|^2 = \mu/(\mu - \lambda), \quad \|Y_\gamma(t_0)\|^2 = -\lambda/(\mu - \lambda).$$

We now consider on the domain where $\rho_\gamma(t) \neq 0$. We then have

$$\begin{cases}(1-\rho_\gamma^2)(\kappa\rho_\gamma - \lambda)X_\gamma = (\lambda - \mu)\langle\phi X_\gamma, Y_\gamma\rangle\phi Y_\gamma,\\ (1-\rho_\gamma^2)(\kappa\rho_\gamma - \mu)Y_\gamma = (\lambda - \mu)\langle\phi X_\gamma, Y_\gamma\rangle\phi X_\gamma.\end{cases}$$

These equalities show that Y_γ is parallel to ϕX_γ. Hence we see $\langle\phi X_\gamma, Y_\gamma\rangle^2 = \|X_\gamma\|^2\|Y_\gamma\|^2$. Since $\rho_\gamma \neq \pm 1$, we note that either X_γ does not vanish or Y_γ does not vanish at each point. If $X_\gamma(t_1) = 0$ at some t_1, by the second equality we have $\kappa\rho_\gamma(t_1) = \mu$ and $\|Y_\gamma(t_1)\|^2 = 1 - \rho_\gamma^2(t_1)$. Similarly, if $Y_\gamma(t_1) = 0$ at some t_1, then we have $\kappa\rho_\gamma(t_1) = \lambda$ and $\|X_\gamma(t_1)\|^2 = 1 - \rho_\gamma^2(t_1)$. If both X_γ and Y_γ do not vanish, by taking inner products of both sides of the first equality with X_γ and with ϕY_γ, we obtain

$$\|X_\gamma\|^2 = \frac{(1-\rho_\gamma^2)(\kappa\rho_\gamma - \mu)}{\lambda - \mu}, \quad \|Y_\gamma\|^2 = \frac{(1-\rho_\gamma^2)(\lambda - \kappa\rho_\gamma)}{\lambda - \mu}.$$

We can see that this expression include the cases ρ_γ vanishes, $\|X_\gamma\|$ vanishes and $\|Y_\gamma\|$ vanishes. We hence get the conclusion. □

As a direct consequence of the above Proposition, we can obtain a result on non-existence of trajectories which are curves of order 2 on real hypersurfaces of type (B) in a complex hyperbolic space.

Proposition 16.2. *If the strength of a Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a real hypersurface $R(r)$ of type (B) in $\mathbb{C}H^n(c)$ satisfies* $0 < |\kappa| \le (\sqrt{|c|}/2)\tanh(\sqrt{|c|}\, r/2)$*, then there are no $\boldsymbol{F}_\kappa$-trajectories which are curves of order* 2 *and are not geodesics.*

Proof. By Lemma 6.1, if we have $\rho_\gamma(t_0) = \pm 1$ at some point t_0, then we find that $\rho_\gamma \equiv \pm 1$ and γ is a geodesic. Therefore we only need to treat the case $|\rho_\gamma| < 1$. The equalities (16.2) in Proposition 16.1 show that the structure torsion ρ_γ of a trajectory γ which is also a curve of order 2 satisfies $\lambda \ge \kappa\rho_\gamma \ge \mu$ (> 0). Since $|\rho_\gamma| < 1$, we find $|\kappa| > \mu$ and get the conclusion. □

For a real hypersurfaces of type (B) in a complex projective space, we can not conclude such a non-existence theorem only by Proposition 16.1 because two principal curvatures have different signatures.

16.2. **Behaviors of structure torsions on hypersurfaces of type (B) in $\mathbb{C}H^n$.** For now we can not conclude whether there exists a trajectory which are also curves of order 2 on real hypersurfaces in $\mathbb{C}M^n(c)$. We further our study on structure torsions under the hypothesis of existence of such trajectories.

Let γ be a trajectory which are also curve of order 2 on a real hypersurface $R(r)$ of type (B) in $\mathbb{C}H^n(c)$. By use of the same notations as in Proposition 16.1, we denote as $\dot\gamma = X_\gamma + Y_\gamma + \rho_\gamma\xi \in V_\lambda \oplus V_\mu \oplus \mathbb{R}\xi$ for a trajectory γ which is a curve of order 2 on a real hypersurface of type (B). Recalling the computation on the derivative of its structure torsion ρ_γ in Lemma 6.3, we have

$$\begin{aligned} \rho_\gamma' &= \langle \dot\gamma, \phi A\dot\gamma\rangle = \langle \dot\gamma, \lambda\phi X_\gamma + \mu\phi Y_\gamma\rangle \\ &= \lambda\langle Y_\gamma, \phi X_\gamma\rangle + \mu\langle X_\gamma, \phi Y_\gamma\rangle = (\lambda-\mu)\langle \phi X_\gamma, Y_\gamma\rangle. \end{aligned} \tag{16.3}$$

By Proposition 16.1 we see ρ_γ never vanishes. As we see in the proof of Proposition 16.1, we have $\langle\phi X_\gamma, Y_\gamma\rangle^2 = \|X_\gamma\|^2\|Y_\gamma\|^2$ in this case. Therefore, the above equation turns to

$$\left(\rho_\gamma'\right)^2 = (\lambda-\mu)^2\|X_\gamma\|^2\,\|Y_\gamma\|^2 = (1-\rho_\gamma^2)^2(\lambda-\kappa\rho_\gamma)(\kappa\rho_\gamma-\mu). \tag{16.4}$$

We solve this differential equation.

1) If $|\kappa| > \mu$, we see that $\rho_\gamma \equiv \mu/\kappa$ is the solution of the above equation;

2) if $|\kappa| > \lambda$, then $\rho_\gamma \equiv \lambda/\kappa$ and $\rho_\gamma \equiv \mu/\kappa$ are solutions of the above equation.

We should note that $\rho = \lambda/\kappa$ on some interval and $\rho = \mu/\kappa$ on some other interval can not occur because trajectories for Sasakian magnetic fields are not bifurcated. In those cases, $X_\gamma \equiv 0$ and $\|Y_\gamma\|^2 \equiv 1-\rho_\gamma^2$ when $\rho_\gamma \equiv \mu/\kappa$, and $Y_\gamma \equiv 0$ and $\|X_\gamma\|^2 \equiv 1-\rho_\gamma^2$ when $\rho_\gamma \equiv \lambda/\kappa$. In view of structure torsions of trajectories on real hypersurfaces of type (A), such cases likely occur. We shall discuss this later, and here we focus our mind on behaviors of their structure torsions.

We study other solutions. We first consider the case $\kappa > 0$. The equation (16.4) guarantees that $\mu/\kappa \le \rho_\gamma \le \lambda/\kappa$. We modify the equation

$$dt = \frac{\pm d\rho_\gamma}{(1-\rho_\gamma^2)\sqrt{(\lambda-\kappa\rho_\gamma)(\kappa\rho_\gamma-\mu)}}$$

by changing variable. Here, the double sign corresponds to the signature of ρ'_γ. We put $y = \sqrt{(\lambda - \kappa\rho_\gamma)/(\kappa\rho_\gamma - \mu)}$. As we have

$$\rho_\gamma = \frac{\mu y^2 + \lambda}{\kappa(y^2+1)} \quad \text{and} \quad \frac{d\rho_\gamma}{dy} = -\frac{2y(\lambda-\mu)}{\kappa(y^2+1)^2},$$

we get

$$\begin{aligned}
\mathcal{I} :&= \int \frac{d\rho_\gamma}{(1-\rho_\gamma^2)\sqrt{(\lambda-\kappa\rho_\gamma)(\kappa\rho_\gamma-\mu)}} \\
&= \int \frac{1}{\left\{1-\left(\frac{\mu y^2+\lambda}{\kappa(y^2+1)}\right)^2\right\} y\left(\frac{\mu y^2+\lambda}{y^2+1}-\mu\right)} \times \frac{2y(\mu-\lambda)}{\kappa(y^2+1)^2}\, dy \\
&= \int \frac{2\kappa(y^2+1)}{(\mu y^2+\lambda)^2 - \kappa^2(y^2+1)^2}\, dy \\
&= \int \left\{ \frac{1}{(\mu-\kappa)y^2+(\lambda-\kappa)} - \frac{1}{(\mu+\kappa)y^2+(\lambda+\kappa)} \right\} dy.
\end{aligned}$$

In order to continue our computation, we divide the situation into 3 cases. When $\lambda > \kappa > \mu$, we have

$$\begin{aligned}
\mathcal{I} &= -\frac{1}{\sqrt{(\kappa+\mu)(\lambda+\kappa)}} \tan^{-1}\sqrt{\frac{\kappa+\mu}{\lambda+\kappa}}\, y \\
&\quad + \frac{1}{2\sqrt{\lambda-\kappa}} \int \left\{ \frac{1}{\sqrt{\kappa-\mu}\, y + \sqrt{\lambda-\kappa}} - \frac{1}{\sqrt{\kappa-\mu}\, y - \sqrt{\lambda-\kappa}} \right\} dy \\
&= -\frac{1}{\sqrt{(\kappa+\mu)(\lambda+\kappa)}} \tan^{-1}\sqrt{\frac{(\kappa+\mu)(\lambda-\kappa\rho_\gamma)}{(\lambda+\kappa)(\kappa\rho_\gamma-\mu)}} \\
&\quad + \frac{1}{2\sqrt{(\lambda-\kappa)(\kappa-\mu)}} \log\left| \frac{\sqrt{(\kappa-\mu)(\lambda-\kappa\rho_\gamma)} + \sqrt{(\lambda-\kappa)(\kappa\rho_\gamma-\mu)}}{\sqrt{(\kappa-\mu)(\lambda-\kappa\rho_\gamma)} - \sqrt{(\lambda-\kappa)(\kappa\rho_\gamma-\mu)}} \right|.
\end{aligned}$$

When $\kappa = \lambda$, we have

$$\begin{aligned}
\mathcal{I} &= \frac{1}{(\kappa-\mu)y} - \frac{1}{\sqrt{(\kappa+\mu)(\lambda+\kappa)}} \tan^{-1}\sqrt{\frac{\kappa+\mu}{\lambda+\kappa}}\, y \\
&= \frac{\sqrt{\kappa\rho_\gamma-\mu}}{(\lambda-\mu)\sqrt{\lambda-\kappa\rho_\gamma}} - \frac{1}{\sqrt{(\kappa+\mu)(\lambda+\kappa)}} \tan^{-1}\sqrt{\frac{(\kappa+\mu)(\lambda-\kappa\rho_\gamma)}{(\lambda+\kappa)(\kappa\rho_\gamma-\mu)}}.
\end{aligned}$$

When $\kappa > \lambda$, we have

$$\begin{aligned}\mathcal{I} &= -\frac{1}{\sqrt{(\kappa-\mu)(\kappa-\lambda)}}\tan^{-1}\sqrt{\frac{\kappa-\mu}{\kappa-\lambda}}\,y - \frac{1}{\sqrt{(\kappa+\mu)(\lambda+\kappa)}}\tan^{-1}\sqrt{\frac{\kappa+\mu}{\lambda+\kappa}}\,y \\ &= -\frac{1}{\sqrt{(\kappa-\mu)(\kappa-\lambda)}}\tan^{-1}\sqrt{\frac{(\kappa-\mu)(\lambda-\kappa\rho_\gamma)}{(\kappa-\lambda)(\kappa\rho_\gamma-\mu)}} \\ &\quad - \frac{1}{\sqrt{(\kappa+\mu)(\lambda+\kappa)}}\tan^{-1}\sqrt{\frac{(\kappa+\mu)(\lambda-\kappa\rho_\gamma)}{(\lambda+\kappa)(\kappa\rho_\gamma-\mu)}}.\end{aligned}$$

Summarizing up, by solving the differential equation (16.4), we obtain the following:

$$t + C = -\mathrm{sgn}\big(\rho_\gamma'(t)\big)\left\{\frac{1}{\sqrt{(\kappa+\mu)(\kappa+\lambda)}}\tan^{-1}\sqrt{\frac{(\kappa+\mu)(\lambda-\kappa\rho_\gamma)}{(\kappa+\lambda)(\kappa\rho_\gamma-\mu)}} - f_\kappa(\rho_\gamma)\right\}, \tag{16.5}$$

where the constant C is determined by initial condition, the function f_κ is given as

$$f_\kappa(\rho_\gamma) = \begin{cases} \dfrac{1}{2\sqrt{(\kappa-\mu)(\lambda-\kappa)}}\log\left|\dfrac{\sqrt{(\kappa-\mu)(\lambda-\kappa\rho_\gamma)}+\sqrt{(\lambda-\kappa)(\kappa\rho_\gamma-\mu)}}{\sqrt{(\kappa-\mu)(\lambda-\kappa\rho_\gamma)}-\sqrt{(\lambda-\kappa)(\kappa\rho_\gamma-\mu)}}\right|, & \text{when } \mu < \kappa < \lambda, \\ \dfrac{\sqrt{\kappa\rho_\gamma-\mu}}{(\lambda-\mu)\sqrt{\lambda-\kappa\rho_\gamma}}, & \text{when } \kappa = \lambda, \\ \dfrac{-1}{\sqrt{(\kappa-\mu)(\kappa-\lambda)}}\tan^{-1}\sqrt{\dfrac{(\kappa-\mu)(\lambda-\kappa\rho_\gamma)}{(\kappa-\lambda)(\kappa\rho_\gamma-\mu)}}, & \text{when } \kappa > \lambda, \end{cases}$$

and $\mathrm{sgn}\big(\rho_\gamma'(t)\big)$ denotes the signature of $\rho_\gamma'(t)$. This means that we need to treat separately on the interval where ρ_γ' is positive and on the interval where ρ_γ' is negative.

We now study the behavior of the function ρ_γ. We first study the case $\mu < \kappa \le \lambda$. Suppose $\mu/\kappa < \rho_\gamma(t_*) < 1$ $(\le \lambda/\kappa)$ at some t_*. By (16.4) we see $\rho_\gamma'(t_*) \ne 0$, hence have an interval (t_-, t_+) with $t_- < t_* < t_+$ satisfying that $\rho_\gamma'(t) \ne 0$ on this interval and $\lim_{t\downarrow t_-}\rho_\gamma'(t) = 0 = \lim_{t\uparrow t_+}\rho_\gamma'(t)$. When $\rho_\gamma'(t_*) > 0$, as $\rho_\gamma < 1$, we find $t_+ = \infty$ and $\lim_{t\to\infty}\rho_\gamma(t) = 1$. As a matter of fact, if we suppose $t_+ < \infty$, then we have $\rho_\gamma(t_+) = 1$ by (16.4). By Lemma 6.1, we find γ is a geodesic and satisfies $\rho_\gamma \equiv 1$, which is a contradiction. In this case, we also find $\lim_{t\downarrow t_-}\rho_\gamma(t) = \mu/\kappa$ by (16.4). In view of the equality (16.5), the right hand side goes to a finite value $-\pi/(2\sqrt{(\kappa+\mu)(\kappa+\lambda)})$ when we make $t \downarrow t_-$. Therefore we find $t_- > -\infty$. By homogeneity of M and the fact that trajectories can not be bifurcated, there is $t_\flat$ ($< t_-$) such that $\rho'_\gamma(t) < 0$ on the interval $(t_\flat, t_-)$ and $\lim_{t\downarrow t_\flat} \rho'_\gamma(t) = 0$. Again as $\rho_\gamma < 1$, we find $t_\flat = -\infty$ and $\lim_{t\to-\infty} \rho_\gamma(t) = 1$. When $\rho'_\gamma(t_*) < 0$, by the same argument we have $\rho'_\gamma(t) < 0$ on the interval $(-\infty, t_+)$ and $\rho'_\gamma(t) > 0$ on the interval (t_+, ∞), and have $\lim_{t\to-\infty} \rho_\gamma(t) = \lim_{t\to\infty} \rho_\gamma(t) = 1,\ \rho_\gamma(t_+) = \mu/\kappa$. The image can be seen in Figure 2. This is just an image and is not the graph of the function ρ_γ.

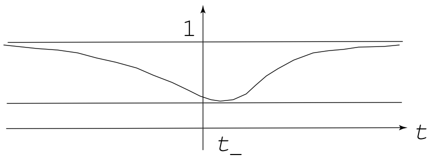


FIGURE 2. $\mu < \kappa \le \lambda$

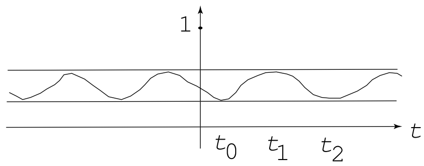


FIGURE 3. $\kappa > \lambda$

We next consider the case $\kappa > \lambda$. Suppose $\mu/\kappa < \rho_\gamma(t_*) < \lambda/\kappa$ at some t_*. We then have an interval (t_-, t_+) with $t_- < t_* < t_+$ satisfying that $\rho'_\gamma(t) \neq 0$ on this interval and $\lim_{t\downarrow t_-} \rho'_\gamma(t) = 0 = \lim_{t\uparrow t_+} \rho'_\gamma(t)$. When $\rho'_\gamma(t_*) > 0$, we set $t_- = t_0$ and $t_+ = t_1$. By (16.4), we have $\lim_{t\uparrow t_1} \rho_\gamma(t) = \lambda/\kappa$ and $\lim_{t\downarrow t_0} \rho_\gamma(t) = \mu/\kappa$. In view of the equality (16.5), the right hand side goes to a finite value 0 when we make $t \uparrow t_1$. We hence find that $t_1 < \infty$. By the same argument as in the case $\mu < \kappa \le \lambda$, we also find that $t_0 > -\infty$. Repeating the same argument, we have a sequence $\{t_i\}_{-\infty<i<\infty}$ such that $\rho'_\gamma(t) < 0$ on the interval (t_{2j-1}, t_{2j}) and $\rho'_\gamma(t) > 0$ on the interval (t_{2j}, t_{2j+1}), and $\rho_\gamma(t_{2j}) = \mu/\kappa,\ \rho_\gamma(t_{2j+1}) = \lambda/\kappa$. We also note that the interval is

$$t_{i+1} - t_i = \frac{\pi}{2\sqrt{(\kappa+\mu)(\kappa+\lambda)}} + \frac{\pi}{2\sqrt{(\kappa-\mu)(\kappa-\lambda)}}.$$

Hence it is periodic with period $\big(\pi/\sqrt{(\kappa+\mu)(\kappa+\lambda)}\big) + \big(\pi/\sqrt{(\kappa-\mu)(\kappa-\lambda)}\big)$. When $\rho'_\gamma(t_*) < 0$, by setting $t_- = t_{-1}$ and $t_+ = t_0$, we get such a sequence. The image can be seen in Figure 3.

We finally consider the case $\kappa < 0$. In this case we reverse the orientation of the trajectory γ. That is we consider a curve σ which is defined by $\sigma(t) = \gamma(-t)$. Then it is a trajectory for $\boldsymbol{F}_{-\kappa}$ and is a curve of order 2. Summarizing up we have the following.

