名古屋工業大学博士論文
甲第２１９号（課程修了による）
平成１０年３月２４日授与

RENORMALIZATION GROUP THEORY FOR TURBULENCE WITH A PASSIVE SCALAR

by

Yoshihiro Itazu

B.S., Nagoya Institute of Technology, 1993
M.S., Nagoya Institute of Technology, 1995

Doctoral Dissertation

Submitted to
Nagoya Institute of Technology
in partial fulfillment of
the requirements for the degree of
Doctor of Engineering

January 1998

Acknowledgments

First of all, the author would like to express his gratitude to the Examining Committee members, Professor Y. Nagano, Professor T. Tsuji and Professor O. Kitoh for carefully reviewing the manuscript and for making appropriate suggestions.

The author wishes to express his sincere gratitude to his supervisor, Professor Y. Nagano, for his encouragement, guidance, and many helpful suggestions in the progress of this investigation, and for suggesting the subject of this research.

During the author's Ph.D. program at the Nagoya Institute of Technology (NIT), he received much help from the members of the Heat Transfer Laboratory in the Department of Mechanical Engineering. Special thanks are due to Professor T. Tsuji, Dr. O. Iida, and Mr. H. Hattori for their valuable discussions and helpful comments. The author wishes to thank Dr. M. Tagawa for his valuable suggestions, as well as the graduate students of the Heat Transfer Laboratory for their generous assistance and discussions.

Thanks are also due to Professor A. Yoshizawa of the Institute of Industrial Science at the University of Tokyo, and to Dr. R. Rubinstein of the Institute for Computer Applications in Science and Engineering at NASA Langley Research Center for their helpful discussions and incisive comments.

Finally, the author would like to express his gratitude and appreciation to his family for their continuous encouragement and support.

Abstract

In this study, a renormalization group (RNG) theory for turbulence with a passive scalar is investigated to close the turbulent moment hierarchy in a turbulent shear flow in order to formulate rational turbulence models.

RNG theory, as a tool for resolving physical matters, has been chiefly developed in the analysis of phase transition phenomena, and applied to the study of isotropic turbulence by means of the Navier-Stokes equation with a random external force in the inertial range at a high Reynolds number limit. The RNG theory for isotropic turbulence was primarily intended for calculation of scaling laws for asymptotic turbulent energy spectra and reduction of the number of degrees of freedom of turbulence.

Then, the RNG theory has been modified to close the moment hierarchy for an inhomogeneous turbulent shear flow. In particular, the RNG theory with the ϵ-expansion technique proposed by Yakhot and Orszag (1986) and later revised by Yakhot and Smith (1992), and believed to be the application to an inhomogeneous turbulent flow based on that for Wilson's ϵ-expansion theory, is well-known as representative. It yields a $\overline{K}$-$\overline{\varepsilon}$ two-equation turbulence model for a high Reynolds number flow, the Smagorinsky model for large eddy simulation, the eddy diffusivity for thermal field, and the equation for the turbulent Prandtl number, each of which has been used in predicting some cases of turbulent flows. It has been frequently emphasized that the strong point of this kind of theory is its ability to derive turbulence models with their model constants in Fourier space with the aid of the Kolmogorov $-5/3$ power law for the spectrum of the turbulent kinetic energy.

However, some researchers pointed out that the RNG theory of Yakhot, Orszag and Smith has some algebraic errors and theoretical problems. The latter have been very controversial and now open to question.

On the other hand, the procedure to eliminate the fluctuating components in sequence in wave number space is similar to the concept of large eddy simulations and convenient to derive turbulence models at a high Reynolds number limit. Thus, the RNG by an iterative averaging method is considered to be more schematic and applicable to constructing turbulence models for a turbulent shear flow rather than by the ϵ-expansion technique. In the present study, by using the iterative averaging RNG method for inhomogeneous turbulence, an eddy-viscosity type turbulence model, which corresponds well to the Boussinesq postulate, is obtained with the aid of the Kolmogorov $-5/3$ power law spectrum for the turbulent energy. The model constant thus obtained becomes the function of the Kolmogorov constant in the inertial range.

By the same manner, an eddy diffusivity for heat and the equation for the turbulent Prandtl number Pr_t are derived. The equation shows that the turbulent Prandtl number Pr_t is the function of the molecular Prandtl number Pr and the turbulent Reynolds Re_t or Peclet number Pe_t. The result is in fairly good agreement with the data of some experiments and direct numerical simulation with the change of the molecular Prandtl number. The obtained turbulent Prandtl number at a high Reynolds number limit converges on $Pr_t = 0.79$ with any molecular Prandtl number.

CONTENTS

LIST OF FIGURES

NOMENCLATURE

C_K	Kolmogorov constant in inertial range
C_μ	turbulence model constant
D_0	amplitude of spectrum for f_i
d	number of spatial dimension
$E(k,\tau)$	three-dimensional turbulent kinetic energy spectrum
$E_{ij}(\boldsymbol{k})$	spectrum tensor
f_i	Gaussian stirring force in i-direction
$G(\hat{k})$	Green function for velocity field
$g(\hat{k})$	Green function for thermal field
H	Hamiltonian
K	instantaneous turbulent kinetic energy, $u_i u_i/2$
$\overline{K}$	mean turbulent kinetic energy, $\overline{u_i u_i}/2$
k	absolute value of $\boldsymbol{k}$, $\lvert\boldsymbol{k}\rvert$
$\boldsymbol{k}$	wave vector, (k_1, k_2, k_3)
k_d	Kolmogorov dissipation wave number, $(\overline{\varepsilon}/\nu_0^3)^{1/4}$
k_e	wave number in range of energy-containing eddies, $\overline{\varepsilon}/\overline{K}^{3/2}$
k_i	wave number in i-direction or position coordinate in k-space
L	spatial length scale
n	integer
P	static pressure
Pe_t	turbulent Peclet number, $Pr(\nu_t/\nu_0)$
$P_{ij}(\boldsymbol{k})$	projection operator, $\delta_{ij} - k_i k_j/k^2$
$P_{imn}(\boldsymbol{k})$	compound projection operator, $k_m P_{in}(\boldsymbol{k}) + k_n P_{im}(\boldsymbol{k})$

P_K	production of $\overline{K}$, $-\overline{u_i u_j}\partial\overline{U}_i/\partial x_j$
Pr	molecular Prandtl number, ν_0/α_0
Pr_t	turbulent Prandtl number, ν_t/α_t
p	fluctuation of static pressure
$Q(k,\tau)$	spectral density, $E(k,\tau)/(4\pi k^2)$
$Q_{ij}(\boldsymbol{k},\tau)$	spectrum tensor, $P_{ij}(\boldsymbol{k})Q(k,\tau)$
q	absolute value of $\boldsymbol{q}$, $\|\boldsymbol{q}\|$
$\boldsymbol{q}$	wave vector, (q_1,q_2,q_3)
Re_t	turbulent Reynolds number, $\overline{K}/(\nu_0\overline{\varepsilon})$
S	strain-rate parameter, $(2S_{ij}S_{ij})^{1/2}$
S_d	superficial area of d-dimensional unit sphere
S_{ij}	strain-rate tensor, $(\partial\overline{U}_i/\partial x_j + \partial\overline{U}_j/\partial x_i)/2$
T	temperature of fluid, $\overline{T}+t$
$\overline{T}$	mean temperature
t	fluctuating temperature
U_i	instantaneous velocity in i-direction, $\overline{U}_i + u_i$
$\overline{U}_i$	mean velocity in i-direction
u_i	fluctuating velocity in i-direction
x_i	Eulerian Cartesian coordinate in i-direction
$\boldsymbol{x}$	spatial position vector, (x_1,x_2,x_3)
y	scaling parameter of spectrum for f_i
$z(\Lambda)$	inverse of effective Prandtl number in Chapter 3, $\alpha(\Lambda)/\nu(\Lambda)$

Greek symbols

α_0	molecular diffusivity in thermal field
α_t	eddy diffusivity in thermal field
$\alpha(r)$	inverse of effective Prandtl number in Chapter 2, $\chi(r)/\nu(r)$
β	turbulence model constant
δ_{ij}	Kronecker delta
$\delta(k)$	Dirac delta function
ϵ	expansion parameter

ε	instantaneous dissipation rate of turbulent kinetic energy, $\nu_0(\partial u_i/\partial x_j)^2$
$\overline{\varepsilon}$	mean dissipation rate of turbulent kinetic energy, $\nu_0\overline{(\partial u_i/\partial x_j)^2}$
η, η_0	nondimensional strain-rate parameter, $S\overline{K}/\overline{\varepsilon}$
Λ	cutoff wave number in renormalization
Λ_e, Λ_{et}	lowest wave number in renormalization
λ	band width parameter
$\lambda(r)$	coupling constant in renormalization
$\bar{\lambda}(r)$	nondimensional coupling constant in renormalization
ν_0	molecular kinematic viscosity
$\nu(r), \nu_n$	renormalized effective viscosity
ν_t	eddy viscosity
ρ	density of fluid
τ	time
χ_0	molecular diffusivity in scalar field
$\chi(r)$	renormalized effective diffusivity in scalar field
Ω, ω	frequency with respect to time

Subscripts

0	initial value in renormalization
K	variables relevant to K
n	renormalized value at nth step
ε	variables relevant to ε
$*$	renormalized value at fixed point

Special symbols

$(\ )^0$	initial value in renormalization
$(\ )^<$	rest value at each renormalization step
$(\ )^>$	value to be eliminated at each renormalization step
$\langle\ \rangle_c$	conditional-averaged value in wave number space
$\langle\ \rangle$	total-averaged value in wave number space
$(\hat{\ })$	$d+1$-dimensional wave-frequency vector
$(\ )^*$	renormalized value at fixed point

Chapter 1

INTRODUCTION

1.1 Background

Renormalization group (RNG) theory was primarily used to eliminate divergences in field theory, and later became famous because of successful applications to critical phenomena in the 1970s (Wilson & Kogut 1974; Ma & Mazenko 1975).

RNG theories for critical phenomena

The subject of critical phenomena deals with matters in the vicinity of a phase transition. Some examples are a liquid gas system near the critical point or a ferromagnet at the Curie point. The application of RNG theory to magnetism is often interpreted as giving a quantitative meaning to the concept of block spins. Kadanoff (1966) proposed an RNG theory in order to explain the observed self-similarity of certain thermodynamic relationships under scaling transformations.

The corresponding RNG theory that which starts with an interaction Hamiltonian H_0, which is associated with two spins at a distance L_0 (i.e., the lattice spacing). Then one calculates an effective Hamiltonian H_1, based on a region of size $2L_0$, which means averaging over the effects of scales L_0. Next, we calculate H_2, based on a region of size $4L_0$, with the effects of scales less than or equal to $2L_0$ averaged out. Thereafter, the general expression can be obtained by calculation of the Hamiltonian H_n, associated with a region of size $2^n L_0$, and the elimination of scales less than or equal to $2^{n-1}L_0$.

The above process can be expressed in terms of a renormalization operator R, which is applied repeatedly:

$$R(H_0) = H_1, \quad R(H_1) = H_2, \quad R(H_2) = H_3, \cdots. \tag{1.1}$$

At each renormalization stage, the length scales of the system are changed as

$$L_0 \to 2L_0, \quad 2L_0 \to 4L_0 \cdots, \tag{1.2}$$

and the spin variables are rescaled in an appropriate fashion such that the Hamiltonian always seemingly looks the same in scaled coordinates. It is this rescaling which produces renormalization and the transformations Eq. (1.1) define a simple group (i.e., renormalization group). And iterating the transformation leads to the result:

$$H_{n+1} = H_n, \tag{1.3}$$

where $H_{n+1} = T(H_n)$, then $H_n = H_N$, is called a fixed point which corresponds to the critical point of the system. Intuitively, this can be understood in terms of the fact that the fluctuations of infinite wavelength (which occur at the critical point) will be invariant under scaling transformations. Thus, the procedure of RNG is to move the system along a trajectory, with the sequence of scaling operations playing the part of time. The resulting fixed point is determined by the solution of the following equation:

$$R(H_N) = H_*, \tag{1.4}$$

and is a property of the operator R rather than the initial condition H_0. This is associated with the idea of universality of critical behavior. *In the case of turbulence, the corresponding property would be that the renormalized effective viscosity ν which would not depend on the molecular viscosity ν_0.*

RNG theory by epsilon-expansion

Concerning the above discussion with respect to wave number, the Fourier transformation of the spin variables (or the spin field) is indispensable, and this expression enable us to consider the space dimension d as a variable parameter. The calculation of H_1 from H_0

is done by integrating over the band $k \geq 2\pi/L_0$, and the other modes are eliminated by turns in terms of the bands $\pi/L_0 \leq k \leq 2\pi/L_0$, $\pi/2L_0 \leq k \leq \pi/L_0, \cdots$ in order to obtain the corresponding Hamiltonians, $H_2, H_3, \cdots$. Then, the problem at the fixed point can be solved for $d = 4$ because the critical dimension of the Ising model of a ferromagnet is 4; furthermore, the solutions for $d = 4 - \epsilon$, where ϵ is considered to be a small parameter, are obtained by the expansion for ϵ. Thus, ϵ is set equal to unity and the solution for $d = 3$ is obtained.

RNG theory for turbulence

The application of RNG theory to turbulence is classified into three: transition from laminar to turbulent flow; calculation of scaling laws for asymptotic turbulent energy spectra in a randomly stirred fluid; and reduction of the number of degree of freedom of fully developed turbulent flow. The first of these topics, which involves a transition from quasi-periodic behavior to chaos under the influence of external noise inputs, is beyond the subject of this study. The second matter has been primarily investigated by Forster et al. (1977), who studied the behavior of a randomly stirred fluid by the ϵ-expansion technique. The third has been mainly investigated by McComb (1990), the concept of which is that averaging over the shortest period smooths out the part of the field which corresponds to the highest frequency fluctuations. Then the mean effect of these fluctuations is calculated from the averaged-equation and eliminated from the equation for the rest of the velocity field. This procedure is repeated in terms of a narrow band in the whole wave number range to derive an effective viscosity.

Following Forster et al. (1977), Yakhot & Orszag (1986) proposed the RNG theory for turbulence which is applicable to an inhomogeneous turbulent flow and the RNG-based turbulence models: $\overline{K}$-$\bar{\varepsilon}$ two-equation model; the Smagorinsky model for large eddy simulation; and the equation for the turbulent Prandtl number Pr_t. These models have been used for predicting some cases of turbulent flows (Yakhot et al. 1987; Piomelli 1989; Yakhot et al. 1992; Orszag et al. 1993). Rubinstein & Barton (1992) applied the RNG theory by Yakhot & Orszag (1986) to modeling of the transport equation for the Reynolds stress $-\overline{u_i u_j}$. Giles (1994a, 1994b) applied the RNG theory to the probability distribution of the velocity field, and proposed a $\overline{K}$-$\overline{\varepsilon}$ two-equation model with the numerical constants. However, the validity of the RNG by the ϵ-expansion for turbulence has been discussed (Kraichnan 1987; Avellaneda & Majda 1990; Avellaneda & Majda 1992; Carati & Chriaa 1993; Lesieur 1993; Eyink 1994; Frisch 1995; Rubinstein 1996), and now becomes a controversial problem in the study of turbulence.

1.2 Objectives

In view of the background described in the previous section, the present study has the following main objectives:

1. To assess the RNG theory proposed by Yakhot et al. (Yakhot & Orszag 1986; Yakhot & Smith 1992) in order to reveal all the problems in their derivation of turbulence models;

2. To apply an iterative averaging RNG method to an inhomogeneous turbulent shear flow with a passive scalar in order to formulate an eddy viscosity, an eddy diffusivity, and the equation for the turbulent Prandtl number without the problems occurred in the ϵ-expansion technique.

1.3 Organization of Dissertation

The subject of this thesis consists of two main parts, one concerned with an assessment of the Yakhot-Orszag-Smith RNG theory (Yakhot & Orszag 1986; Yakhot & Smith 1992), the other with refinement of the iterative averaging RNG based on McComb's method (McComb 1990; McComb & Watt 1990) so as to derive turbulence models in an inhomogeneous shear flow with a passive scalar (Itazu & Nagano 1997a; Itazu & Nagano 1997b; Itazu & Nagano 1997c; Nagano & Itazu 1997b; Itazu & Nagano 1998). Chapter 2 is related to the former study, and Chapter 3 with the latter one.

Chapter 2 refers to the RNG theory proposed by Yakhot & Orszag (1986) and later revised by Yakhot & Smith (1992). In this chapter, the estimation of their RNG theory is carried out to clear up all the problems in turbulence models they derived (Nagano & Itazu 1995; Itazu & Nagano 1996; Nagano & Itazu 1997a). Firstly, the application of the RNG to the Navier-Stokes equation with a stirring force is discussed with the validity of the ϵ-expansion and the obtained numerical constants in the inertial range. Then, the modeling of the $\overline{K}$-$\overline{\varepsilon}$ two-equation turbulence model by the RNG theory is reexamined. The problems in deriving this model are pointed out in order.

Chapter 3 refers to the development of the iterative averaging RNG theory to analyze an inhomogeneous turbulent shear flow with a passive scalar. This method is explained in this chapter. An eddy viscosity type turbulence model is formulated by using the exact Navier-Stokes equation with no stirring force apart from the problem appearing in the ϵ-expansion technique. In the same manner, an eddy diffusivity for heat and the equation for the turbulent Prandtl number Pr_t are formulated.

The conclusions are summarized in Chapter 4.

The detailed calculations through Chapters 2 and 3 are shown in Appendices A and B.

Chapter 2

ASSESSMENT OF THE YAKHOT-ORSZAG-SMITH THEORY

Here the author goes into details of the Yakhot & Orszag (hereinafter referred to as YO) and Yakhot & Smith (YS) theory for turbulence according to their papers (Yakhot & Orszag 1986; Yakhot & Smith 1992), clearly points out all the problems in turns, and comments on them.

2.1 Basic Equations

In the inertial range, one can consider the following forced Navier-Stokes equation and incompressible continuity equation:

$$\frac{\partial u_i}{\partial \tau} + u_j \frac{\partial u_i}{\partial x_j} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu_0 \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \tag{2.1}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2.2}$$

where $u_i(\boldsymbol{x}, \tau)$ is the fluctuating velocity component, ρ is the density, p is the pressure, and ν_0 is the molecular kinematic viscosity. The equation of motion for the fluctuating velocity is generalized by the addition of an external random force. The random force f_i is assumed to be Gaussian, white noise in time, isotropic in space, and homogeneous in time. The $(d+1)$-dimensional Fourier transform of its two-point correlation is defined as follows:

$$\left\langle f_i(\hat{k})f_j(\hat{k}')\right\rangle = \begin{cases} 2D(k)\,(2\pi)^{d+1}P_{ij}(\boldsymbol{k})\,\delta(\hat{k}+\hat{k}') & : \quad \Lambda_e < k < \Lambda_0 \\ 0 & : \quad \text{otherwise} \end{cases}, \tag{2.3}$$

where

$$D(k) = D_0 k^{-y}, \tag{2.4}$$

$$P_{ij}(\boldsymbol{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}. \tag{2.5}$$

Here, $\hat{k} \equiv (\boldsymbol{k},\omega)$ is the $(d+1)$-dimensional wave-frequency vector, D_0 is a dimensional coefficient, and the parameter y is chosen to describe the Kolmogorov form of the energy spectrum in three dimensions ($y = d = 3$). The projection operator $P_{ij}(\boldsymbol{k})$ makes the random force statistically isotropic and divergence free. The initial cutoff wave number in Eq. (2.3) is $\Lambda_0 = O\left[(\bar{\varepsilon}/\nu_0^3)^{1/4}\right]$, and $\Lambda_e = O(\pi/L)$.

We use the $(d+1)$-dimensional Fourier transform:

$$u_i(\boldsymbol{x},\tau) = \int_{k<\Lambda_0}\int_{-\infty}^{+\infty} u_i(\boldsymbol{k},\omega)\exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}. \tag{2.6}$$

The space-time Fourier transformed Navier-Stokes equation for Eq. (2.1) and continuity equation for Eq. (2.2) are

$$u_i(\hat{k}) = G_0(\hat{k})f_i(\hat{k}) - \frac{i\lambda_0}{2}G_0(\hat{k})P_{imn}(\boldsymbol{k})\int_{q<\Lambda_0} u_m(\hat{q})u_n(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^{d+1}}, \tag{2.7}$$

$$k_i u_i(\hat{k}) = 0, \tag{2.8}$$

where

$$G_0(\hat{k}) = (-i\omega + \nu_0 k^2)^{-1}, \tag{2.9}$$

$$P_{imn}(\boldsymbol{k}) = k_m P_{in}(\boldsymbol{k}) + k_n P_{im}(\boldsymbol{k}), \tag{2.10}$$

$$\hat{q} = (\boldsymbol{q},\Omega) = (q_1,q_2,q_3,\Omega), \tag{2.11}$$

$$d\hat{q} = dq_1 dq_2 dq_3 d\Omega. \tag{2.12}$$

In Eq. (2.7), $\lambda_0(=1)$ is added as a bookkeeping parameter in front of the nonlinear term, and Eq. (2.7) is defined on the domain $0 < k < \Lambda_0$, $0 < q < \Lambda_0$, $-\infty < \omega < +\infty$, and $-\infty < \Omega < +\infty$.