Lemma 16.1. *Let $\boldsymbol{F}_\kappa$ be a non-trivial Sasakian magnetic field on a real hypersurface $R(r)$ of type (B) in $\mathbb{C}H^n(c)$. Suppose there exist a $\boldsymbol{F}_\kappa$-trajectory γ which is also a curve of order 2. Then its structure torsion ρ_γ satisfies the following with $\lambda = (\sqrt{|c|}/2)\coth\big(\sqrt{|c|}\,r/2\big)$ and $\mu = (\sqrt{|c|}/2)\tanh\big(\sqrt{|c|}\,r/2\big)$.*

(I) *When $\mu < \kappa \leq \lambda$,*

1) $\rho_\gamma \equiv \pm 1$, (γ *is a geodesic*);
2) $\rho_\gamma \equiv \mu/\kappa$, ($X_\gamma \equiv 0$, $\|Y_\gamma\| = 1 - \rho_\gamma^2$);
3) $\lim_{t\to-\infty} \rho_\gamma(t) = \lim_{t\to\infty} \rho_\gamma(t) = 1$ *and there is t_0 satisfying that ρ_γ is strictly monotone decreasing on the interval $(-\infty, t_0)$ and strictly monotone increasing on the interval (t_0, ∞), and $\rho_\gamma(t_0) = \mu/\kappa$.*

(II) *When $-\mu > \kappa \geq -\lambda$,*

1) $\rho_\gamma \equiv \pm 1$, (γ *is a geodesic*);
2) $\rho_\gamma \equiv \mu/\kappa$, ($X_\gamma \equiv 0$, $\|Y_\gamma\| = 1 - \rho_\gamma^2$);
3) $\lim_{t\to-\infty} \rho_\gamma(t) = \lim_{t\to\infty} \rho_\gamma(t) = -1$ *and there is t_0 satisfying that ρ_γ is strictly monotone increasing on the interval $(-\infty, t_0)$ and strictly monotone decreasing on the interval (t_0, ∞), and $\rho_\gamma(t_0) = \mu/\kappa$.*

(III) *When $|\kappa| > \lambda$,*

1) $\rho_\gamma \equiv \pm 1$, (γ *is a geodesic*);
2) $\rho_\gamma \equiv \mu/\kappa$, ($X_\gamma \equiv 0$, $\|Y_\gamma\| = 1 - \rho_\gamma^2$);
3) $\rho_\gamma \equiv \lambda/\kappa$, ($Y_\gamma \equiv 0$, $\|X_\gamma\| = 1 - \rho_\gamma^2$);
4) *ρ_γ is a periodic function satisfying $\mu \leq \kappa\rho_\gamma \leq \lambda$. In this case $\kappa\rho_\gamma$ takes all the values in the interval $[\mu, \lambda]$.*

16.3. **Trajectories which are curves of order 2 on hypersurfaces of type (B) in $\mathbb{C}H^n$.** We now consider whether these cases in Lemma 16.1 really occur. For this sake we consider $R(r)$ through a Hopf fibration $\varpi : H_1^{2n+1} \to \mathbb{C}H^n$ in the case $c = -4$. As we mentioned in §5.3, we may consider that

$$\varpi^{-1}\big(R(r)\big) = \left\{ \hat{z} = (z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \;\middle|\; \begin{matrix} -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = -1, \\ |-z_0^2 + z_1^2 + \cdots + z_n^2| = \cosh 2r \end{matrix} \right\}$$
$$= \left\{ e^{\sqrt{-1}\theta} \begin{pmatrix} x_0 + \sqrt{-1}y_0 \\ x_1 + \sqrt{-1}y_1 \\ \vdots \\ x_n + \sqrt{-1}y_n \end{pmatrix}^t \in \mathbb{C}^{n+1} \;\middle|\; \begin{matrix} \theta \in \mathbb{R}, \\ -x_0y_0 + x_1y_1 + \cdots + x_ny_n = 0, \\ -x_0^2 + x_1^2 + \cdots + x_n^2 = -\cosh^2 r, \\ -y_0^2 + y_1^2 + \cdots + y_n^2 = \sinh^2 r \end{matrix} \right\}.$$

We take a point $\hat{z} \in \widehat{M} = \varpi^{-1}\big(R(r)\big)$ which is represented as above with $\theta = 0$. The horizontal part $\mathcal{H}_{\hat{z}}$ of the tangent space $T_{\hat{z}}\widehat{M}$ at $\hat{z}$ is represented as

$$\mathcal{H}_{\hat{z}} = \left\{ (\hat{z}, \hat{w}) \in \{\hat{z}\} \times \mathbb{C}^{n+1} \;\middle|\; \begin{matrix} -x_0u_0 + x_1u_1 + \cdots + x_nu_n = 0, \\ -y_0v_0 + y_1v_1 + \cdots + y_nv_n = 0, \\ -x_0v_0 + x_1v_1 + \cdots + x_nv_n \\ \quad -y_0u_0 + y_1u_1 + \cdots + y_nu_n = 0 \end{matrix} \right\},$$

where we denote $\hat{w} = \big(u_0 + \sqrt{-1}v_0, u_1 + \sqrt{-1}v_1, \dots, u_n + \sqrt{-1}v_n\big)$. The horizontal lift $\widehat{\mathcal{N}}_{\hat{z}}$ of the unit normal vector $\mathcal{N}_{\varpi(\hat{z})}$ of $R(r)$ in $\mathbb{C}H^n(-4)$, which is a unit normal vector of $\widehat{M}$ in H_1^{2n+1}, is represented as

$$\widehat{\mathcal{N}}_{\hat{z}} = \left(\hat{z}, \begin{pmatrix} -\tanh r x_0 - \sqrt{-1}\coth r y_0 \\ -\tanh r x_1 - \sqrt{-1}\coth r y_1 \\ \vdots \\ -\tanh r x_n - \sqrt{-1}\coth r y_n \end{pmatrix}^t \right) \in \{\hat{z}\} \times \mathbb{C}^{n+1}.$$

We put $\hat{\xi}_{\hat{z}} = -J\widehat{\mathcal{N}}_{\hat{z}}$, which is the horizontal lift of $\xi_{\varpi(\hat{z})}$ and is given as

$$\hat{\xi}_{\hat{z}} = \left(\hat{z}, \begin{pmatrix} -\coth r y_0 + \sqrt{-1}\tanh r x_0 \\ -\coth r y_1 + \sqrt{-1}\tanh r x_1 \\ \vdots \\ -\coth r y_n + \sqrt{-1}\tanh r x_n \end{pmatrix}^t \right) \in \{\hat{z}\} \times \mathbb{C}^{n+1}.$$

Therefore we find that the horizontal lift of $\langle \xi_{\varpi(\hat{z})}\rangle^{\perp} = T^0_{\varpi(\hat{z})}R(r)$ is represented as

$$\langle \hat{\xi}_{\hat{z}}\rangle^{\perp} \cap \mathcal{H}_{\hat{z}} = \left\{ (\hat{z}, \hat{w}) \in \{\hat{z}\} \times \mathbb{C}^{n+1} \;\middle|\; \begin{array}{c} -x_0u_0 + x_1u_1 + \cdots + x_nu_n = 0, \\ -y_0u_0 + y_1u_1 + \cdots + y_nu_n = 0, \\ -y_0v_0 + y_1v_1 + \cdots + y_nv_n = 0, \\ -x_0v_0 + x_1v_1 + \cdots + x_nv_n = 0 \end{array} \right\}$$
$$= \left\{ (\hat{z}, \hat{w}) \in \{\hat{z}\} \times \mathbb{C}^{n+1} \;\middle|\; \begin{array}{c} -z_0w_0 + z_1w_1 + \cdots + z_nw_n = 0, \\ -z_0\overline{w}_0 + z_1\overline{w}_1 + \cdots + z_n\overline{w}_n = 0 \end{array} \right\}.$$

The horizontal lifts of the subspaces $V_{\lambda,\varpi(\hat{z})}$ and $V_{\mu,\varpi(\hat{z})}$ of principal curvature vectors are given as

$$\widehat{V}_{\lambda,\hat{z}} = \left\{ (\hat{z}, \hat{w}) \in \{\hat{z}\} \times \mathbb{C}^{n+1} \;\middle|\; \begin{array}{c} -x_0v_0 + x_1v_1 + \cdots + x_nv_n = 0, \\ -y_0v_0 + y_1v_1 + \cdots + y_nv_n = 0, \\ u_0 = u_1 = \cdots = u_n = 0 \end{array} \right\},$$
$$\widehat{V}_{\mu,\hat{z}} = \left\{ (\hat{z}, \hat{w}) \in \{\hat{z}\} \times \mathbb{C}^{n+1} \;\middle|\; \begin{array}{c} -x_0u_0 + x_1u_1 + \cdots + x_nu_n = 0, \\ -y_0u_0 + y_1u_1 + \cdots + y_nu_n = 0, \\ v_0 = v_1 = \cdots = v_n = 0 \end{array} \right\}.$$

In order to simplify the description, for $x = (x_0, x_1, \ldots, x_n),\ x = (x'_0, x'_1, \ldots, x'_n) \in \mathbb{R}^{n+1}$, we set

$$\langle\!\langle x, x' \rangle\!\rangle = -x_0x'_0 + x_1x'_1 + \cdots + x_nx'_n.$$

Though it is the same notation as of the Hermitian form on $\mathbb{C}^{n+1}_1$, one may do not confuse them.

Let γ be a trajectory for $\boldsymbol{F}_\kappa$ which is also a curve of order 2. We suppose $\kappa\rho_\gamma(t_0) = \mu$ at some point t_0. We take its horizontal lift $\hat{\gamma}$ satisfying that $\hat{\gamma}(t_0) \in \widehat{M}$ is represented with $\theta = 0$ in our representation. We denote as $\hat{\gamma}(t_0) = \hat{z}_\sharp = x_\sharp + \sqrt{-1}y_\sharp \in \mathbb{C}^{n+1}$ and $\hat{\gamma}'(t_0) = (\hat{z}_\sharp, u_\sharp + \sqrt{-1}v_\sharp) \in \mathcal{H}_{\hat{z}_\sharp} \subset \{\hat{z}_\sharp\} \times \mathbb{C}^{n+1}$ with some $x_\sharp, y_\sharp, u_\sharp, v_\sharp \in \mathbb{R}^{n+1}$. By Proposition 16.1 (see also Lemma 16.1), we see

$$\begin{aligned} \widehat{V}_{\mu,\hat{z}_\sharp} &\ni \hat{\gamma}'(t_0) - \rho_\gamma(t_0)\hat{\xi}_{\hat{\gamma}(t_0)} \\ &= \Big(\hat{z}_\sharp, \big(u_\sharp + \rho_\gamma(t_0)\coth r\, y_\sharp\big) + \sqrt{-1}\big(v_\sharp - \rho_\gamma(t_0)\tanh r\, x_\sharp\big)\Big). \end{aligned}$$

We therefore obtain

$$\text{(16.6)} \qquad \begin{cases} v_\sharp = \rho_\gamma(t_0)\tanh r\, x_\sharp, \\ \langle\!\langle u_\sharp, u_\sharp\rangle\!\rangle + 2\rho_\gamma(t_0)\coth r \langle\!\langle u_\sharp, y_\sharp\rangle\!\rangle + \rho_\gamma^2(t_0)\coth^2 r \langle\!\langle y_\sharp, y_\sharp\rangle\!\rangle = 1 - \rho_\gamma^2(t_0), \end{cases}$$

As $\hat{\gamma}'(t_0) \in \mathcal{H}_{\hat{\gamma}(t_0)}$, we have $\langle\!\langle u_\sharp, y_\sharp \rangle\!\rangle + \langle\!\langle v_\sharp, x_\sharp \rangle\!\rangle = 0$. Hence, by the definition of structure torsions, we have

$$\begin{aligned} \rho_\gamma(t_0) = \langle \dot{\gamma}(t_0), \xi_{\gamma(t_0)} \rangle &= \langle\!\langle u_\sharp, -\coth r\, y_\sharp \rangle\!\rangle + \langle\!\langle v_\sharp, \tanh r\, x_\sharp \rangle\!\rangle \\ &= -(\coth r + \tanh r)\langle\!\langle u_\sharp, y_\sharp \rangle\!\rangle. \end{aligned} \tag{16.7}$$

Since $\langle\!\langle x_\sharp, x_\sharp \rangle\!\rangle = -\cosh^2 r$, $\langle\!\langle y_\sharp, y_\sharp \rangle\!\rangle = \sinh^2 r$, by (16.6) and (16.7) we obtain

$$\langle\!\langle u_\sharp, u_\sharp \rangle\!\rangle = 1 - \rho_\gamma^2(t_0)(1 + \cosh^2 r) + \frac{2\rho_\gamma^2(t_0)\coth r}{\coth r + \tanh r}$$

We therefore get

$$\begin{aligned} 1 = \|\dot{\gamma}(t_0)\|^2 &= \langle\!\langle u_\sharp, u_\sharp \rangle\!\rangle + \langle\!\langle v_\sharp, v_\sharp \rangle\!\rangle \\ &= 1 - \rho_\gamma^2(t_0)(1 + \cosh^2 r) + \frac{2\rho_\gamma^2(t_0)\cosh^2 r}{\cosh^2 r + \sinh^2 r} - \rho_\gamma^2(t_0)\sinh^2 r \\ &= 1 - \frac{4\rho_\gamma^2(t_0)\cosh^2 r \sinh^2 r}{\cosh^2 r + \sinh^2 r}. \end{aligned}$$

This shows $\rho_\gamma(t_0) = 0$, which is a contradiction.

We next suppose $\kappa\rho_\gamma(t_0) = \lambda$ at some point t_0. By using the same notations we see by Proposition 16.1 (see also Lemma 16.1) that

$$\begin{aligned} \widehat{V}_{\lambda, \hat{z}_\sharp} \ni \hat{\gamma}'(t_0) - \rho_\gamma(t_0)\hat{\xi}_{\hat{\gamma}(t_0)} \\ = \Big(\hat{z}_\sharp, \big(u_\sharp + \rho_\gamma(t_0)\coth r\, y_\sharp\big) + \sqrt{-1}\big(v_\sharp - \rho_\gamma(t_0)\tanh r\, x_\sharp\big)\Big). \end{aligned}$$

We hence obtain

$$\begin{cases} u_\sharp = -\rho_\gamma(t_0)\coth r\, y_\sharp, \\ \langle\!\langle v_\sharp, v_\sharp \rangle\!\rangle - 2\rho_\gamma(t_0)\tanh r \langle\!\langle v_\sharp, x_\sharp \rangle\!\rangle + \rho_\gamma^2(t_0)\tanh^2 r \langle\!\langle x_\sharp, x_\sharp \rangle\!\rangle = 1 - \rho_\gamma^2(t_0), \end{cases} \tag{16.8}$$

By (16.7) and (16.8), we obtain

$$\langle\!\langle v_\sharp, v_\sharp \rangle\!\rangle = 1 + \rho_\gamma^2(t_0)(\sinh^2 r - 1) + \frac{2\rho_\gamma^2(t_0)\tanh r}{\coth r + \tanh r}.$$

We therefore get

$$\begin{aligned} 1 = \|\dot{\gamma}(t_0)\|^2 &= \langle\!\langle u_\sharp, u_\sharp \rangle\!\rangle + \langle\!\langle v_\sharp, v_\sharp \rangle\!\rangle \\ &= 1 + \rho_\gamma^2(t_0)(\sinh^2 r - 1) + \frac{2\rho_\gamma^2(t_0)\sinh^2 r}{\cosh^2 r + \sinh^2 r} + \rho_\gamma^2(t_0)\cosh^2 r \\ &= 1 + \frac{4\rho_\gamma^2(t_0)\sinh^2 r \cosh^2 r}{\cosh^2 r + \sinh^2 r} \end{aligned}$$

This shows $\rho_\gamma(t_0) = 0$, which is a contradiction. Thus we can conclude that if a trajectory $\boldsymbol{F}_\kappa$ is also a curve of order 2 then its structure torsion satisfies neither $\kappa\rho_\gamma(t_0) = \mu$ nor $\kappa\rho_\gamma(t_0) = \lambda$ at arbitrary t_0.

Combining the above argument with Lemma 16.1 and Proposition 16.2 we get the following.

Theorem 16.1. *On a real hypersurface $R(r)$ of type (B) in $\mathbb{C}H^n(c)$ there are no trajectories for Sasakian magnetic fields which are curves of order 2 and are not geodesics.*

Remark 16.1. For non-trivial Sasakian magnetic fields on a real hypersurface $R(r)$ of type (B) in $\mathbb{C}H^n(c)$, trajectories are geodesic if and only if their structure torsions satisfy $\rho_\gamma \equiv \pm 1$.

We here make mention of congruence classes of geodesic trajectories.

Proposition 16.3. *On a real hypersurface $R(r)$ of type* (B) *in $\mathbb{C}H^n(c)$, all geodesic trajectories for non-trivial Sasakian magnetic fields are congruent to each other in strong sense.*

We can obtain this by the following Lemma. Let $T_zR(r) = V_{\lambda,z} \oplus V_{\mu,z} \oplus \mathbb{R}\xi_z$ be the splitting of the tangent space at $z \in R(r)$ into subspaces of principal curvature vectors.

Lemma 16.2. *Let $z, z' \in R(r)$ be arbitrary points on a hypersurface $R(r)$ of type (B) in $\mathbb{C}H^n(c)$. Given unit tangent vectors $w \in V_{\lambda,z}$, $\varsigma \in V_{\mu,z}$ and $w' \in V_{\lambda,z'}$, $\varsigma' \in V_{\mu,z'}$, we have isometries $\tilde{\varphi}_\lambda^+$, $\tilde{\varphi}_\lambda^-$, $\tilde{\varphi}_\mu^+$, $\tilde{\varphi}_\mu^-$ of $\mathbb{C}H^n$ satisfying the following conditions*:

i) $\tilde{\varphi}_\lambda^+\big(R(r)\big) = \tilde{\varphi}_\mu^+\big(R(r)\big) = \tilde{\varphi}_\lambda^-\big(R(r)\big) = \tilde{\varphi}_\mu^-\big(R(r)\big) = R(r)$;

ii) $\tilde{\varphi}_\lambda^+(z) = \tilde{\varphi}_\mu^+(z) = \tilde{\varphi}_\lambda^-(z) = \tilde{\varphi}_\mu^-(z) = z'$;

iii) $d\tilde{\varphi}_\lambda^+(\xi_z) = d\tilde{\varphi}_\mu^+(\xi_z) = \xi_{z'}$ *and* $d\tilde{\varphi}_\lambda^-(\xi_z) = d\tilde{\varphi}_\mu^-(\xi_z) = -\xi_{z'}$;

iv) $d\tilde{\varphi}^+_\lambda(w) = d\tilde{\varphi}^-_\lambda(w) = w'$, $d\tilde{\varphi}^+_\mu(\varsigma) = d\tilde{\varphi}^-_\mu(\varsigma) = \varsigma'$,

v) $d\tilde{\varphi}^+_\lambda \circ J = J \circ d\tilde{\varphi}^+_\lambda$, $d\tilde{\varphi}^+_\mu \circ J = J \circ d\tilde{\varphi}^+_\mu$,

vi) $d\tilde{\varphi}^-_\lambda \circ J = -J \circ d\tilde{\varphi}^-_\lambda$, $d\tilde{\varphi}^-_\mu \circ J = -J \circ d\tilde{\varphi}^-_\mu$.

Proof. We consider the case $n = 3$ and $c = -4$. We may consider that

$$\varpi^{-1}\big(R(r)\big) = \left\{ e^{\sqrt{-1}\theta} \begin{pmatrix} x_0+\sqrt{-1}y_0 \\ x_1+\sqrt{-1}y_1 \\ x_2+\sqrt{-1}y_2 \\ x_3+\sqrt{-1}y_3 \end{pmatrix}^t \in \mathbb{C}^4 \;\middle|\; \begin{matrix} \theta \in \mathbb{R}, \\ -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0, \\ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -\cosh^2 r, \\ -y_0^2 + y_1^2 + y_2^2 + y_3^2 = \sinh^2 r \end{matrix} \right\}.$$

We take a point $\hat{z} \in \widehat{M} = \varpi^{-1}\big(R(r)\big)$ which is represented as above with $\theta = 0$. At this point the horizontal lift $\hat{\xi}_{\hat{z}}$ of $\xi_{\varpi(\hat{z})}$ is given as

$$\hat{\xi}_{\hat{z}} = \left(\hat{z}, \begin{pmatrix} -\coth r y_0+\sqrt{-1}\tanh r x_0 \\ -\coth r y_1+\sqrt{-1}\tanh r x_1 \\ -\coth r y_2+\sqrt{-1}\tanh r x_2 \\ -\coth r y_3+\sqrt{-1}\tanh r x_3 \end{pmatrix}^t \right) \in \{\hat{z}\} \times \mathbb{C}^4,$$

and the horizontal lifts of the subspaces $V_{\lambda,z}$ and $V_{\mu,z}$ of principal curvature vectors are

$$\widehat{V}_{\lambda,\hat{z}} = \left\{ (\hat{z}, \hat{w}) \in \{\hat{z}\} \times \mathbb{C}^4 \;\middle|\; \begin{matrix} -x_0v_0 + x_1v_1 + x_2v_2 + x_3v_3 = 0, \\ -y_0v_0 + y_1v_1 + y_2v_2 + y_3v_3 = 0, \\ u_0 = u_1 = u_2 = u_3 = 0 \end{matrix} \right\},$$
$$\widehat{V}_{\mu,\hat{z}} = \left\{ (\hat{z}, \hat{w}) \in \{\hat{z}\} \times \mathbb{C}^4 \;\middle|\; \begin{matrix} -x_0u_0 + x_1u_1 + x_2u_2 + x_3u_3 = 0, \\ -y_0u_0 + y_1u_1 + y_2u_2 + y_3u_3 = 0, \\ v_0 = v_1 = v_2 = v_3 = 0 \end{matrix} \right\}.$$

We now take a point $\hat{z}_* = (\cosh r, \sqrt{-1}\sinh r, 0, 0) \in \widehat{M}$ and unit tangent vectors $\hat{w}_* = \big(\hat{z}_*, (0, 0, \sqrt{-1}, 0)\big) \in \widehat{V}_{\lambda,\hat{z}_*}$, $\hat{\varsigma}_* = \big(\hat{z}_*, (0, 0, 0, 1)\big) \in \widehat{V}_{\mu,\hat{z}_*}$. We then see $\hat{\xi}_{\hat{z}_*} = \big(\hat{z}_*, (\sqrt{-1}\sinh r, -\cosh r, 0, 0)\big)$. At an arbitrary point $\hat{z} \in \widehat{M}$ which is represented with $\theta = 0$, we take an arbitrary unit tangent vectors $\hat{w} \in \widehat{V}_{\lambda,\hat{z}}$, $\hat{\varsigma} \in \widehat{V}_{\mu,\hat{z}}$. We also take unit tangent vectors $\hat{w}' \in \widehat{V}_{\lambda,\hat{z}}$, $\hat{\varsigma}' \in \widehat{V}_{\mu,\hat{z}}$ satisfying $\langle\langle \hat{w}, \hat{w}' \rangle\rangle = \langle\langle \hat{\varsigma}, \hat{\varsigma}' \rangle\rangle = 0$. Here, we note that we use the symbols $\hat{w}, \hat{w}', \hat{\varsigma}, \hat{\varsigma}'$ both for tangent vectors and for elements of $\mathbb{C}^4$ showing these tangent vectors. We define matrices U^+_λ, $U^+_\mu \in O(3, 1) \subset U(3, 1)$

by

$$U_\lambda^+ = \begin{pmatrix} x_0/\cosh r & y_0/\sinh r & v_0 & v_0' \\ x_1/\cosh r & y_1/\sinh r & v_1 & v_1' \\ x_2/\cosh r & y_2/\sinh r & v_2 & v_2' \\ x_3/\cosh r & y_3/\sinh r & v_3 & v_3' \end{pmatrix}, \ U_\mu^+ = \begin{pmatrix} x_0/\cosh r & y_0/\sinh r & u_0' & u_0 \\ x_1/\cosh r & y_1/\sinh r & u_1' & u_1 \\ x_2/\cosh r & y_2/\sinh r & u_2' & u_2 \\ x_3/\cosh r & y_3/\sinh r & u_3' & u_3 \end{pmatrix}.$$

We also consider a matrix

$$\Psi = \begin{pmatrix} \epsilon & O & O & O \\ O & \epsilon & O & O \\ O & O & \epsilon & O \\ O & O & O & \epsilon \end{pmatrix} \in O(4) \quad \text{with} \quad \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and define U_λ^-, U_μ^- by

$$U_\lambda^- = \begin{pmatrix} x_0/\cosh r & -y_0/\sinh r & -v_0 & v_0' \\ x_1/\cosh r & -y_1/\sinh r & -v_1 & v_1' \\ x_2/\cosh r & -y_2/\sinh r & -v_2 & v_2' \\ x_3/\cosh r & -y_3/\sinh r & -v_3 & v_3' \end{pmatrix} \Psi,$$

$$U_\mu^- = \begin{pmatrix} x_0/\cosh r & -y_0/\sinh r & -u_0' & u_0 \\ x_1/\cosh r & -y_1/\sinh r & -u_1' & u_1 \\ x_2/\cosh r & -y_2/\sinh r & -u_2' & u_2 \\ x_3/\cosh r & -y_3/\sinh r & -u_3' & u_3 \end{pmatrix} \Psi.$$

These four matrices induce linear transformations of $\mathbb{C}_1^4$ which preserves the Hermitian form $\langle\!\langle\ ,\ \rangle\!\rangle$, hence induce isometries $\hat\varphi_{\hat z,\lambda}^+$, $\hat\varphi_{\hat z,\mu}^+$, $\hat\varphi_{\hat z,\lambda}^-$, $\hat\varphi_{\hat z,\mu}^-$ of H_1^7. Since $U_\lambda^\pm$, $U_\mu^\pm \in U(3,1)$, these isometries preserve $\widehat{M}$ and satisfy

$$\hat\varphi_{\hat z,\lambda}^+\big(e^{\sqrt{-1}\theta}\hat p\big) = e^{\sqrt{-1}\theta}\hat\varphi_{\hat z,\lambda}^+(\hat p), \quad \hat\varphi_{\hat z,\mu}^+\big(e^{\sqrt{-1}\theta}\hat p\big) = e^{\sqrt{-1}\theta}\hat\varphi_{\hat z,\mu}^+(\hat p),$$

$$\hat\varphi_{\hat z,\lambda}^-\big(e^{\sqrt{-1}\theta}\hat p\big) = e^{-\sqrt{-1}\theta}\hat\varphi_{\hat z,\lambda}^-(\hat p), \quad \hat\varphi_{\hat z,\mu}^-\big(e^{\sqrt{-1}\theta}\hat p\big) = e^{-\sqrt{-1}\theta}\hat\varphi_{\hat z,\mu}^-(\hat p),$$

for arbitrary $\theta \in [0, 2\pi)$ and $\hat p \in \widehat{M}$. As $U_\lambda^\pm$, $U_\mu^\pm \in O(3,1)$, we have

$$d\hat\varphi_{\hat z,\lambda}^+ \circ \hat\phi = \hat\phi \circ d\hat\varphi_{\hat z,\lambda}^+, \quad d\hat\varphi_{\hat z,\mu}^+ \circ \hat\phi = \hat\phi \circ d\hat\varphi_{\hat z,\mu}^+,$$

$$d\hat\varphi_{\hat z,\lambda}^- \circ \hat\phi = -\hat\phi \circ d\hat\varphi_{\hat z,\lambda}^-, \quad d\hat\varphi_{\hat z,\mu}^- \circ \hat\phi = -\hat\phi \circ d\hat\varphi_{\hat z,\mu}^-$$

on horizontal part $\mathcal{H}$ of the tangent bundle of H_1^7, where $\hat\phi$ denotes the characteristic tensor on H_1^7 in $\mathbb{C}_1^4$. Therefore $\hat\varphi_{\hat z,\lambda}^+$, $\hat\varphi_{\hat z,\mu}^+$, $\hat\varphi_{\hat z,\lambda}^-$ and $\hat\varphi_{\hat z,\mu}^-$ induce isometries $\tilde\varphi_{\hat z,\lambda}^+$, $\tilde\varphi_{\hat z,\mu}^+$, $\tilde\varphi_{\hat z,\lambda}^-$, $\tilde\varphi_{\hat z,\mu}^-$ of $\mathbb{C}H^3(-4)$ satisfying

i) $\tilde\varphi_{\hat z,\lambda}^\pm \circ \varpi = \varpi \circ \hat\varphi_{\hat z,\lambda}^\pm, \quad \tilde\varphi_{\hat z,\mu}^\pm \circ \varpi = \varpi \circ \hat\varphi_{\hat z,\mu}^\pm,$

ii) $\tilde{\varphi}^{\pm}_{\hat{z},\lambda}\big(\varpi(\hat{z}_*)\big) = \tilde{\varphi}^{\pm}_{\hat{z},\mu}\big(\varpi(\hat{z}_*)\big) = \varpi(\hat{z})$,

iii) $d\tilde{\varphi}^{+}_{\hat{z},\lambda}(\xi_{\hat{z}_*}) = d\tilde{\varphi}^{+}_{\hat{z},\mu}(\xi_{\hat{z}_*}) = \xi_{\hat{z}},\ d\tilde{\varphi}^{-}_{\hat{z},\lambda}(\xi_{\hat{z}_*}) = d\tilde{\varphi}^{-}_{\hat{z},\mu}(\xi_{\hat{z}_*}) = -\xi_{\hat{z}}$,

iv) $d\tilde{\varphi}^{\pm}_{\hat{z},\lambda}\big(d\varpi(\hat{w}_*)\big) = d\varpi(\hat{w}), \quad d\tilde{\varphi}^{\pm}_{\hat{z},\mu}\big(d\varpi(\hat{\varsigma}_*)\big) = d\varpi(\hat{\varsigma})$,

v) $d\tilde{\varphi}^{+}_{\hat{z},\lambda} \circ J = J \circ d\tilde{\varphi}^{+}_{\hat{z},\lambda},\ d\tilde{\varphi}^{+}_{\hat{z},\mu} \circ J = J \circ d\tilde{\varphi}^{+}_{\hat{z},\mu}$,

vi) $d\tilde{\varphi}^{-}_{\hat{z},\lambda} \circ J = -J \circ d\tilde{\varphi}^{-}_{\hat{z},\lambda},\ d\tilde{\varphi}^{-}_{\hat{z},\mu} \circ J = -J \circ d\tilde{\varphi}^{-}_{\hat{z},\mu}$.

This leads us to the conclusion for $n \geq 3$. In the case $n = 2$, we have $\varsigma = \pm\phi(w),\ \varsigma' = \pm\phi(w')$. Our argument goes through also in this case. We hence complete the proof. □

In Lemma 16.2, we consider the case that $\langle w, \phi(\varsigma)\rangle = \langle w', \phi(\varsigma')\rangle = 0$. In the above proof we consider matrices $U^+, U^- \in O(3,1)$ given by

$$U^+ = \begin{pmatrix} x_0/\cosh r & y_0/\sinh r & v_0 & u_0 \\ x_1/\cosh r & y_1/\sinh r & v_1 & u_1 \\ x_2/\cosh r & y_2/\sinh r & v_2 & u_2 \\ x_3/\cosh r & y_3/\sinh r & v_3 & u_3 \end{pmatrix},$$

$$U^- = \begin{pmatrix} x_0/\cosh r & -y_0/\sinh r & -v_0 & u_0 \\ x_1/\cosh r & -y_1/\sinh r & -v_1 & u_1 \\ x_2/\cosh r & -y_2/\sinh r & -v_2 & u_2 \\ x_3/\cosh r & -y_3/\sinh r & -v_3 & u_3 \end{pmatrix} \Psi$$

instead of U^+_λ, U^-_λ. We can then obtain the following.

Lemma 16.3. *Let $z, z' \in R(r)$ be arbitrary points on a hypersurface $R(r)$ of type (B) in $\mathbb{C}H^n(c)$. For unit tangent vectors $w \in V_{\lambda,z},\ \varsigma \in V_{\mu,z}$ and $w' \in V_{\lambda,z'},\ \varsigma' \in V_{\mu,z'}$ satisfying $\langle w, \phi\varsigma\rangle = \langle w', \phi\varsigma'\rangle = 0$, there are isometries $\tilde{\varphi}^+$, $\tilde{\varphi}^-$ of $\mathbb{C}H^n$ satisfying*

i) $\tilde{\varphi}^+\big(R(r)\big) = \tilde{\varphi}^-\big(R(r)\big) = R(r)$,

ii) $\tilde{\varphi}^+(z) = \tilde{\varphi}^-(z) = z'$,

iii) $d\tilde{\varphi}^+(\xi_z) = \xi_{z'},\ d\tilde{\varphi}^-(\xi_z) = -\xi_{z'}$,

iv) $d\tilde{\varphi}^+(w) = d\tilde{\varphi}^-(w) = w'$,

v) $d\tilde{\varphi}^+(\varsigma) = d\tilde{\varphi}^-(\varsigma) = \varsigma'$,

vi) $d\tilde{\varphi}^+ \circ J = J \circ d\tilde{\varphi}^+,\ d\tilde{\varphi}^- \circ J = -J \circ d\tilde{\varphi}^-$.

Remark 16.2. Every isometry of a real hypersurface $R(r)$ of type (B) in $\mathbb{C}H^n(c)$ is equivariant.

17. Trajectories which are also curves of order 2 on real hypersurfaces of type (B) in $\mathbb{C}P^n$

In this section we study trajectories for Sasakian magnetic fields on a real hypersurface $R(r)$ of type (B) in $\mathbb{C}P^n(c)$ along the lines in the previous section.

17.1. Behaviors of structure torsions on hypersurfaces of type (B) in $\mathbb{C}P^n$. We fist consider the behavior of structure torsions. We suppose there is a trajectory γ for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on $R(r)$ which are also a curve of order 2 and study the behavior of its structure torsion ρ_γ. Being different from the case in $\mathbb{C}H^n(c)$, as principal curvatures λ, μ of vectors orthogonal to ξ, which are $\lambda = -(\sqrt{c}/2)\cot\bigl(\sqrt{c}\,r/2\bigr)$, $\mu = (\sqrt{c}/2)\tan\bigl(\sqrt{c}\,r/2\bigr)$, satisfy $\lambda < 0 < \mu$ and $|\lambda| > \mu$, we find that the function ρ_γ may satisfy $\rho_\gamma \equiv 0$ on some interval. By homogeneity of M we see $\rho_\gamma \equiv 0$ in this case. In the rest of cases $\rho_\gamma = 0$ on a discrete subset of $\mathbb{R}$. We consider on the open dense subset $\mathcal{T} = \bigl\{t \in \mathbb{R} \bigm| \rho_\gamma(t) \neq 0\bigr\}$ in $\mathbb{R}$. As we see in the proof of Proposition 16.1, we have $\langle \phi X_\gamma, Y_\gamma\rangle^2 = \|X_\gamma\|^2\|Y_\gamma\|^2$ in this case. Therefore, by the same argument as in §16.2, we find ρ_γ also satisfies the differential equation

$$(17.1) \qquad \bigl(\rho_\gamma'\bigr)^2 = (1-\rho_\gamma^2)^2(\mu - \kappa\rho_\gamma)(\kappa\rho_\gamma - \lambda)$$

on $\mathcal{T}$, which is the same equation as (16.4). By smoothness of ρ_γ, we find ρ_γ satisfies this differential equation on whole $\mathbb{R}$ in this case.

We consider this differential equation.

1) If $|\kappa| > \mu$, we see that $\rho_\gamma \equiv \mu/\kappa$ is a solution of the above equation;
2) if $|\kappa| > |\lambda|$, then $\rho_\gamma \equiv \lambda/\kappa$ and $\rho_\gamma \equiv \mu/\kappa$ are solutions of the above equation.

In those cases, by (16.2) we obtain $X_\gamma \equiv 0$ and $\|Y_\gamma\|^2 \equiv 1 - \rho_\gamma^2$ when $\rho_\gamma \equiv \mu/\kappa$, and $Y_\gamma \equiv 0$ and $\|X_\gamma\|^2 \equiv 1 - \rho_\gamma^2$ when $\rho_\gamma \equiv \lambda/\kappa$.

We study other solutions. First we consider the case $\kappa > 0$. By the differential equation (17.1), we have $\lambda/\kappa \le \rho_\gamma \le \mu/\kappa$. We solve the differential equation (17.1) by modifying it as

$$dt = \frac{\pm d\rho_\gamma}{(1-\rho_\gamma^2)\sqrt{(\mu-\kappa\rho_\gamma)(\kappa\rho_\gamma-\lambda)}},$$

where the double sign corresponds to the signature of ρ_γ'. Our argument is almost the same as in §16.2, but we need to take the signatures carefully. We put $y = \sqrt{(\mu-\kappa\rho_\gamma)/(\kappa\rho_\gamma-\lambda)}$. As we have

$$\rho_\gamma = \frac{\lambda y^2+\mu}{\kappa(y^2+1)} \qquad \text{and} \qquad \frac{d\rho_\gamma}{dy} = -\frac{2y(\mu-\lambda)}{\kappa(y^2+1)^2},$$

we get

$$\begin{aligned}
\mathcal{I} := &\int \frac{d\rho_\gamma}{(1-\rho_\gamma^2)\sqrt{(\mu-\kappa\rho_\gamma)(\kappa\rho_\gamma-\lambda)}}\, dt \\
= &\int \frac{1}{\Big\{1-\Big(\frac{\lambda y^2+\mu}{\kappa(y^2+1)}\Big)^2\Big\} y \Big(\frac{\lambda y^2+\mu}{y^2+1}-\lambda\Big)} \times \frac{2y(\lambda-\mu)}{\kappa(y^2+1)^2}\, dy \\
= &\int \frac{2\kappa(y^2+1)}{(\lambda y^2+\mu)^2-\kappa^2(y^2+1)^2}\, dy \\
= &\int \Big\{ \frac{1}{(\lambda-\kappa)y^2+(\mu-\kappa)} - \frac{1}{(\lambda+\kappa)y^2+(\mu+\kappa)} \Big\}\, dy.
\end{aligned}$$

We continue computation by dividing the situation into five cases. When $0 < \kappa < \mu$, we have

$$\begin{aligned}
\mathcal{I} = &\frac{1}{2\sqrt{\mu-\kappa}} \int \Big\{ \frac{1}{\sqrt{\kappa-\lambda}\, y + \sqrt{\mu-\kappa}} - \frac{1}{\sqrt{\kappa-\lambda}\, y - \sqrt{\mu-\kappa}} \Big\}\, dy \\
&+ \frac{1}{2\sqrt{\mu+\kappa}} \int \Big\{ \frac{1}{\sqrt{|\lambda|-\kappa}\, y - \sqrt{\mu+\kappa}} - \frac{1}{\sqrt{|\lambda|-\kappa}\, y + \sqrt{\mu+\kappa}} \Big\}\, dy \\
= &\frac{1}{2\sqrt{(\mu-\kappa)(\kappa-\lambda)}} \log \left| \frac{\sqrt{(\kappa-\lambda)(\mu-\kappa\rho_\gamma)} + \sqrt{(\mu-\kappa)(\kappa\rho_\gamma-\lambda)}}{\sqrt{(\kappa-\lambda)(\mu-\kappa\rho_\gamma)} - \sqrt{(\mu-\kappa)(\kappa\rho_\gamma-\lambda)}} \right| \\
&+ \frac{1}{2\sqrt{(\mu+\kappa)(|\lambda|-\kappa)}} \log \left| \frac{\sqrt{(|\lambda|-\kappa)(\mu-\kappa\rho_\gamma)} - \sqrt{(\mu+\kappa)(\kappa\rho_\gamma-\lambda)}}{\sqrt{(|\lambda|-\kappa)(\mu-\kappa\rho_\gamma)} + \sqrt{(\mu+\kappa)(\kappa\rho_\gamma-\lambda)}} \right|.
\end{aligned}$$

When $\kappa = \mu$, we have

$$\begin{aligned}\mathcal{I} &= \frac{1}{(\kappa-\lambda)y} + \frac{1}{2\sqrt{(\mu+\kappa)(|\lambda|-\kappa)}} \log\left|\frac{\sqrt{(|\lambda|-\kappa)(\mu-\kappa\rho_\gamma)} - \sqrt{(\mu+\kappa)(\kappa\rho_\gamma-\lambda)}}{\sqrt{(|\lambda|-\kappa)(\mu-\kappa\rho_\gamma)} + \sqrt{(\mu+\kappa)(\kappa\rho_\gamma-\lambda)}}\right| \\ &= \frac{\sqrt{\kappa\rho_\gamma-\lambda}}{(\kappa-\lambda)\sqrt{\mu-\kappa\rho_\gamma}} \\ &\quad + \frac{1}{2\sqrt{(\mu+\kappa)(|\lambda|-\kappa)}} \log\left|\frac{\sqrt{(|\lambda|-\kappa)(\mu-\kappa\rho_\gamma)} - \sqrt{(\mu+\kappa)(\kappa\rho_\gamma-\lambda)}}{\sqrt{(|\lambda|-\kappa)(\mu-\kappa\rho_\gamma)} + \sqrt{(\mu+\kappa)(\kappa\rho_\gamma-\lambda)}}\right|.\end{aligned}$$

When $\mu < \kappa < |\lambda|$, we have

$$\begin{aligned}\mathcal{I} &= \frac{-1}{\sqrt{(\kappa-\lambda)(\kappa-\mu)}} \tan^{-1}\sqrt{\frac{\kappa-\lambda}{\kappa-\mu}}\,y \\ &\quad + \frac{1}{2\sqrt{(\mu+\kappa)(|\lambda|-\kappa)}} \log\left|\frac{\sqrt{(|\lambda|-\kappa)(\mu-\kappa\rho_\gamma)} - \sqrt{(\mu+\kappa)(\kappa\rho_\gamma-\lambda)}}{\sqrt{(|\lambda|-\kappa)(\mu-\kappa\rho_\gamma)} + \sqrt{(\mu+\kappa)(\kappa\rho_\gamma-\lambda)}}\right| \\ &= \frac{-1}{\sqrt{(\kappa-\lambda)(\kappa-\mu)}} \tan^{-1}\sqrt{\frac{(\kappa-\lambda)(\mu-\kappa\rho_\gamma)}{(\kappa-\mu)(\kappa\rho_\gamma-\lambda)}} \\ &\quad + \frac{1}{2\sqrt{(\mu+\kappa)(|\lambda|-\kappa)}} \log\left|\frac{\sqrt{(|\lambda|-\kappa)(\mu-\kappa\rho_\gamma)} - \sqrt{(\mu+\kappa)(\kappa\rho_\gamma-\lambda)}}{\sqrt{(|\lambda|-\kappa)(\mu-\kappa\rho_\gamma)} + \sqrt{(\mu+\kappa)(\kappa\rho_\gamma-\lambda)}}\right|.\end{aligned}$$

When $\kappa = |\lambda|$, we have

$$\begin{aligned}\mathcal{I} &= \frac{-1}{\sqrt{(\kappa-\lambda)(\kappa-\mu)}} \tan^{-1}\sqrt{\frac{(\kappa-\lambda)(\mu-\kappa\rho_\gamma)}{(\kappa-\mu)(\kappa\rho_\gamma-\lambda)}} - \frac{y}{\mu+\kappa} \\ &= \frac{-1}{\sqrt{(\kappa-\lambda)(\kappa-\mu)}} \tan^{-1}\sqrt{\frac{(\kappa-\lambda)(\mu-\kappa\rho_\gamma)}{(\kappa-\mu)(\kappa\rho_\gamma-\lambda)}} - \frac{\sqrt{\mu-\kappa\rho_\gamma}}{(\mu+\kappa)\sqrt{\kappa\rho_\gamma-\lambda}}.\end{aligned}$$

When $\kappa > |\lambda|$, we have

$$\begin{aligned}\mathcal{I} = &\frac{-1}{\sqrt{(\kappa-\lambda)(\kappa-\mu)}} \tan^{-1}\sqrt{\frac{(\kappa-\lambda)(\mu-\kappa\rho_\gamma)}{(\kappa-\mu)(\kappa\rho_\gamma-\lambda)}} \\ &- \frac{1}{\sqrt{(\kappa+\lambda)(\kappa+\mu)}} \tan^{-1}\sqrt{\frac{\kappa+\lambda}{\kappa+\mu}}\,y\end{aligned}$$

$$= \frac{-1}{\sqrt{(\kappa-\lambda)(\kappa-\mu)}}\tan^{-1}\sqrt{\frac{(\kappa-\lambda)(\mu-\kappa\rho_\gamma)}{(\kappa-\mu)(\kappa\rho_\gamma-\lambda)}} - \frac{1}{\sqrt{(\kappa+\lambda)(\kappa+\mu)}}\tan^{-1}\sqrt{\frac{(\kappa+\lambda)(\mu-\kappa\rho_\gamma)}{(\kappa+\mu)(\kappa\rho_\gamma-\lambda)}}.$$

Summarizing up, by solving the differential equation (17.1), we obtain the following:

$$t + C = \mathrm{sgn}\big(\rho'_\gamma(t)\big)\{f_\kappa(\rho_\gamma) - g_\kappa(\rho_\gamma)\}, \tag{17.2}$$

where the constant C is determined by initial condition, the functions f_κ, g_κ are given as

$$f_\kappa(\rho_\gamma) = \begin{cases} \dfrac{1}{2\sqrt{(\mu-\kappa)(\kappa-\lambda)}}\log\left|\dfrac{\sqrt{(\kappa-\lambda)(\mu-\kappa\rho_\gamma)}+\sqrt{(\mu-\kappa)(\kappa\rho_\gamma-\lambda)}}{\sqrt{(\kappa-\lambda)(\mu-\kappa\rho_\gamma)}-\sqrt{(\mu-\kappa)(\kappa\rho_\gamma-\lambda)}}\right|, & \text{if } 0<\kappa<\mu, \\ \dfrac{\sqrt{\kappa\rho_\gamma-\lambda}}{(\kappa-\lambda)\sqrt{\mu-\kappa\rho_\gamma}}, & \text{if } \kappa=\mu, \\ \dfrac{-1}{\sqrt{(\kappa-\lambda)(\kappa-\mu)}}\tan^{-1}\sqrt{\dfrac{(\kappa-\lambda)(\mu-\kappa\rho_\gamma)}{(\kappa-\mu)(\kappa\rho_\gamma-\lambda)}}, & \text{if } \kappa>\mu, \end{cases}$$

$$g_\kappa(\rho_\gamma) = \begin{cases} \dfrac{1}{2\sqrt{(\mu+\kappa)(|\lambda|-\kappa)}}\log\left|\dfrac{\sqrt{(|\lambda|-\kappa)(\mu-\kappa\rho_\gamma)}+\sqrt{(\mu+\kappa)(\kappa\rho_\gamma-\lambda)}}{\sqrt{(|\lambda|-\kappa)(\mu-\kappa\rho_\gamma)}-\sqrt{(\mu+\kappa)(\kappa\rho_\gamma-\lambda)}}\right|, & \text{if } 0<\kappa<|\lambda|, \\ \dfrac{\sqrt{\mu-\kappa\rho_\gamma}}{(\mu+\kappa)\sqrt{\kappa\rho_\gamma-\lambda}}, & \text{if } \kappa=|\lambda|, \\ \dfrac{1}{\sqrt{(\kappa+\lambda)(\kappa+\mu)}}\tan^{-1}\sqrt{\dfrac{(\kappa+\lambda)(\mu-\kappa\rho_\gamma)}{(\kappa+\mu)(\kappa\rho_\gamma-\lambda)}}, & \text{if } \kappa>|\lambda|, \end{cases}$$

and $\mathrm{sgn}\big(\rho_\gamma(t)\big)$ denotes the signature of $\rho_\gamma(t)$ at initial.