2.2 Scale Removal Procedure

As shown in Fig. 2.1, we divide the velocity $u_i(\hat{k})$ into two components as follows:

$$u_i(\hat{k}) = \begin{cases} u_i^<(\hat{k}) & : \ \Lambda_e < k < \Lambda(r) \\ u_i^>(\hat{k}) & : \ \Lambda(r) < k < \Lambda_0 \end{cases}, \tag{2.13}$$

$$\Lambda(r) = \Lambda_0 \exp(-r). \tag{2.14}$$

The parameter r is chosen so that $0 < \exp(-r) < 1$. Then the corresponding decomposition of the Navier-Stokes equation can be obtained by substituting Eq. (2.13) into Eq. (2.7):

$$\begin{aligned} u_i^<(\hat{k}) &= G_0^<(\hat{k})\, f_i^<(\hat{k}) - \frac{i\lambda_0}{2}\, G_0^<(\hat{k})\, P_{imn}^<(\boldsymbol{k}) \int_q u_m(\hat{q}) u_n(\hat{k}-\hat{q})\, \frac{d\hat{q}}{(2\pi)^{d+1}} \\ &= G_0^<(\hat{k})\, f_i^<(\hat{k}) \\ &\quad -\frac{i\lambda_0}{2}\, G_0^<(\hat{k})\, P_{imn}^<(\boldsymbol{k}) \int_q \big\{ u_m^<(\hat{q}) u_n^<(\hat{k}-\hat{q}) + 2u_m^>(\hat{q}) u_n^<(\hat{k}-\hat{q}) \\ &\quad + u_m^>(\hat{q}) u_n^>(\hat{k}-\hat{q}) \big\} \frac{d\hat{q}}{(2\pi)^{d+1}} \end{aligned} \tag{2.15}$$

$$\begin{aligned} u_i^>(\hat{k}) &= G_0^>(\hat{k})\, f_i^>(\hat{k}) - \frac{i\lambda_0}{2}\, G_0^>(\hat{k})\, P_{imn}^>(\boldsymbol{k}) \int_q u_m(\hat{q}) u_n(\hat{k}-\hat{q})\, \frac{d\hat{q}}{(2\pi)^{d+1}} \\ &= G_0^>(\hat{k})\, f_i^>(\hat{k}) \\ &\quad -\frac{i\lambda_0}{2}\, G_0^>(\hat{k})\, P_{imn}^>(\boldsymbol{k}) \int_q \big\{ u_m^<(\hat{q}) u_n^<(\hat{k}-\hat{q}) + 2u_m^>(\hat{q}) u_n^<(\hat{k}-\hat{q}) \\ &\quad + u_m^>(\hat{q}) u_n^>(\hat{k}-\hat{q}) \big\} \frac{d\hat{q}}{(2\pi)^{d+1}}. \end{aligned} \tag{2.16}$$

Note that in the equation of motion for $\boldsymbol{u}^<$, the higher wave number modes $\boldsymbol{u}^>$ are

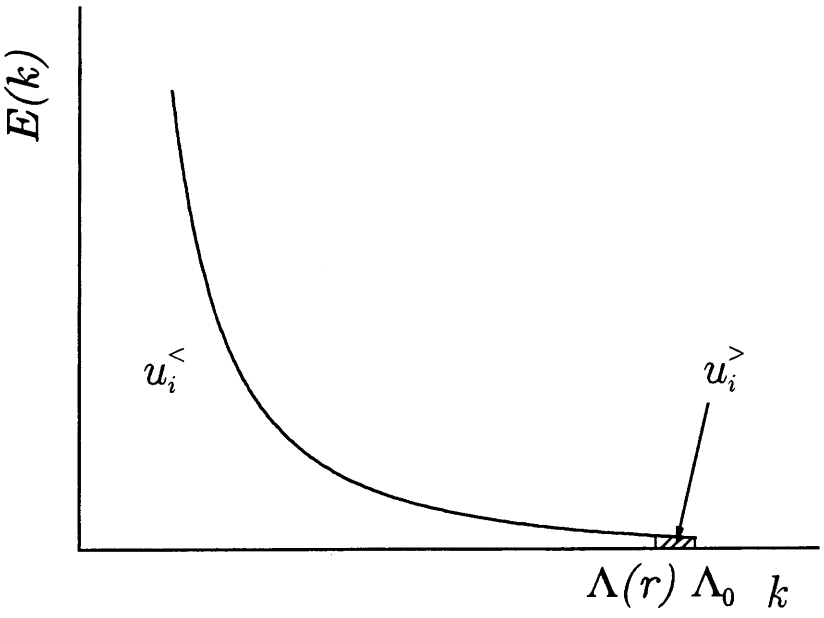

Figure 2.1: Fine scale removal

included in the nonlinear terms.

In order to eliminate $\boldsymbol{u}^>$ from Eq. (2.15), all $\boldsymbol{u}^>$ terms should be removed by the substitution of $u_i^>$ given by Eq. (2.16) into all the modes $\boldsymbol{u}^>$ in the $\boldsymbol{u}^<$ equation [Eq. (2.15)]; thus,

$$\begin{aligned} u_m^>(\hat{q}) &= G_0^>(\hat{q})f_m^>(\hat{q}) - \frac{i\lambda_0}{2}G_0^>(\hat{q})P_{m\alpha\beta}^>(\boldsymbol{q})\int_s \Big\{u_\alpha^<(\hat{s})u_\beta^<(\hat{q}-\hat{s}) + 2u_\alpha^>(\hat{s})u_\beta^<(\hat{q}-\hat{s}) \\ &\quad + u_\alpha^>(\hat{s})u_\beta^>(\hat{q}-\hat{s})\Big\}\frac{d\hat{s}}{(2\pi)^{d+1}} \\ &= G_0^>(\hat{q})f_m^>(\hat{q}) - \frac{i\lambda_0}{2}G_0^>(\hat{q})P_{m\alpha\beta}^>(\boldsymbol{q})\int_s \Big\{u_\alpha^<(\hat{s})u_\beta^<(\hat{q}-\hat{s}) + 2G_0^>(\hat{s})f_\alpha^>(\hat{s})u_\beta^<(\hat{q}-\hat{s}) \\ &\quad + G_0^>(\hat{s})f_\alpha^>(\hat{s})G_0^>(\hat{q}-\hat{s})f_\beta^>(\hat{q}-\hat{s})\Big\}\frac{d\hat{s}}{(2\pi)^{d+1}} \\ &\quad + O(\lambda_0^2) \end{aligned} \tag{2.17}$$

$$u_n^>(\hat{k}-\hat{q}) = G_0^>(\hat{k}-\hat{q})f_n^>(\hat{k}-\hat{q})$$

$$
\begin{aligned}
& -\frac{i\lambda_0}{2} G_0^>(\hat{k}-\hat{q}) P_{n\gamma\delta}^>(\boldsymbol{k}-\boldsymbol{q}) \int_r \left\{ u_\gamma^<(\hat{r}) u_\delta^<(\hat{k}-\hat{q}-\hat{r}) \right. \\
& \left. +2u_\gamma^>(\hat{r}) u_\delta^<(\hat{k}-\hat{q}-\hat{r}) + u_\gamma^>(\hat{r}) u_\delta^>(\hat{k}-\hat{q}-\hat{r}) \right\} \frac{d\hat{r}}{(2\pi)^{d+1}} \\
= \; & G_0^>(\hat{k}-\hat{q}) f_n^>(\hat{k}-\hat{q}) \\
& -\frac{i\lambda_0}{2} G_0^>(\hat{k}-\hat{q}) P_{n\gamma\delta}^>(\boldsymbol{k}-\boldsymbol{q}) \int_r \left\{ u_\gamma^<(\hat{r}) u_\delta^<(\hat{k}-\hat{q}-\hat{r}) \right. \\
& \left. +2G_0^>(\hat{r}) f_\gamma^>(\hat{r}) u_\delta^<(\hat{k}-\hat{q}-\hat{r}) + G_0^>(\hat{r}) f_\gamma^>(\hat{r}) G_0^>(\hat{k}-\hat{q}-\hat{r}) f_\delta^>(\hat{k}-\hat{q}-\hat{r}) \right\} \frac{d\hat{r}}{(2\pi)^{d+1}} \\
& +O(\lambda_0^2). \qquad (2.18)
\end{aligned}
$$

This procedure generates an infinite expansion for $\boldsymbol{u}^<$ in powers of λ_0 in which the modes $\boldsymbol{u}^>$ do not formally appear:

$$(-i\omega + \nu_0 k^2) u_i^<(\hat{k}) = f_i^<(\hat{k}) + O\left(\lambda_0^1\right) + O\left(\lambda_0^2\right) \cdots, \qquad (2.19)$$

where

$$O(\lambda_0^1) = -\frac{i\lambda_0}{2} P_{imn}^<(\boldsymbol{k}) \int_q u_m^<(\hat{q}) u_n^<(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}}, \qquad (2.20)$$

$$
\begin{aligned}
O(\lambda_0^2) \; = \; & 4D_0 \left(\frac{i\lambda_0}{2}\right)^2 P_{imn}^<(\boldsymbol{k}) \int_q |G_0^>(\hat{q})|^2 \, G_0^>(\hat{k}-\hat{q}) \\
& \times P_{n\alpha\beta}^>(\boldsymbol{k}-\boldsymbol{q}) P_{m\alpha}^>(\boldsymbol{q}) q^{-y} \frac{d\hat{q}}{(2\pi)^{d+1}} \, u_\beta^<(\hat{k}) \\
& +4D_0 \left(\frac{i\lambda_0}{2}\right)^2 P_{imn}^<(\boldsymbol{k}) \int_q \left|G_0^>(\hat{k}-\hat{q})\right|^2 G_0^>(\hat{q}) P_{n\alpha\beta}^>(\boldsymbol{q}) \\
& \times P_{m\alpha}^>(\boldsymbol{k}-\boldsymbol{q}) \left|\boldsymbol{k}-\boldsymbol{q}\right|^{-y} \frac{d\hat{q}}{(2\pi)^{d+1}} u_\beta^<(\hat{k}). \qquad (2.21)
\end{aligned}
$$

The equation for $\boldsymbol{u}^<$ is averaged over the fine-scale $\Lambda_0 \exp(-r) < q < \Lambda_0$ [use Eq. (2.3) for $\boldsymbol{f}^>$]. Integration of the higher wave-number components such as $G_0^>(\hat{q})$ and $P_{imn}^>(\boldsymbol{q})$ over the higher wave-frequency vector $\hat{q} \equiv (\boldsymbol{q}, \Omega)$ is carried out on the assumption of the *distant-interaction* limit ($|\boldsymbol{k}| \ll |\boldsymbol{q}|$). Keeping terms to $O(\lambda_0^2)$, we obtain

$$\begin{aligned} O(\lambda_0^2) &= -\frac{\lambda_0^2 D_0}{2\nu_0^2(2\pi)^d}\frac{d^2-y-4}{d(d+2)}\frac{S_d\exp(\epsilon r)-1}{\epsilon\Lambda_0^\epsilon}k^2u_i^<(\hat{k}) \\ &= -\frac{\lambda_0^2 D_0}{2\nu_0^2(2\pi)^d}\frac{d^2-d-\epsilon}{d(d+2)}\frac{S_d\exp(\epsilon r)-1}{\epsilon\Lambda_0^\epsilon}k^2u_i^<(\hat{k}), \end{aligned} \tag{2.22}$$

and

$$\left(-i\omega+\nu_0k^2\right)u_i^<(\hat{k}) = f_i^<(\hat{k}) + O\left(\lambda_0\right) - \Delta\nu(r)k^2u_i^<(\hat{k}) \tag{2.23}$$

or

$$\left\{-i\omega+\left(\nu_0+\Delta\nu\right)k^2\right\}u_i^<(\hat{k}) = f_i^<(\hat{k}) + O\left(\lambda_0\right), \tag{2.24}$$

where

$$\Delta\nu(r) = A_d\,\nu_0\,\bar{\lambda}_0^2\frac{\exp(\epsilon r)-1}{\epsilon}, \tag{2.25}$$

$$A_d = \frac{1}{2}\frac{S_d}{(2\pi)^d}\frac{d^2-d-\epsilon}{d(d+2)}, \tag{2.26}$$

$$\epsilon = 4+y-d, \tag{2.27}$$

$$\bar{\lambda}_0^2 = \frac{\lambda_0^2D_0}{\nu_0^3\Lambda_0^\epsilon} \tag{2.28}$$

(see Appendix A). Equation (2.24) can be rewritten as

$$\begin{aligned} u_i^<(\hat{k}) &= G_r(\hat{k})f_i^<(\hat{k}) \\ &\quad -\frac{i\lambda_0}{2}G_r(\hat{k})P_{imn}^<(\boldsymbol{k})\int_{q<\Lambda_0\exp(-r)}u_m^<(\hat{q})u_n^<(\hat{k}-\hat{q})\,\frac{d\hat{q}}{(2\pi)^{d+1}}, \end{aligned} \tag{2.29}$$

where

$$G_r(\hat{k}) = \left\{-i\omega+\nu(r)k^2\right\}^{-1}, \tag{2.30}$$

$$\begin{aligned} \nu(r) &= \nu_0+\Delta\nu(r) \\ &= \nu_0\left\{1+A_d\bar{\lambda}_0^2\frac{\exp(\epsilon r)-1}{\epsilon}\right\}. \end{aligned} \tag{2.31}$$

In Eq. (2.26), S_d is the area of a d-dimensional unit sphere (i.e., $S_3 = 4\pi$). Equation (2.25) denotes the effect of the eliminated mode $\boldsymbol{u}^>$, and is added to the bare viscosity ν_0. Note that the expansion parameter ϵ in Eq. (2.27) is determined not only by the number of dimensions d but also by the decay parameter y defined by Eq. (2.4). Thus, the expansion parameter ϵ in the YO theory is never determined by only a dimension d. It also depends on the scaling parameter y that describes the random force spectrum. And the rescaling process is not done. Actually, the YO theory is not Wilson-type (ϵ-expansion) RNG at all.

The effective viscosity is represented as

$$\nu(r) = \nu_0 + \int_0^r \frac{d\nu(l)}{dl}\, dl, \tag{2.32}$$

and the effect of the nonlinear terms is treated as an increment to the viscosity by increasing r.

2.3 Differential Relation for Effective Viscosity

Here we clarify the methodology of how to derive the differential relation for $\nu(r)$. To obtain the derivative of $\nu(r)$ with respect to r, YO argued that *it can be obtained by taking the limit* $r \to 0$ *in Eq. (2.31)*. However, this approach does not lead to the correct renormalized equation for $d\nu(r)/dr$. Hence, from our point of view, the correct procedure should be as follows.

From Eq. (2.25), the effect of eliminated modes is represented as

$$\Delta\nu(r) = \frac{A_d D_0 \lambda_0^2}{\nu_0^2 \epsilon}\left[\frac{1}{\{\Lambda_0 \exp(-r)\}^\epsilon} - \frac{1}{\Lambda_0^\epsilon}\right], \tag{2.33}$$

so that the renormalized viscosity becomes

$$\begin{aligned} \nu\left(\Lambda - \Delta\Lambda\right) &= \nu\left(\Lambda\right) + \Delta\nu\left(\Lambda\right) \\ &= \nu\left(\Lambda\right) + \frac{A_d D_0 \lambda_0^2}{\nu^2\left(\Lambda\right)\epsilon}\left\{\frac{1}{\left(\Lambda - \Delta\Lambda\right)^\epsilon} - \frac{1}{\Lambda^\epsilon}\right\}, \end{aligned} \tag{2.34}$$

where

$$\Lambda = \Lambda_0 \exp(-r). \tag{2.35}$$

Accordingly, the differential equation for $\nu(\Lambda)$ is represented as

$$\begin{aligned}
\frac{d\nu(\Lambda)}{d\Lambda} &= \lim_{\Delta\Lambda\to 0} \frac{\nu(\Lambda-\Delta\Lambda)-\nu(\Lambda)}{(\Lambda-\Delta\Lambda)-\Lambda} \\
&= \lim_{\Delta\Lambda\to 0} \frac{\frac{A_d D_0 \lambda_0^2}{\nu^2(\Lambda)\epsilon}\left\{\frac{1}{(\Lambda-\Delta\Lambda)^\epsilon}-\frac{1}{\Lambda^\epsilon}\right\}}{-\Delta\Lambda} \\
&= -\lim_{\Delta\Lambda\to 0} \frac{A_d D_0 \lambda_0^2}{\nu^2(\Lambda)\epsilon} \frac{(\Lambda-\Delta\Lambda)^{-\epsilon}-\Lambda^{-\epsilon}}{\Delta\Lambda} \\
&= -\frac{A_d D_0 \lambda_0^2}{\nu^2(\Lambda)\epsilon}\left[-\frac{d\Lambda^{-\epsilon}}{d\Lambda}\right] \\
&= -\frac{A_d D_0 \lambda_0^2}{\nu^2(\Lambda)\Lambda^{\epsilon+1}}.
\end{aligned} \tag{2.36}$$

The integration of Eq. (2.36) gives

$$\int_{\nu_0}^{\nu} \nu^2 d\nu = -\int_{\Lambda_0}^{k} A_d D_0 \lambda_0^2 \Lambda^{-\epsilon-1} d\Lambda, \tag{2.37}$$

which yields

$$\nu(k) = \nu_0 \left\{1 + \frac{3A_d D_0 \lambda_0^2}{\epsilon}\left(k^{-\epsilon}-\Lambda_0^{-\epsilon}\right)\right\}^{\frac{1}{3}} \tag{2.38}$$

or

$$\nu(r) = \nu_0 \left\{1 + 3A_d D_0 \lambda_0^2 \frac{\exp(\epsilon r)-1}{\epsilon\Lambda_0^\epsilon}\right\}^{\frac{1}{3}}. \tag{2.39}$$

Note that the renormalized viscosity is obtained without any help of an ϵ-expansion though YS alluded to some relationship with the ϵ-expansion.

In the limit case $k \ll \Lambda_0$, YO suggested that

$$\nu(k) \simeq \left(\frac{3A_d D_0}{\epsilon}\right)^{\frac{1}{3}} k^{-\frac{\epsilon}{3}}. \tag{2.40}$$

And, they made use of the following relation:

$$\frac{2D_0 S_d}{(2\pi)^d} = 1.59\,\bar{\varepsilon}, \tag{2.41}$$

and related D_0 to the turbulent energy dissipation rate $\bar{\varepsilon}$. [Lam (1992) pointed out that the constant 1.59 is in error, and 1.57 is correct.] Then the effective viscosity has the spectrum form:

$$\begin{aligned}
\nu(k) &= \left\{\frac{3\times 1.59}{2\epsilon}\,\frac{d^2-d-\epsilon}{2(d^2+2d)}\right\}^{\frac{1}{3}} \bar{\varepsilon}^{\frac{1}{3}} k^{-\frac{\epsilon}{3}} \\
&= \left\{\frac{3\times 1.59}{2\times(4)}\,\frac{1}{2}\,\frac{3^2-3-(0)}{3^2+2\times 3}\right\}^{\frac{1}{3}} \bar{\varepsilon}^{\frac{1}{3}} k^{-\frac{(4)}{3}} \\
&= 0.49\,\bar{\varepsilon}^{\frac{1}{3}} k^{-\frac{4}{3}}. \qquad (2.42)
\end{aligned}$$

We emphasize here that the numerical constant is invalid because they evaluated it by putting $\epsilon = 0$ and $\epsilon = 4$ (it describes the Kolmogorov spectrum) in Eq. (2.42) in an illogical manner. If we substitute $\epsilon = 4$ for Eq. (2.42), we obtain a value of 0.34 instead of 0.49.

2.4 Renormalized Energy Spectrum

Using the following zeroth-order ($\lambda_0 = 0$) equation for $u_i^<(\hat{k})$ on the condition $\Lambda_e < k \ll \Lambda_0$:

$$\left\{-i\omega + \nu(k)k^2\right\} u_i^<(\hat{k}) = f_i^<(\hat{k}), \qquad (2.43)$$

we obtain the energy spectrum:

$$E(k) = \frac{S_d D_0}{(2\pi)^d \nu(k)} k^{1-\epsilon}. \qquad (2.44)$$

Substituting Eq. (2.40) and Eq. (2.41) into Eq. (2.44), the renormalized energy spectrum is represented as

$$E(k) = C_K\,\bar{\varepsilon}^{\frac{2}{3}}\,k^{1-\frac{2}{3}\epsilon}, \qquad (2.45)$$

and the parameter $\epsilon = 4$ (or $y = 3$) is necessary to describe the Kolmogorov $-5/3$ power-law spectrum; besides, the Kolmogorov constant is determined as

$$C_K = \left(\frac{1.59}{2}\right)^{\frac{2}{3}} \left\{\frac{3}{2\epsilon}\,\frac{d^2-d-\epsilon}{d(d+2)}\right\}^{-\frac{1}{3}}$$

$$\begin{aligned} &= \left(\frac{1.59}{2}\right)^{\frac{2}{3}} \left\{\frac{3}{2\times(4)} \times \frac{3^2-3-(0)}{3\times(3+2)}\right\}^{-\frac{1}{3}} \\ &= 1.615 \end{aligned} \tag{2.46}$$

by using $\epsilon = 0$ and $\epsilon = 4$ at different points in the same equation. If we use a value of $\epsilon = 4$ consistently, the Kolmogorov constant becomes $C_K = 1.113$, which is out of range of the measurement $C_K = 1.4 \sim 2.0$ (Sreenivasan 1995).