We now study the behavior of ρ_γ. We first study the case $0<\kappa\le\mu$ $(<|\lambda|)$. Suppose $(-1\le)$ $\lambda/\kappa<\rho_\gamma(t_*)<1$ $(\le\mu/\kappa)$ and $\rho_\gamma(t_*)\ne 0$ at some t_*. By (17.1) we see $\rho'_\gamma(t_*)\ne 0$, hence have an interval (t_-,t_+) with $t_-<t_*<t_+$ satisfying that $\rho'_\gamma(t)\ne 0$ on this interval and $\lim_{t\downarrow t_-}\rho'_\gamma(t)=0=\lim_{t\uparrow t_+}\rho'_\gamma(t)$. When $\rho'_\gamma(t_*)>0$, as $-1<\rho_\gamma<1$, we find $t_+=\infty$, $t_-=-\infty$ and $\lim_{t\to\infty}\rho_\gamma(t)=1$, $\lim_{t\to-\infty}\rho_\gamma(t)=-1$.

This means that ρ_γ is monotone increasing (see Figure 4). As a matter of fact, we can get the above conclusion by just the same way as in §16.2. If we suppose $t_- > -\infty$, then we have $\rho_\gamma(t_-) = -1$ by (17.1). Thus we see γ is a geodesic and satisfies $\rho_\gamma \equiv -1$ by Lemma 6.1, which is a contradiction. Similarly, when $\rho'_\gamma(t_*) < 0$, we find $t_+ = \infty$, $t_- = -\infty$ and $\lim_{t\to\infty} \rho_\gamma(t) = -1$, $\lim_{t\to-\infty} \rho_\gamma(t) = 1$. This means that ρ_γ is monotone decreasing (see Figure 4).

Next we consider the case $\mu < \kappa \leq |\lambda|$. Suppose $(\lambda/\kappa \leq)\ -1 < \rho_\gamma(t_*) < \mu/\kappa\ (< 1)$ at some t_*. By (17.1) we see $\rho'_\gamma(t_*) \neq 0$, hence have an interval (t_-, t_+) with $t_- < t_* < t_+$ satisfying that $\rho'_\gamma(t) \neq 0$ on this interval and $\lim_{t\downarrow t_-} \rho'_\gamma(t) = 0 = \lim_{t\uparrow t_+} \rho'_\gamma(t)$. When $\rho'_\gamma(t_*) > 0$, as $-1 < \rho_\gamma$, we find that $t_- = -\infty$ and $\lim_{t\to-\infty} \rho_\gamma(t) = -1$. We also find that $\lim_{t\uparrow t_+} \rho_\gamma(t) = \mu/\kappa$ by (17.1). Since the right hand side of (17.2) goes to a finite value 0 when we make $t \uparrow t_+$, we find $t_+ < \infty$. Thus, there is $t_\sharp\ (> t+)$ such that $\rho'_\gamma(t) < 0$ on the interval $(t_+, t_\sharp)$ and $\lim_{t\uparrow t_\sharp} \rho'_\gamma(t) = 0$. Again, as $\rho_\gamma > -1$, we find that $t_\sharp = \infty$ and $\lim_{t\to\infty} \rho_\gamma(t) = -1$. When $\rho'_\gamma(t_*) < 0$, by the same argument we obtain that ρ_γ satisfies the similar property (see Figure 5).

In the case $\kappa > |\lambda|$, we find that the argument in §16.2 goes through by changing λ to $|\lambda|$. We have a sequence $\{t_i\}_{-\infty<i<\infty}$ such that $\rho'_\gamma(t) > 0$ on the interval (t_{2j-1}, t_{2j}) and $\rho'_\gamma(t) < 0$ on the interval (t_{2j}, t_{2j+1}), and $\rho_\gamma(t_{2j}) = \mu/\kappa$, $\rho_\gamma(t_{2j+1}) = \lambda/\kappa$, $\rho'_\gamma(t_i) = 0$ (see Figure 6). In this case, the lengths of intervals are

$$t_{i+1} - t_i = \frac{\pi}{2\sqrt{(\kappa+\mu)(\kappa+\lambda)}} + \frac{\pi}{2\sqrt{(\kappa-\mu)(\kappa-\lambda)}}.$$

Hence it is periodic with period $\big(\pi/\sqrt{(\kappa+\mu)(\kappa+\lambda)}\big) + \big(\pi/\sqrt{(\kappa-\mu)(\kappa-\lambda)}\big)$.

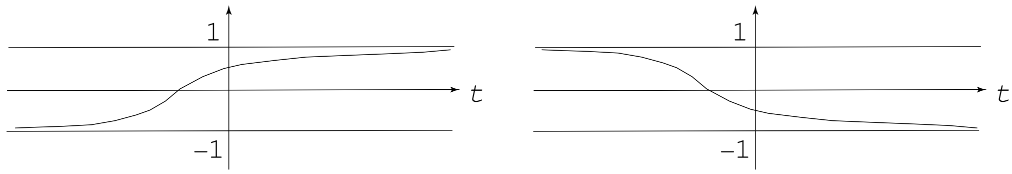


FIGURE 4. $0 < \kappa \leq \mu$

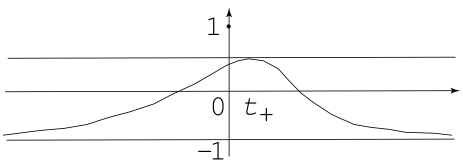


FIGURE 5. $\mu < \kappa \le |\lambda|$

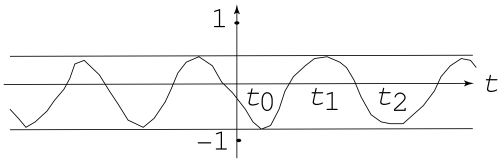


FIGURE 6. $\kappa > |\lambda|$

When $\kappa < 0$, by changing the orientation of γ and defining $\sigma(t) = \gamma(-t)$, we have a trajectory σ for $\boldsymbol{F}_{-\kappa}$. As $\rho_\sigma(t) = -\rho_\gamma(-t)$, we obtain the following.

Lemma 17.1. *Let $\boldsymbol{F}_\kappa$ be a non-trivial Sasakian magnetic field on a real hypersurface $R(r)$ of type (B) in $\mathbb{C}P^n(c)$. Suppose there exist a $\boldsymbol{F}_\kappa$-trajectory γ which is also a curve of order 2. Then its structure torsion ρ_γ satisfies the following with $\lambda = -(\sqrt{c}/2)\cot\big(\sqrt{c}\,r/2\big)$ and $\mu = (\sqrt{c}/2)\tan\big(\sqrt{c}\,r/2\big)$.*

(I) *When $0 < |\kappa| \le \mu$,*

1) $\rho_\gamma \equiv \pm 1$, (γ *is a geodesic*);
2) $\rho_\gamma \equiv 0$;
3) ρ_γ *is strictly monotone increasing and satisfies* $\lim_{t\to-\infty} \rho_\gamma(t) = -1$ *and* $\lim_{t\to\infty} \rho_\gamma(t) = 1$;
4) ρ_γ *is strictly monotone decreasing and satisfies* $\lim_{t\to-\infty} \rho_\gamma(t) = 1$ *and* $\lim_{t\to\infty} \rho_\gamma(t) = -1$.

(II) *When $\mu < \kappa \le |\lambda|$,*

1) $\rho_\gamma \equiv \pm 1$, (γ *is a geodesic*);
2) $\rho_\gamma \equiv 0$;
3) $\rho_\gamma \equiv \mu/\kappa$, ($X_\gamma \equiv 0$, $\|Y_\gamma\| = 1 - \rho_\gamma^2$);
4) $\lim_{t\to-\infty} \rho_\gamma(t) = \lim_{t\to\infty} \rho_\gamma(t) = -1$ *and there is t_0 satisfying that ρ_γ is strictly monotone increasing on the interval $(-\infty, t_0)$ and strictly monotone decreasing on the interval (t_0, ∞), and $\rho_\gamma(t_0) = \mu/\kappa$.*

(III) *When $-\mu > \kappa \ge \lambda$,*

1) $\rho_\gamma \equiv \pm 1$, (γ *is a geodesic*);

2) $\rho_\gamma \equiv 0$;

3) $\rho_\gamma \equiv \mu/\kappa$, $(X_\gamma \equiv 0,\ \|Y_\gamma\| = 1 - \rho_\gamma^2)$;

4) $\lim_{t\to-\infty} \rho_\gamma(t) = \lim_{t\to\infty} \rho_\gamma(t) = 1$ *and there is* t_0 *satisfying that* ρ_γ *is strictly monotone decreasing on the interval* $(-\infty, t_0)$ *and strictly monotone increasing on the interval* (t_0, ∞), *and* $\rho_\gamma(t_0) = \mu/\kappa$.

(Ⅳ) *When* $|\kappa| > |\lambda|$,

1) $\rho_\gamma \equiv \pm 1$, (γ *is a geodesic*);

2) $\rho_\gamma \equiv 0$;

3) $\rho_\gamma \equiv \mu/\kappa$, $(X_\gamma \equiv 0,\ \|Y_\gamma\| = 1 - \rho_\gamma^2)$;

4) $\rho_\gamma \equiv \lambda/\kappa$, $(Y_\gamma \equiv 0,\ \|X_\gamma\| = 1 - \rho_\gamma^2)$;

5) ρ_γ *is a periodic function satisfying* $\lambda \le \kappa\rho_\gamma \le \mu$. *In this case* $\kappa\rho_\gamma$ *takes all the values in the interval* $[\lambda, \mu]$.

17.2. **Circular trajectories on hypersurfaces of type (B) in $\mathbb{C}P^n$.** Just like the case of real hypersurfaces of type (B) in $\mathbb{C}H^n$, we now consider whether these cases in Lemma 17.1 occur. For this sake we consider $R(r)$ through a Hopf fibration $\varpi : S^{2n+1} \to \mathbb{C}P^n$ in the case $c = 4$. As we mentioned in §5.2, we may consider that

$$\varpi^{-1}\big(R(r)\big) = \left\{ z = (z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \;\middle|\; \begin{array}{c} |z_0|^2 + \cdots + |z_n|^2 = 1, \\ |z_0^2 + \cdots + z_2^2| = \cos 2r \end{array} \right\}$$
$$= \left\{ e^{\sqrt{-1}\theta} \begin{pmatrix} x_0 + \sqrt{-1}y_0 \\ \vdots \\ x_2 + \sqrt{-1}y_2 \end{pmatrix}^t \in \mathbb{C}^{n+1} \;\middle|\; \begin{array}{c} \theta \in \mathbb{R}, \\ x_0y_0 + \cdots + x_ny_n = 0, \\ x_0^2 + \cdots + x_n^2 = \cos^2 r, \\ y_0^2 + \cdots + y_n^2 = \sin^2 r \end{array} \right\}.$$

We take a point $\hat{z} \in \widehat{M} = \varpi^{-1}\big(R(r)\big)$ which is represented as above with $\theta = 0$. The horizontal part $\mathcal{H}_{\hat{z}}$ of the tangent space $T_{\hat{z}}\widehat{M}$ at $\hat{z}$ is represented as

$$\mathcal{H}_{\hat{z}} = \left\{ (\hat{z}, \hat{w}) \in \{\hat{z}\} \times \mathbb{C}^{n+1} \;\middle|\; \begin{array}{c} x_0u_0 + \cdots + x_nu_n = 0, \\ y_0v_0 + \cdots + y_nv_n = 0, \\ x_0v_0 + \cdots + x_nv_n + y_0u_0 + \cdots y_nu_n = 0 \end{array} \right\},$$

where we denote $\hat{w} = \big(u_0+\sqrt{-1}v_0, \ldots, u_n+\sqrt{-1}v_n\big)$. The horizontal lift $\widehat{\mathcal{N}}_{\hat{z}}$ of the unit normal vector $\mathcal{N}_{\varpi(\hat{z})}$ of $R(r)$ in $\mathbb{C}P^n(4)$, which is a unit normal vector of $\widehat{M}$ in S^{2n+1}, is represented as

$$\widehat{\mathcal{N}}_{\hat{z}} = \Big(\hat{z}, \big(-\tan rx_0+\sqrt{-1}\cot ry_0, \ldots, -\tan rx_n+\sqrt{-1}\cot ry_n\big)\Big) \in \{\hat{z}\} \times \mathbb{C}^{n+1}.$$

We put $\hat{\xi}_{\hat{z}} = -J\widehat{\mathcal{N}}_{\hat{z}}$, which is a horizontal lift of $\xi_{\varpi(\hat{z})}$ and is given by

$$\hat{\xi}_{\hat{z}} = \Big(\hat{z}, \big(\cot ry_0+\sqrt{-1}\tan rx_0, \ldots, \cot ry_n+\sqrt{-1}\tan rx_n\big)\Big) \in \{\hat{z}\} \times \mathbb{C}^{n+1}.$$

We therefore find that the horizontal lift of $\langle \xi_{\varpi(\hat{z})}\rangle^{\perp} = T^0_{\varpi(\hat{z})}R(r)$ is represented as

$$\begin{aligned}\langle\hat{\xi}_{\hat{z}}\rangle^{\perp} \cap \mathcal{H}_{\hat{z}} &= \left\{ (\hat{z}, \hat{w}) \in \{\hat{z}\} \times \mathbb{C}^{n+1} \;\middle|\; \begin{array}{c} x_0u_0 + \cdots + x_nu_n = 0, \\ y_0u_0 + \cdots + y_nu_n = 0, \\ y_0v_0 + \cdots + y_nv_n = 0, \\ x_0v_0 + \cdots + x_nv_n = 0 \end{array} \right\} \\ &= \left\{ (\hat{z}, \hat{w}) \in \{\hat{z}\} \times \mathbb{C}^{n+1} \;\middle|\; \begin{array}{c} z_0w_0 + \cdots + z_nw_n = 0, \\ z_0\overline{w}_0 + \cdots + z_n\overline{w}_n = 0 \end{array} \right\}.\end{aligned}$$

The horizontal lifts of the subspaces $V_{\lambda,\varpi(\hat{z})}$, $V_{\mu,\varpi(\hat{z})}$ of principal curvature vectors are

$$\widehat{V}_{\lambda,\hat{z}} = \left\{ (\hat{z}, \hat{w}) \in \{\hat{z}\} \times \mathbb{C}^{n+1} \;\middle|\; \begin{array}{c} x_0v_0 + \cdots + x_nv_n = 0, \\ y_0v_0 + \cdots + y_nv_n = 0, \\ u_0 = \cdots = u_n = 0 \end{array} \right\},$$

$$\widehat{V}_{\mu,\hat{z}} = \left\{ (\hat{z}, \hat{w}) \in \{\hat{z}\} \times \mathbb{C}^{n+1} \;\middle|\; \begin{array}{c} x_0u_0 + \cdots + x_nu_n = 0, \\ y_0u_0 + \cdots + y_nu_n = 0, \\ v_0 = \cdots = v_2 = 0 \end{array} \right\}.$$

In order to simplify the description, we denote by $(\!(\;,\;)\!)$ the standard inner product on $\mathbb{R}^{n+1}$, that is, for $x = (x_0, x_1, \ldots, x_n)$, $x = (x_0', x_1', \ldots, x_n') \in \mathbb{R}^{n+1}$, we set

$$(\!(x, x')\!) = x_0y_0 + x_1y_1 + \cdots + x_ny_n.$$

Let γ be a trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_{\kappa}$ on $R(r)$ which is also a curve of order 2. We suppose $\kappa\rho_{\gamma}(t_0) = \mu$ at some point t_0. We take its horizontal lift $\hat{\gamma}$ satisfying that $\hat{\gamma}(t_0) \in \widehat{M}$ is represented with $\theta = 0$ in our representation. We denote as $\hat{\gamma}(t_0) = \hat{z}_{\sharp} = x_{\sharp} + \sqrt{-1}y_{\sharp} \in \mathbb{C}^{n+1}$ and $\hat{\gamma}'(t_0) =$

$(\hat{z}_\sharp, u_\sharp + \sqrt{-1}v_\sharp) \in \mathcal{H}_{\hat{z}_\sharp} \subset \{\hat{z}_\sharp\} \times \mathbb{C}^{n+1}$ with some $x_\sharp, y_\sharp, u_\sharp, v_\sharp \in \mathbb{R}^{n+1}$. By Proposition 16.1 (see also Lemma 17.1), we see

$$\begin{aligned}\widehat{V}_{\mu,\hat{z}_\sharp} &\ni \hat{\gamma}'(t_0) - \rho_\gamma(t_0)\hat{\xi}_{\hat{\gamma}(t_0)} \\ &= \Big(\hat{z}_\sharp, \big(u_\sharp - \rho_\gamma(t_0)\cot r\, y_\sharp\big) + \sqrt{-1}\big(v_\sharp - \rho_\gamma(t_0)\tan r\, x_\sharp\big)\Big).\end{aligned}$$

We therefore obtain

$$\tag{17.3} \begin{cases} v_\sharp = \rho_\gamma(t_0)\tan r\, x_\sharp, \\ \langle\!\langle u_\sharp, u_\sharp\rangle\!\rangle - 2\rho_\gamma(t_0)\cot r\langle\!\langle u_\sharp, y_\sharp\rangle\!\rangle + \rho_\gamma^2(t_0)\cot^2 r\langle\!\langle y_\sharp, y_\sharp\rangle\!\rangle = 1 - \rho_\gamma^2(t_0). \end{cases}$$

As $\hat{\gamma}'(t_0) \in \mathcal{H}_{\hat{\gamma}(t_0)}$, we have $\langle\!\langle u_\sharp, y_\sharp\rangle\!\rangle + \langle\!\langle v_\sharp, x_\sharp\rangle\!\rangle = 0$. Hence, by the definition of structure torsions, we have

$$\tag{17.4} \begin{aligned}\rho_\gamma(t_0) = \langle\dot{\gamma}(t_0), \xi_{\gamma(t_0)}\rangle &= \langle\!\langle u_\sharp, \cot r\, y_\sharp\rangle\!\rangle + \langle\!\langle v_\sharp, \tan r\, x_\sharp\rangle\!\rangle \\ &= (\cot r - \tan r)\langle\!\langle u_\sharp, y_\sharp\rangle\!\rangle.\end{aligned}$$

Since $\langle\!\langle x_\sharp, x_\sharp\rangle\!\rangle = \cos^2 r$, $\langle\!\langle y_\sharp, y_\sharp\rangle\!\rangle = \sin^2 r$, by (16.6) and (16.7) we obtain

$$\langle\!\langle u_\sharp, u_\sharp\rangle\!\rangle = 1 - \rho_\gamma^2(t_0)(1+\cos^2 r) + \frac{2\rho_\gamma^2(t_0)\cot r}{\cot r - \tan r}.$$

We therefore get

$$\begin{aligned}1 = \|\dot{\gamma}(t_0)\|^2 &= \langle\!\langle u_\sharp, u_\sharp\rangle\!\rangle + \langle\!\langle v_\sharp, v_\sharp\rangle\!\rangle \\ &= 1 - \rho_\gamma^2(t_0)(1+\cos^2 r) + \frac{2\rho_\gamma^2(t_0)\cos^2 r}{\cos^2 r - \sin^2 r} + \rho_\gamma^2(t_0)\sin^2 r \\ &= 1 - \frac{4\rho_\gamma^2(t_0)\cos^2 r\sin^2 r}{\cos^2 r - \sin^2 r}.\end{aligned}$$

This shows $\rho_\gamma(t_0) = 0$, which is a contradiction because we supposed that $\kappa\rho_\gamma(t_0) = \mu$.