After the elimination of the modes in the inertial range, one can set the wave number k to Λ_e in Eq. (2.40). On the other hand, the averaged turbulent kinetic energy evaluated from Eq. (2.45) is

$$\begin{aligned} \overline{K} &= \int_0^\infty E(k)dk \simeq \int_{\Lambda_e}^\infty C_K\, \overline{\varepsilon}^{\frac{2}{3}}\, k^{-\frac{5}{3}} \\ &= \frac{3}{2} C_K\, \overline{\varepsilon}^{\frac{2}{3}} \Lambda_e^{-\frac{2}{3}}, \end{aligned} \tag{2.47}$$

and from this relation, the effective viscosity given by Eq. (2.40) is reformulated as

$$\begin{aligned} \nu(\Lambda_e) &= \frac{\left(\frac{1}{2}\,\frac{1.59\times 3}{2\epsilon}\,\frac{d^2-d-\epsilon}{d^2+2d}\right)^{\frac{1}{3}}}{\left(\frac{3}{2}C_K\right)^2}\,\frac{\overline{K}^2}{\overline{\varepsilon}} \\ &= \frac{\left(\frac{1}{2}\,\frac{1.59\times 3}{2\times(4)} \times \frac{3^2-3-(0)}{3^2+2\times 3}\right)^{\frac{1}{3}}}{\left(\frac{3}{2}\times 1.615\right)^2}\,\frac{\overline{K}^2}{\overline{\varepsilon}} \\ &= 0.085\frac{\overline{K}^2}{\overline{\varepsilon}}. \end{aligned} \tag{2.48}$$

For the numerical constant 0.085, YO emphasized that this result is in good agreement with the eddy viscosity of the standard $\overline{K}$-$\overline{\varepsilon}$ model. But this would seem to be only sheer coincidence. There are the following inconsistencies in the derivation of Eq. (2.48): the use of the renormalized spectrum [Eq. (2.45)], which is derived under the condition $\Lambda_e < k \ll \Lambda_0$, for Eq. (2.47), and the computation of the numerical constants using

$\epsilon = 0$ and $\epsilon = 4$ in the same equation. [It can be argued that this numerical constant might be obtained with a lowest order approximation $C_K = A_0/\epsilon$ to the inertial range constant in the form: $C_K = A(\epsilon)/\epsilon = A_0/\epsilon + A_1 + A_2\epsilon + \cdots$, as shown by Woodruff (1994); and ϵ is never set to any value other than 4, either in evaluating exponents or amplitudes, although the quantitative accuracy of this approximation is not proven mathematically. Any improvement on $C_K = A_0/\epsilon$ would require evaluation of a higher order approximation, which would be their objection to setting $C_K = A(4)/4$ (Rubinstein 1996).] If one use a consistent value of $\epsilon = 4$, the numerical constant in Eq. (2.48) becomes 0.122. This value differs from the value of 0.09 used in the standard $\overline{K}$-$\overline{\varepsilon}$ model.

2.5 Modeling of $\overline{K}$-Equation

Next, we examine the derivation of their RNG $\overline{K}$-equation. The procedure is the same as for the Navier-Stokes equations: All the modes in the interval $\Lambda_e < k < \Lambda_0$ are removed, and represented as an increment of a diffusion term.

The transport for the turbulent kinetic energy K is

$$\frac{\partial K}{\partial \tau} + u_i \frac{\partial K}{\partial x_i} = -\varepsilon + \nu_0 \frac{\partial^2 K}{\partial x_i \partial x_i} - \frac{\partial}{\partial x_i}\left(u_i \frac{p}{\rho}\right) + u_i f_i, \tag{2.49}$$

where $K = 1/2 u_i u_i$ and $\varepsilon = \nu_0(\partial u_i/\partial x_j)^2$. We use the definition of the Fourier transform:

$$K(\boldsymbol{x}, \tau) = \int_{k<\Lambda_0} \int_{-\infty}^{+\infty} K(\boldsymbol{k}, \omega) \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau) \frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}, \tag{2.50}$$

$$K(\boldsymbol{k}, \omega) \equiv K(\hat{k}) = \frac{1}{2} \int_{q<\Lambda_0} u_i(\hat{q}) u_i(\hat{k} - \hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}}, \tag{2.51}$$

and obtain the transport equation for K in Fourier space:

$$(-i\omega + \chi_K^0 k^2) K(\hat{k}) = -i\lambda_0 k_i \int_{q<\Lambda_0} u_i(\hat{q}) K(\hat{k} - \hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} - \varepsilon(\hat{k}). \tag{2.52}$$

Here $\chi_K^0 (= \nu_0)$ is the bare diffusivity. The pressure-strain correlation term and the force-strain correlation term are assumed to be negligibly small. To eliminate the higher wave number modes in the interval $\Lambda_0 \exp(-r) < k < \Lambda_0$, we decompose $K(\hat{k})$ into $K^<(\hat{k})$ and

$K^>(\hat{k})$ in the same way as in Section 2.2, and substitute $K^>(\hat{k})$ and $\boldsymbol{u}^>$ determined from each transport equation into the nonlinear term in the transport equation for $K^<(\hat{k})$.

The resultant renormalized equation for $K^<$ becomes

$$\left\{-i\omega + \chi_K(r)k^2\right\} K^<(\hat{k}) = -i\lambda_0 k_i \int_{q<\Lambda_0 \exp(-r)} u_i^<(\hat{q})K^<(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^{d+1}} - \varepsilon^<(\hat{k}), \quad (2.53)$$

where the renormalized diffusivity is

$$\chi_K(r) = \chi_K^0 \left\{1 + \frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{\bar{\lambda}_0^2\nu_0^2}{\chi_K^0+\nu_0}\frac{\exp(\epsilon r)-1}{\epsilon\chi_K^0}\right\}. \quad (2.54)$$

In what follows, we attempt to derive the differential relation for χ_K in the same way as in Section 2.2, because YO made few comments on its derivation, only mentioning that *taking* $r \to 0$. However, by simply taking $r \to 0$, we cannot get the correct equation. The correct procedure should be as follows. From Eq. (2.54), according to the renormalization theory, the renormalized diffusivity has the following relation:

$$\begin{aligned} \chi_K(\Lambda - \Delta\Lambda) &= \chi_K(\Lambda) + \Delta\chi_K(\Lambda) \\ &= \chi_K(\Lambda) \\ &\quad + \frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{D_0\lambda_0^2}{\nu(\Lambda)\{\chi_K(\Lambda)+\nu(\Lambda)\}\epsilon}\left\{\frac{1}{(\Lambda-\Delta\Lambda)^\epsilon} - \frac{1}{\Lambda^\epsilon}\right\}. \end{aligned} \quad (2.55)$$

Hence, the differential relation for χ_K is obtained as follows:

$$\begin{aligned} \frac{d\chi_K(\Lambda)}{d\Lambda} &= \lim_{\Delta\Lambda\to 0}\frac{\chi_K(\Lambda-\Delta\Lambda) - \chi_K(\Lambda)}{(\Lambda-\Delta\Lambda)-\Lambda} \\ &= \lim_{\Delta\Lambda\to 0}\frac{\frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{D_0\lambda_0^2}{\nu(\Lambda)\{\nu(\Lambda)+\chi_K(\Lambda)\}\epsilon}\left\{\frac{1}{(\Lambda-\Delta\Lambda)^\epsilon}-\frac{1}{\Lambda^\epsilon}\right\}}{-\Delta\Lambda} \\ &= -\lim_{\Delta\Lambda\to 0}\frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{D_0\lambda_0^2}{\nu(\Lambda)\{\nu(\Lambda)+\chi_K(\Lambda)\}\epsilon}\frac{(\Lambda-\Delta\Lambda)^{-\epsilon}-\Lambda^{-\epsilon}}{\Delta\Lambda} \\ &= -\frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{D_0\lambda_0^2}{\nu(\Lambda)\{\nu(\Lambda)+\chi_K(\Lambda)\}\epsilon}\left[-\frac{d\Lambda^{-\epsilon}}{d\Lambda}\right] \end{aligned}$$

$$= -\frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{D_0\lambda_0^2}{\nu(\Lambda)\{\nu(\Lambda)+\chi_K(\Lambda)\}\Lambda^{\epsilon+1}}, \tag{2.56}$$

and replacing Λ with $\Lambda_0 \exp(-r)$ gives

$$\frac{d\chi_K(r)}{dr} = \frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{D_0}{\nu(r)\Lambda_0^\epsilon}\frac{\lambda_0^2}{\chi_K(r)+\nu(r)}\exp(\epsilon r) \tag{2.57}$$

or

$$\frac{d\chi_K(r)}{dr} = \frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{\bar{\lambda}^2(r)\nu^2(r)}{\chi_K(r)+\nu(r)}, \tag{2.58}$$

where

$$\bar{\lambda}^2(r) = \frac{\lambda_0^2 D_0}{\nu^3(r)\Lambda_0^\epsilon}\exp(\epsilon r). \tag{2.59}$$

Equation (2.58) is identical to that used in the YO theory. Again, there is no ϵ-expansion in mathematical treatment.

Next, YO define the parameter $\alpha_K(r)$ as $\alpha_K(r) = \chi_K(r)/\nu(r)$, write its differential equation as

$$\begin{aligned}\frac{d\alpha_K(r)}{dr} &= \frac{1}{\nu(r)}\frac{d\chi_K(r)}{dr} - \frac{\chi_K(r)}{\nu^2(r)}\frac{d\nu(r)}{dr} \\ &= \frac{S_d}{(2\pi)^d}\bar{\lambda}^2(r)\left\{\frac{d-1}{d}\frac{1}{1+\alpha_K(r)} - \frac{1}{2}\frac{d^2-d-\epsilon}{d(d+2)}\alpha_K(r)\right\},\end{aligned} \tag{2.60}$$

and solve this under the initial condition $\alpha_K(0) = \alpha_0$:

$$\left|\frac{\alpha_K - a}{\alpha_0 - a}\right|^{\frac{a+1}{a+b}}\left|\frac{\alpha_K + b}{\alpha_0 + b}\right|^{\frac{b-1}{a+b}} = \frac{\nu_0}{\nu(r)}, \tag{2.61}$$

where they estimate a and b as follows:

$$\begin{aligned} a &= \frac{1}{2}\left\{-1 + \left(1 + \frac{4\epsilon}{3}\frac{d(d+2)}{d^2-d-\epsilon}\right)^{\frac{1}{2}}\right\} \\ &= \frac{1}{2}\left\{-1 + \left(1 + \frac{4\times(4)}{3}\times\frac{3\times(3+2)}{3^2-3-(0)}\right)^{\frac{1}{2}}\right\} \\ &= 1.3929, \end{aligned} \tag{2.62}$$

$$
\begin{aligned}
b &= \frac{1}{2}\left\{1+\left(1+\frac{4\epsilon}{3}\frac{d(d+2)}{d^2-d-\epsilon}\right)^{\frac{1}{2}}\right\} \\
&= \frac{1}{2}\left\{1+\left(1+\frac{4\times(4)}{3}\times\frac{3\times(3+2)}{3^2-3-(0)}\right)^{\frac{1}{2}}\right\} \\
&= 2.3929. \qquad (2.63)
\end{aligned}
$$

From Eq. (2.61), α_K approaches the constant $\alpha_K^* = a = 1.3929 \simeq 1.39$ when $\nu_0 \ll \nu(r)$. But this value $\alpha_K^* = 1.39$ is also computed by putting $\epsilon = 0$ and $\epsilon = 4$ in the same equation [see Eq. (2.62)].

In actual turbulent shear flow, the production term and the convection term would appear from the nonlinear term. To account for this fact, YS add these terms to the initial governing equation:

$$u_j\frac{\partial K}{\partial x_j} \to u_j\frac{\partial K}{\partial x_j} + u_iu_j\frac{\partial \overline{U}_i}{\partial x_j} + \overline{U}_j\frac{\partial K}{\partial x_j}. \qquad (2.64)$$

The averaged value for the turbulent kinetic energy and its dissipation rate are defined as $\lim_{\hat{k}\to 0} K(\hat{k}) = \overline{K}$ and $\lim_{\hat{k}\to 0} \varepsilon(\hat{k}) = \overline{\varepsilon}$, respectively. Hence the production term is

$$\lim_{\hat{k}\to 0} u_iu_j\frac{\partial \overline{U}_i}{\partial x_j} = \overline{u_iu_j}\frac{\partial \overline{U}_i}{\partial x_j}, \qquad (2.65)$$

and the convection term is

$$\lim_{\hat{k}\to 0} \overline{U}_j\frac{\partial K}{\partial x_j} = \overline{U}_j\frac{\partial \overline{K}}{\partial x_j}. \qquad (2.66)$$

Although the following $\overline{K}$-equation is obtained, it is not directly derived by the RNG theory. It is apparent from the foregoing discussions that their RNG theory predicts only the modified diffusion term and the numerical constants are invalid.

$$\frac{\partial \overline{K}}{\partial \tau} + \overline{U}_i\frac{\partial \overline{K}}{\partial x_i} = -\overline{\varepsilon} + P_K + \frac{\partial}{\partial x_i}\left\{\alpha_K^*\nu(\Lambda_e)\frac{\partial \overline{K}}{\partial x_i}\right\} \qquad (2.67)$$

$$P_K = -\overline{u_iu_j}\frac{\partial \overline{U}_i}{\partial x_j} \qquad (2.68)$$

$$\nu(\Lambda_e) = 0.085\,\frac{\overline{K}^2}{\overline{\varepsilon}} \qquad (2.69)$$

$$\alpha_K^* = 1.39 \qquad (2.70)$$

2.6 Modeling of $\bar{\varepsilon}$-Equation

The transport equation for ε is obtained from Eq. (2.1).

$$\begin{aligned}\frac{\partial \varepsilon}{\partial \tau} + u_i \frac{\partial \varepsilon}{\partial x_i} &= 2\nu_0 \frac{\partial u_i}{\partial x_j}\frac{\partial f_i}{\partial x_j} - 2\nu_0 \frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_j}\frac{\partial u_i}{\partial x_l} - 2\nu_0^2 \left(\frac{\partial^2 u_i}{\partial x_j \partial x_l}\right)^2 \\ &\quad -2\frac{\nu_0}{\rho}\frac{\partial u_i}{\partial x_j}\frac{\partial^2 p}{\partial x_i \partial x_j} + \nu_0 \frac{\partial^2 \varepsilon}{\partial x_i \partial x_i}\end{aligned} \tag{2.71}$$

Using the definition of the Fourier transform:

$$\varepsilon(\boldsymbol{x}, \tau) = \int_{k<\Lambda_0}\int_{-\infty}^{+\infty} \varepsilon(\boldsymbol{k}, \omega) \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}, \tag{2.72}$$

and

$$\varepsilon(\boldsymbol{k}, \omega) \equiv \varepsilon(\hat{k}) = -\nu_0 \int_{q<\Lambda_0} q_j(k-q)_j u_i(\hat{q}) u_i(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^{d+1}}, \tag{2.73}$$

the equation for ε in Fourier space is

$$\begin{aligned}(-i\omega + \chi_\varepsilon^0 k^2)\varepsilon(\hat{k}) &= -i\lambda_0 k_i \int_{q<\Lambda_0} u_i(\hat{q})\varepsilon(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^{d+1}} \\ &\quad -Y_1(\hat{k}) - Y_2(\hat{k}),\end{aligned} \tag{2.74}$$

where $\chi_\varepsilon^0 (= \nu_0)$ is the bare diffusivity, the turbulence production term for dissipation $\varepsilon(\hat{k})$ is

$$Y_1(\hat{k}) = -2i\nu_0 \int_{q<\Lambda_0}\int_{r<\Lambda_0} q_j r_j (k-q-r)_l u_i(\hat{q}) u_l(\hat{r}) u_i(\hat{k}-\hat{q}-\hat{r})\frac{d\hat{r}d\hat{q}}{(2\pi)^{2d+2}}, \tag{2.75}$$

and destruction of the dissipation is

$$Y_2(\hat{k}) = 2\nu_0^2 \int_{q<\Lambda_0} q^2 |\boldsymbol{k}-\boldsymbol{q}|^2 u_i(\hat{q}) u_i(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^{d+1}}. \tag{2.76}$$

Smith and Reynolds (1992) showed that the pressure-strain correlation term and the force-strain correlation term are neglected as a result of a renormalized procedure.

However, we consider that Eq. (2.74) with Eq. (2.75) and Eq. (2.76) does not describe the turbulent energy dissipation rate exactly because it is not defined in the dissipation range which exists in much higher wave number region ($k > \Lambda_0$).

After eliminating the modes in the interval $\Lambda_e < k < \Lambda_0$, the renormalized ε-equation is

$$\left\{-i\omega + \chi_\varepsilon(\Lambda_e)k^2\right\}\varepsilon^<(\hat{k}) = -i\lambda_0 k_i \int_{q<\Lambda_e} u_i^<(\hat{q})\varepsilon^<(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^{d+1}} - Y_1^<(\hat{k}) - Y_2^<(\hat{k}), \quad (2.77)$$

where the renormalized diffusivity is represented as

$$\chi_\varepsilon(\Lambda_e) = \alpha_\varepsilon^* \nu(\Lambda_e), \quad (2.78)$$

$$\alpha_\varepsilon^* = 1.39, \quad (2.79)$$

and the diffusion term is modified as

$$\frac{\partial}{\partial x_j}\left(\nu_0 \frac{\partial \varepsilon}{\partial x_j} - u_j \varepsilon\right) \to \frac{\partial}{\partial x_j}\left\{\alpha_\varepsilon^* \nu(\Lambda_e)\frac{\partial \varepsilon}{\partial x_j}\right\}. \quad (2.80)$$

In homogeneous flow, the following terms, which include the mean velocity, are added to the production term of the dissipation in the YS analysis:

$$2\nu_0 \frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_j}\frac{\partial u_i}{\partial x_l} \to \underbrace{2\nu_0 \frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_j}\frac{\partial u_i}{\partial x_l}}_{T_1} + \underbrace{2\nu_0 \frac{\partial \overline{U}_i}{\partial x_j}\frac{\partial u_l}{\partial x_j}\frac{\partial u_i}{\partial x_l}}_{T_3}$$

$$+ \underbrace{2\nu_0 \frac{\partial u_i}{\partial x_j}\frac{\partial \overline{U}_l}{\partial x_j}\frac{\partial u_i}{\partial x_l}}_{T_4} + \underbrace{2\nu_0 \frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_j}\frac{\partial \overline{U}_i}{\partial x_l}}_{T_5}. \quad (2.81)$$

Smith and Reynolds (1992) showed that T_1, which corresponds $Y_1(\hat{k})$ given by Eq. (2.75), can be neglected as a result of the YO renormalization procedure. On the other hand, T_3 vanishes for homogeneity, and T_4 is modeled as follows (Yakhot et al. 1992):

$$\overline{T}_4 = \frac{C_\mu \eta^3 (1-\eta/\eta_0)}{1+\beta\eta^3}\frac{\overline{\varepsilon}^2}{\overline{K}}, \quad (2.82)$$

$$\eta = \frac{S\overline{K}}{\overline{\varepsilon}}, \quad (2.83)$$

$$S = (2S_{ij}S_{ij})^{\frac{1}{2}}, \; S_{ij} = \frac{1}{2}\left(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i}\right), \quad (2.84)$$

where $C_\mu = 0.085$ [see Eq. (2.48)], $\eta_0 = 4.38$, $\beta = 0.012$.

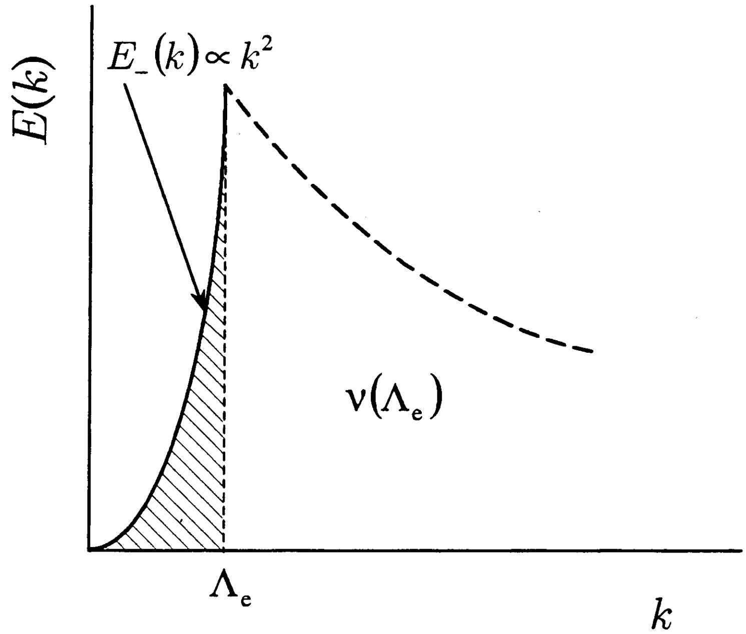

Figure 2.2: Energy spectrum in lower wave-numbers range

In the limit $\hat{k} \rightarrow 0$, T_5 is estimated in the lower wave-numbers range:

$$\begin{aligned}\overline{T}_5 &= 2\nu_0 \frac{\partial \overline{U}_i}{\partial x_j} \int_{q<\Lambda_0} q^2 u_i(\hat{q}) u_j(-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\ &= 2\nu\left(\Lambda_e\right) \frac{\partial \overline{U}_i}{\partial x_j} \int_{q<\Lambda_e} q^2 u_i^<(\hat{q}) u_j^<(-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}}. \end{aligned} \tag{2.85}$$

In the YO and YS theory, it is further assumed that the spectrum tensor $E_{ij}(\boldsymbol{k})$ is proportional to k^2 in the region $k < \Lambda_e$ (see Fig. 2.2), the Reynolds stress is thus given by

$$\begin{aligned}\overline{u_i u_j} &= \int_{k<\Lambda_e} u_i(\hat{k}) u_j(-\hat{k}) \frac{d\hat{k}}{(2\pi)^{d+1}} \\ &= \int_{k<\Lambda_e} A k^2 dk = \frac{A}{3}\,\Lambda_e^3, \end{aligned} \tag{2.86}$$

and substituting its relation,

$$A = \frac{3}{\Lambda_e^3}\,\overline{u_i u_j}, \tag{2.87}$$

into Eq. (2.85), the following result is obtained:

$$\begin{aligned}\overline{T}_5 &= 2\nu(\Lambda_e)\frac{\partial \overline{U}_i}{\partial x_j}\int_{q<\Lambda_e} q^2 u_i^<(\hat{q})u_j^<(-\hat{q})\frac{d\hat{q}}{(2\pi)^{d+1}} \\ &= 2\nu(\Lambda_e)\frac{\partial \overline{U}_i}{\partial x_j}\int_{q<\Lambda_e} Aq^4 dq \\ &= \frac{6}{5}\nu(\Lambda_e)\Lambda_e^2\overline{u_i u_j}\frac{\partial \overline{U}_i}{\partial x_j}. \end{aligned} \tag{2.88}$$

Note that by eliminating $\overline{\varepsilon}$ from Eq. (2.47) and Eq. (2.48) it follows that

$$\overline{K} = 10\nu^2(\Lambda_e)\Lambda_e^2. \tag{2.89}$$

Using Eq. (2.48) and Eq. (2.89), the above $\overline{T}_5$ is ultimately modeled in the YO and YS theory as

$$\overline{T}_5 = -1.42 P_K \frac{\overline{\varepsilon}}{\overline{K}}. \tag{2.90}$$

Similarly, in the limits $\omega \to 0$ and $k \to 0$, YS set the destruction term as $\overline{T}_2 = \lim_{\hat{k}\to 0} Y_2^<(\hat{k})$, and integrate this over the lower range $0 < k < \Lambda_e$, which yields

$$\overline{T}_2 = 1.68\frac{\overline{\varepsilon}^2}{\overline{K}}. \tag{2.91}$$

It should be noted that Eq. (2.91) is obtained by the use of the following relations:

$$\overline{T}_2 = 4\nu^2(\Lambda_e)\int_0^{\Lambda_e} q^4 E_-(q)dq, \tag{2.92}$$

$$\overline{\varepsilon} \simeq 2\nu(\Lambda_e)\int_0^{\Lambda_e} q^2 E_-(q)dq, \tag{2.93}$$

$$E_-(k) = Ak^2. \tag{2.94}$$

The existing theory of turbulence does not permit our estimating $\overline{T}_2$ and $\overline{\varepsilon}$ by using the lower wave number spectrum [Eq. (2.94)].