We next suppose $\kappa\rho_\gamma(t_0) = \lambda$ at some point t_0. By using the same notations we see by Proposition 16.1 (see also Lemma 17.1) that

$$\begin{aligned}\widehat{V}_{\lambda,\hat{z}_\sharp} &\ni \hat{\gamma}'(t_0) - \rho_\gamma(t_0)\hat{\xi}_{\hat{\gamma}(t_0)} \\ &= \Big(\hat{z}_\sharp, \big(u_\sharp - \rho_\gamma(t_0)\cot r\, y_\sharp\big) + \sqrt{-1}\big(v_\sharp - \rho_\gamma(t_0)\tan r\, x_\sharp\big)\Big).\end{aligned}$$

We hence obtain

$$\tag{17.5} \begin{cases} u_\sharp = \rho_\gamma(t_0)\cot r\, y_\sharp, \\ \langle\!\langle v_\sharp, v_\sharp\rangle\!\rangle - 2\rho_\gamma(t_0)\tan r\langle\!\langle v_\sharp, x_\sharp\rangle\!\rangle + \rho_\gamma^2(t_0)\tan^2 r\langle\!\langle x_\sharp, x_\sharp\rangle\!\rangle = 1 - \rho_\gamma^2(t_0). \end{cases}$$

By (17.4) and (17.5), we obtain

$$\langle\!\langle v_\sharp, v_\sharp \rangle\!\rangle = 1 - \rho_\gamma^2(t_0)(1+\sin^2 r) - \frac{2\rho_\gamma^2(t_0)\tan r}{\cot r - \tan r}.$$

We therefore get

$$\begin{aligned} 1 = \|\dot\gamma(t_0)\|^2 &= \langle\!\langle u_\sharp, u_\sharp \rangle\!\rangle + \langle\!\langle v_\sharp, v_\sharp \rangle\!\rangle \\ &= 1 - \rho_\gamma^2(t_0)(1+\sin^2 r) - \frac{2\rho_\gamma^2(t_0)\sin^2 r}{\cos^2 r - \sin^2 r} + \rho_\gamma^2(t_0)\cos^2 r \\ &= 1 - \frac{4\rho_\gamma^2(t_0)\sin^2 r\cos^2 r}{\cos^2 r - \sin^2 r}. \end{aligned}$$

This shows $\rho_\gamma(t_0) = 0$, which is a contradiction because we supposed $\kappa\rho_\gamma(t_0) = \lambda$. Thus we can conclude that if a trajectory for non-trivial $\boldsymbol{F}_\kappa$ is also a curve of order 2 then its structure torsion satisfies neither $\kappa\rho_\gamma(t_0) = \mu$ nor $\kappa\rho_\gamma(t_0) = \lambda$ at arbitrary t_0.

We also consider the case that $\rho_\gamma \equiv 0$. Recalling Lemma 6.2 we have

$$0 = \rho_\gamma' = \langle \phi A\dot\gamma, \dot\gamma \rangle = \big\langle \phi(\lambda X_\gamma + \mu Y_\gamma), X_\gamma + Y_\gamma + \rho_\gamma \xi \big\rangle = (\mu - \lambda)\langle X_\gamma, \phi Y_\gamma \rangle$$

because $\phi(V_\mu) = V_\lambda$. This shows that $\langle X_\gamma, \phi Y_\gamma \rangle = 0$. When $n = 2$, as $\dim(V_{\lambda,p}) = \dim(V_{\mu,p}) = 1$ at each point $p \in R(r)$ $\big(\subset \mathbb{C}P^2(c)\big)$ and $\|X_\gamma\|^2 = \mu/(\mu-\lambda)$, $\|Y_\gamma\|^2 = -\lambda/(\mu-\lambda)$, this is a contradiction. Hence we find that the case that $\rho_\gamma \equiv 0$ also does not occur when $n = 2$.

Combining the above argument with Lemma 17.1 we get the following.

Theorem 17.1. *On a real hypersurface $R(r)$ of type (B) in $\mathbb{C}P^2(c)$, there are no circular trajectories for Sasakian magnetic fields.*

Theorem 17.2. *On a real hypersurface $R(r)$ of type (B) in $\mathbb{C}P^n(c)$ with $n \geq 3$, there are no circular trajectories for Sasakian magnetic fields having non-null structure torsions.*

Theorem 17.3. *Let $\boldsymbol{F}_\kappa$ be a non-trivial Sasakian magnetic field on a real hypersurface $R(r)$ of type* (B) *in $\mathbb{C}P^n(c)$. When $|\kappa| > (\sqrt{c}/2)\tan\big(\sqrt{c}\,r/2\big)$, we have the following.*

(1) *There are no trajectories for $\boldsymbol{F}_\kappa$ which are curve of order 2, are not geodesics and have non-null structure torsions.*

(2) *A trajectory for $\boldsymbol{F}_\kappa$ is a geodesic if and only if $\rho_\gamma \equiv \pm 1$.*

The author guesses that even if $|\kappa| \leq (\sqrt{c}/2)\tan\big(\sqrt{c}\,r/2\big)$ the above non-existence theorem holds. Though we have no idea for this by now, we can say the following.

Proposition 17.1. *Let $\boldsymbol{F}_\kappa$ be a Sasakian magnetic field whose strength satisfies $0 < |\kappa| \leq (\sqrt{c}/2)\tan\big(\sqrt{c}\,r/2\big)$ on a real hypersurface $R(r)$ of type* (B) *in $\mathbb{C}P^n(c)$.*

(1) *When $n = 2$, there are at most 2 congruence classes of trajectories for $\boldsymbol{F}_\kappa$ which are curve of order 2 and are not geodesics.*

(2) *When $n \geq 3$, there are at most 2 congruence classes of trajectories for $\boldsymbol{F}_\kappa$ which are curve of order 2, are not geodesics and do not have null structure torsions.*

Proposition 17.2. *On a real hypersurface $R(r)$ of type* (B) *in $\mathbb{C}P^n(c)$, all geodesic trajectories for non-trivial Sasakian magnetic fields are congruent to each other.*

In order to show this we need the following results on isometries. Let $T_zR(r) = V_{\lambda,z} \oplus V_{\mu,z} \oplus \mathbb{R}\xi_z$ be the splitting of the tangent space at $z \in R(r)$ into subspaces of principal curvature vecoters.

Lemma 17.2. *Let $z, z' \in R(r)$ be arbitrary points on a hypersurface $R(r)$ of type (B) in $\mathbb{C}P^n(c)$. Given unit tangent vectors $w \in V_{\lambda,z}$, $\varsigma \in V_{\mu,z}$ and $w' \in V_{\lambda,z'}$, $\varsigma' \in V_{\mu,z'}$, we have isometries $\tilde{\varphi}_\lambda^+$, $\tilde{\varphi}_\mu^+$, $\tilde{\varphi}_\lambda^-$, $\tilde{\varphi}_\mu^-$ of $\mathbb{C}P^n$ satisfying the following conditions*:

i) $\tilde{\varphi}_\lambda^+\big(R(r)\big) = \tilde{\varphi}_\mu^+\big(R(r)\big) = \tilde{\varphi}_\lambda^-\big(R(r)\big) = \tilde{\varphi}_\mu^-\big(R(r)\big) = R(r)$;

ii) $\tilde\varphi_\lambda^+(z) = \tilde\varphi_\mu^+(z) = \tilde\varphi_\lambda^-(z) = \tilde\varphi_\mu^-(z) = z'$;

iii) $d\tilde\varphi_\lambda^+(\xi_z) = d\tilde\varphi_\mu^+(\xi_z) = \xi_{z'}$, $d\tilde\varphi_\lambda^-(\xi_z) = d\tilde\varphi_\mu^-(\xi_z) = -\xi_{z'}$;

iv) $d\tilde\varphi_\lambda^+(w) = d\tilde\varphi_\lambda^-(w) = w'$, $d\tilde\varphi_\mu^+(\varsigma) = d\tilde\varphi_\mu^-(\varsigma) = \varsigma'$,

v) $d\tilde\varphi_\lambda^+ \circ J = J \circ d\tilde\varphi_\lambda^+$, $d\tilde\varphi_\mu^+ \circ J = J \circ d\tilde\varphi_\mu^+$,

vi) $d\tilde\varphi_\lambda^- \circ J = -J \circ d\tilde\varphi_\lambda^-$, $d\tilde\varphi_\mu^- \circ J = -J \circ d\tilde\varphi_\mu^-$.

Proof. We consider the case $n = 3$ and $c = 4$. We may consider

$$\varpi^{-1}\big(R(r)\big) = \left\{ e^{\sqrt{-1}\theta} \begin{pmatrix} x_0+\sqrt{-1}y_0 \\ x_1+\sqrt{-1}y_1 \\ x_2+\sqrt{-1}y_2 \\ x_3+\sqrt{-1}y_3 \end{pmatrix}^t \in \mathbb{C}^4 \;\middle|\; \begin{array}{c} \theta \in \mathbb{R}, \\ x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0, \\ x_0^2 + x_1^2 + x_2^2 + x_3^2 = \cos^2 r, \\ y_0^2 + y_1^2 + y_2^2 + y_3^2 = \sin^2 r \end{array} \right\}.$$

We take a point $\hat z \in \widehat{M} = \varpi^{-1}\big(R(r)\big)$ which is represented as above with $\theta = 0$. At this point the horizontal lift $\hat\xi_{\hat z}$ of $\xi_{\varpi(\hat z)}$ is given as

$$\widehat{\xi}_{\hat z} = \left(\hat z, \begin{pmatrix} \cot r y_0+\sqrt{-1}\tan r x_0 \\ \cot r y_1+\sqrt{-1}\tan r x_1 \\ \cot r y_2+\sqrt{-1}\tan r x_2 \\ \cot r y_3+\sqrt{-1}\tan r x_3 \end{pmatrix}^t\right) \in \{\hat z\} \times \mathbb{C}^4.$$

and the horizontal lifts of the subspaces $V_{\lambda,z}$ and $V_{\mu,z}$ of principal curvature vectors are

$$\widehat{V}_{\lambda,\hat z} = \left\{ (\hat z, \hat w) \in \{\hat z\} \times \mathbb{C}^4 \;\middle|\; \begin{array}{c} x_0v_0 + x_1v_1 + x_2v_2 + x_3v_3 = 0, \\ y_0v_0 + y_1v_1 + y_2v_2 + y_3v_3 = 0, \\ u_0 = u_1 = u_2 = u_3 = 0 \end{array} \right\},$$

$$\widehat{V}_{\mu,\hat z} = \left\{ (\hat z, \hat w) \in \{\hat z\} \times \mathbb{C}^4 \;\middle|\; \begin{array}{c} x_0u_0 + x_1u_1 + x_2u_2 + x_3u_3 = 0, \\ y_0u_0 + y_1u_1 + y_2u_2 + y_3u_3 = 0, \\ v_0 = v_1 = v_2 = v_3 = 0 \end{array} \right\}.$$

We now take a point $\hat z_* = (\cos r, \sqrt{-1}\sin r, 0, 0) \in \widehat{M}$ and unit tangent vectors $\hat w_* = \big(\hat z_*, (0, 0, \sqrt{-1}, 0)\big) \in \widehat{V}_{\lambda,\hat z_*}$, $\hat\varsigma_* = \big(\hat z_*, (0,0,0,1)\big) \in \widehat{V}_{\mu,\hat z_*}$. We then see $\hat\xi_{\hat z_*} = \big(\hat z_*, (\sqrt{-1}\sin r, \cos r, 0, 0)\big)$. At an arbitrary point $\hat z \in \widehat{M}$ which is represented with $\theta = 0$ in our expression, we take an arbitrary unit tangent vectors $\hat w \in \widehat{V}_{\lambda,\hat z}$, $\hat\varsigma \in \widehat{V}_{\mu,\hat z}$. We also take unit tangent vectors $\hat w' \in \widehat{V}_{\lambda,\hat z}$, $\hat\varsigma' \in \widehat{V}_{\mu,\hat z}$ satisfying $\langle\!\langle \hat w, \hat w' \rangle\!\rangle = \langle\!\langle \hat\varsigma, \hat\varsigma' \rangle\!\rangle =$

0. We define orthogonal matrices U_λ^+, $U_\mu^+ \in O(4) \subset U(4)$ by

$$U_\lambda^+ = \begin{pmatrix} x_0/\cos r & y_0/\sin r & v_0 & v_0' \\ x_1/\cos r & y_1/\sin r & v_1 & v_1' \\ x_2/\cos r & y_2/\sin r & v_2 & v_2' \\ x_2/\cos r & y_2/\sin r & v_3 & v_3' \end{pmatrix}, \ U_\mu^+ = \begin{pmatrix} x_0/\cos r & y_0/\sin r & u_0' & u_0 \\ x_1/\cos r & y_1/\sin r & u_1' & u_1 \\ x_2/\cos r & y_2/\sin r & u_2' & u_2 \\ x_2/\cos r & y_2/\sin r & u_3' & u_3 \end{pmatrix}.$$

We also consider a matrix

$$\Psi = \begin{pmatrix} \epsilon & O & O & O \\ O & \epsilon & O & O \\ O & O & \epsilon & O \\ O & O & O & \epsilon \end{pmatrix} \in O(4) \quad \text{with} \quad \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and define U_λ^-, U_μ^- by

$$U_\lambda^- = \begin{pmatrix} x_0/\cos r & -y_0/\sin r & -v_0 & v_0' \\ x_1/\cos r & -y_1/\sin r & -v_1 & v_1' \\ x_2/\cos r & -y_2/\sin r & -v_2 & v_2' \\ x_2/\cos r & -y_2/\sin r & -v_3 & v_3' \end{pmatrix} \Psi,$$

$$U_\mu^- = \begin{pmatrix} x_0/\cos r & -y_0/\sin r & -u_0' & u_0 \\ x_1/\cos r & -y_1/\sin r & -u_1' & u_1 \\ x_2/\cos r & -y_2/\sin r & -u_2' & u_2 \\ x_2/\cos r & -y_2/\sin r & -u_3' & u_3 \end{pmatrix} \Psi.$$

These four matrices induce linear transformations of $\mathbb{C}^4$ which preserves the Hermitian inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathbb{C}^4$, hence induce isometries $\hat\varphi^+_{\hat z,\lambda}$, $\hat\varphi^+_{\hat z,\mu}$, $\hat\varphi^-_{\hat z,\lambda}$, $\hat\varphi^-_{\hat z,\mu}$ of S^7. Since $U_\lambda^\pm$, $U_\mu^\pm$ are orthogonal, these isometries preserve $\widehat{M}$ and satisfy

$$\hat\varphi^+_{\hat z,\lambda}\big(e^{\sqrt{-1}\theta}\hat p\big) = e^{\sqrt{-1}\theta}\hat\varphi^+_{\hat z,\lambda}(\hat p), \quad \hat\varphi^+_{\hat z,\mu}\big(e^{\sqrt{-1}\theta}\hat p\big) = e^{\sqrt{-1}\theta}\hat\varphi^+_{\hat z,\mu}(\hat p),$$

$$\hat\varphi^-_{\hat z,\lambda}\big(e^{\sqrt{-1}\theta}\hat p\big) = e^{-\sqrt{-1}\theta}\hat\varphi^-_{\hat z,\lambda}(\hat p), \quad \hat\varphi^-_{\hat z,\mu}\big(e^{\sqrt{-1}\theta}\hat p\big) = e^{-\sqrt{-1}\theta}\hat\varphi^-_{\hat z,\mu}(\hat p)$$

for arbitrary $\theta \in [0, 2\pi)$ and $\hat p \in \widehat{M}$. As $U_\lambda^\pm$, $U_\mu^\pm \in O(4)$, we have

$$d\hat\varphi^+_{\hat z,\lambda} \circ \hat\phi = \hat\phi \circ d\hat\varphi^+_{\hat z,\lambda}, \quad d\hat\varphi^+_{\hat z,\mu} \circ \hat\phi = \hat\phi \circ d\hat\varphi^+_{\hat z,\mu},$$

$$d\hat\varphi^-_{\hat z,\lambda} \circ \hat\phi = -\hat\phi \circ d\hat\varphi^-_{\hat z,\lambda}, \quad d\hat\varphi^-_{\hat z,\mu} \circ \hat\phi = -\hat\phi \circ d\hat\varphi^-_{\hat z,\mu}$$

on horizontal part $\mathcal{H}$ of the tangent bundle of S^7, where $\hat\phi$ denotes the characteristic tensor on S^7 in $\mathbb{C}^4$. Therefore $\hat\varphi^+_{\hat z,\lambda}$, $\hat\varphi^+_{\hat z,\mu}$, $\hat\varphi^-_{\hat z,\lambda}$ and $\hat\varphi^-_{\hat z,\mu}$ induce isometries $\tilde\varphi^+_{\hat z,\lambda}$, $\tilde\varphi^+_{\hat z,\mu}\tilde\varphi^+_{\hat z,\lambda}$, $\tilde\varphi^-_{\hat z,\mu}$ of $\mathbb{C}P^3(4)$ satisfying

i) $\tilde\varphi^\pm_{\hat z,\lambda} \circ \varpi = \varpi \circ \hat\varphi^\lambda_{\hat z}$, $\quad \tilde\varphi^\pm_{\hat z,\mu} \circ \varpi = \varpi \circ \hat\varphi^\pm_{\hat z,\mu}$,

ii) $\tilde{\varphi}^{\pm}_{\hat{z},\lambda}\big(\varpi(\hat{z}_*)\big) = \tilde{\varphi}^{\mu}_{\hat{z}}\big(\varpi(\hat{z}_*)\big) = \varpi(\hat{z})$,

iii) $d\tilde{\varphi}^{+}_{\hat{z},\lambda}(\xi_{\hat{z}_*}) = d\tilde{\varphi}^{+}_{\hat{z},\mu}(\xi_{\hat{z}_*}) = \xi_{\hat{z}}, \quad d\tilde{\varphi}^{+}_{\hat{z},\lambda}(\xi_{\hat{z}_*}) = d\tilde{\varphi}^{+}_{\hat{z},\mu}(\xi_{\hat{z}_*}) = -\xi_{\hat{z}}$,

iv) $d\tilde{\varphi}^{\pm}_{\hat{z},\lambda}\big(d\varpi(\hat{w}_*)\big) = d\varpi(\hat{w}), \quad d\tilde{\varphi}^{\pm}_{\hat{z},\mu}\big(d\varpi(\hat{\varsigma}_*)\big) = d\varpi(\hat{\varsigma})$,

v) $d\tilde{\varphi}^{+}_{\hat{z},\lambda} \circ J = J \circ d\tilde{\varphi}^{+}_{\hat{z},\lambda},\ d\tilde{\varphi}^{+}_{\hat{z},\mu} \circ J = J \circ d\tilde{\varphi}^{+}_{\hat{z},\mu}$,

vi) $d\tilde{\varphi}^{-}_{\hat{z},\lambda} \circ J = -J \circ d\tilde{\varphi}^{-}_{\hat{z},\lambda},\ d\tilde{\varphi}^{-}_{\hat{z},\mu} \circ J = -J \circ d\tilde{\varphi}^{-}_{\hat{z},\mu}$.

This leads us to the conclusion for $n \geq 3$. In the case $n = 2$, we have $\varsigma = \pm\phi(w),\ \varsigma' = \pm\phi(w')$. Our argument goes through also in this case. We hence complete the proof. □

In Lemma 17.2, we consider the case that $\langle w, \phi(\varsigma)\rangle = \langle w', \phi(\varsigma')\rangle = 0$. In the above proof we consider matrices $U^+, U^- \in O(3,1)$ given by

$$U^+ = \begin{pmatrix} x_0/\cos r & y_0/\sin r & v_0 & u_0 \\ x_1/\cos r & y_1/\sin r & v_1 & u_1 \\ x_2/\cos r & y_2/\sin r & v_2 & u_2 \\ x_3/\cos r & y_3/\sin r & v_3 & u_3 \end{pmatrix},$$

$$U^- = \begin{pmatrix} x_0/\cos r & -y_0/\sin r & -v_0 & u_0 \\ x_1/\cos r & -y_1/\sin r & -v_1 & u_1 \\ x_2/\cos r & -y_2/\sin r & -v_2 & u_2 \\ x_3/\cos r & -y_3/\sin r & -v_3 & u_3 \end{pmatrix} \Psi$$

instead of U^+_λ, U^-_λ. We can then obtain the following.

Lemma 17.3. *Let $z, z' \in R(r)$ be arbitrary points on a hypersurface $R(r)$ of type (B) in $\mathbb{C}P^n(c)$. For unit tangent vectors $w \in V_{\lambda,z},\ \varsigma \in V_{\mu,z}$ and $w' \in V_{\lambda,z'},\ \varsigma' \in V_{\mu,z'}$ satisfying $\langle w, \phi\varsigma\rangle = \langle w', \phi\varsigma'\rangle = 0$, there are isometries $\tilde{\varphi}^+, \tilde{\varphi}^-$ of $\mathbb{C}P^n$ satisfying*

i) $\tilde{\varphi}^+\big(R(r)\big) = \tilde{\varphi}^-\big(R(r)\big) = R(r)$,

ii) $\tilde{\varphi}^+(z) = \tilde{\varphi}^+(z) = z'$,

iii) $d\tilde{\varphi}^+(\xi_z) = \xi_{z'},\ d\tilde{\varphi}^-(\xi_z) = -\xi_{z'}$,

iv) $d\tilde{\varphi}^+(w) = d\tilde{\varphi}^+(w) = w'$,

v) $d\tilde{\varphi}^+(\varsigma) = d\tilde{\varphi}^+(\varsigma) = \varsigma'$.

vi) $d\tilde{\varphi}^+ \circ J = J \circ d\tilde{\varphi}^+,\ d\tilde{\varphi}^- \circ J = -J \circ d\tilde{\varphi}^-$.