On the other hand, in order to represent actual turbulent flows, YS add $\overline{U}_i\partial\varepsilon/\partial x_i$ to the nonlinear term in the initial governing equation with the stirring forces as follows:

$$u_i\frac{\partial\varepsilon}{\partial x_i} \to u_i\frac{\partial\varepsilon}{\partial x_i} + \overline{U}_i\frac{\partial\varepsilon}{\partial x_i}. \tag{2.95}$$

Finally, the high Reynolds number form of the RNG-based $\overline{\varepsilon}$-equation is written as

$$\frac{\partial \overline{\varepsilon}}{\partial \tau} + \overline{U}_i \frac{\partial \overline{\varepsilon}}{\partial x_i} = 1.42 P_K \frac{\overline{\varepsilon}}{\overline{K}} - 1.68 \frac{\overline{\varepsilon}^2}{\overline{K}} + \frac{C_\mu \eta^3 (1 - \eta/\eta_0)}{1 + \beta \eta^3} \frac{\overline{\varepsilon}^2}{\overline{K}} + \frac{\partial}{\partial x_i} \left\{ \alpha_\varepsilon^* \nu(\Lambda_e) \frac{\partial \overline{\varepsilon}}{\partial x_i} \right\}. \quad (2.96)$$

On the face of it, it seems to be in good agreement with the current standard $\overline{K}$-$\overline{\varepsilon}$ model (see Nagano & Hishida 1987; Nagano & Tagawa 1990). But the numerical constants $\alpha_\varepsilon^* = 1.39$ and $C_\mu = 0.085$ are invalid, the destruction of ε is estimated in the lower wave number range, and the turbulent energy dissipation rate itself is also underestimated. In addition, the direct numerical simulation of turbulence (Kim et al. 1987; Mansour et al. 1988; Mansour et al. 1989) has revealed that it is the $Y_1(\hat{k})$ term that dominates the production of ε. The term, however, vanishes in the Yakhot-Orszag-Smith theory.

2.7 Concluding Remarks

The RNG theory developed by Yakhot & Orszag (1986) and reformulated by Yakhot & Smith (1992) has been confirmed. It became evident that their theory is only a scale removal procedure rather than a renormalization group theory because any ϵ-expansion does not appear in their theory.

The forced Navier-Stokes equation is considered to be unsuitable for deriving the turbulence model identified with the Reynolds-averaged model. The exact Navier-Stokes equation with mean shear should be used as a governing equation.

Chapter 3

ITERATIVE AVERAGING RNG THEORY

3.1 Iterative Averaging Method for Velocity Field

3.1.1 Basic equations

The equation of motion for an incompressible fluid is

$$\frac{\partial U_i}{\partial \tau} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu_0 \frac{\partial^2 U_i}{\partial x_j \partial x_j}, \tag{3.1}$$

and the continuity equation is

$$\frac{\partial U_i}{\partial x_i} = 0. \tag{3.2}$$

Here U_i is the instantaneous velocity component, P is the pressure, ρ is the density of the fluid, and ν_0 is the molecular kinematic viscosity. The mean velocity $\overline{U}_i$ and the fluctuating velocity u_i are represented in a Fourier series as follows (Giles 1994a; Giles 1994b):

$$\overline{U}_i(\boldsymbol{x}, \tau) = \sum_{k \ll \Lambda_e} \overline{U}_i(\boldsymbol{k}, \tau) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \tag{3.3}$$

$$u_i(\boldsymbol{x}, \tau) = \sum_{k \geq \Lambda_e} u_i(\boldsymbol{k}, \tau) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}), \tag{3.4}$$

where the wave number Λ_e is the order of that for the energy-containing eddies, the wave vector $\boldsymbol{k}$ is given by

$$\boldsymbol{k} = \frac{2\pi}{L}\{n_1, n_2, n_3\}, \tag{3.5}$$

and n_1, n_2 and n_3 are integers, each of which is summed over the range from $-\infty$ to $+\infty$; we begin by considering the turbulent fluid to occupy a cubic box of size L. In the present formulation, it should be noted that the effect of fluctuating velocity u_i defined in the lowest wave number range $k < \Lambda_e$ is neglected so that:

$$\int_0^{\Lambda_e} E(k)dk \ll \int_{\Lambda_e}^{\infty} E(k)dk \tag{3.6}$$

at high Reynolds number (see Fig. 3.1). Then, the turbulent kinetic energy $\overline{K}$ is represented using the Kolmogorov spectrum, $E(k) = C_K\overline{\varepsilon}^{2/3}k^{-5/3}$, as follows:

$$\begin{aligned} K &= \int_{\Lambda_e}^{\infty} C_K\overline{\varepsilon}^{\frac{2}{3}}k^{-\frac{5}{3}}dk \\ &\simeq \int_{\Lambda_e}^{\Lambda_0} C_K\overline{\varepsilon}^{\frac{2}{3}}k^{-\frac{5}{3}}dk \\ &\simeq \frac{3C_K}{2}\overline{\varepsilon}^{\frac{2}{3}}\Lambda_e^{-\frac{2}{3}}, \end{aligned} \tag{3.7}$$

where C_K is the Kolmogorov constant, $\overline{\varepsilon}$ is the dissipation rate of $\overline{K}$, and $\Lambda_e \ll \Lambda_0$ (Λ_0 is a sufficiently large wave number which marks the dissipation range of turbulent kinetic energy as defined later), so that Λ_e is given by

$$\Lambda_e = \left(\frac{3C_K}{2}\right)^{\frac{3}{2}}\frac{\overline{\varepsilon}}{\overline{K}^{\frac{3}{2}}} = \left(\frac{3C_K}{2}\right)^{\frac{3}{2}}k_e. \tag{3.8}$$

Note that the wave number $k_e(=\overline{\varepsilon}/\overline{K}^{3/2})$ represents the range of the energy-containing eddies. Hence, the fluctuating velocity u_i satisfies the following condition:

$$\overline{u}_i = 0. \tag{3.9}$$

Accordingly, we can consider that averaging according to Eqs. (3.3) and (3.4) is equivalent to the Reynolds decomposition on the assumption that Λ_e is a sufficiently small value, i.e., the turbulence Reynolds number $Re_t(=\overline{K}^2/\nu_0\overline{\varepsilon})$ is sufficiently large since $(\overline{K}^{1/2}/\nu_0)/\Lambda_e =$

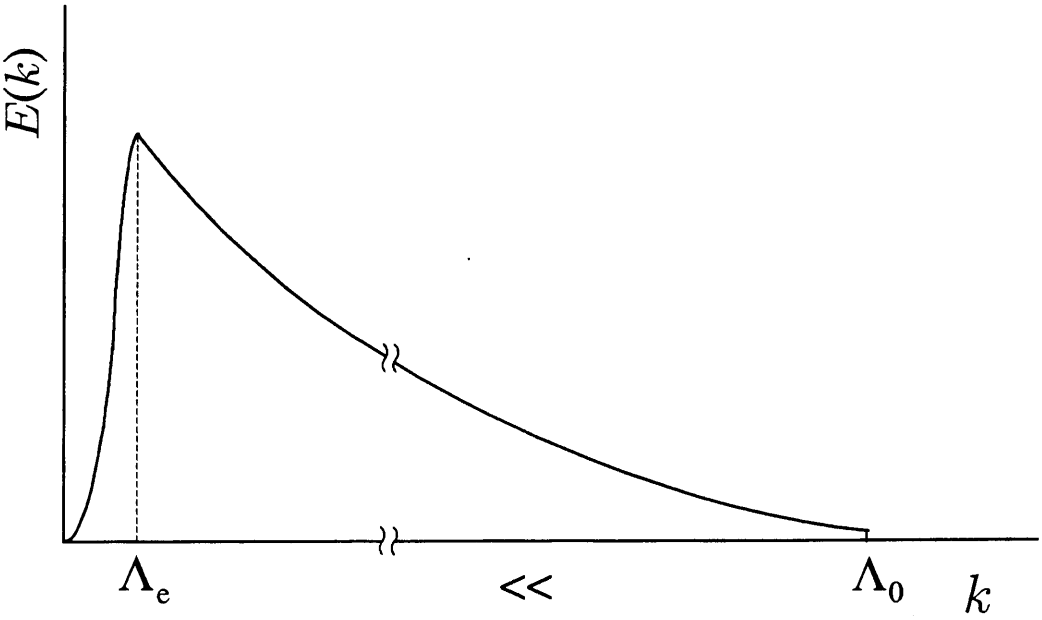

Figure 3.1: Energy spectrum at large Reynolds number

$(2/3C_K)^{3/2}Re_t$. As a result, the application of Eqs. (3.3) and (3.4) to Eq. (3.1) gives the following two Navier-Stokes equations:

$$\frac{\partial \overline{U}_i}{\partial \tau} + \overline{U}_j \frac{\partial \overline{U}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} + \nu_0 \frac{\partial^2 \overline{U}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\overline{u_i u_j}, \tag{3.10}$$

and

$$\frac{\partial u_i}{\partial \tau} + u_j \frac{\partial \overline{U}_i}{\partial x_j} + \overline{U}_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu_0 \frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\left(u_i u_j - \overline{u_i u_j}\right). \tag{3.11}$$

Then we begin by transforming Eq. (3.11) to Fourier space as follows:

$$\begin{aligned} \left(\frac{\partial}{\partial \tau} + \nu_0 k^2\right) u_i(\boldsymbol{k}, \tau) &= -\frac{i}{2} P_{imn}(\boldsymbol{k}) \sum_q u_m(\boldsymbol{q}, \tau) u_n(\boldsymbol{k}-\boldsymbol{q}, \tau) \\ & \quad -i P_{imn}(\boldsymbol{k}) \sum_q \overline{U}_m(\boldsymbol{q}, \tau) u_n(\boldsymbol{k}-\boldsymbol{q}, \tau), \end{aligned} \tag{3.12}$$

where

$$P_{imn}(\boldsymbol{k}) = k_m P_{in}(\boldsymbol{k}) + k_n P_{im}(\boldsymbol{k}) \tag{3.13}$$

and

$$P_{ij}(\boldsymbol{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}. \tag{3.14}$$

The continuity equation for u_i also becomes

$$k_i u_i(\boldsymbol{k}, \tau) = 0. \tag{3.15}$$

Here the compound projection operator $P_{imn}(\boldsymbol{k})$ results from the elimination of the pressure by using the continuity condition, and a variable wave number k is defined in the interval $\Lambda_e \leq k < \Lambda_0$.

The maximum wave number Λ_0 is defined approximately through the dissipation integral (McComb 1990):

$$\bar{\varepsilon} = \int_0^\infty 2\nu_0 k^2 E(k) dk \simeq \int_{\Lambda_e}^{\Lambda_0} 2\nu_0 k^2 E(k) dk. \tag{3.16}$$

Apparently Λ_0 should be the same order of magnitude as the Kolmogorov dissipation wave number $k_d = (\bar{\varepsilon}/\nu_0^3)^{1/4}$. If we assume the Kolmogorov spectrum $E(k) = C_K \bar{\varepsilon}^{2/3} k^{-5/3}$, Eq. (3.16) yields

$$\Lambda_0 \simeq \left(\frac{2}{3C_K}\right)^{\frac{3}{4}} k_d. \tag{3.17}$$

The estimates given by Eqs. (3.8) and (3.17) are most appropriate in the case where a value of the Reynolds number is so high that the range of the energy-containing eddies and the range of maximum dissipation are sufficiently wide apart, i.e., $k_e \ll k_d$ or $\Lambda_e \ll \Lambda_0$. The technicalities of Fourier-transforming Eq. (3.11) to Eq. (3.12) are discussed in Appendix B. In addition, multiplying each side of Eq. (3.12) by k_i with Eq. (3.15) gives an important property of $P_{ij}(\boldsymbol{k})$:

$$k_i P_{ij}(\boldsymbol{k}) = 0, \tag{3.18}$$

which is of great use in Section 3.1.2.

Similarly, we can represent the Reynolds stress $\overline{u_i u_j}$ in the Fourier series as follows:

$$\overline{u_i u_j} = \sum_{k \ll \Lambda_e} R_{ij}(\boldsymbol{k}, \tau) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}), \tag{3.19}$$

where

$$R_{ij}(\boldsymbol{k}, \tau) = \sum_q u_i(\boldsymbol{q}, \tau) u_j(\boldsymbol{k} - \boldsymbol{q}, \tau), \tag{3.20}$$

and its wave numbers are defined in the interval $k \ll \Lambda_e$, $\Lambda_e \leq q < \Lambda_0$, $\Lambda_e \leq |\boldsymbol{k}-\boldsymbol{q}| < \Lambda_0$. In particular, it should be noted that the wave number k is sufficiently small.

3.1.2 Elimination of small scales

In this section, we will develop the eddy-viscosity type turbulence model with the aid of RNG by iterative averaging. First of all, we begin by dividing the velocity component into two components in terms of the second cutoff wave number $\Lambda_1(\Lambda_e \ll \Lambda_1 < \Lambda_0)$ as shown in Fig. 3.2:

$$u_i(\boldsymbol{q},\tau) = \begin{cases} u_i^<(\boldsymbol{q},\tau) & : \ \Lambda_e \leq q < \Lambda_1 \\ u_i^>(\boldsymbol{q},\tau) & : \ \Lambda_1 \leq q < \Lambda_0 \end{cases}, \tag{3.21}$$

where the interval $\Lambda_1 \leq q < \Lambda_0$ is assumed to be a narrow band near the initial cutoff wave number Λ_0, so that Λ_1 may be written using the bandwidth parameter λ as

$$\Lambda_1 = \Lambda_0(1-\lambda)\ , \ \ 0 < \lambda \ll 1. \tag{3.22}$$

According to Eq. (3.21), the r.h.s. of Eq. (3.20) is expanded as follows:

$$\begin{aligned} \sum_q u_i(\boldsymbol{q},\tau)u_j(\boldsymbol{k}-\boldsymbol{q},\tau) &= \sum_q u_i^<(\boldsymbol{q},\tau)u_j^<(\boldsymbol{k}-\boldsymbol{q},\tau) \\ &\quad + \sum_q u_i^>(\boldsymbol{q},\tau)u_j^<(\boldsymbol{k}-\boldsymbol{q},\tau) \\ &\quad + \sum_q u_i^<(\boldsymbol{q},\tau)u_j^>(\boldsymbol{k}-\boldsymbol{q},\tau) \\ &\quad + \sum_q u_i^>(\boldsymbol{q},\tau)u_j^>(\boldsymbol{k}-\boldsymbol{q},\tau). \end{aligned} \tag{3.23}$$

Here we assume that the probability distribution of $\boldsymbol{u}^>$ in the band $\Lambda_1 \leq q < \Lambda_0$ is near-Gaussian, and introduce the operation of partial averaging $\langle \quad \rangle_c$ over fluctuations in the band $\Lambda_1 \leq q < \Lambda_0$, where the subscript serves to distinguish it from the total (global) average, defined by $\langle \quad \rangle$. By applying this average to Eq. (3.23), the first term of the r.h.s. of Eq. (3.23) becomes

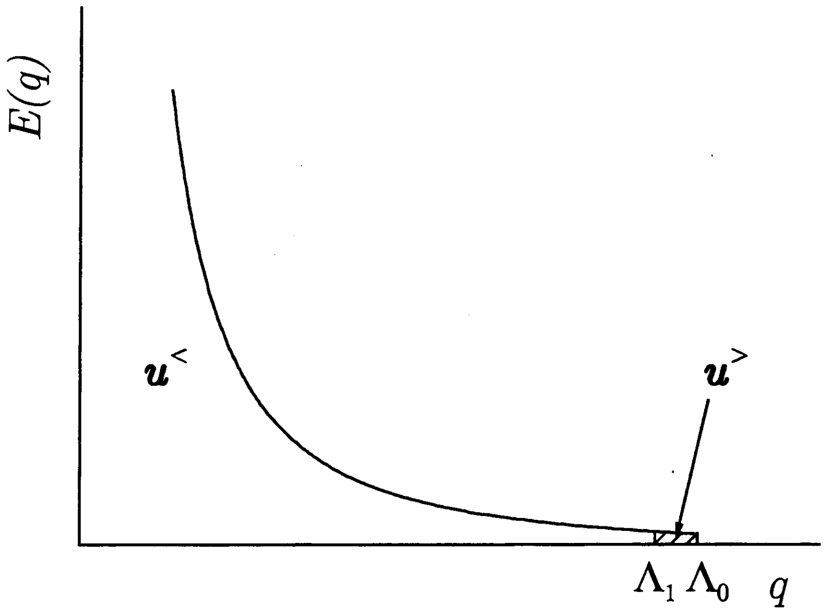

Figure 3.2: Elimination of velocity components in the band $\Lambda_1 \leq q < \Lambda_0$

$$\sum_q \Big\langle u_i^<(\boldsymbol{q},\tau)u_j^<(\boldsymbol{k}-\boldsymbol{q},\tau)\Big\rangle_c = \sum_q u_i^<(\boldsymbol{q},\tau)u_j^<(\boldsymbol{k}-\boldsymbol{q},\tau) \tag{3.24}$$

because this term has no components defined in the range $\Lambda_1 \leq q < \Lambda_0$. Thus, it is not influenced under partial averaging. For the second and the third terms, we have the problem that the $\boldsymbol{u}^>$ mode in real turbulence seems to be not independent of the $\boldsymbol{u}^<$ mode because these modes are coupled each other through the nonlinear term in the Navier-Stokes equation, which conflicts in part with the concept of partial averaging, i. e., the straightforward partial average is equivalent to a conditional average in which $\boldsymbol{u}^<$ is held constant while $\boldsymbol{u}^>$ is averaged (McComb 1990). Therefore, we take advantage of McComb's idea (McComb & Watt 1990) writing the higher wave number mode in terms of another velocity field $\boldsymbol{v}^>$:

$$u_i^>(\boldsymbol{q},\tau) = v_i^>(\boldsymbol{q},\tau) + \Delta_i^>(\boldsymbol{q},\tau), \tag{3.25}$$

where $\boldsymbol{v}^{>}$ is a field of the same type as $\boldsymbol{u}^{>}$ except that it is not coupled to the $\boldsymbol{u}^{<}$ modes. The properties of $\boldsymbol{v}^{>}$ under total averaging, denoted by $\langle \quad \rangle$, are the same as those of $\boldsymbol{u}^{>}$; thus,

$$\langle v_i^{>}(\boldsymbol{q},\tau)\rangle = \langle u_i^{>}(\boldsymbol{q},\tau)\rangle = 0 \tag{3.26}$$

and

$$\left\langle v_i^{>}(\boldsymbol{q},\tau)v_j^{>}(\boldsymbol{q}',\tau)\right\rangle = \left\langle u_i^{>}(\boldsymbol{q},\tau)u_j^{>}(\boldsymbol{q}',\tau)\right\rangle . \tag{3.27}$$

The function $\Delta^{>}$ in Eq. (3.25) represents the part coupled to the $\boldsymbol{u}^{<}$ modes. Thus, from Eqs. (3.26) and (3.27), its properties under total averaging become

$$\langle \Delta_i^{>}(\boldsymbol{q},\tau)\rangle = 0 \tag{3.28}$$

and

$$\left\langle \Delta_i^{>}(\boldsymbol{q},\tau)\Delta_j^{>}(\boldsymbol{q}',\tau)\right\rangle = 0. \tag{3.29}$$

Accordingly, in view of the above points, the second and the third terms of the r.h.s. in Eq. (3.23) are assumed to be

$$\begin{aligned}
\sum_q \left\langle u_i^{>}(\boldsymbol{q},\tau)u_j^{<}(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c &= \sum_q \langle v_i^{>}(\boldsymbol{q},\tau)\rangle_c u_j^{<}(\boldsymbol{k}-\boldsymbol{q},\tau) + \sum_q \langle \Delta_i^{>}(\boldsymbol{q},\tau)\rangle_c u_j^{<}(\boldsymbol{k}-\boldsymbol{q},\tau) \\
&= \sum_q \langle \Delta_i^{>}(\boldsymbol{q},\tau)\rangle_c u_j^{<}(\boldsymbol{k}-\boldsymbol{q},\tau) \\
&\simeq O(\lambda^m) \simeq 0 \ , \ m \geq 1
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
\sum_q \left\langle u_i^{<}(\boldsymbol{q},\tau)u_j^{>}(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c &= \sum_q u_i^{<}(\boldsymbol{q},\tau)\left\langle v_j^{>}(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c + \sum_q u_i^{<}(\boldsymbol{q},\tau)\left\langle \Delta_j^{>}(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c \\
&= \sum_q u_i^{<}(\boldsymbol{q},\tau)\left\langle \Delta_j^{>}(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c \\
&\simeq O(\lambda^m) \simeq 0 \ , \ m \geq 1
\end{aligned} \tag{3.31}$$

on the condition that the interval $\Lambda_1 \leq q < \Lambda_0$, or the bandwidth parameter λ, is sufficiently small. Note that Eqs. (3.30) and (3.31) amount to a so-called statistical scale-separation assumption.

Lastly, for the fourth term $\sum_q \left\langle u_i^>(\boldsymbol{q},\tau)u_j^>(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c$, we cannot replace it directly with the following energy spectrum tensor for homogeneous isotropic turbulence:

$$\langle u_\alpha(\boldsymbol{k},\tau)u_\beta(\boldsymbol{k}',\tau)\rangle = \left(\frac{2\pi}{L}\right)^3 Q(k,\tau)P_{\alpha\beta}(\boldsymbol{k})\delta_{k+k'}, \tag{3.32}$$

$$E(k,\tau) = 4\pi k^2 Q(k,\tau) \tag{3.33}$$

$$\delta_{k+k'} = \left(\frac{1}{L}\right)^3 \int_L\int_L\int_L \exp\{-i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}\}\, dx_1 dx_2 dx_3 \tag{3.34}$$

because the wave number k (or wave vector $\boldsymbol{k}$) in Eq. (3.23) is small but *not zero.* If the wave number k in Eq. (3.23) is set to 0, we obtain the result, which is the case of homogeneous field but out of our interest. Therefore, we begin by constructing the transport equation for $u_i^>(\boldsymbol{q},\tau)\ u_j^>(\boldsymbol{k}-\boldsymbol{q},\tau)$ obtained from multiplying the transport equation for $u_i^>(\boldsymbol{q},\tau)$ equivalent to Eq. (3.12) by $u_j^>(\boldsymbol{k}-\boldsymbol{q},\tau)$ and that for $u_j^>(\boldsymbol{k}-\boldsymbol{q},\tau)$ by $u_i^>(\boldsymbol{q},\tau)$, and adding these equations:

$$\begin{aligned}
&\left(\frac{\partial}{\partial\tau} + \nu_0 q^2 + \nu_0|\boldsymbol{k}-\boldsymbol{q}|^2\right) u_i^>(\boldsymbol{q},\tau)u_j^>(\boldsymbol{k}-\boldsymbol{q},\tau)\\
&= -iP_{jmn}^>(\boldsymbol{k}-\boldsymbol{q})\sum_r \overline{U}_m(\boldsymbol{r},\tau)u_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u_i^>(\boldsymbol{q},\tau)\\
&-iP_{imn}^>(\boldsymbol{q})\sum_r \overline{U}_m(\boldsymbol{r},\tau)u_n(\boldsymbol{q}-\boldsymbol{r},\tau)u_j^>(\boldsymbol{k}-\boldsymbol{q},\tau)\\
&-\frac{i}{2}P_{imn}^>(\boldsymbol{q})\sum_r u_m(\boldsymbol{r},\tau)u_n(\boldsymbol{q}-\boldsymbol{r},\tau)u_j^>(\boldsymbol{k}-\boldsymbol{q},\tau)\\
&-\frac{i}{2}P_{jmn}^>(\boldsymbol{k}-\boldsymbol{q})\sum_r u_m(\boldsymbol{r},\tau)u_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u_i^>(\boldsymbol{q},\tau).
\end{aligned}\tag{3.35}$$

Then substitution of Eq. (3.35) into $\sum_q \left\langle u_i^>(\boldsymbol{q},\tau)u_j^>(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c$ becomes (see Appendix B):

$$\begin{aligned}
&\sum_q \left\langle u_i^>(\boldsymbol{q},\tau)u_j^>(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c\\
=\ &-i\sum_q\sum_r \left(\frac{\partial}{\partial\tau} + \nu_0 q^2 + \nu_0|\boldsymbol{k}-\boldsymbol{q}|^2\right)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&\times\left\{P^{>}_{jmn}(\boldsymbol{k}-\boldsymbol{q})\overline{U}_m(\boldsymbol{r},\tau)\left\langle u_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_i(\boldsymbol{q},\tau)\right\rangle_c\right.\\
&+P^{>}_{imn}(\boldsymbol{q})\overline{U}_m(\boldsymbol{r},\tau)\left\langle u_n(\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_j(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c\\
&+\frac{1}{2}P^{>}_{imn}(\boldsymbol{q})\left\langle u_m(\boldsymbol{r},\tau)u_n(\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_j(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c\\
&\left.+\frac{1}{2}P^{>}_{jmn}(\boldsymbol{k}-\boldsymbol{q})\left\langle u_m(\boldsymbol{r},\tau)u_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_i(\boldsymbol{q},\tau)\right\rangle_c\right\}\\
\simeq\ & -i\sum_q\sum_r\left(\frac{\partial}{\partial\tau}+\nu_0q^2+\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2\right)^{-1}\\
&\times\left\{P^{>}_{jmn}(\boldsymbol{k}-\boldsymbol{q})\overline{U}_m(\boldsymbol{r},\tau)\left\langle v^{>}_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)v^{>}_i(\boldsymbol{q},\tau)\right\rangle_c\right.\\
&\left.+P^{>}_{imn}(\boldsymbol{q})\,\overline{U}_m(\boldsymbol{r},\tau)\left\langle v^{>}_n(\boldsymbol{q}-\boldsymbol{r},\tau)v^{>}_j(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c\right\}. \qquad (3.36)
\end{aligned}
$$

And the r.h.s. of Eq. (3.36) can be expanded infinitely by replacing the second-order moment of $\boldsymbol{u}^{>}$ repeatedly with its transport equation. Thus, as shown in Appendix B, keep terms to the order of $\overline{\boldsymbol{U}}^1$, since these terms are the leading ones. This corresponds, in effect, to a moment closure hypothesis. As a result, the fourth term of the r.h.s. of Eq. (3.23) can be reformed as follows:

$$
\begin{aligned}
&\sum_q\left\langle u^{>}_i(\boldsymbol{q},\tau)u^{>}_j(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c\\
&\simeq -i\sum_q\left(\frac{2\pi}{L}\right)^3\frac{1}{\nu_0q^2+\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2}\left\{P^{>}_{jmn}(\boldsymbol{k}-\boldsymbol{q})P^{>}_{in}(\boldsymbol{q})Q^{>}_v(q)\right.\\
&\left.+P^{>}_{imn}(\boldsymbol{q})P^{>}_{jn}(\boldsymbol{k}-\boldsymbol{q})Q^{>}_v(|\boldsymbol{k}-\boldsymbol{q}|)\right\}\overline{U}_m(\boldsymbol{k},\tau). \qquad (3.37)
\end{aligned}
$$

It should be noted that $\partial/\partial\tau$ can be neglected on account of universality of the inertial range. So this idea is, as it were, a simple case called a Markovian approximation. By assuming the relation $|\boldsymbol{k}-\boldsymbol{q}|\simeq q$ and $Q^{>}_v(|\boldsymbol{k}-\boldsymbol{q}|)\simeq Q^{>}_v(q)$ in terms of the narrow band $\Lambda_1\le q<\Lambda_0$, Eq. (3.37) is rewritten as:

$$
\sum_q\left\langle u^{>}_i(\boldsymbol{q},\tau)u^{>}_j(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c
$$

$$\simeq -i\sum_q\left(\frac{2\pi}{L}\right)^3\frac{1}{2\nu_0 q^2}\left\{P^>_{jmn}(\boldsymbol{k}-\boldsymbol{q})P^>_{in}(\boldsymbol{q})+P^>_{imn}(\boldsymbol{q})P^>_{jn}(\boldsymbol{k}-\boldsymbol{q})\right\}$$

$$\times Q^>_v(q)\overline{U}_m(\boldsymbol{k},\tau). \tag{3.38}$$

In addition, the projection operators in Eq. (3.38) expanded to $O(k)$ are also reformed simply by using the relation $|\boldsymbol{k}-\boldsymbol{q}|\simeq q$:

$$P^>_{imn}(\boldsymbol{q})P^>_{jn}(\boldsymbol{k}-\boldsymbol{q})+P^>_{jmn}(\boldsymbol{k}-\boldsymbol{q})P^>_{in}(\boldsymbol{q})$$

$$\simeq k_j\delta_{im}+k_i\delta_{jm}-\frac{k_jq_iq_m}{q^2}-\frac{k_iq_jq_m}{q^2}-\frac{k_nq_jq_n}{q^2}\delta_{im}-\frac{k_nq_iq_n}{q^2}\delta_{jm}$$

$$+2\frac{k_nq_iq_jq_mq_n}{q^4}. \tag{3.39}$$

Here, to make the transition to the infinite system, we replace sums over wave vectors in Eq. (3.38) by integrals according to

$$\lim_{L\to\infty}\sum_q\left(\frac{2\pi}{L}\right)^3\equiv\int_q d\boldsymbol{q}. \tag{3.40}$$

As a result, we can obtain the renormalized form for the Reynolds stress so that the effects of all the eliminated components $\boldsymbol{u}^>$ in the band $\Lambda_1\le q<\Lambda_0$ are replaced by ν_1 as follows:

$$\sum_q\langle u_i(\boldsymbol{q},\tau)u_j(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c \simeq \sum_q u^<_i(\boldsymbol{q},\tau)u^<_j(\boldsymbol{k}-\boldsymbol{q},\tau)$$

$$-\nu_1\left\{ik_i\overline{U}_j(\boldsymbol{k},\tau)+ik_j\overline{U}_i(\boldsymbol{k},\tau)\right\}, \tag{3.41}$$

where

$$\nu_1=\Delta\nu_0 \tag{3.42}$$

$$\Delta\nu_0=\frac{7}{30\nu_0}\int_{\Lambda_1}^{\Lambda_0}\frac{E(q)}{q^2}dq. \tag{3.43}$$

Note that ν_1 represents the apparent eddy viscosity due to the nonlinear coupling in the band $\Lambda_1\le q<\Lambda_0$. The numerical constant 7/30 arises from transforming the integration from the wave vector $\boldsymbol{q}$ to the wave number $q(=|\boldsymbol{q}|)$. The details of its derivation are given in Appendix B.

3.1.3 Recursion relation

After eliminating the higher wave number modes defined in the band $\Lambda_1 \le q < \Lambda_0$, we relabel $u_i^< \to u_i$, and then divide u_i again into $u_i^<$ and $u_i^>$ in terms of the new cutoff wave number Λ_2 ($\Lambda_e \ll \Lambda_2 < \Lambda_1$).

$$u_i(\boldsymbol{q},\tau) = \begin{cases} u_i^<(\boldsymbol{q},\tau) & : \quad \Lambda_e \le q < \Lambda_2 \\ u_i^>(\boldsymbol{q},\tau) & : \quad \Lambda_2 \le q < \Lambda_1 \end{cases} \tag{3.44}$$

Hence, the higher wave-number modes in the new band $\Lambda_2 \le q < \Lambda_1$ are eliminated by the procedure shown above in Section 3.1.2:

$$\begin{aligned} \sum_{q<\Lambda_0} \langle u_i(\boldsymbol{q},\tau)u_j(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c \quad \simeq \quad & \sum_{q<\Lambda_2} u_i^<(\boldsymbol{q},\tau)u_j^<(\boldsymbol{k}-\boldsymbol{q},\tau) \\ & -\nu_2\left\{ik_i\overline{U}_j(\boldsymbol{k},\tau) + ik_j\overline{U}_i(\boldsymbol{k},\tau)\right\}, \end{aligned} \tag{3.45}$$

where

$$\nu_2 = \nu_1 + \Delta\nu_1 \tag{3.46}$$

and

$$\Delta\nu_1 = \frac{7}{30\nu_1}\int_{\Lambda_2}^{\Lambda_1} \frac{E(q)}{q^2}dq. \tag{3.47}$$

Carrying out the same procedure for successive bands $\Lambda_{n+1} < \Lambda_n < \cdots\cdots \Lambda_2 < \Lambda_1 < \Lambda_0$, we obtain the recursion relation for ν_{n+1} as follows (see Fig. 3.3):

$$\begin{aligned} \sum_{q<\Lambda_0} \langle u_i(\boldsymbol{q},\tau)u_j(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c \quad \simeq \quad & \sum_{q<\Lambda_{n+1}} u_i^<(\boldsymbol{q},\tau)u_j^<(\boldsymbol{k}-\boldsymbol{q},\tau) \\ & -\nu_{n+1}\left\{ik_i\overline{U}_j(\boldsymbol{k},\tau) + ik_j\overline{U}_i(\boldsymbol{k},\tau)\right\}, \end{aligned} \tag{3.48}$$

where

$$\nu_{n+1} = \nu_n + \Delta\nu_n \tag{3.49}$$

and

$$\Delta\nu_n = \frac{7}{30\nu_n}\int_{\Lambda_{n+1}}^{\Lambda_n} \frac{E(q)}{q^2}dq. \tag{3.50}$$

Using the relations given by Eqs. (3.48)–(3.50), we obtain the renormalized expression to Eq. (3.48) in which all the fluctuating velocity components [or the first term of the r.h.s.

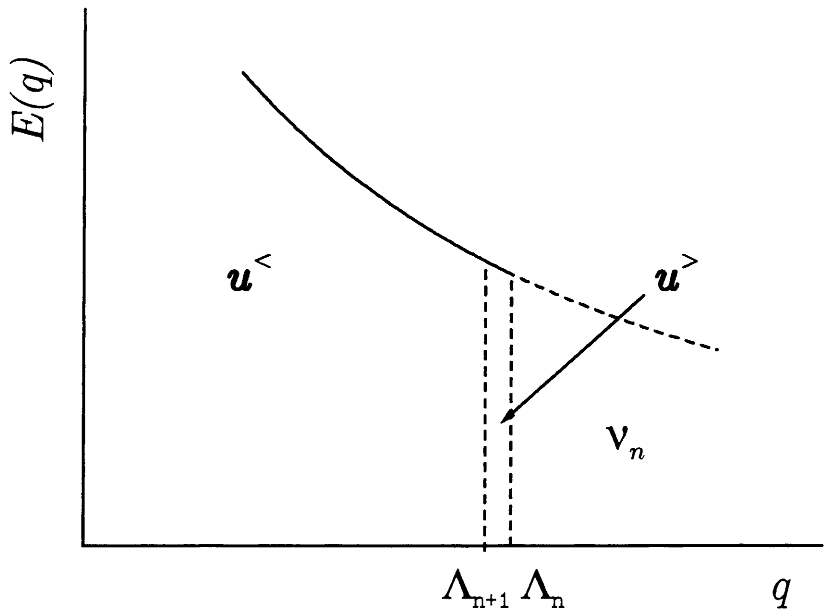

Figure 3.3: Recursion relation for ν_n

of Eq. (3.48)] are removed. Therefore, we begin by summing over the square of each side of Eq. (3.49) up to an arbitrary number $N-1$:

$$\sum_{i=0}^{N-1} \nu_{i+1}^2 = \sum_{i=0}^{N-1} \left\{ \nu_i^2 + 2\nu_i \Delta\nu_i + (\Delta\nu_i)^2 \right\} \tag{3.51}$$

or

$$\sum_{i=0}^{N-1} \left(\nu_{i+1}^2 - \nu_i^2 \right) = 2 \sum_{i=0}^{N-1} \nu_i \Delta\nu_i + \sum_{i=0}^{N-1} (\Delta\nu_i)^2 , \tag{3.52}$$

where

$$\sum_{i=0}^{N-1} \left(\nu_{i+1}^2 - \nu_i^2 \right) = \nu_N^2, \tag{3.53}$$

$$2 \sum_{i=0}^{N-1} \nu_i \Delta\nu_i = \frac{7}{15} \int_{\Lambda_N}^{\Lambda_0} \frac{E(q)}{q^2} dq, \tag{3.54}$$

and

$$\sum_{i=0}^{N-1} (\Delta\nu_i)^2 = \sum_{i=0}^{N-1} \left\{ \frac{7}{30\nu_i} \int_{\Lambda_{i+1}}^{\Lambda_i} \frac{E(q)}{q^2} dq \right\}^2 . \tag{3.55}$$

Here, replacing $E(q)$ in Eqs. (3.54) and (3.55) with the Kolmogorov $-5/3$ power law spectrum gives

$$2\sum_{i=0}^{N-1}\nu_i\Delta\nu_i = \frac{7C_K}{40}\bar{\varepsilon}^{\frac{2}{3}}\left(\Lambda_N^{-\frac{8}{3}} - \Lambda_0^{-\frac{8}{3}}\right) \tag{3.56}$$

and

$$\begin{aligned}\sum_{i=0}^{N-1}(\Delta\nu_i)^2 &= \sum_{i=0}^{N-1}\left(\frac{7C_K}{80}\right)^2 \frac{\bar{\varepsilon}^{\frac{4}{3}}}{\nu_i^2\Lambda_i^{\frac{16}{3}}}\left\{(1-\lambda)^{-\frac{8}{3}} - 1\right\}^2 \\ &= \sum_{i=0}^{N-1}\left(\frac{7C_K}{80}\right)^2 \frac{\bar{\varepsilon}^{\frac{4}{3}}}{\nu_i^2\Lambda_i^{\frac{16}{3}}}\left(\frac{8}{3}\lambda + \frac{44}{9}\lambda^2 + \cdots\right)^2 \\ &= O(\lambda^2) + O(\lambda^3) + \cdots .\end{aligned} \tag{3.57}$$

As a result, with respect to an arbitrary cutoff wave number Λ_N, ν_N can be written as

$$\nu_N \simeq \sqrt{\frac{7C_K}{40}\bar{\varepsilon}^{\frac{1}{3}}\left(\Lambda_N^{-\frac{8}{3}} - \Lambda_0^{-\frac{8}{3}}\right)^{\frac{1}{2}}} \tag{3.58}$$

on condition that the bandwidth parameter λ is sufficiently small.

Obviously, in the case of $\Lambda_N \ll \Lambda_0$, ν_N becomes independent of the initial cutoff wave number Λ_0, and Eq. (3.58) is simplified as

$$\nu_N \simeq \sqrt{\frac{7C_K}{40}}\,\bar{\varepsilon}^{\frac{1}{3}}\,\Lambda_N^{-\frac{4}{3}}. \tag{3.59}$$

3.1.4 Derivation of eddy viscosity

Naturally, the result of renormalizing all the components in the inertial range is also obtained by setting $\Lambda_N = \Lambda_e$ in Eq. (3.59):

$$\nu_e \simeq \sqrt{\frac{7C_K}{40}}\,\bar{\varepsilon}^{\frac{1}{3}}\,\Lambda_e^{-\frac{4}{3}}. \tag{3.60}$$

Moreover, the well-known representation:

$$\nu_e = C_\mu\,\frac{\overline{K}^2}{\bar{\varepsilon}}, \tag{3.61}$$

where

$$C_\mu = \sqrt{\frac{7C_K}{40}}\left(\frac{2}{3C_K}\right)^2 \tag{3.62}$$

results from replacing Λ_e in Eq. (3.60) with the turbulent kinetic energy $\overline{K}$ by using Eq. (3.8).

Finally, from Eq. (3.61), the Reynolds stress is modeled as:

$$\begin{aligned}
\overline{u_i u_j} &= \sum_{k \ll \Lambda_e} R_{ij}(\boldsymbol{k}, \tau) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \\
&= \sum_{k \ll \Lambda_e} \left[\frac{2}{3} \overline{K}(\boldsymbol{k}, \tau)\, \delta_{ij} - \nu_e \left\{ ik_i \overline{U}_j(\boldsymbol{k}, \tau) + ik_j \overline{U}_i(\boldsymbol{k}, \tau) \right\} \right] \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \\
&= \frac{2}{3} \overline{K} \delta_{ij} - \nu_t \left(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i} \right), \qquad (3.63)
\end{aligned}$$

where

$$\nu_t = C_\mu \frac{\overline{K}^2}{\overline{\varepsilon}}. \qquad (3.64)$$

Here, δ_{ij} is the Kronecker delta, and the diagonal components of the Reynolds stress $\overline{u_i u_j}$ $(i = j)$ are related to the turbulent kinetic energy:

$$K(\boldsymbol{k}, \tau) = \frac{1}{2} \sum_q u_i(\boldsymbol{q}, \tau) u_i(\boldsymbol{k} - \boldsymbol{q}, \tau), \qquad (3.65)$$

which yields

$$\overline{K} = \frac{1}{2} \overline{u_i u_i} = \sum_{k \ll \Lambda_e} \overline{K}(\boldsymbol{k}, \tau) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}). \qquad (3.66)$$

The formulation given by Eq. (3.64) is identical with the Boussinesq postulate for the eddy viscosity (Yoshizawa 1984), and our model constant C_μ is determined from the Kolmogorov constant C_K. With the typical value of $C_K = 1.6$ for the Kolmogorov constant (Kraichnan 1965; McComb 1990; Sreenivasan 1995), Eq. (3.62) gives $C_\mu = 0.092$. This value is in good agreement with that for the standard $\overline{K}$-$\overline{\varepsilon}$ model (Nagano & Tagawa 1990).