Remark 17.1. Every isometry of a real hypersurface $R(r)$ of type (B) in $\mathbb{C}P^n(c)$ is equivariant.

Proof of Proposition 17.1. If γ is a trajectory for $\boldsymbol{F}_\kappa$ which is also a curve of order 2, is not a geodesic and does not have null structure torsion, then we see that its structure torsion ρ_γ takes all values in the interval $(-1,1)$ and that $Y_\gamma(0) = \pm\dfrac{\lambda-\kappa\rho_\gamma(0)}{\kappa\rho_\gamma(0)-\mu}\,\phi X_\gamma(0)$, where we decompose $\dot\gamma$ as $\dot\gamma = X_\gamma + Y_\gamma + \rho_\gamma\xi_\gamma \in V_\lambda \oplus V_\mu \oplus \mathbb{R}\xi$. We say γ has positive principal torsion if the signature in the expression of $Y_\gamma(0)$ is positive, and say negative principal torsion if the signature is negative.

(2) We take two trajectories γ_1, γ_2 for $\boldsymbol{F}_\kappa$ which are also curves of order 2, are not geodesics and do not have null structure torsions. Then we have t_0 satisfying $\rho_{\gamma_2}(t_0) = \rho_{\gamma_1}(0)$. By Lemma 17.2, there is an isometry φ of $R(r)$ satisfying $\varphi\big(\gamma_1(0)\big) = \gamma_2(t_0)$, $d\varphi\big(X_{\gamma_1}(0)\big) = X_{\gamma_2}(t_0)$. If both γ_1, γ_2 have positive principal torsions, then we have

$$Y_{\gamma_2}(t_0) = \phi X_{\gamma_2}(t_0) = d\varphi\big(\phi X_{\gamma_1}(0)\big) = d\varphi\big(Y_{\gamma_1}(0)\big).$$

Similarly, if both of them have negative principal torsions, we have

$$Y_{\gamma_2}(t_0) = -\phi X_{\gamma_2}(t_0) = -d\varphi\big(\phi X_{\gamma_1}(0)\big) = d\varphi\big(-\phi X_{\gamma_1}(0)\big) = d\varphi\big(Y_{\gamma_1}(0)\big).$$

Thus they are congruent to each other in each of these cases.

(1) As we do not have trajectories having null structure torsions in this case, we get the conclusion by the above argument. □

Remark 17.2. If a trajectory γ for $\boldsymbol{F}_\kappa$ with $0<|\kappa|\le(\sqrt{c}/2)\tan\big(\sqrt{c}\,r/2\big)$ on $R(r)$ in $\mathbb{C}P^n(c)$ which is also a curve of order 2 and is not a geodesic exists, then its structure torsion satisfies the following:

(1) it is monotone increasing when γ has positive principal torsion;

(2) it is monotone decreasing when γ has negative principal torsion;

(3) it is constantly 0 when γ has null principal torsion.

Proof of Proposition 17.2. As a trajectory γ is geodesic, we have $0 = \nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma = \kappa(\phi Y_\gamma + \phi X_\gamma)$, hence get $X_\gamma = Y_\gamma = 0$ and $\rho_\gamma = \pm 1$. On the other hand, if $\rho_\gamma = \pm 1$ then γ is a geodesic by Lemma 6.1. We get the assertion by Lemma 17.2. □

18. **Structure torsions of trajectories on real hypersurfaces of exceptional type in $\mathbb{C}P^n$**

We devote this section to study structure torsions of trajectories for Sasakian magnetic fields on real hypersurfaces of exceptional types in $\mathbb{C}P^n(c)$, that is, hypersurfaces of types (C), (D) and (E) in a complex projective space. For a real hypersurface M of type one of (C), (D) and (E), the holomorphic distribution T^0M of its tangent bundle TM splits into four subbundles of principal curvature vectors which are orthogonal to ξ as $T^0M = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4}$. Here, if M is a homogeneous real hypersurface of radius r given in §5, the subbundles $V_{\lambda_1}, V_{\lambda_2}, V_{\lambda_3}, V_{\lambda_4}$ correspond to the principal curvatures

$$\lambda_1 = \big(\sqrt{c}/2\big)\cot\big(\sqrt{c}r/2\big), \qquad \lambda_2 = -\big(\sqrt{c}/2\big)\tan\sqrt{c}r/2\big),$$
$$\lambda_3 = \big(\sqrt{c}/2\big)\cot\big(\sqrt{c}r/2 - \pi/4\big), \quad \lambda_4 = \big(\sqrt{c}/2\big)\cot\big(\sqrt{c}r/2 + \pi/4\big),$$

respectively. We note that the principal curvature of ξ is $\nu = \sqrt{c}\cot\sqrt{c}r$. The characteristic tensor ϕ acts on $TM = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus \mathbb{R}\xi$ as

$$\phi(V_{\lambda_1}) = V_{\lambda_1},\ \phi(V_{\lambda_2}) = V_{\lambda_2},\ \phi(V_{\lambda_3}) = V_{\lambda_4},\ \phi(V_{\lambda_4}) = V_{\lambda_3},\ \phi(\mathbb{R}\xi) = 0.$$

In order to make clear the radius r of M as a tube, we shall denote it as $M(r)$ in this section.

Proposition 18.1. *Let γ be a trajectory for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ on a real hypersurface M of type one of (C), (D) and (E) in $\mathbb{C}P^n(c)$. Suppose γ is also a curve of order 2 and $|\rho_\gamma| < 1$. If we decompose its velocity vector as $\dot\gamma = X_\gamma + Y_\gamma + Z_\gamma + W_\gamma + \rho_\gamma\xi \in V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus \mathbb{R}\xi$, we find one of the following holds*:

1) $\rho_\gamma \equiv 0$,
2) $\kappa\rho_\gamma \equiv \lambda_1,\ Y_\gamma = Z_\gamma = W_\gamma \equiv 0$ *and* $\|X_\gamma\| \equiv \sqrt{1-\rho_\gamma^2}$,
3) $\kappa\rho_\gamma \equiv \lambda_2,\ X_\gamma = Z_\gamma = W_\gamma \equiv 0$ *and* $\|Y_\gamma\| \equiv \sqrt{1-\rho_\gamma^2}$,

4) $X_\gamma = Y_\gamma \equiv 0$ *and the following equalities*

$$(18.1)\qquad \begin{cases} (1-\rho_\gamma^2)(\lambda_3-\kappa\rho_\gamma)Z_\gamma = (\lambda_3-\lambda_4)\langle Z_\gamma, \phi W_\gamma\rangle \phi W_\gamma, \\ (1-\rho_\gamma^2)(\lambda_4-\kappa\rho_\gamma)W_\gamma = (\lambda_3-\lambda_4)\langle Z_\gamma, \phi W_\gamma\rangle \phi Z_\gamma, \\ \kappa\rho_\gamma(1-\rho_\gamma^2) = \lambda_3\|Z_\gamma\|^2 + \lambda_4\|W_\gamma\|^2, \end{cases}$$

hold. In particular, Z_γ *and* W_γ *are parallel and have*

$$\|Z_\gamma\|^2 = \frac{(1-\rho_\gamma^2)(\lambda_4-\kappa\rho_\gamma)}{\lambda_4-\lambda_3}, \quad \|W_\gamma\|^2 = \frac{(1-\rho_\gamma^2)(\kappa\rho_\gamma-\lambda_3)}{\lambda_4-\lambda_3},$$

and ρ_γ *satisfies* $\lambda_3 \le \kappa\rho_\gamma \le \lambda_3$.

In the case 1) *the situations* $\|Y_\gamma(t)\| = \|Z_\gamma(t)\| = 0$ *and* $\|X_\gamma(t)\| = \|W_\gamma(t)\| = 0$ *do not occur at any point* t.

Proof. By use of the decomposition $\dot\gamma = X_\gamma + Y_\gamma + Z_\gamma + W_\gamma + \rho_\gamma\xi \in V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus \mathbb{R}\xi$, we find that the trajectory γ satisfies the following:

$$\begin{aligned} \nabla_{\dot\gamma}\dot\gamma &= \kappa\phi\dot\gamma = \kappa\big(\phi X_\gamma + \phi Y_\gamma + \phi W_\gamma + \phi Z_\gamma\big), \\ \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma &= \kappa\big\{(\nabla_{\dot\gamma}\phi)\dot\gamma + \phi\nabla_{\dot\gamma}\dot\gamma\big\} = \kappa\big\{\rho_\gamma A\dot\gamma - \langle A\dot\gamma, \dot\gamma\rangle\xi + \kappa\phi^2\dot\gamma\big\} \\ &= \kappa\big\{(\rho_\gamma\lambda_1-\kappa)X_\gamma + (\rho_\gamma\lambda_2-\kappa)Y_\gamma + (\rho_\gamma\lambda_3-\kappa)Z_\gamma + (\rho_\gamma\lambda_4-\kappa)W_\gamma \\ &\qquad - \big(\lambda_1\|X_\gamma\|^2 + \lambda_2\|Y_\gamma\|^2 + \lambda_3\|Z_\gamma\|^2 + \lambda_4\|W_\gamma\|^2\big)\xi\big\}. \end{aligned}$$

Since γ is also a curve of order 2, we substitute these into the equality

$$\|\nabla_{\dot\gamma}\dot\gamma\|^2\big(\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \|\nabla_{\dot\gamma}\dot\gamma\|^2\dot\gamma\big) = \langle\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma\rangle\nabla_{\dot\gamma}\dot\gamma.$$

As we have $\|\nabla_{\dot\gamma}\dot\gamma\| = |\kappa|\sqrt{1-\rho_\gamma^2}$, considering the action of ϕ, we get

$$\begin{aligned} &\kappa^3(1-\rho_\gamma^2)\big\{\rho_\gamma(\lambda_1-\kappa\rho_\gamma)X_\gamma + \rho_\gamma(\lambda_2-\kappa\rho_\gamma)Y_\gamma + \rho_\gamma(\lambda_3-\kappa\rho_\gamma)Z_\gamma + \rho_\gamma(\lambda_4-\kappa\rho_\gamma)W_\gamma \\ &\qquad + \big(\kappa\rho_\gamma(1-\rho_\gamma^2) - \lambda_1\|X_\gamma\|^2 - \lambda_2\|Y_\gamma\|^2 - \lambda_3\|Z_\gamma\|^2 - \lambda_4\|W_\gamma\|^2\big)\xi\big\} \\ &= \kappa^3\rho_\gamma(\lambda_3-\lambda_4)\langle Z_\gamma, \phi W_\gamma\rangle(\phi X_\gamma + \phi Y_\gamma + \phi W_\gamma + \phi Z_\gamma). \end{aligned}$$

Comparing each components of subbundles of principal curvature vectors, we find the following hold:

$$
(18.2)\quad \begin{cases}
\rho_\gamma(1-\rho_\gamma^2)(\lambda_1-\kappa\rho_\gamma)X_\gamma = \rho_\gamma(\lambda_3-\lambda_4)\langle Z_\gamma, \phi W_\gamma\rangle \phi X_\gamma, & \cdots\cdots\cdots ① \\
\rho_\gamma(1-\rho_\gamma^2)(\lambda_2-\kappa\rho_\gamma)Y_\gamma = \rho_\gamma(\lambda_3-\lambda_4)\langle Z_\gamma, \phi W_\gamma\rangle \phi Y_\gamma, & \cdots\cdots\cdots ② \\
\rho_\gamma(1-\rho_\gamma^2)(\lambda_3-\kappa\rho_\gamma)Z_\gamma = \rho_\gamma(\lambda_3-\lambda_4)\langle Z_\gamma, \phi W_\gamma\rangle \phi W_\gamma, & \cdots\cdots\cdots ③ \\
\rho_\gamma(1-\rho_\gamma^2)(\lambda_4-\kappa\rho_\gamma)W_\gamma = \rho_\gamma(\lambda_3-\lambda_4)\langle Z_\gamma, \phi W_\gamma\rangle \phi Z_\gamma, & \cdots\cdots\cdots ④ \\
\kappa\rho_\gamma(1-\rho_\gamma^2) = \lambda_1\|X_\gamma\|^2+\lambda_2\|Y_\gamma\|^2+\lambda_3\|Z_\gamma\|^2+\lambda_4\|W_\gamma\|^2. & \cdots ⑤
\end{cases}
$$

Since $\lambda_1 > \lambda_4 > 0 > \lambda_2 > \lambda_3$, we find that the function ρ_γ may vanishes on some interval. By homogeneity of M we see $\rho_\gamma \equiv 0$ in this case. In the rest of cases it might occur $\rho_\gamma = 0$ on some discrete subset of $\mathbb{R}$. We consider on the open dense subset $\mathcal{T} = \{t \in \mathbb{R} \mid \rho_\gamma(t) \neq 0\}$ in $\mathbb{R}$. We choose a point $t_0 \in \mathcal{T}$. If $X_\gamma(t_0) \neq 0$, then we have $\kappa\rho_\gamma(t_0) = \lambda_1$ and $\langle Z_\gamma(t_0), \phi W_\gamma(t_0)\rangle = 0$ by the equality ① in (18.2). We then obtain $Y_\gamma(t_0) = Z_\gamma(t_0) = W_\gamma(t_0) = 0$ by the equalities ②, ③, ④, hence obtain $\|X_\gamma(t_0)\| = \sqrt{1-\rho_\gamma^2(t_0)}$. By smoothness of $\|X_\gamma\|$ we find that $\kappa\rho_\gamma \equiv \lambda_1$ and $Y_\gamma = Z_\gamma = W_\gamma \equiv 0$ on $\mathbb{R}$. Similarly, if $Y_\gamma(t_0) \neq 0$, then we have $\kappa\rho_\gamma \equiv \lambda_2$ and $X_\gamma = Z_\gamma = W_\gamma \equiv 0$ on $\mathbb{R}$. We next consider the case $X_\gamma = Y_\gamma \equiv 0$. Considering smoothness of $\rho_\gamma, Z_\gamma, W_\gamma$, we obtain the relations (18.1) hold on $\mathbb{R}$:

$$
\begin{cases}
(1-\rho_\gamma^2)(\kappa\rho_\gamma-\lambda_3)Z_\gamma = (\lambda_4-\lambda_3)\langle Z_\gamma, \phi W_\gamma\rangle \phi W_\gamma, \\
(1-\rho_\gamma^2)(\kappa\rho_\gamma-\lambda_4)W_\gamma = (\lambda_4-\lambda_3)\langle Z_\gamma, \phi W_\gamma\rangle \phi Z_\gamma, \\
\lambda_3\|Z_\gamma\|^2+\lambda_4\|W_\gamma\|^2 = \kappa\rho_\gamma(1-{\rho_\gamma}^2).
\end{cases}
$$

These equalities show that Z_γ is parallel to ϕW_γ. We hence see $\langle Z_\gamma, \phi W_\gamma\rangle^2 = \|Z_\gamma\|^2\|W_\gamma\|^2$. Since $\rho_\gamma \neq \pm 1$, we see either Z_γ or W_γ does not vanish. If $Z_\gamma(t_0) = 0$ the second equality shows that $\kappa\rho_\gamma(t_0) = \lambda_4$. Similarly, if $W_\gamma(t_0) = 0$ the first equality shows that $\kappa\rho_\gamma(t_0) = \lambda_3$. If $Z_\gamma(t_0) \neq 0$, $W_\gamma(t_0) \neq 0$, by taking inner products of both sides of the first equality with Z_γ and with ϕW_γ, we obtain

$$
\|Z_\gamma(t_0)\|^2 = \frac{(1-\rho_\gamma^2)(\lambda_4-\kappa\rho_\gamma)}{\lambda_4-\lambda_3}, \quad \|W_\gamma(t_0)\|^2 = \frac{(1-\rho_\gamma^2)(\kappa\rho_\gamma-\lambda_3)}{\lambda_4-\lambda_3}.
$$

We can see that this expression include the cases when $\|Z_\gamma\|$ vanishes and when $\|W_\gamma\|$ vanishes. As $\lambda_4 > 0 > \lambda_3$, these expressions guarantee that $\lambda_3 \le \kappa\rho_\gamma \le \lambda_4$.

We finally study the case $\rho_\gamma \equiv 0$. The fifth equation in (18.2) and the definition of ρ_γ show that

$$\begin{cases} \lambda_1\|X_\gamma\|^2 + \lambda_2\|Y_\gamma\|^2 + \lambda_3\|Z_\gamma\|^2 + \lambda_4\|W_\gamma\|^2 = 0, \\ \|X_\gamma\|^2 + \|Y_\gamma\|^2 + \|Z_\gamma\|^2 + \|W_\gamma\|^2 = 1. \end{cases}$$

As $\lambda_1 > \lambda_4 > 0 > \lambda_2 > \lambda_3$, we can conclude that situations $\|Y_\gamma\| = \|Z_\gamma\| = 0$ and $\|X_\gamma\| = \|W_\gamma\| = 0$ do not occur. This complete the proof. □

We now compare two systems of equations (16.1) and (18.2). As they are essentially the same, we can apply the argument in §17.1 on trajectories on real hypersurfaces of exceptional type which are also curves of order 2. We here note that principal curvatures satisfy $\lambda_1 > \lambda_4 > 0 > \lambda_2 > \lambda_3$ and

$$\begin{cases} \lambda_1 > |\lambda_3| > \lambda_4 > |\lambda_2|, & \text{if } 0 < r < \pi/(4\sqrt{c}), \\ \lambda_1 = |\lambda_3| > \lambda_4 = |\lambda_2|, & \text{if } r = \pi/(4\sqrt{c}), \\ |\lambda_3| > \lambda_1 > |\lambda_2| > \lambda_4, & \text{if } \pi/(4\sqrt{c})r < \pi/(2\sqrt{c}). \end{cases}$$

In view of Lemma 17.1, we can conclude the following propositions.

Proposition 18.2. *Let $M(r)$ be a real hypersurface of exceptional type and of radius $r < \pi/(4\sqrt{c})$ in $\mathbb{C}P^n(c)$. Suppose there exist a trajectory γ for a non-trivial Sasakian magnetic field $\boldsymbol{F}_\kappa$ which is also a curve of order 2 and is not a geodesic on $M(r)$. Then its structure torsion ρ_γ satisfies the following with*

$$\lambda_1 = \big(\sqrt{c}/2\big)\cot\big(\sqrt{c}r/2\big), \qquad \lambda_2 = -\big(\sqrt{c}/2\big)\tan\big(\sqrt{c}r/2\big),$$

$$\lambda_3 = \big(\sqrt{c}/2\big)\cot\big(\sqrt{c}r/2 - \pi/4\big), \quad \lambda_4 = \big(\sqrt{c}/2\big)\cot\big(\sqrt{c}r/2 + \pi/4\big).$$

(I) *When* $0 < |\kappa| \le |\lambda_2|$,

1) $\rho_\gamma \equiv 0$;

2) ρ_γ *is strictly monotone increasing and satisfies* $\lim_{t\to-\infty}\rho_\gamma(t) = -1$ *and* $\lim_{t\to\infty}\rho_\gamma(t) = 1$;

3) ρ_γ *is strictly monotone decreasing and satisfies* $\lim_{t\to-\infty}\rho_\gamma(t)=1$ *and* $\lim_{t\to\infty}\rho_\gamma(t)=-1$.

(Ⅱ) *When* $|\lambda_2|<|\kappa|\le\lambda_4$,

1) $\rho_\gamma\equiv 0$;

2) $\rho_\gamma\equiv\lambda_2/\kappa$, $(X_\gamma=Z_\gamma=W_\gamma\equiv 0,\ \|Y_\gamma\|^2=1-\rho_\gamma^2)$;

3) ρ_γ *is strictly monotone increasing and satisfies* $\lim_{t\to-\infty}\rho_\gamma(t)=-1$ *and* $\lim_{t\to\infty}\rho_\gamma(t)=1$;

4) ρ_γ *is strictly monotone decreasing and satisfies* $\lim_{t\to-\infty}\rho_\gamma(t)=1$ *and* $\lim_{t\to\infty}\rho_\gamma(t)=-1$.

(Ⅲ) *When* $\lambda_4<\kappa\le|\lambda_3|$,

1) $\rho_\gamma\equiv 0$;

2) $\rho_\gamma\equiv\lambda_2/\kappa$, $(X_\gamma=Z_\gamma=W_\gamma\equiv 0,\ \|Y_\gamma\|^2=1-\rho_\gamma^2)$;

3) $\rho_\gamma\equiv\lambda_4/\kappa$, $(X_\gamma=Y_\gamma=Z_\gamma\equiv 0,\ \|W_\gamma\|=1-\rho_\gamma^2)$;

4) $\lim_{t\to-\infty}\rho_\gamma(t)=\lim_{t\to\infty}\rho_\gamma(t)=-1$ *and there is* t_0 *satisfying that* ρ_γ *is strictly monotone increasing on the interval* $(-\infty,t_0)$ *and strictly monotone decreasing on the interval* (t_0,∞), *and* $\rho_\gamma(t_0)=\lambda_4/\kappa$.