3.2 Iterative Averaging Method for Thermal Field

3.2.1 Energy equation

The energy equation for an incompressible fluid is

$$\frac{\partial T}{\partial \tau} + U_i \frac{\partial T}{\partial x_i} = \alpha_0 \frac{\partial^2 T}{\partial x_i \partial x_i}, \tag{3.67}$$

where T is the instantaneous fluid temperature, U_i is the instantaneous velocity component, and α_0 is the thermal diffusivity. The mean components and the fluctuating components are represented by means of a Fourier series (Giles 1994a; Giles 1994b):

$$\overline{U}_i(\boldsymbol{x}, \tau) = \sum_{k \ll \Lambda_e} \overline{U}_i(\boldsymbol{k}, \tau) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}), \tag{3.68}$$

$$u_i(\boldsymbol{x}, \tau) = \sum_{k \geq \Lambda_e} u_i(\boldsymbol{k}, \tau) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}), \tag{3.69}$$

$$\overline{T}(\boldsymbol{x}, \tau) = \sum_{k \ll \Lambda_{et}} \overline{T}(\boldsymbol{k}, \tau) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}), \tag{3.70}$$

$$t(\boldsymbol{x}, \tau) = \sum_{k \geq \Lambda_{et}} t(\boldsymbol{k}, \tau) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}). \tag{3.71}$$

Here, we can assume $\Lambda_{et} = O(\Lambda_e)$ because the behavior of the temperature fluctuation spectrum at large wave numbers mainly depends on the Prandtl number (Tennekes & Lumley 1972; Hinze 1975; Townsend 1976). Thus, we hereafter specify the wave number Λ_e to the thermal field as a counterpart of Λ_{et}. This implies that the fluctuating components u_i and t satisfy the following condition:

$$\bar{u}_i = 0, \tag{3.72}$$

$$\bar{t} = 0. \tag{3.73}$$

Accordingly, the application of Eqs. (3.68)–(3.71) to Eq. (3.67) yields the transport equations for the mean temperature $\overline{T}$ and the temperature fluctuation t:

$$\frac{\partial \overline{T}}{\partial \tau} + \overline{U}_i \frac{\partial \overline{T}}{\partial x_i} = \alpha_0 \frac{\partial^2 \overline{T}}{\partial x_i \partial x_i} - \frac{\partial}{\partial x_i} \overline{u_i t}, \tag{3.74}$$

$$\frac{\partial t}{\partial \tau} + u_i \frac{\partial \overline{T}}{\partial x_i} + \overline{U}_i \frac{\partial t}{\partial x_i} = \alpha_0 \frac{\partial^2 t}{\partial x_i \partial x_i} - \frac{\partial}{\partial x_i}\left(u_i t - \overline{u_i t}\right). \tag{3.75}$$

Then, the transformation of Eq. (3.75) to Fourier space becomes

$$\begin{aligned}\left(\frac{\partial}{\partial \tau} + \alpha_0 k^2\right) t(\boldsymbol{k},\tau) &= -ik_i \sum_q u_i(\boldsymbol{q},\tau) t(\boldsymbol{k}-\boldsymbol{q},\tau) - ik_i \sum_q \overline{U}_i(\boldsymbol{q},\tau) t(\boldsymbol{k}-\boldsymbol{q},\tau) \\ &\quad -ik_i \sum_q u_i(\boldsymbol{q},\tau) \overline{T}(\boldsymbol{k}-\boldsymbol{q},\tau), \end{aligned} \tag{3.76}$$

where a variable wave number k is defined in the range $\Lambda_e \leq k < \Lambda_0$, and the initial cutoff wave number Λ_0 is the same as that for the fluctuating velocity component u_i: the order of magnitude of the Kolmogorov dissipation wave number $k_d [= (\overline{\varepsilon}/\nu_0^3)^{1/4}]$. Furthermore, we assume the condition $\Lambda_e \ll \Lambda_0$, since the turbulent Reynolds number $Re_t (= \overline{K}^2/\nu_0 \overline{\varepsilon})$ is sufficiently large.

Similarly, the turbulent heat flux $-\overline{u_i t}$ in Fourier space can be written as follows:

$$-\overline{u_i t} = \sum_{k \ll \Lambda_e} R_{it}(\boldsymbol{k},\tau) \exp(i\boldsymbol{k}\cdot\boldsymbol{x}), \tag{3.77}$$

$$R_{it}(\boldsymbol{k},\tau) = -\sum_q u_i(\boldsymbol{q},\tau) t(\boldsymbol{k}-\boldsymbol{q},\tau), \tag{3.78}$$

where the wave numbers are defined in the range $k \ll \Lambda_e$, $\Lambda_e \leq q < \Lambda_0$, $\Lambda_e \leq |\boldsymbol{k}-\boldsymbol{q}| < \Lambda_0$, and we note that the wave number k is sufficiently small.

3.2.2 Elimination of small scales

In accordance with the iterative averaging procedure, we begin by dividing the fluctuations u_i and t into two modes about the second cutoff wave number $\Lambda_1 (\Lambda_e \ll \Lambda_1 < \Lambda_0)$:

$$t(\boldsymbol{q},\tau) = \begin{cases} t^<(\boldsymbol{q},\tau) & : \quad \Lambda_e \leq q < \Lambda_1 \\ t^>(\boldsymbol{q},\tau) & : \quad \Lambda_1 \leq q < \Lambda_0, \end{cases} \tag{3.79}$$

and

$$u_i(\boldsymbol{q},\tau) = \begin{cases} u_i^<(\boldsymbol{q},\tau) & : \quad \Lambda_e \leq q < \Lambda_1 \\ u_i^>(\boldsymbol{q},\tau) & : \quad \Lambda_1 \leq q < \Lambda_0, \end{cases} \tag{3.80}$$

where the interval $\Lambda_1 \leq q < \Lambda_0$ is assumed to be a sufficiently small band in the higher wave-number range. Hence, the application of Eqs. (3.79) and (3.80) to the r.h.s. of Eq. (3.78) yields

$$\begin{aligned} -\sum_q u_i(\boldsymbol{q},\tau)t(\boldsymbol{k}-\boldsymbol{q},\tau) &= -\sum_q u_i^<(\boldsymbol{q},\tau)t^<(\boldsymbol{k}-\boldsymbol{q},\tau) \\ &\quad -\sum_q u_i^>(\boldsymbol{q},\tau)t^<(\boldsymbol{k}-\boldsymbol{q},\tau) \\ &\quad -\sum_q u_i^<(\boldsymbol{q},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau) \\ &\quad -\sum_q u_i^>(\boldsymbol{q},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau). \end{aligned} \tag{3.81}$$

Here we assume that the joint probability distribution of $\boldsymbol{u}^>$ and $t^>$ is Gaussian (Nagano & Tagawa 1988), and introduce the operation of conditional partial averaging $\langle\ \ \rangle_c$ over these fluctuations in the narrow band $\Lambda_1 \leq q < \Lambda_0$. By applying this operation to the r.h.s. of Eq. (3.81), the first term becomes

$$-\sum_q \langle u_i^<(\boldsymbol{q},\tau)t^<(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c = -\sum_q u_i^<(\boldsymbol{q},\tau)t^<(\boldsymbol{k}-\boldsymbol{q},\tau) \tag{3.82}$$

because the conditional average $\langle\ \ \rangle_c$ does not affect the terms $u_i^<$ and $t^<$. Also, as in Section 3.1, the second and third term of the r.h.s. in Eq. (3.81) can be assumed to be negligibly small as compared to the other terms (see Appendix B), so Eq. (3.81) may be approximately represented as:

$$\begin{aligned} -\sum_q u_i(\boldsymbol{q},\tau)t(\boldsymbol{k}-\boldsymbol{q},\tau) &\simeq -\sum_q u_i^<(\boldsymbol{q},\tau)t^<(\boldsymbol{k}-\boldsymbol{q},\tau) \\ &\quad -\sum_q \langle u_i^>(\boldsymbol{q},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c . \end{aligned} \tag{3.83}$$

For the second term of the r.h.s. of Eq. (3.83), we can not replace the correlation $\langle u_i^> t^>\rangle_c$ with its spectrum directly, because the functional formalism for that has never been specified analytically; so we begin by expanding the term $\langle u_i^> t^>\rangle_c$ by means of a renormalization approximation, and form an eddy diffusivity representation in Fourier space. We replace the correlation $u_i^> t^>$ with its transport equation which is formed in multiplying the transport equation for $u_i^>$:

$$\left(\frac{\partial}{\partial\tau} + \nu_0 q^2\right) u_i^>(\boldsymbol{q},\tau) = -\frac{i}{2} P_{imn}^>(\boldsymbol{q}) \sum_r u_m(\boldsymbol{r},\tau) u_n(\boldsymbol{q}-\boldsymbol{r},\tau) - i P_{imn}^>(\boldsymbol{q}) \sum_r \overline{U}_m(\boldsymbol{r},\tau) u_n(\boldsymbol{q}-\boldsymbol{r},\tau), \tag{3.84}$$

$$P_{imn}(\boldsymbol{k}) = k_m P_{in}(\boldsymbol{k}) + k_n P_{im}(\boldsymbol{k}), \tag{3.85}$$

$$P_{ij}(\boldsymbol{k}) = \delta_{ij} - \frac{k_i k_j}{k^2} \tag{3.86}$$

by $t^>$ and Eq. (3.76) for $t^>$ by $u_i^>$, and adding these equations:

$$\begin{aligned} &\left(\frac{\partial}{\partial\tau} + \nu_0 q^2 + \alpha_0 |\boldsymbol{k}-\boldsymbol{q}|^2\right) u_i^>(\boldsymbol{q},\tau) t^>(\boldsymbol{k}-\boldsymbol{q},\tau) \\ = &-i(k-q)_j \sum_r u_j(\boldsymbol{r},\tau) \overline{T}(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau) u_i^>(\boldsymbol{q},\tau) \\ &-i(k-q)_j \sum_r \overline{U}_j(\boldsymbol{r},\tau) t(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau) u_i^>(\boldsymbol{q},\tau) \\ &-i(k-q)_j \sum_r u_j(\boldsymbol{r},\tau) t(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau) u_i^>(\boldsymbol{q},\tau) \\ &-\frac{i}{2} P_{imn}^>(\boldsymbol{q}) \sum_r u_m(\boldsymbol{r},\tau) u_n(\boldsymbol{q}-\boldsymbol{r},\tau) t^>(\boldsymbol{k}-\boldsymbol{q},\tau) \\ &-i P_{imn}^>(\boldsymbol{q}) \sum_r \overline{U}_m(\boldsymbol{r},\tau) u_n(\boldsymbol{q}-\boldsymbol{r},\tau) t^>(\boldsymbol{k}-\boldsymbol{q},\tau). \end{aligned} \tag{3.87}$$

Hence, the second term of the r.h.s. in Eq. (3.83) is expanded infinitely by substituting $\langle u_i^> t^> \rangle_c$ into its transport equation [Eq. (3.87)] repeatedly. Performing a moment closure hypothesis (see Appendix B), we keep the terms to the order of $\overline{T}^1$; then the fourth term becomes

$$-\sum_q \langle u_i^>(\boldsymbol{q},\tau) t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c \simeq i \sum_q \left(\frac{2\pi}{L}\right)^3 \frac{(k_j - q_j) P_{ij}^>(\boldsymbol{q}) Q^>(q)}{\nu_0 q^2 + \alpha_0 |\boldsymbol{k}-\boldsymbol{q}|^2} \overline{T}(\boldsymbol{k},\tau). \tag{3.88}$$

Here, it should be noted that $\partial/\partial\tau$ can be neglected because of the universality of the inertial range. By assuming the relation $|\boldsymbol{k}-\boldsymbol{q}| \simeq q$ relevant to the sufficiently narrow band $\Lambda_1 \leq q < \Lambda_0$, Eq. (3.88) is rewritten as:

$$-\sum_q \langle u_i^>(\boldsymbol{q},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c \simeq i\sum_q \left(\frac{2\pi}{L}\right)^3 \frac{(k_j-q_j)P_{ij}^>(\boldsymbol{q})Q^>(q)}{(\nu_0+\alpha_0)q^2}\overline{T}(\boldsymbol{k},\tau). \tag{3.89}$$

To make the transition to the infinite system, we replace sums over wave vectors in Eq. (3.89) by integrals according to

$$\lim_{L\to\infty}\sum_q \left(\frac{2\pi}{L}\right)^3 \equiv \int_q d\boldsymbol{q}. \tag{3.90}$$

Finally, we obtain the renormalized form for the turbulent heat flux corresponding to an eddy-diffusivity representation, so that the effect of the eliminated components in the range $\Lambda_1 \leq q < \Lambda_0$ is shown as

$$-\sum_q \langle u_i^>(\boldsymbol{q},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c \simeq i\Delta\alpha_0 k_i\overline{T}(\boldsymbol{k},\tau), \tag{3.91}$$

$$\Delta\alpha_0 = \frac{d-1}{d}\frac{1}{\nu_0+\alpha_0}\int_{\Lambda_1}^{\Lambda_0}\frac{E(q)}{q^2}dq, \tag{3.92}$$

and

$$-\sum_{q<\Lambda_0}\langle u_i(\boldsymbol{q},\tau)t(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c \simeq -\sum_{q<\Lambda_1} u_i^<(\boldsymbol{q},\tau)t^<(\boldsymbol{k}-\boldsymbol{q},\tau) + i\Delta\alpha_0 k_i\overline{T}(\boldsymbol{k},\tau), \tag{3.93}$$

where d is the space dimension. Hence, the r.h.s. of Eq. (3.91) is considered to be an increment to the diffusion term in Eq. (3.74). Thus, we can also rewrite it as

$$-\sum_{q<\Lambda_0}\langle u_i(\boldsymbol{q},\tau)t(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c + i\alpha_0 k_i\overline{T}(\boldsymbol{k},\tau)$$

$$\simeq -\sum_{q<\Lambda_1} u_i^<(\boldsymbol{q},\tau)t^<(\boldsymbol{k}-\boldsymbol{q},\tau) + i\alpha_1 k_i\overline{T}(\boldsymbol{k},\tau), \tag{3.94}$$

$$\alpha_1 = \alpha_0 + \Delta\alpha_0, \tag{3.95}$$

in which all the wave numbers are defined in the range $\Lambda_e \leq |\boldsymbol{k}-\boldsymbol{q}| < \Lambda_1$, $\Lambda_e \leq |\boldsymbol{q}| < \Lambda_1$.

3.2.3 Derivation of thermal eddy diffusivity

After eliminating the higher wave-number modes with respect to the sufficiently small band $\Lambda_1 \leq q < \Lambda_0$, we relabel $u_i^< \to u_i$, $t^< \to t$, and then divide these fluctuating components again into two modes with the next cutoff wave number $\Lambda_2(<\Lambda_1)$:

$$u_i(\boldsymbol{q},\tau)=\begin{cases} u_i^<(\boldsymbol{q},\tau) & : \quad \Lambda_e \le q < \Lambda_2 \\ u_i^>(\boldsymbol{q},\tau) & : \quad \Lambda_2 \le q < \Lambda_1, \end{cases} \tag{3.96}$$

$$t(\boldsymbol{q},\tau)=\begin{cases} t^<(\boldsymbol{q},\tau) & : \quad \Lambda_e \le q < \Lambda_2 \\ t^>(\boldsymbol{q},\tau) & : \quad \Lambda_2 \le q < \Lambda_1. \end{cases} \tag{3.97}$$

As shown in Section 3.2.2, the first term of the r.h.s. in Eq. (3.94) is divided into four counterparts, each of which is averaged over the higher wave-number range $\Lambda_2 \le q < \Lambda_1$; and the term $\langle u_i^> t^> \rangle_c$ is added as an increment to the second term of the r.h.s. in Eq. (3.94). Equation (3.94) is renormalized to

$$-\sum_{q<\Lambda_0}\langle u_i(\boldsymbol{q},\tau)t(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c + i\alpha_0 k_i\overline{T}(\boldsymbol{k},\tau)$$

$$\simeq -\sum_{q<\Lambda_2} u_i^<(\boldsymbol{q},\tau)t^<(\boldsymbol{k}-\boldsymbol{q},\tau) + i\alpha_2 k_i\overline{T}(\boldsymbol{k},\tau), \tag{3.98}$$

$$\alpha_2 = \alpha_1 + \Delta\alpha_1, \tag{3.99}$$

$$\Delta\alpha_1 = \frac{d-1}{d}\frac{1}{\nu_1+\alpha_1}\int_{\Lambda_2}^{\Lambda_1}\frac{E(q)}{q^2}dq. \tag{3.100}$$

Accordingly, carrying out this procedure for successive bands $\Lambda_{n+1} < \Lambda_n < \cdots \Lambda_2 < \Lambda_1 < \Lambda_0$ in sequence, we can obtain the recursion relation for α_{n+1} as follows:

$$-\sum_{q<\Lambda_{n+1}}\langle u_i^<(\boldsymbol{q},\tau)t^<(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c + i\alpha_{n+1} k_i\overline{T}(\boldsymbol{k},\tau), \tag{3.101}$$

where

$$\alpha_{n+1} = \alpha_n + \Delta\alpha_n \tag{3.102}$$

and

$$\Delta\alpha_n = \frac{d-1}{d}\frac{1}{\nu_n+\alpha_n}\int_{\Lambda_{n+1}}^{\Lambda_n}\frac{E(q)}{q^2}dq. \tag{3.103}$$

This implies that progressive elimination of the fluctuating components by turns completes an eddy-diffusivity representation in Fourier space, i.e., $\alpha_{n+1} \to \alpha_t + \alpha_0$ as $\Lambda_{n+1} \to \Lambda_e$. Making use of Eqs. (3.101)–(3.103), we write α_{n+1} as a function of the cutoff wave number Λ_{n+1} to obtain its functional formalism $\alpha(\Lambda)$ about an arbitrary cutoff wave number Λ.

Hence, we begin by transforming from the discrete representation α_{n+1} to the continuous one:

$$\begin{aligned}\frac{d\alpha(\Lambda)}{d\Lambda} &= \lim_{\Lambda_{n+1}\to\Lambda_n} \frac{\Delta\alpha_n}{\Lambda_{n+1}-\Lambda_n} \\ &= -\lim_{\Delta\Lambda\to 0} \frac{d-1}{d} \frac{1}{\{\nu(\Lambda)+\alpha(\Lambda)\}\,\Delta\Lambda} \int_{\Lambda-\Delta\Lambda}^{\Lambda} \frac{E(q)}{q^2} dq, \end{aligned} \tag{3.104}$$

and replace the energy spectrum $E(q)$ with the well-known Kolmogorov spectrum in the inertial range:

$$E(q) = C_K\, \bar{\varepsilon}^{\frac{2}{3}}\, q^{-\frac{5}{3}}, \tag{3.105}$$

which yields

$$\frac{d\alpha(\Lambda)}{d\Lambda} = -\frac{2C_K}{3} \frac{\bar{\varepsilon}^{\frac{2}{3}}}{\{\nu(\Lambda)+\alpha(\Lambda)\}\,\Lambda^{\frac{11}{3}}}, \tag{3.106}$$

where $d = 3$. Following Yakhot & Orszag (1986), we introduce a new variable:

$$z(\Lambda) = \frac{\alpha(\Lambda)}{\nu(\Lambda)} \tag{3.107}$$

and its differential equation:

$$\frac{dz(\Lambda)}{d\Lambda} = \frac{1}{\nu(\Lambda)} \frac{d\alpha(\Lambda)}{d\Lambda} - \frac{\alpha(\Lambda)}{\nu^2(\Lambda)} \frac{d\nu(\Lambda)}{d\Lambda}, \tag{3.108}$$

which is very convenient to find the solution of $\alpha(\Lambda)$. For the r.h.s. of Eq. (3.108), $d\alpha(\Lambda)/d\Lambda$ is replaced with its differential equation [Eq. (3.106)], and $d\nu(\Lambda)/d\Lambda$ is replaced with

$$\frac{d\nu\,(\Lambda)}{d\Lambda} = -\frac{7C_K}{30} \frac{\varepsilon^{\frac{2}{3}}}{\nu\,(\Lambda)\,\Lambda^{\frac{11}{3}}}, \tag{3.109}$$

which is obtained from iterative averaging for the Reynolds stress $\overline{u_i u_j}$ (Itazu & Nagano 1997b; Nagano & Itazu 1997b). Then, Eq. (3.108) becomes

$$\begin{aligned}\frac{dz(\Lambda)}{d\Lambda} &= -\frac{2C_K}{3} \frac{\bar{\varepsilon}^{\frac{2}{3}}}{\nu(\Lambda)\,\{\nu(\Lambda)+\alpha(\Lambda)\}\,\Lambda^{\frac{11}{3}}} + \frac{7C_K}{30} \frac{z(\Lambda)\bar{\varepsilon}^{\frac{2}{3}}}{\nu^2(\Lambda)\Lambda^{\frac{11}{3}}} \\ &= \left(\frac{10}{7}\frac{1}{z+1} - \frac{1}{2}z\right) \frac{1}{\nu^2(\Lambda)} \frac{d\nu^2(\Lambda)}{d\Lambda}, \end{aligned} \tag{3.110}$$

which can be simplified to

$$\frac{dz}{\left(\frac{10}{7}\frac{1}{1+z}-\frac{z}{2}\right)}=\frac{d\nu^2(\Lambda)}{\nu^2(\Lambda)}. \tag{3.111}$$

Finally, the solution of Eq. (3.111) is

$$\left|\frac{z(\Lambda)-a}{z_0-a}\right|^c\left|\frac{z(\Lambda)-b}{z_0-b}\right|^d=\frac{\nu}{\nu(\Lambda)}, \tag{3.112}$$

where

$$\left.\begin{aligned} a&=-\frac{1}{2}+\frac{1}{2}\sqrt{\frac{87}{7}}\simeq 1.2627\\ b&=-\frac{1}{2}-\frac{1}{2}\sqrt{\frac{87}{7}}\simeq -2.2627\\ c&=\frac{1}{2}+\frac{1}{2}\sqrt{\frac{7}{87}}\simeq 0.6418\\ d&=\frac{1}{2}-\frac{1}{2}\sqrt{\frac{7}{87}}\simeq 0.3582 \end{aligned}\right\}. \tag{3.113}$$

Hence, the result of renormalizing all the components in the range $\Lambda_e \le q < \Lambda_0$ is obtained by setting $\Lambda = \Lambda_e$ in Eq. (3.112):

$$\left|\frac{z^*-1.2627}{z_0-1.2627}\right|^{0.64}\left|\frac{z^*+2.2627}{z_0+2.2627}\right|^{0.36}=\frac{1}{1+\frac{\nu_t}{\nu_0}}, \tag{3.114}$$

where

$$z^*\equiv\frac{\alpha(\Lambda_e)}{\nu(\Lambda_e)}=\frac{\alpha_0+\alpha_t}{\nu_0+\nu_t}=\frac{1/Pr+(\nu_t/\nu_0)/Pr_t}{1+\nu_t/\nu_0}, \tag{3.115}$$

and

$$z_0\equiv\frac{\alpha_0}{\nu_0}=\frac{1}{Pr}. \tag{3.116}$$

This equation shows how the turbulent Prandtl number Pr_t changes with the molecular Prandtl number Pr and the eddy viscosity ν_t.

As shown in Fig. 3.4, compared to the data of some experiments (Blackwell et al. 1972; Zukauskas & Slanciauskas 1987; Kays & Crawford 1993) and DNS (Kasagi et al. 1992; Kasagi & Ohtsubo 1992), the turbulent Prandtl number Pr_t obtained from Eq. (3.114) is suitable for turbulent flow with both a low and high Prandtl number Pr; and $Pr_t \to 0.79$ as the turbulent Peclet number Pe_t is sufficiently large.

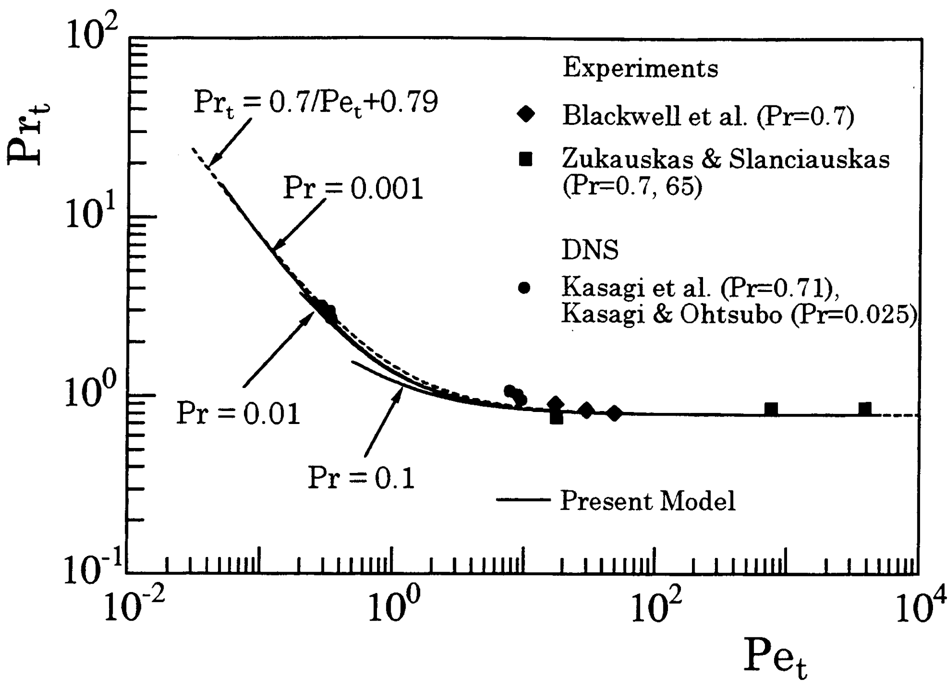

Figure 3.4: Analytic solutions for Pr_t

In case of a high Prandtl number, although there is a viscous convective range at high wave numbers $k > \Lambda_0$ which affects the temperature variance $\bar{t^2}$ (Tennekes & Lumley 1972; Hinze 1975; Townsend 1976), there is an exponential drop-off of the spectrum for the fluctuation u_i in the dissipation range $k > \Lambda_0$, which indicates that the correlation $\sum_{k<\Lambda_e} u_i(\boldsymbol{q},\tau)t(\boldsymbol{k}-\boldsymbol{q},\tau)$, consisting of $t(\boldsymbol{k}-\boldsymbol{q},\tau)$ and $u_i(\boldsymbol{q},\tau)$ in the range $q > \Lambda_0, |\boldsymbol{k}-\boldsymbol{q}| > \Lambda_0$, can be neglected in deriving an eddy diffusivity turbulence model, since it contributes little quantitatively to the turbulent heat flux $-\overline{u_i t}$ (see Fig. 3.5). On the other hand, there is an exponential decrease of the spectrum for t in the range $\Lambda_e \ll |\boldsymbol{k}-\boldsymbol{q}| < \Lambda_0$, i.e., inertial diffusive subrange if the Prandtl number is low (Tennekes & Lumley 1972; Hinze 1975; Townsend 1976); and this implies that only the correlation $\sum_{k<\Lambda_e} u_i(\boldsymbol{q},\tau)t(\boldsymbol{k}-\boldsymbol{q},\tau)$ with respect to $t(\boldsymbol{k}-\boldsymbol{q},\tau)$ in the inertial convective range is valid for deriving the eddy diffusivity turbulence model by iterative averaging. Hence, our theory is applicable to the change of the molecular Prandtl number Pr. It is revealed that the behavior of the spectra for the fluctuations ($\boldsymbol{u}$ and t) at lower wave-numbers largely dominates the turbulent heat flux $-\overline{u_i t}$ at high Reynolds number flows, and the property of Eq. (3.114) may respond to the assumption $\Lambda_{\mathrm{et}} = O(\Lambda_{\mathrm{e}})$.

Following Kays (1994), we can also introduce an empirical equation as a counterpart of Eq. (3.114):

$$Pr_t = 0.7/Pe_t + 0.79. \tag{3.117}$$

To sum up, the turbulent heat flux $-\overline{u_i t}$ is modeled, based on the iterative averaging RNG, as:

$$\begin{aligned} -\overline{u_i t} &= \sum_{k \ll \Lambda_{\mathrm{e}}} i \left\{ \alpha(\Lambda_{\mathrm{e}}) - \alpha_0 \right\} k_i \overline{T}(\boldsymbol{k}, \tau) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \\ &= \sum_{k \ll \Lambda_{\mathrm{e}}} i \alpha_t k_i \overline{T}(\boldsymbol{k}, \tau) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \\ &= \alpha_t \frac{\partial \overline{T}}{\partial x_i}, \end{aligned} \tag{3.118}$$

where α_t is given by Eq. (3.114).

3.3 Concluding Remarks

The drawback of the YO theory is the misleading ϵ-expansion technique. From our point of view, a renormalization group theory based on the ϵ-expansion is not suitable for investigation of turbulence models in comparison with critical phenomena.

The eddy-viscosity type turbulence model has been directly formed with the aid of RNG by iterative averaging. It has become evident that the result is in perfect agreement with the Boussinesq postulate; moreover, its proportional constant C_μ becomes a suitable value if the Kolmogorov constant C_K is within the normally acceptable value.

The thermal eddy diffusivity has been formed with the aid of RNG theory on the basis of iterative averaging in Fourier space, in which the inconsistency of ϵ-expansion due to the Yakhot-Orszag theory is completely excluded. The equation for the turbulent Prandtl number Pr_t as a function of the molecular Prandtl number Pr is also obtained. Compared to the data of some experiments and DNS, the present formulation is considered to be mostly valid for turbulent flow with a variety of Prandtl numbers.

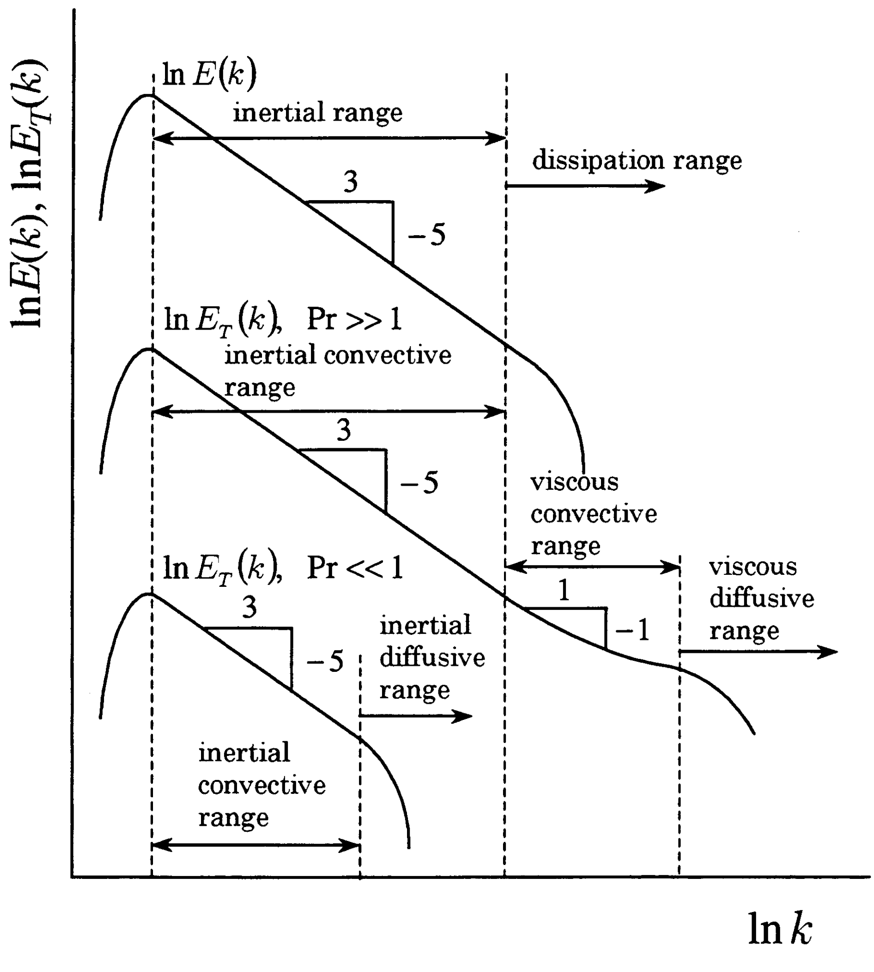

Figure 3.5: Spectra of temperature variance with large and small Prandtl numbers

Chapter 4

CONCLUSIONS

The investigation of the renormalization group theory for turbulence has been carried out as a fundamental approach for modeling a turbulent shear flow with a passive scalar. The conclusions through this study are summarized herewith.

The results obtained from Chapter 2 are as follows:

1. It became evident that the RNG theory for turbulence developed by Yakhot & Orszag (1986) and reformed by Yakhot & Smith (1992) is a scale removal procedure has nothing to do with the exact RNG by the ϵ-expansion technique for phase transition phenomena.

2. The obtained numerical constants in their theory are problematic because the quantitative accuracy of the ϵ-expansion is not proven.

3. The performance of their theory to derive turbulence models is contrary to the result of DNS on modeling the production terms in $\bar{\varepsilon}$-equation.

In Chapter 3, the improved iterative averaging RNG theory is applied to an inhomogeneous turbulent shear flow with a passive scalar in order to formulate turbulence models. The main results are as follows:

1. By applying the iterative averaging RNG theory to the exact Navier-Stokes equation without a stirring force, the eddy-viscosity type turbulence model is formulated with

its model constant which is determined by the Kolmogorov constant. The derived model corresponds well to that of the current standard $\overline{K}$-$\overline{\varepsilon}$ model.

2. The eddy diffusivity for heat and the equation for the turbulent Prandtl number are formulated with the aid of the iterative averaging RNG method. The model is in good agreement with the data of experiments and DNS with the change of the molecular Prandtl number.

BIBLIOGRAPHY

Avellaneda, M. & Majda, A. J. 1990 Mathematical models with exact renormalization for turbulent transport. *Commun. Math. Phys.* **131**, 381–429.

Avellaneda, M. & Majda, A. J. 1992 Approximate and exact renormalization theories for a model for turbulent transport. *Phys. Fluids A* **4**, 41–57.

Blackwell, B. F., Kays, W. M. & Moffat, R. J. 1972 The turbulent boundary layer on a porous plate: An experimental study of the heat transfer behavior with adverse pressure gradients. *HMT-16, Thermosciences Division, Department of Mechanical Engineering*, Stanford University. Stanford.

Carati, D. & Chriaa, K. 1993 Scale invariance and ϵ expansion in the RNG theory of stirred fluids. *Phys. Fluids A* **5**, 3023–3025.

Eyink, G. L. 1994 The renormalization group method in statistical hydrodynamics. *Phys. Fluids* **6**, 3063–3078.

Forster, D., Nelson, D. R. & Stephen, M. J. 1977 Large-distance and long-time properties of a randomly stirred fluid. *Phys. Rev. A* **16**, 732–749.

Frisch, U. 1995 *Turbulence*, pp. 235–240. Cambridge University Press.

Giles, M. J. 1994a Turbulence renormalization group calculations using statistical mechanics methods. *Phys. Fluids* **6**, 595–604.

Giles, M. J. 1994b Statistical mechanics renormalization group calculations for inhomogeneous turbulence. *Phys. Fluids* **6**, 3750–3764.

Hinze, J. O. 1975 *Turbulence*, Second Edition, pp. 290–299. McGraw-Hill.

Itazu, Y. & Nagano, Y. 1996 Renormalization group theory for turbulence (1st report, assessment of Yakhot-Orszag-Smith method for deriving turbulence models). *Trans. Japan Soc. Mech. Engrs: Ser. B* **62**, 999–1005.

Itazu, Y. & Nagano, Y. 1997a Derivation of thermal eddy diffusivity with iterative-averaging RNG. In *Proc. 2nd International Symposium on Turbulence, Heat and Mass Transfer*, pp. 291–300, Delft University of Technology, Delft.

Itazu, Y. & Nagano, Y. 1997b Renormalization group theory for turbulence (2nd report, formulation of eddy viscosity model with iterative-averaging renormalization). *Trans. Japan Soc. Mech. Engrs: Ser. B* **63**, 2957–2962.

Itazu, Y. & Nagano, Y. 1997c Modeling of thermal eddy diffusivity based on renormalization theory. *Trans. Japan Soc. Mech. Engrs: Ser. B* **63**, 3072–3077.

Itazu, Y. & Nagano, Y. 1998 RNG modeling of turbulent heat flux and its application to wall shear flows. *Int. J. Japan Soc. Mech. Engrs: Ser. B*, to be appeared.

Kadanoff, L. P. 1966 Scalings Laws for Ising models near T_c. *Physics* **2**, 263–272.

Kasagi, N. & Ohtsubo, Y. 1992 Direct numerical simulation of low Prandtl number thermal field in a turbulent channel flow. In *Turbulent Shear Flows 8* (eds. F. Durst et al.), pp. 97–119. Springer-Verlag. Berlin.

Kasagi, N., Tomita, Y. & Kuroda, A. 1992 Direct numerical simulation of passive scalar field in a turbulent channel flow. *Trans. ASME: J. Heat Transfer* **114**, 598–606.

Kays, W. M. 1994 Turbulent Prandtl number-where are we? *Trans. ASME: J. Heat Transfer* **116**, 284–295.

Kays, W. M. & Crawford, M. E. 1993 *Convective Heat and Mass Transfer*, Third Edition, pp. 260–265. McGraw-Hill.

Kim, J., Moin, P. & Moser, R. 1987 Turbulence statistics in fully developed channel flow at low Reynolds number. *J. Fluid Mech.* **177**, 133–166.

Kraichnan, R. H. 1965 Lagrangian-history closure approximation for turbulence. *Phys. Fluids* **8**, 575–598.

Kraichnan, R. H. 1987 An interpretation of the Yakhot-Orszag turbulence theory. *Phys. Fluids* **30**, 2400–2405.

Lam, S. H. 1992 On the RNG theory of turbulence. *Phys. Fluids A* **4**, 1007–1017.

Lesieur, M. 1993 *Turbulence in Fluids*, Second Revised Edition, pp. 197–204. Kluwer Academic Publishers.

Ma, S. K. & Mazenko, G. F. 1975 Critical dynamics of ferromagnets in 6-ϵ dimensions: General discussion and detailed calculation. *Phys. Rev. B* **11**, 4077–4099.

Mansour, N. N., Kim, J. & Moin, P. 1988 Reynolds-stress and dissipation-rate budgets in a turbulent channel flow. *J. Fluid Mech.* **194**, 15–44.

Mansour, N. N., Kim, J. & Moin, P. 1989 Near-wall k-ϵ turbulence modeling. *AIAA J.* **27**, 1068–1073.

McComb, W. D. 1990 *The Physics of Fluid Turbulence*, pp. 346–380. Oxford University Press.

McComb, W. D. & Watt, A. G. 1990 Conditional averaging procedure for the elimination of the small-scale modes from incompressible fluid turbulence at high Reynolds numbers. *Phys. Rev. Lett.* **65**, 3281–3284.

Nagano, Y. & Hishida, M. 1987 Improved form of the k-ϵ model for wall turbulent shear flows. *Trans. ASME: J. Fluids Eng.* **109**, 156–160.

Nagano, Y. & Itazu, Y. 1995 Renormalization group theory for turbulence. In *Proc. International Symposium on Mathematical Modelling of Turbulent Flows*, pp. 251–256. The University of Tokyo, Tokyo.

Nagano, Y. & Itazu, Y. 1997a Renormalization group theory for turbulence: Assessment of the Yakhot-Orszag-Smith theory. *Fluid Dynamics Research* **20**, 157–172.

Nagano, Y. & Itazu, Y. 1997b Renormalization group theory for turbulence: Eddy-viscosity type model based on an iterative averaging method. *Phys. Fluids* **9**, 143–153.

Nagano, Y. & Tagawa, M. 1990 An improved k-ϵ model for boundary layer flows. *Trans.*

ASME: J. Fluids Eng. **112**, 33–39.

Nagano, Y. & Tagawa, M. 1988 Statistical characteristics of wall turbulence with a passive scalar. *J. Fluid Mech.* **196**, 157–185.

Orszag, S. A., Yakhot, V., Flannery, W. S., Boysan, F., Choudhury, D., Maruzewski, J. & Patel, B. 1993 Renormalization group modeling and turbulence simulations. *Near-Wall Turbulent Flows* (eds. R.M.C. So et al.), pp. 1031–1046. Elsevier.

Piomelli, U. 1989 Application of renormalization group theory to the large eddy simulation of transitional boundary layers. *Instab. Transition* **2**, 480–496.

Rubinstein, R. & Barton, J. M. 1992 Renormalization group analysis of the Reynolds stress transport equation. *Phys. Fluids A* **4**, 1759–1766.

Rubinstein, R. 1996 (private communication).

Sreenivasan, K. R. 1995 On the universality of the Kolmogorov constant. *Phys. Fluids* **7**, 2778–2784.

Smith, L. M. & Reynolds, W. C. 1992 On the Yakhot-Orszag renormalization group method for deriving turbulence statistics and models. *Phys. Fluids A* **4**, 364–390.

Tennekes, H. & Lumley, J. L. 1972 *A First Course in Turbulence*, pp. 281–286. MIT Press. Cambridge.

Townsend, A. A. 1976 *The structure of turbulent shear flow*, Second Edition, pp. 342–347. Cambridge University Press.

Wilson, K. G. & Kogut, J. 1974 The renormalization group and the ϵ expansion. *Phys. Rep. C* **12**, 75–200.

Woodruff, S. L. 1994 A similarity solution for the direct interaction approximation and its relationship to renormalization-group analyses of turbulence. *Phys. Fluids* **6**, 3051–3062.

Yakhot, V. & Orszag, S. A. 1986 Renormalization group analysis of turbulence. *J. Sci. Comput.* **1**, 3–51.

Yakhot, V., Orszag, S. A. & Yakhot, A. 1987 Heat transfer in turbulent fluids–I. Pipe flow. *Int. J. Heat Mass Transfer* **30**, 15–22.

Yakhot, V. & Smith, L. M. 1992 The renormalization group, the ε-expansion and derivation of turbulence models. *J. Sci. Comput.* **7**, 35–61.

Yakhot, V., Orszag, S. A., Thangam, S., Gatski, T. B. & Speziale, C. G. 1992 Development of turbulence models for shear flows by a double expansion technique. *Phys. Fluids A* **4**, 1510–1520.

Yoshizawa, A. 1984 Statistical analysis of the deviation of the Reynolds stress from its eddy-viscosity representation. *Phys. Fluids* **27**, 1377–1387.

Zukauskas, A. & Slanciauskas, A. 1987 *Heat Transfer in Turbulent Fluid Flows*, p. 131. Hemisphere Publishing Corp. Washington, DC.

Appendix A

CALCULATIONS BASED ON YAKHOT-ORSZAG THEORY

Here, the author delves into calculations as a supplementary explanation of Chapter 2, and shows how to derive some unfamiliar equations. The equations partly overlap those in Chapter 2, allowing a better understanding of this theory.