(Ⅳ) *When* $-\lambda_4>\kappa\ge\lambda_3$,

1) $\rho_\gamma\equiv 0$;

2) $\rho_\gamma\equiv\lambda_2/\kappa$, $(X_\gamma=Z_\gamma=W_\gamma\equiv 0,\ \|Y_\gamma\|^2=1-\rho_\gamma^2)$;

3) $\rho_\gamma\equiv\lambda_4/\kappa$, $(X_\gamma=Y_\gamma=Z_\gamma\equiv 0,\ \|W_\gamma\|^2=1-\rho_\gamma^2)$;

4) $\lim_{t\to-\infty}\rho_\gamma(t)=\lim_{t\to\infty}\rho_\gamma(t)=1$ *and there is* t_0 *satisfying that* ρ_γ *is strictly monotone decreasing on the interval* $(-\infty,t_0)$ *and strictly monotone increasing on the interval* (t_0,∞), *and* $\rho_\gamma(t_0)=\lambda_4/\kappa$.

(V) *When* $|\lambda_3|<|\kappa|\le\lambda_1$,

1) $\rho_\gamma\equiv 0$;

2) $\rho_\gamma\equiv\lambda_2/\kappa$, $(X_\gamma=Z_\gamma=W_\gamma\equiv 0,\ \|Y_\gamma\|^2=1-\rho_\gamma^2)$;

3) $\rho_\gamma \equiv \lambda_3/\kappa$, $(X_\gamma = Y_\gamma = W_\gamma \equiv 0,\ \|Z_\gamma\|^2 = 1-\rho_\gamma^2)$;

4) $\rho_\gamma \equiv \lambda_4/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0,\ \|W_\gamma\|^2 = 1-\rho_\gamma^2)$;

5) ρ_γ *is a periodic function satisfying* $\lambda_3 \leq \kappa\rho_\gamma \leq \lambda_4$.

(VI) *When* $|\kappa| > \lambda_1$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv \lambda_1/\kappa$, $(Y_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|X_\gamma\|^2 = 1-\rho_\gamma^2)$;

3) $\rho_\gamma \equiv \lambda_2/\kappa$, $(X_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|Y_\gamma\|^2 = 1-\rho_\gamma^2)$;

4) $\rho_\gamma \equiv \lambda_3/\kappa$, $(X_\gamma = Y_\gamma = W_\gamma \equiv 0,\ \|Z_\gamma\|^2 = 1-\rho_\gamma^2)$;

5) $\rho_\gamma \equiv \lambda_4/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0,\ \|W_\gamma\|^2 = 1-\rho_\gamma^2)$;

6) ρ_γ *is a periodic function satisfying* $\lambda_3 \leq \kappa\rho_\gamma \leq \lambda_4$.

Proposition 18.3. *Let* $M\big(\pi/(4\sqrt{c})\big)$ *be a real hypersurface of exceptional type of radius* $r = \pi/(4\sqrt{c})$ *in* $\mathbb{C}P^n(c)$. *Suppose there exist a trajectory* γ *for a non-trivial Sasakian magnetic field* $\boldsymbol{F}_\kappa$ *which is also a curve of order* 2 *and is not a geodesic on* $M\big(\pi/(4\sqrt{c})\big)$. *Then its structure torsion* ρ_γ *satisfies the following with*

$$\lambda_1 = -\lambda_3 = \sqrt{c}\big(\sqrt{2}+1\big)/2, \quad -\lambda_2 = \lambda_4 = \sqrt{c}\big(\sqrt{2}-1\big)/2.$$

(I) *When* $0 < |\kappa| \leq \lambda_4$,

1) $\rho_\gamma \equiv 0$;

2) ρ_γ *is strictly monotone increasing and satisfies* $\lim_{t\to-\infty}\rho_\gamma(t) = -1$ *and* $\lim_{t\to\infty}\rho_\gamma(t) = 1$;

3) ρ_γ *is strictly monotone decreasing and satisfies* $\lim_{t\to-\infty}\rho_\gamma(t) = 1$ *and* $\lim_{t\to\infty}\rho_\gamma(t) = -1$.

(II) *When* $\lambda_4 < \kappa \leq \lambda_1$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv -\lambda_4/\kappa$, $(X_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|Y_\gamma\|^2 = 1-\rho_\gamma^2)$;

3) $\rho_\gamma \equiv \lambda_4/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0,\ \|W_\gamma\| = 1-\rho_\gamma^2)$;

4) $\lim_{t\to-\infty}\rho_\gamma(t) = \lim_{t\to\infty}\rho_\gamma(t) = -1$ *and there is* t_0 *satisfying that* ρ_γ *is strictly monotone increasing on the interval* $(-\infty, t_0)$ *and strictly monotone decreasing on the interval* (t_0, ∞), *and* $\rho_\gamma(t_0) = \lambda_4/\kappa$.

(Ⅲ) *When* $-\lambda_4 > \kappa \geq -\lambda_1$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv -\lambda_4/\kappa, \quad (X_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|Y_\gamma\|^2 = 1 - \rho_\gamma^2)$;

3) $\rho_\gamma \equiv \lambda_4/\kappa, \quad (X_\gamma = Y_\gamma = Z_\gamma \equiv 0,\ \|W_\gamma\| = 1 - \rho_\gamma^2)$;

4) $\lim_{t\to-\infty}\rho_\gamma(t) = \lim_{t\to\infty}\rho_\gamma(t) = 1$ *and there is* t_0 *satisfying that* ρ_γ *is strictly monotone decreasing on the interval* $(-\infty, t_0)$ *and strictly monotone increasing on the interval* (t_0, ∞), *and* $\rho_\gamma(t_0) = \lambda_4/\kappa$.

(Ⅳ) *When* $|\kappa| > \lambda_1$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv \lambda_1/\kappa, \quad (Y_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|X_\gamma\|^2 = 1 - \rho_\gamma^2)$;

3) $\rho_\gamma \equiv -\lambda_4/\kappa, \quad (X_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|Y_\gamma\|^2 = 1 - \rho_\gamma^2)$;

4) $\rho_\gamma \equiv -\lambda_1/\kappa, \quad (X_\gamma = Y_\gamma = W_\gamma \equiv 0,\ \|Z_\gamma\|^2 = 1 - \rho_\gamma^2)$;

5) $\rho_\gamma \equiv \lambda_4/\kappa, \quad (X_\gamma = Y_\gamma = Z_\gamma \equiv 0,\ \|W_\gamma\|^2 = 1 - \rho_\gamma^2)$;

6) ρ_γ *is a periodic function satisfying* $-\lambda_1 \leq \kappa\rho_\gamma \leq \lambda_4$.

Proposition 18.4. *Let* $M(r)$ *be a real hypersurface of exceptional type and of radius* $r > \pi/(4\sqrt{c})$ *in* $\mathbb{C}P^n(c)$ *Suppose there exist a trajectory* γ *for a non-trivial Sasakian magnetic field* $\boldsymbol{F}_\kappa$ *which is also a curve of order* 2 *and is not a geodesic on* $M(r)$. *Then its structure torsion* ρ_γ *satisfies the following with*

$$\lambda_1 = \big(\sqrt{c}/2\big)\cot\big(\sqrt{c}r/2\big), \qquad \lambda_2 = -\big(\sqrt{c}/2\big)\tan\big(\sqrt{c}r/2\big),$$
$$\lambda_3 = \big(\sqrt{c}/2\big)\cot\big(\sqrt{c}r/2 - \pi/4\big), \quad \lambda_4 = \big(\sqrt{c}/2\big)\cot\big(\sqrt{c}r/2 + \pi/4\big).$$

(I) *When* $0 < |\kappa| \leq \lambda_4$,

1) $\rho_\gamma \equiv 0$;

2) ρ_γ *is strictly monotone increasing and satisfies* $\lim_{t\to-\infty}\rho_\gamma(t) = -1$ *and* $\lim_{t\to\infty}\rho_\gamma(t) = 1$;

3) ρ_γ *is strictly monotone decreasing and satisfies* $\lim_{t\to-\infty}\rho_\gamma(t) = 1$ *and* $\lim_{t\to\infty}\rho_\gamma(t) = -1$.

(Ⅱ) *When* $\lambda_4 < \kappa \le |\lambda_2|$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv \lambda_4/\kappa$, $\quad (X_\gamma = Y_\gamma = Z_\gamma \equiv 0,\ \|W_\gamma\| = 1 - \rho_\gamma^2)$;

3) $\lim_{t\to-\infty}\rho_\gamma(t) = \lim_{t\to\infty}\rho_\gamma(t) = -1$ *and there is* t_0 *satisfying that* ρ_γ *is strictly monotone increasing on the interval* $(-\infty, t_0)$ *and strictly monotone decreasing on the interval* (t_0, ∞), *and* $\rho_\gamma(t_0) = \lambda_4/\kappa$.

(Ⅲ) *When* $-\lambda_4 > \kappa \ge \lambda_2$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv \lambda_4/\kappa$, $\quad (X_\gamma = Y_\gamma = Z_\gamma \equiv 0,\ \|W_\gamma\|^2 = 1 - \rho_\gamma^2)$;

3) $\lim_{t\to-\infty}\rho_\gamma(t) = \lim_{t\to\infty}\rho_\gamma(t) = 1$ *and there is* t_0 *satisfying that* ρ_γ *is strictly monotone decreasing on the interval* $(-\infty, t_0)$ *and strictly monotone increasing on the interval* (t_0, ∞), *and* $\rho_\gamma(t_0) = \lambda_4/\kappa$.

(Ⅲ) *When* $|\lambda_2| < \kappa \le \lambda_1$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv \lambda_2/\kappa$, $\quad (X_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|Y_\gamma\|^2 = 1 - \rho_\gamma^2)$;

3) $\rho_\gamma \equiv \lambda_4/\kappa$, $\quad (X_\gamma = Y_\gamma = Z_\gamma \equiv 0,\ \|W_\gamma\| = 1 - \rho_\gamma^2)$;

4) $\lim_{t\to-\infty}\rho_\gamma(t) = \lim_{t\to\infty}\rho_\gamma(t) = -1$ *and there is* t_0 *satisfying that* ρ_γ *is strictly monotone increasing on the interval* $(-\infty, t_0)$ *and strictly monotone decreasing on the interval* (t_0, ∞), *and* $\rho_\gamma(t_0) = \lambda_4/\kappa$.

(Ⅳ) *When* $\lambda_2 > \kappa \ge -\lambda_1$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv \lambda_2/\kappa$, $\quad (X_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|Y_\gamma\|^2 = 1 - \rho_\gamma^2)$;

3) $\rho_\gamma \equiv \lambda_4/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0,\ \|W_\gamma\|^2 = 1 - \rho_\gamma^2)$;

4) $\lim_{t\to-\infty} \rho_\gamma(t) = \lim_{t\to\infty} \rho_\gamma(t) = 1$ *and there is* t_0 *satisfying that* ρ_γ *is strictly monotone decreasing on the interval* $(-\infty, t_0)$ *and strictly monotone increasing on the interval* (t_0, ∞), *and* $\rho_\gamma(t_0) = \lambda_4/\kappa$.

(V) *When* $\lambda_1 < \kappa \leq |\lambda_3|$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv \lambda_1/\kappa$, $(Y_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|X_\gamma\|^2 = 1 - \rho_\gamma^2)$;

3) $\rho_\gamma \equiv \lambda_2/\kappa$, $(X_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|Y_\gamma\|^2 = 1 - \rho_\gamma^2)$;

4) $\rho_\gamma \equiv \lambda_4/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0,\ \|W_\gamma\| = 1 - \rho_\gamma^2)$;

5) $\lim_{t\to-\infty} \rho_\gamma(t) = \lim_{t\to\infty} \rho_\gamma(t) = -1$ *and there is* t_0 *satisfying that* ρ_γ *is strictly monotone increasing on the interval* $(-\infty, t_0)$ *and strictly monotone decreasing on the interval* (t_0, ∞), *and* $\rho_\gamma(t_0) = \lambda_4/\kappa$.

(VI) *When* $-\lambda_1 > \kappa \geq \lambda_3$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv \lambda_1/\kappa$, $(Y_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|X_\gamma\|^2 = 1 - \rho_\gamma^2)$;

3) $\rho_\gamma \equiv \lambda_2/\kappa$, $(X_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|Y_\gamma\|^2 = 1 - \rho_\gamma^2)$;

4) $\rho_\gamma \equiv \lambda_4/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0,\ \|W_\gamma\|^2 = 1 - \rho_\gamma^2)$;

5) $\lim_{t\to-\infty} \rho_\gamma(t) = \lim_{t\to\infty} \rho_\gamma(t) = 1$ *and there is* t_0 *satisfying that* ρ_γ *is strictly monotone decreasing on the interval* $(-\infty, t_0)$ *and strictly monotone increasing on the interval* (t_0, ∞), *and* $\rho_\gamma(t_0) = \lambda_4/\kappa$.

(VII) *When* $|\kappa| > |\lambda_3|$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv \lambda_1/\kappa$, $(Y_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|X_\gamma\|^2 = 1 - \rho_\gamma^2)$;

3) $\rho_\gamma \equiv \lambda_2/\kappa$, $(X_\gamma = Z_\gamma = W_\gamma \equiv 0,\ \|Y_\gamma\|^2 = 1 - \rho_\gamma^2)$;

4) $\rho_\gamma \equiv \lambda_3/\kappa$, $(X_\gamma = Y_\gamma = W_\gamma \equiv 0,\ \|Z_\gamma\|^2 = 1 - \rho_\gamma^2)$;

5) $\rho_\gamma \equiv \lambda_4/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0,\ \|W_\gamma\|^2 = 1 - \rho_\gamma^2)$;

6) ρ_γ *is a periodic function satisfying* $\lambda_3 \leq \kappa\rho_\gamma \leq \lambda_4$.

We set

$$\lambda(r,c) = \begin{cases} \big(\sqrt{c}/2\big)\tan\big(\sqrt{c}r/2\big), & \text{if } 0 < r \leq \pi/(4\sqrt{c}), \\ \big(\sqrt{c}/2\big)\cot\big(\sqrt{c}r/2 + \pi/4\big), & \text{if } \pi/(4\sqrt{c}) < r < \pi/(2\sqrt{c}). \end{cases}$$

As a consequence of these propositions we can conclude the following.

Theorem 18.1. *Let* $\boldsymbol{F}_\kappa$ *be a non-trivial Sasakian magnetic field on a real hypersurface* $M(r)$ *of exceptional type in* $\mathbb{C}P^n(c)$. *When* $0 < |\kappa| \leq \lambda(r,c)$, *there are no circular trajectories for* $\boldsymbol{F}_\kappa$ *having non-null structure torsions.*

The author considers that similar result as for real hypersurfaces of type (B) holds on the number of congruence classes of circular trajectories. We shall discuss this in future.

19. **Some characterizations of real hypersurfaces of type (A) in a nonflat complex space form**

As an application of our study on circular trajectories for Sasakian magnetic fields, we give some characterizations of hypersurfaces of type (A) in this section. Even on a real hypersurface of type (A), if we fix a Sasakian magnetic field, then its circular trajectories have definite directions. We therefore consider all Sasakian magnetic fields on a real hypersurface, and study the amount of circular trajectories.

Theorem 19.1. *Let M be a real hypersurface in a nonflat complex space form $\mathbb{C}M^n(c)$ $(c \neq 0)$. It is of type (A_0) or of type (A_1) if and only if for every unit tangent vector $v \in UM$ which is neither orthogonal to ξ nor parallel to ξ there is a circular trajectory for some non-trivial Sasakian magnetic field on M whose initial vector is v.*

Proof. "Only if" part. We take an arbitrary $v \in UM$ satisfying $0 < |\eta(v)| < 1$. By Theorems 7.1, 11.1, 12.1 and 13.1 (or by Propositions 7.1, 11.1, 12.1 and 13.1), a trajectory for a Sasakian magnetic field $\boldsymbol{F}_{\lambda/\eta(v)}$ with initial vector v is circular. Here, λ denotes the principal curvature for vectors orthogonal to ξ. This shows that the "only if" part holds.

"If" part. For an arbitrary $v \in UM$ with $0 < |\eta(v)| < 1$ we take a circular trajectory γ whose initial vector is v. Since γ is circular, we particularly have ρ_γ is constant along γ. By Lemma 6.1, we have $\langle (A\phi - \phi A)v, v\rangle = 2\rho_\gamma'(0) = 0$. By continuity of the Riemannian metric, the shape operator and the characteristic tensor field and by their linearity, we obtain $\langle (A\phi - \phi A)u, u\rangle = 0$ for arbitrary tangent vector $u \in TM$. Since $A\phi - \phi A$ is symmetric, we find that

$$\begin{aligned} 0 &= \langle (A\phi - \phi A)(u+w), u+w\rangle \\ &= \langle (A\phi - \phi A)u, u\rangle + \langle (A\phi - \phi A)u, w\rangle + \langle (A\phi - \phi A)w, u\rangle + \langle (A\phi - \phi A)w, w\rangle \\ &= 2\langle (A\phi - \phi A)u, w\rangle \end{aligned}$$

for arbitrary $u, w \in TM$. If we take $w = (A\phi - \phi A)u$, this shows $(A\phi - \phi A)u = 0$. We hence get $A\phi - \phi A = 0$ and find that M is of type (A) by Lemma 5.4. As hypersurfaces of type (A_2) do not satisfy the condition. For a trajectory γ on a hypersurface of type (A_2), its principal torsion satisfies $0 \le \tau_\gamma \le \sqrt{1-\rho_\gamma^2}$. But when $\rho_\gamma \ne 0$ principal torsions of circular trajectories should satisfy $\tau_\gamma = 0$ or $\tau_\gamma = \sqrt{1-\rho_\gamma^2}$. Thus we get the conclusion. □

The above result distinguish between hypersurfaces of types (A_0), (A_1) and of type (A_2). In [37] Maeda-Adachi characterized hypersurfaces of type (A) by constancy of structure torsion of trajectories. But as all trajectories on hypersurfaces of type (A) have constant structure torsion, we can not distinguish these hypersurfaces. Thus we may say that the above result improve their result. But as we only use the property that structure torsions of circular trajectories are constant, we are interested in getting more information by existence of circular trajectories.

In order to give other characterization of homogeneous real hypersurfaces in a complex space form, we restrict ourselves to the class of Hopf hypersurfaces. For a real hypersurface M we consider its holomorphic distribution $T^0M = \big\{v \in TM \mid \langle v, \xi\rangle = 0\big\}$, and denote by $\mathrm{Proj}_0 : TM \to T^0M$ the projection.

First we consider the constancy of structure torsions of trajectories.

Lemma 19.1. *On a Hopf hypersurface M, the structure torsion ρ_γ is constant if and only if $\big\langle (\phi A - A\phi)\mathrm{Proj}_0(\dot\gamma), \mathrm{Proj}_0(\dot\gamma)\big\rangle \equiv 0$.*

Proof. For a tangent vector $u \in TM$ we decompose it as $u = v + \eta(u)\xi$ with $v \in T^0M$. We then have $\phi A u = \phi A\big(v + \eta(u)\xi\big) = \phi\big(Av + \eta(u)\nu\xi\big) = \phi A v$. We therefore have

$$\langle(\phi A - A\phi)u, u\rangle = \langle(\phi A - A\phi)v, u\rangle = \langle(\phi A - A\phi)v, v\rangle.$$

Hence we get our conclusion by Lemma 6.2. □

Next we consider the conditions that trajectories for Sasakian magnetic fields to be circular. On a Hopf hypersurface M, the condition (2)-ii) in Lemma 6.3 that a trajectory γ for $\boldsymbol{F}_\kappa$ is circular turns to

$$\begin{cases} \rho_\gamma A\mathrm{Proj}_0(\dot\gamma) = \kappa\rho_\gamma^2\mathrm{Proj}_0(\dot\gamma), \\ \kappa\rho_\gamma(1-\rho_\gamma^2) + \rho_\gamma^2\langle A\xi,\xi\rangle - \langle A\dot\gamma,\dot\gamma\rangle = 0, \end{cases} \tag{19.1}$$

if we decompose the equality in the condition (2)-ii) in Lemma 6.3 into the component parallel to ξ and the component in T^0M. Under the condition $\rho_\gamma = 0$, they are equivalent to $\langle A\dot\gamma,\dot\gamma\rangle = 0$, and under the condition $\rho_\gamma \neq 0$, they are equivalent to $A\mathrm{Proj}_0(\dot\gamma) = \kappa\rho_\gamma\mathrm{Proj}_0(\dot\gamma)$. In this case, this shows that $\mathrm{Proj}_0(\dot\gamma)$ is a principal vector with principal curvature $\kappa\rho_\gamma$.

Lemma 19.2. *If γ is a circular trajectory for a Sasakian magnetic field on a Hopf hypersurface, then either $\rho_\gamma \equiv 0$ or $\mathrm{Proj}_0\big(\dot\gamma(t)\big)$ is principal at each t.*

We note that the converse of the above lemma holds.

Lemma 19.3. *Let γ be a trajectory for a Sasakian magnetic field on a Hopf hypersurface. If $\mathrm{Proj}_0\big(\dot\gamma(t)\big)$ is principal on an interval I, then ρ_γ is constant on this interval.*

Proof. If $A\mathrm{Proj}_0\big(\dot\gamma(t)\big) = \alpha(t)\mathrm{Proj}_0\big(\dot\gamma(t)\big)$, by use of the computation in Lemma 6.2, we have

$$\begin{aligned} \rho_\gamma'(t) &= \langle\dot\gamma(t), \phi A\dot\gamma(t)\rangle = \big\langle\dot\gamma(t), \phi A\big(\mathrm{Proj}_0\big(\dot\gamma(t)\big)\big) + \phi A\big(\rho_\gamma(t)\xi\big)\big\rangle \\ &= \big\langle\dot\gamma(t), \alpha(t)\phi\mathrm{Proj}_0\big(\dot\gamma(t)\big) + \nu\rho_\gamma(t)\phi\xi\big\rangle = \alpha(t)\langle\dot\gamma(t), \phi\dot\gamma(t)\rangle = 0, \end{aligned}$$

hence get the conclusion. □

We here consider to reduce the number of circular trajectories in the assumption to characterize hypersurfaces of type (A).

Theorem 19.2. *Let M be a Hopf hypersurface in $\mathbb{C}M^n$. Then M is congruent to one of hypersurfaces of types (A_0) and (A_1) if and only if at each point $p \in M$ there exist tangent vectors $v_1, \ldots, v_{2n-2} \in T_p^0 M$ satisfying the following properties*:

i) $v_1, \ldots, v_{2n-2}$ *span* $T_p^0 M$;

ii) *For each i $(1 \le i \le 2n{-}2)$, there is a circular trajectory γ_i for some Sasakian magnetic field satisfying that $\rho_{\gamma_i} \neq 0$ and $\mathrm{Proj}_0\big(\dot{\gamma}_i(0)\big)$ is parallel to v_i*;

iii) *For each j $(2 \le j \le 2n{-}2)$, there is a circular trajectory γ_{1j} satisfying that $\rho_{\gamma_{1j}} \neq 0$ and $\mathrm{Proj}_0\big(\dot{\gamma}_{1j}(0)\big)$ is parallel to $v_1{+}v_j$.*

Proof. "Only if" part. We take κ satisfying $\kappa > \lambda$, where λ is the principal curvature for vectors orthogonal to ξ. For a Sasakian magnetic field $\boldsymbol{F}_\kappa$ and for an arbitrary unit vector $v \in T_p^0 M$, a trajectory γ with initial vector $\big(\sqrt{\kappa^2 - \lambda^2}\, v + \lambda\xi\big)/\kappa$ is circular, by Theorems 7.1, 11.1, 12.1 and 13.1. Thus we find that the conditions hold.

"If" part. By the conditions ii) and iii), we find that $v_1, \ldots v_{2n-2}$ and $v_1 + v_2, \ldots, v_1 + v_{2n-2}$ are principal curvature vectors by Lemma 19.2. Since $v_1, \ldots v_{2n-2}$ are linearly independent, we see they have the same principal curvatures. Thus $T_p^0 M$ is a vector subspace of principal curvature vectors associated with one principal curvature. Since $T_p^0 M$ is invariant under the action of ϕ, we find $A\phi = \phi A$, which lead us to that M is of type (A). As the bundle $T_p^0 M$ of a hypersurface of type (A_2) is divided into two principal subbundles, we get the conclusion. □

We here consider to drop the third condition in Theorem 19.2, which may seem a bit artificial.

Proposition 19.1. *Let M be a Hopf hypersurface in $\mathbb{C}M^n$. Then M is a hypersurface of type (A) if and only if at each point $p \in M$ there exist unit tangent vectors $v_1, \ldots, v_{n-1} \in T_p^0 M$ satisfying the following properties*:

i) $v_1, \phi v_1, \ldots, v_{n-1}, \phi v_{n-1}$ *span* $T_x^0 M$;

ii) *For each* i $(1 \leq i \leq n-1)$, *there are circular trajectories* γ_i^+, γ_i^- *for some Sasakian magnetic fields* $\boldsymbol{F}_{\kappa_i^+}, \boldsymbol{F}_{\kappa_i^-}$ *satisfying that*

a) $\rho_{\gamma_i^+}(0) \neq 0,\ \rho_{\gamma_i^-}(0) \neq 0,$

b) $\kappa_i^+ \rho_{\gamma_i^+}(0) = \kappa_i^- \rho_{\gamma_i^-}(0),$

c) $\mathrm{Proj}_0\big(\dot\gamma_i^+(0)\big)$ *is parallel to* v_i *and* $\mathrm{Proj}_0\big(\dot\gamma_i^-(0)\big)$ *is parallel to* ϕv_i;

Proof. "Only if" part. When M is a real hypersurface of types (A_0) and (A_1), as we pointed out in the proof of Theorem 19.2, we have desirable tangent vectors with given κ $(> \lambda)$ and $\rho_\gamma = \lambda/\kappa$. When M is a real hypersurface of type (A_2) in $\mathbb{C}P^n(c)$, the holomorphic distribution T^0M splits into subbundles $V_\lambda \oplus V_\mu$ of principal curvature vectors associated with $\lambda = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2),\ \mu = -(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$. We take κ with $\kappa > \max\{\lambda, |\mu|\}$. By Proposition 7.4 (or Theorems 7.2, 7.3, 7.4), we find the following:

- For an arbitrary unit vector $v \in V_\lambda$ a trajectory γ for $\boldsymbol{F}_\kappa$ with initial vector $\big(\sqrt{\kappa^2-\lambda^2}\, v + \lambda\xi\big)/\kappa$ is circular;
- For an arbitrary unit vector $w \in V_\mu$ a trajectory γ for $\boldsymbol{F}_\kappa$ with initial vector $\big(\sqrt{\kappa^2-\lambda^2}\, w + \mu\xi\big)/\kappa$ is circular.

Since V_λ, V_μ are invariant under the action of ϕ, we also have desirable tangent vectors in this case.

When M is a real hypersurface of type (A_2) in $\mathbb{C}H^n(c)$, the holomorphic distribution T^0M splits into subbundles $V_\lambda \oplus V_\mu$ of principal curvature vectors associated with $\lambda = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2),\ \mu = (\sqrt{|c|}/2)\tanh(\sqrt{|c|}\,r/2)$. We take κ with $\kappa > \lambda$ $(> \mu)$. By Proposition 14.1 (or by Theorem 14.1), we find that M has the same properties as for real hypersurfaces of type (A_2) in $\mathbb{C}P^n$. Since V_λ, V_μ are also invariant under the action of ϕ, we have desirable tangent vectors also in this case.

"If" part. By the condition ii) and Lemma 19.2 show that v_i and ϕv_i are principal curvature vectors associated with the same principal curvature $\kappa_i^+ \rho_{\gamma_i^+}(0) =$

$\kappa_i^- \rho_{\gamma_i^-}(0)$. We hence find that each vector subspace of principal curvature vectors in $T_x^0 M$ is invariant under the action of ϕ. Thus we find $A\phi = \phi A$ and M is of type (A). □

The proof of the above Theorem shows the following.

Proposition 19.2. *Let M be a Hopf hypersurface in $\mathbb{C}M^n$. Then M is congruent to one of real hypersurfaces of types (A_0) and (A_1) if and only if at each point $p \in M$ there exist unit tangent vectors $v_1, \dots, v_{2n-2} \in T_p^0 M$ and a nonzero constant $\alpha = \alpha(p)$ satisfying the following properties*:

i) $v_1, \dots, v_{2n-2}$ *span* $T_p^0 M$;

ii) *For each i $(1 \le i \le 2n–2)$, there are circular trajectories γ_i for some Sasakian magnetic fields $\boldsymbol{F}_{\kappa_i}$ satisfying that $\kappa_i \rho_{\gamma_i}(0) = \alpha$ and* $\mathrm{Proj}_0\big(\dot\gamma_i(0)\big)$ *is parallel to v_i.*

Proof. The "only if" part was proved in Theorem 19.2. We check the "if" part. By Lemma 19.2, we see that $v_1, \dots, v_{2n-2}$ are principal curvature vectors with principal curvature α. Thus, $T^0 M$ is the bundle of principal curvature vectors associated with α by the condition i). In particular, we have $A\phi = \phi A$. Thus we obtain that M is of type (A_0) or of type (A_1). □

We next consider to drop the condition on structure torsions in Theorem 19.2.

Theorem 19.3. *Let M be a Hopf hypersurface in $\mathbb{C}P^n$. Then M is congruent to one of hypersurfaces of type (A) if and only if at each point $p \in M$ there exist tangent vectors $v_1, \dots, v_{2n-2} \in T_p^0 M$ satisfying the following properties*:

i) $v_1, \dots, v_{2n-2}$ *span* $T_p^0 M$;

ii) *For each i $(1 \le i \le 2n–2)$, there is a circular trajectory γ_i for some Sasakian magnetic field satisfying that* $\mathrm{Proj}_0\big(\dot\gamma_i(0)\big)$ *is parallel to v_i*;

iii) *For each* i, j $(2 \leq i < j \leq 2n{-}2)$, *there is a circular trajectory* γ_{ij} *satisfying that* $\mathrm{Proj}_0\big(\dot\gamma_{ij}(0)\big)$ *is parallel to* $v_i{+}v_j$.

Theorem 19.4. *Let* M *be a Hopf hypersurface in* $\mathbb{C}H^n$. *Then* M *is congruent to one of hypersurfaces of type* (A_0) *or* (A_1) *if and only if at each point* $p \in M$ *there exist tangent vectors* $v_1, \ldots, v_{2n-2} \in T_p^0 M$ *satisfying the following properties*:

i) $v_1, \ldots, v_{2n-2}$ *span* $T_p^0 M$;

ii) *For each* i $(1 \leq i \leq 2n{-}2)$, *there is a circular trajectory* γ_i *for some Sasakian magnetic field satisfying that* $\mathrm{Proj}_0\big(\dot\gamma_i(0)\big)$ *is parallel to* v_i;

iii) *For each* i, j $(2 \leq i < j \leq 2n{-}2)$, *there is a circular trajectory* γ_{ij} *satisfying that* $\mathrm{Proj}_0\big(\dot\gamma_{ij}(0)\big)$ *is parallel to* $v_i{+}v_j$.

Proof of Theorems 19.3, 19.4 . The "only if" part is given in the proof of Theorem 19.2 and in the proof of Proposition 19.1. We hence consider the "if" part along the lines in the proof of Theorem 19.1. By Lemma 19.1, the conditions ii) and iii) show that $\langle(\phi A - A\phi)v_j, v_j\rangle = 0$ for $j = 1, \ldots, 2n{-}2$ and $\langle(\phi A - A\phi)(v_i + v_j), v_i + v_j\rangle = 0$ for $1 \leq i < j \leq 2n{-}2$. We hence have $\langle(\phi A - A\phi)v_i, v_j\rangle = 0$ for $1 \leq i, j \leq 2n{-}2$, because $\phi A - A\phi$ is symmetric. By the condition i), we get $(\phi A - A\phi)v_i = 0$ for $i = 1, \ldots, 2n{-}2$. We then find $A\phi = \phi A$ on $T^0 M$, and hence M is of type (A). In $\mathbb{C}H^n$, by Proposition 14.1, real hypersurfaces of type (A_2) do not satisfy the condition iii). Thus we get the conclusion. □

Since we give some characterization of real hypersurfaces of types (A_0) and (A_1), we next consider real hypersurfaces of type (A_2). In view of Theorem 19.2, we can characterize hypersurfaces of type (A_2) in the following manner.

Proposition 19.3. *Let* M *be a Hopf hypersurface in* $\mathbb{C}M^n$ *and* ℓ *be an integer with* $1 \leq \ell \leq 2n{-}1$. *Then* M *is congruent to a tube around of a totally geodesic* $\mathbb{C}M^\ell$ *or* *a tube around of a totally geodesic* $\mathbb{C}M^{n-\ell}$ *if and only if at each point* $p \in M$ *there exist unit tangent vectors* $v_1, \ldots, v_{2n-2} \in T_p^0 M$ *satisfying the following properties*:

i) $v_1, \ldots, v_{2n-2}$ *span* $T_p^0 M$;

ii) *For each* i $(1 \le i \le 2n{-}2)$, *there is a circular trajectory* γ_i *for some Sasakian magnetic field satisfying that* $\rho_{\gamma_i} \neq 0$ *and* $\mathrm{Proj}_0\big(\dot\gamma_i(0)\big)$ *is parallel to* v_i;

iii) *For each* i, j $(1 \le i < j \le 2n{-}2\ell{-}2)$, *there is a circular trajectory* γ_{ij} *satisfying that* $\mathrm{Proj}_0\big(\dot\gamma_{ij}(0)\big)$ *is parallel to* $v_i{+}v_j$;

iv) *For each* i, j $(2n{-}2\ell{-}1 \le i < j \le 2n{-}2)$, *there is a circular trajectory* γ_{ij} *satisfying that* $\mathrm{Proj}_0\big(\dot\gamma_{ij}(0)\big)$ *is parallel to* $v_i{+}v_j$;

v) *Every trajectory* γ *satisfying that* $\rho_\gamma \neq 0$ *and that* $\mathrm{Proj}_0\big(\dot\gamma(0)\big)$ *is parallel to* $v_1{+}v_{2n-2}$ *is not circular.*

Proof. The "only if" part is proved in the proof of Proposition 19.1. We hence show the "if" part. We denote by V_1 $(\subset T_p^0 M)$ the subspace spanned by $v_1, \ldots, v_{2n-2\ell-2}$, and by V_2 $(\subset T_p^0 M)$ the subspace spanned by $v_{2n-2\ell-1}, \ldots, v_{2n-2}$. By the conditions ii) and iii) and by Lemma 19.1, we have $\langle(\phi A - A\phi)v_j, v_j\rangle = 0$ for $i = 1, \ldots, 2n{-}2$ and $\langle(\phi A - A\phi)(v_i + v_j), v_i + v_j\rangle = 0$ for $1 \le i < j \le 2n{-}2\ell{-}2$. Thus we have $\langle(\phi A - A\phi)v_i, v_j\rangle = 0$ for $1 \le i, j \le 2n{-}2\ell{-}2$. This shows that $\phi A - A\phi = 0$ on V_1. Similarly, by the conditions ii) and iv) we find that $\phi A - A\phi = 0$ on V_2. As $T^0 M = V_1 + V_2$, we see $\phi A = A\phi$ on $T^0 M$. This shows that M is of type (A). The condition (v) shows that M is not of types (A_0) and (A_1). We get the conclusion. □

Remark 19.1. In Proposition 19.3, if M is a Hopf hypersurface in $\mathbb{C}H^n$, then we can drop the condition $\rho_\gamma \neq 0$ in the condition v).

References

[1] T. Adachi, *Kähler magnetic flows on a manifold of constant holomorphic sectional curvature*, Tokyo J. Math. 18(1995), 473–483.

[2] T. Adachi, *A comparison theorem for magnetic Jacobi fields*, Proc. Edinburgh Math. Soc. 40(1997), 293–308.

[3] T. Adachi, *Trajectories on geodesic spheres in a non-flat complex space form*, J. Geom. 90(2008), 163–172.

[4] T. Adachi, *Geodesics on real hypersurfaces of type A_2 in a complex space form*, Mon. Math. 153(2008), 283–293.

[5] T. Adachi, *Moderate Killing helices of proper order four on a complex projective space*, Tokyo J. Math. 33(2010), 435–452.

[6] T. Adachi, *Essential Killing helices of order less than five on a non-flat complex space form*, J. Math. Soc. Japan 64(2011), 1–20.

[7] T. Adachi, *Magnetic Jacobi fields for Kähler magnetic fields*, in *Recent Progress in Differential Geometry and its Related Fields*, Proceedings of the 2nd International Colloquium on Differential Geometry and its Related Fields, World Scientific 2011, 41–53.

[8] T. Adachi, *A theorem of Hadamard-Cartan type for Kähler magnetic fields*, to appear in J. Math. Soc. Japan.

[9] T. Adachi, T. Bao and S. Maeda, *Congrence classes of minimal ruled real hypersurfaces in a nonflat complex space form*, preprint.

[10] T. Adachi and S. Maeda, *Length spectrum of circles in a complex projective space*, Osaka J. Math. 35(1998), 553-565.

[11] T. Adachi, M. Kameda and S. Maeda, *Geometric meaning of Sasakian space forms from the viewpoint of submanifold theory*, Kodai Math. J. 33(2010), 383–397.

[12] T. Adachi, M. Kimura and S. Maeda, *Real hypersurfaces some of whose geodedsics are plane curves in nonflat complex space form*, Tohoku Math. J. 57(2005), 223–230.

[13] T. Adachi, S. Maeda and S. Udagawa, *Circles in a complex projective space*, Osaka J. Math. 32(1995), 709-719.

[14] T. Adachi, S. Maeda and M. Yamagishi, *Length spectrum of geodesic spheres in a non-flat complex space form*, J. Math Soc Japan 54(2002), 373-408.

[15] T. Bao, *Sasakian magnetic fields on real hypersurfaces in a complex hyperbolic space*, in *Recent Progress in Differential Geometry and its Related Fields*, T. Adachi, H. Hashimoto and M. Hristov eds., World Scientific, Singapole (2011),73–90.

[16] T. Bao and T. Adachi, *Circular trajectories on real hypersurfaces in a nonflat complex space form*, J. Geom. 96(2009), 41–55.

[17] T. Bao and T. Adachi, *Lengths of circular trajectories on geodesic spheres in a complex projective space*, Kodai Math. J. 34(2011), 257–271.

[18] T. Bao and T. Adachi, *Trajectories for Sasakian magnetic fields on homogeneous real hypersurfaces of type (B) in a complex hyperbolic space*, Diff. Geom. its Appl. 29(2011) sup1, S28–S32.

[19] T. Bao and T. Adachi, *Behavior of circular trajectories on hypersurfaces of type (A_1) in a complex hyperbolic space*, Kodai Math. J. 34(2011), 485–504.

[20] J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*, J. reine angew. Math. 395(1989), 132–141.

[21] J. Berndt, *Real hypersurfaces with constant principal curvatures in complex space forms*, Geometry and topology of submanifolds II (Avignon, 1988), 10–19, World Sci. Publ. Teaneck, NJ, 1990.

[22] J. Berndt and H. Tamaru, *Cohomogeneity one actions on noncompact symmetric spaces of rank one*, Trans. Amer. Math. Soc. 359(2007), 3425–3438.
[23] J. Berndt and H. Tamaru, *Homogeneous hypersurfaces in complex hyperbolic spaces*, Geom. Dedicata 138(2009), 129–150.
[24] D.E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math. 509, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
[25] B-y. Chen, *Geometry of Submanifolds*, Pure Appl. Math. Series Monographs and Textbook 22, Dekker, New York, 1973.
[26] A. Comtet, *On the Landau levels on the hyperbolic plane*, Ann. Phys. **173** (1987), 185-209.
[27] J. Cheeger and D.G. Ebin, *Comparison Theorems in Riemannian Geometry*, North-Holland Publ. Co. 1975.
[28] P.B. Eberlein, *Geometry of Nonpositively Curved Manifolds*, The Univ. Chicago Press, 1996, Chicago and London.
[29] P.B. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math. 46(1973), 45–109.
[30] N. Gouda, *Magnetic flows of Anosov type*, Tohoku Math. J. 49(1997), 165–183.
[31] N. Gouda, *The theorem of E. Hopf under uniform magnetic fields*, J. Math. Soc. Japan 50(1998), 767–779.
[32] O. Ikawa, Motion of electric charged particles in homogeneous Kähler manifolds and homogeneous Sasakian manifolds, Far East J. Math Sci. **14**(2004), 283–302.
[33] U-H. Ki, H.S. Kim and H. Nakagawa, *A characterization of a real hypersurface of type* (B), Tsukuba J. Math. 14(1980), 9–26.
[34] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. 296(1986), 137–149.
[35] M. Kimura, *Real hypersurfaces of a complex projective space*, Bull. Austral. Math. Soc. 33(1986), 383–387.
[36] S. Maeda and T. Adachi, *Holomorphic herices in a complex space form*, Proc. A.M.S. 125(1997), 1197–1202.
[37] S. Maeda and T. Adachi, *Sasakian curves on hypersurfaces in a nonflat complex space form*, Result. Math. 56(2009), 489–499.
[38] S. Maeda and Y. Ohnita, *Helical geodesic immersions into complex space forms*, Geom. Dedicata 30(1989), 93–114.
[39] S. Montiel, *Real hypersurfaces of a complex hyperbolic space*, J. Math. Soc. Japan 37(1985), 515–535.
[40] H. Naitoh, *Isotropic immersions with parallel second fundamental form in* $P^m(c)$, Osaka J. Math. 18(1981), 427–464.
[41] R. Niebergall and P.J. Ryan, *Real hypersurfaces in complex space forms*, in *Tight and taut submanifolds*, MSRI Publ. 32(1997), 233–305.
[42] Nomizu and K. Yano, *On circles and spheres in Riemannian geometry*, Math. Ann. 210(1974), 163–170.
[43] T. Sakai, *Riemannian Geometry*, Syokabo 1992 (in Japanese) and Translations Math. Monographs 149, A.M.S. 1996
[44] T. Sunada, *Magnetic flows on a Riemann surface*, Proc. KAIST Math. Workshop 8(1993), 93–108.
[45] K. Suizu, S. Maeda and T. Adachi, *Study of submanifolds by curves of order 2*, Mem. Fac. Sci. Eng. Shimane Univ. Ser. B 34(2001), 61–85.
[46] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. 10(1973), 495–506.

[47] R. Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures*, J. Math. Soc. Japan 27(1975), 43–53.
[48] R. Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures II*, J. Math. Soc. Japan 27(1975), 507–516.

Division of Mathematics and Mathematical Science,
Department of Computer Science and Engineering
Graduate School of Engineering,
Nagoya Institute of Technology,
Nagoya, 466-8555, JAPAN