A.1 Fourier Transformation of Basic Equations

The basic equations for Yakhot-Orszag theory consists of the continuity equation for an incompressible fluid and the Navier-Stokes equation with a random force:

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{A.1}$$

$$\frac{\partial u_i}{\partial \tau} + u_j \frac{\partial u_i}{\partial x_j} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu_0 \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \tag{A.2}$$

which are defined in the inertial range. The random force f_i is needed to compensate for the dissipation of the turbulent kinetic energy so that the equation becomes that for stationary turbulence. The function of f_i satisfies the following conditions: the continuity condition given by

$$\frac{\partial f_i}{\partial x_i} = 0, \tag{A.3}$$

and the two-point correlation in Fourier space defined by

$$\langle f_i(\boldsymbol{k},\omega) f_j(\boldsymbol{k}',\omega')\rangle = 2D_0 k^{-y}(2\pi)^{d+1} P_{ij}(\boldsymbol{k})\,\delta(\boldsymbol{k}+\boldsymbol{k}')\,\delta(\omega+\omega'), \tag{A.4}$$

where D_0 is the amplitude of the forces and y is an arbitrary parameter to determine the decay rate of the energy spectrum for f_i. The projection operator

$$P_{ij}(\boldsymbol{k}) = \delta_{ij} - \frac{k_i k_j}{k^2} \tag{A.5}$$

is the isotropic tensor arising from the continuity condition of the velocity field and is divergence free. The homogeneity in space and time is guaranteed by the Delta function:

$$(2\pi)^{d+1}\delta(\boldsymbol{k}+\boldsymbol{k}')\,\delta(\omega+\omega') = \left(\frac{1}{L}\right)^3 \frac{1}{\Delta\tau}\int_L\int_L\int_L\int_{\Delta\tau} \exp(-i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega\tau)\,d\tau dx_1 dx_2 dx_3. \tag{A.6}$$

Then, the velocity component in physical space $u_i(\boldsymbol{x},\tau)$ is transferred to that in Fourier space by means of the $d+1$-dimensional Fourier integral form:

$$u_i(\boldsymbol{x},\tau) = \int_{k<\Lambda_0}\int_{-\infty}^{+\infty} u_i(\boldsymbol{k},\omega)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}. \tag{A.7}$$

The velocity component $u_i(\boldsymbol{k},\omega)$ is defined in the range $\Lambda_e < k < \Lambda_0$, where the wave number $\Lambda_e[= O(\pi/L)]$ is the order of the energy-containing eddies and the wave number $\Lambda_0[= O(k_d)]$ is the order of the Kolmogorov dissipation scale. This range is assumed to be very large at a high Reynolds number, i.e., $\Lambda_e \ll \Lambda_0$.

Now our discussion moves to the Fourier transformation of Eqs. (A.1) and (A.2). The continuity equation for the velocity field is transferred as follows:

$$\begin{aligned}\frac{\partial u_i}{\partial x_i} &= \frac{\partial}{\partial x_i}\int_{k<\Lambda_0}\int_{-\infty}^{+\infty} u_i(\boldsymbol{k},\omega)\,\exp(i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}\\ &= \int_{k<\Lambda_0}\int_{-\infty}^{+\infty} ik_i u_i(\boldsymbol{k},\omega)\,\exp(i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega\tau)\,\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}\\ &= 0,\end{aligned} \tag{A.8}$$

which leads to

$$k_i u_i(\boldsymbol{k},\omega) = 0. \tag{A.9}$$

Each term in the Navier-Stokes equation is transferred to Fourier space in sequence by means of Eq. (A.7). The derivative with respect to time becomes

$$\begin{aligned}\frac{\partial u_i}{\partial \tau} &= \frac{\partial}{\partial \tau}\int_{k<\Lambda_0}\int_{-\infty}^{+\infty} u_i\,(\boldsymbol{k},\omega)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}} \\ &= \int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\{-i\omega u_i(\boldsymbol{k},\omega)\}\exp\left(i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega\tau\right)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}. \qquad \text{(A.10)}\end{aligned}$$

The nonlinear term is

$$\begin{aligned}u_j\frac{\partial u_i}{\partial x_j} &= \int_{k'<\Lambda_0}\int_{-\infty}^{+\infty} u_j\,(\boldsymbol{k}',\omega')\exp(i\boldsymbol{k}'\cdot\boldsymbol{x}-i\omega'\tau)\frac{d\omega' d\boldsymbol{k}'}{(2\pi)^{d+1}} \\ &\quad\times\frac{\partial}{\partial x_j}\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} u_i\,(\boldsymbol{q},\Omega)\exp(i\boldsymbol{q}\cdot\boldsymbol{x}-i\Omega\tau)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}} \\ &= \int_{k'<\Lambda_0}\int_{-\infty}^{+\infty} u_j\,(\boldsymbol{k}',\omega')\exp(i\boldsymbol{k}'\cdot\boldsymbol{x}-i\omega'\tau)\frac{d\omega' d\boldsymbol{k}'}{(2\pi)^{d+1}} \\ &\quad\times\int_{q<\Lambda_0}\int_{-\infty}^{+\infty}\{iq_j u_i(\boldsymbol{q},\Omega)\}\exp(i\boldsymbol{q}\cdot\boldsymbol{x}-i\Omega\tau)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}} \\ &= \int_{k'<\Lambda_0}\int_{-\infty}^{+\infty}\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} iq_j u_i(\boldsymbol{q},\Omega)u_j(\boldsymbol{k}',\omega') \\ &\quad\times\exp\left\{i(\boldsymbol{k}'+\boldsymbol{q})\cdot\boldsymbol{x}-i(\omega'+\Omega)\tau\right\}\frac{d\Omega d\boldsymbol{q} d\omega' d\boldsymbol{k}'}{(2\pi)^{2d+2}}, \qquad \text{(A.11)}\end{aligned}$$

where the replacements $\boldsymbol{k}'+\boldsymbol{q}\equiv\boldsymbol{k}$ and $\omega'+\Omega\equiv\omega$ yield

$$\begin{aligned}u_j\frac{\partial u_i}{\partial x_j} &= \int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\int_{k'<\Lambda_0}\int_{-\infty}^{+\infty}\left\{\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} i\,(k_j-k_j')u_j(\boldsymbol{k}',\omega')u_i(\boldsymbol{q},\Omega)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}}\right\} \\ &\quad\times\exp(i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega\tau)\delta(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{q})\delta(\omega-\omega'-\Omega)\frac{d\omega' d\boldsymbol{k}'}{(2\pi)^{d+1}}d\omega d\boldsymbol{k} \\ &= \int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\int_{k'<\Lambda_0}\int_{-\infty}^{+\infty}\left\{\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} ik_j u_j(\boldsymbol{k}',\omega')u_i(\boldsymbol{q},\Omega)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}}\right\} \\ &\quad\times\exp(i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega\tau)\delta(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{q})\delta(\omega-\omega'-\Omega)\frac{d\omega' d\boldsymbol{k}'}{(2\pi)^{d+1}}d\omega d\boldsymbol{k} \\ &= \int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\left\{\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} ik_j u_j(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega)u_i(\boldsymbol{q},\Omega)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}}\right\}\end{aligned}$$

$$\times \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\,\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}. \tag{A.12}$$

The random force is

$$f_i(\boldsymbol{x},\tau) = \int_{k<\Lambda_0}\int_{-\infty}^{+\infty} f_i(\boldsymbol{k},\omega)\exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\,\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}. \tag{A.13}$$

The pressure is transferred as

$$\begin{aligned} -\frac{1}{\rho}\frac{\partial p}{\partial x_i} &= -\frac{1}{\rho}\frac{\partial}{\partial x_i}\int_{k<\Lambda_0}\int_{-\infty}^{+\infty} p(\boldsymbol{k},\omega)\exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}} \\ &= -\int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\left\{\frac{i}{\rho}k_i p(\boldsymbol{k},\omega)\right\}\exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}. \end{aligned} \tag{A.14}$$

Here $p(\boldsymbol{k},\omega)$ can be replaced with the velocity correlation by using the Poisson equation:

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_m^2} + \frac{\partial u_m}{\partial x_n}\frac{\partial u_n}{\partial x_m} = 0. \tag{A.15}$$

Each term in Eq. (A.15) is represented as

$$\begin{aligned} \frac{1}{\rho}\frac{\partial^2 p}{\partial x_m^2} &= \frac{1}{\rho}\frac{\partial^2}{\partial x_m^2}\int_{k<\Lambda_0}\int_{-\infty}^{+\infty} p(\boldsymbol{k},\omega)\exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}} \\ &= \int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\left\{-\frac{k^2}{\rho}p(\boldsymbol{k},\omega)\right\}\exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}} \end{aligned} \tag{A.16}$$

and

$$\begin{aligned} \frac{\partial u_m}{\partial x_n}\frac{\partial u_n}{\partial x_m} &= \int_{k'<\Lambda_0}\int_{-\infty}^{+\infty} ik'_n u_m(\boldsymbol{k}',\omega')\exp(i\boldsymbol{k}'\cdot\boldsymbol{x} - i\omega'\tau)\frac{d\omega' d\boldsymbol{k}'}{(2\pi)^{d+1}} \\ &\quad\times\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} iq_m u_n(\boldsymbol{q},\Omega)\exp(i\boldsymbol{q}\cdot\boldsymbol{x} - i\Omega\tau)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}} \\ &= -\int_{k'<\Lambda_0}\int_{-\infty}^{+\infty}\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} k'_n q_m u_m(\boldsymbol{k}',\omega')u_n(\boldsymbol{q},\Omega) \\ &\quad\times\exp\{i(\boldsymbol{k}'+\boldsymbol{q})\cdot\boldsymbol{x} - i(\omega'+\Omega)\tau\}\,\frac{d\Omega d\boldsymbol{q} d\omega' d\boldsymbol{k}'}{(2\pi)^{2d+2}}. \end{aligned} \tag{A.17}$$

The replacements $\boldsymbol{k}'+\boldsymbol{q} \equiv \boldsymbol{k}$ and $\omega'+\Omega \equiv \omega$ yield

$$-\int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\int_{k'<\Lambda_0}\int_{-\infty}^{+\infty}\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} k'_n q_m u_m(\boldsymbol{k}',\omega')u_n(\boldsymbol{q},\Omega)$$

$$\times \exp\{i(\boldsymbol{k}' + \boldsymbol{q}) \cdot \boldsymbol{x} - i(\omega' + \Omega)\tau\} \, \delta(\boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{q})\delta(\omega - \omega' - \Omega) \frac{d\Omega d\boldsymbol{q} d\omega' d\boldsymbol{k}'}{(2\pi)^{2d+2}} d\omega d\boldsymbol{k}$$

$$= -\int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\int_{q<\Lambda_0}\int_{-\infty}^{+\infty}(k_n - q_n)q_m u_n(\boldsymbol{q},\Omega)u_m(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega)$$

$$\times \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\frac{d\Omega d\boldsymbol{q} d\omega d\boldsymbol{k}}{(2\pi)^{2d+2}}$$

$$= -\int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\left\{\int_{q<\Lambda_0}\int_{-\infty}^{+\infty}(k_n - q_n)(k_m - q_m - k_m)u_n(\boldsymbol{q},\Omega)\right. \tag{A.18}$$

$$\left.\times u_m(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}}\right\}\exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}$$

$$= \int_{k<\Lambda_0}\int_{-\infty}^{+\infty} k_m k_n \left\{\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} u_n(\boldsymbol{q},\Omega)u_m(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}}\right\}$$

$$\times \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}. \tag{A.19}$$

The Poisson equation in Fourier space becomes

$$\int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\left\{\frac{1}{\rho}k^2 p(\boldsymbol{k},\omega) + k_m k_n \int_{q<\Lambda_0}\int_{-\infty}^{+\infty} u_n(\boldsymbol{q},\Omega)u_m(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}}\right\}$$

$$\times \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}$$

$$= 0 \tag{A.20}$$

or

$$\frac{1}{\rho}\,p(\boldsymbol{k},\omega) = -\frac{k_m k_n}{k^2}\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} u_n(\boldsymbol{q},\Omega)u_m(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}}. \tag{A.21}$$

Then, the pressure is replaced with the velocity correlation:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x_i} = \int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\left\{ik_i\frac{k_m k_n}{k^2}\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} u_n(\boldsymbol{q},\Omega)u_m(\boldsymbol{k}-\boldsymbol{q},\Omega-\omega)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}}\right\}$$

$$\times \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\,\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}. \tag{A.22}$$

Finally, the molecular viscous diffusion term becomes

$$\nu_0\frac{\partial^2 u_i}{\partial x_j \partial x_j} = \nu_0\frac{\partial^2}{\partial x_j\partial x_j}\int_{k<\Lambda_0}\int_{-\infty}^{+\infty} u_i(\boldsymbol{k},\omega)\exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}$$

$$= -\nu_0 \int_{k<\Lambda_0} \int_{-\infty}^{+\infty} k^2 u_i(\boldsymbol{k},\omega) \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau) \frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}. \tag{A.23}$$

By summing up all the terms, the Fourier integral form of the Navier-Stokes equation is represented as

$$\begin{aligned} &\int_{k<\Lambda_0}\int_{-\infty}^{+\infty} \Big\{ -i\omega u_i(\boldsymbol{k},\omega) + ik_j \int_{q<\Lambda_0}\int_{-\infty}^{+\infty} u_j(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega) u_i(\boldsymbol{q},\Omega) \frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}} \\ &-f_i(\boldsymbol{k},\omega) - ik_i \frac{k_m k_n}{k^2} \int_{q<\Lambda_0}\int_{-\infty}^{+\infty} u_m(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega) u_n(\boldsymbol{q},\Omega) \frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}} \\ &+\nu_0 k^2 u_i(\boldsymbol{k},\omega) \Big\} \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau) \frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}} \\ = \ & 0 \end{aligned} \tag{A.24}$$

or

$$\begin{aligned} &(-i\omega + \nu_0 k^2) u_i(\boldsymbol{k},\omega) \\ &-f_i(\boldsymbol{k},\omega) + i\Big\{ k_j \int_{q<\Lambda_0}\int_{-\infty}^{+\infty} u_j(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega) u_i(\boldsymbol{q},\Omega) \frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}} \\ &- \frac{k_i k_m k_n}{k^2} \int_{q<\Lambda_0}\int_{-\infty}^{+\infty} u_m(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega) u_n(\boldsymbol{q},\Omega) \frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}} \Big\} \\ = \ & 0. \end{aligned} \tag{A.25}$$

Following the notation

$$[u]_{ij} \equiv \int_{q<\Lambda_0}\int_{-\infty}^{+\infty} u_i(\boldsymbol{q},\Omega) u_j(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega) \frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}}, \tag{A.26}$$

the last two terms of the l.h.s. in Eq. (A.25) are rewritten as

$$\begin{aligned} &i\Big\{ k_j \int_{q<\Lambda_0}\int_{-\infty}^{+\infty} u_j(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega) u_i(\boldsymbol{q},\Omega) \frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}} \\ &- \frac{k_i k_m k_n}{k^2} \int_{q<\Lambda_0}\int_{-\infty}^{+\infty} u_m(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega) u_n(\boldsymbol{q},\Omega) \frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}} \Big\} \\ = \ & i\left\{ k_j [u]_{ij} - \frac{k_i k_m k_n}{k^2} [u]_{mn} \right\}. \end{aligned} \tag{A.27}$$

The index j is interchangeable so that Eq. (A.27) becomes

$$\begin{aligned} i\left\{k_j[u]_{ij} - \frac{k_i k_m k_n}{k^2}[u]_{mn}\right\} &= i\left\{k_n[u]_{mn}\delta_{im} - \frac{k_i k_m k_n}{k^2}[u]_{mn}\right\} \\ &= ik_n P_{im}(\boldsymbol{k})[u]_{mn} \\ &= \frac{i}{2}P_{imn}(\boldsymbol{k})[u]_{mn}. \end{aligned} \tag{A.28}$$

Another interpretation is as follows: the index $i = 1$ leads to

$$\begin{aligned} & k_j[u]_{ij} - k_i\frac{k_m k_n}{k^2}[u]_{mn} = k_j[u]_{1j} - k_1\frac{k_m k_n}{k^2}[u]_{mn} \\ = \; & k_1[u]_{11} - k_1\frac{k_1^2}{k^2}[u]_{11} - k_1\frac{k_1 k_2}{k^2}[u]_{12} - k_1\frac{k_1 k_3}{k^2}[u]_{13} \\ & +k_2[u]_{12} - k_1\frac{k_2 k_1}{k^2}[u]_{21} - k_1\frac{k_2^2}{k^2}[u]_{22} - k_1\frac{k_2 k_3}{k^2}[u]_{23} \\ & +k_3[u]_{13} - k_1\frac{k_3 k_1}{k^2}[u]_{31} - k_1\frac{k_3 k_2}{k^2}[u]_{32} - k_1\frac{k_3^2}{k^2}[u]_{33} \\ = \; & k_1\left(1 - \frac{k_1^2}{k^2}\right)[u]_{11} + k_2\left(1 - \frac{k_1^2}{k^2}\right)[u]_{12} + k_3\left(1 - \frac{k_1^2}{k^2}\right)[u]_{13} - k_1\frac{k_2 k_1}{k^2}[u]_{21} \\ & -k_1\frac{k_2^2}{k^2}[u]_{22} - k_1\frac{k_2 k_3}{k^2}[u]_{23} - k_1\frac{k_3 k_1}{k^2}[u]_{31} - k_1\frac{k_3 k_2}{k^2}[u]_{32} - k_1\frac{k_3^2}{k^2}[u]_{33} \\ = \; & k_1\left(\delta_{11} - \frac{k_1^2}{k^2}\right)[u]_{11} + k_2\left(\delta_{11} - \frac{k_1^2}{k^2}\right)[u]_{12} \\ & + k_3\left(\delta_{11} - \frac{k_1^2}{k^2}\right)[u]_{13} + k_1\left(\delta_{21} - \frac{k_2 k_1}{k^2}\right)[u]_{21} \\ & + k_2\left(\delta_{21} - \frac{k_2 k_1}{k^2}\right)[u]_{22} + k_3\left(\delta_{21} - \frac{k_2 k_1}{k^2}\right)[u]_{23} \\ & + k_1\left(\delta_{31} - \frac{k_3 k_1}{k^2}\right)[u]_{31} + k_2\left(\delta_{31} - \frac{k_3 k_1}{k^2}\right)[u]_{32} \\ & + k_3\left(\delta_{31} - \frac{k_3 k_1}{k^2}\right)[u]_{33} \end{aligned}$$

$$
\begin{aligned}
&= k_1P_{11}(\boldsymbol{k})[u]_{11} + k_2P_{11}(\boldsymbol{k})[u]_{12} + k_3P_{11}(\boldsymbol{k})[u]_{13} \\
&\quad + k_1P_{12}(\boldsymbol{k})[u]_{21} + k_2P_{12}(\boldsymbol{k})[u]_{22} + k_3P_{12}(\boldsymbol{k})[u]_{23} \\
&\quad + k_1P_{13}(\boldsymbol{k})[u]_{31} + k_2P_{13}(\boldsymbol{k})[u]_{32} + k_3P_{13}(\boldsymbol{k})[u]_{33} \\
&= \frac{1}{2}\left\{k_1P_{11}(\boldsymbol{k}) + k_1P_{11}(\boldsymbol{k})\right\}[u]_{11} + \frac{1}{2}\left\{k_1P_{12}(\boldsymbol{k}) + k_2P_{11}(\boldsymbol{k})\right\}[u]_{12} \\
&\quad + \frac{1}{2}\left\{k_1P_{13}(\boldsymbol{k}) + k_3P_{11}(\boldsymbol{k})\right\}[u]_{13} + \frac{1}{2}\left\{k_2P_{11}(\boldsymbol{k}) + k_1P_{12}(\boldsymbol{k})\right\}[u]_{21} \\
&\quad + \frac{1}{2}\left\{k_2P_{12}(\boldsymbol{k}) + k_2P_{12}(\boldsymbol{k})\right\}[u]_{22} + \frac{1}{2}\left\{k_2P_{13}(\boldsymbol{k}) + k_3P_{12}(\boldsymbol{k})\right\}[u]_{23} \\
&\quad + \frac{1}{2}\left\{k_1P_{13}(\boldsymbol{k}) + k_3P_{11}(\boldsymbol{k})\right\}[u]_{31} + \frac{1}{2}\left\{k_2P_{13}(\boldsymbol{k}) + k_3P_{12}(\boldsymbol{k})\right\}[u]_{32} \\
&\quad + \frac{1}{2}\left\{k_3P_{13}(\boldsymbol{k}) + k_3P_{13}(\boldsymbol{k})\right\}[u]_{33} \\
&= \frac{1}{2}P_{1mn}(\boldsymbol{k})[u]_{mn},
\end{aligned}
\tag{A.29}
$$

and the equations for other indices $i = 2,\ 3$ are also obtainable in the same manner. Thus, the sum of these becomes

$$
i\left\{k_j[u]_{ij} - \frac{k_ik_mk_n}{k^2}[u]_{mn}\right\} = \frac{i}{2}P_{imn}(\boldsymbol{k})[u]_{mn}. \tag{A.30}
$$

Hereafter the wave-frequency vector $\hat{k}$ consists of the wave vector $\boldsymbol{k}$ and the frequency ω is used.

$$
\int_{k<\Lambda_0}\int_{-\infty}^{+\infty} d\omega d\boldsymbol{k} \longrightarrow \int_{k<\Lambda_0} d\hat{k} \tag{A.31}
$$

The Green function (or the bare propagator in renormalization) is defined as

$$
G_0(\hat{k}) \equiv (-i\omega + \nu_0 k^2)^{-1}. \tag{A.32}
$$

Then, the Navier-Stokes equation with a random force in Fourier space is written as

$$
u_i(\hat{k}) = G_0(\hat{k})f_i(\hat{k}) - \frac{i\lambda_0}{2}G_0(\hat{k})P_{imn}(\boldsymbol{k})\int_{q<\Lambda_0} u_m(\hat{q})u_n(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^{d+1}}, \tag{A.33}
$$

where $\Lambda_e < k(=|\boldsymbol{k}|) < \Lambda_0$, $\Lambda_e < q(=|\boldsymbol{q}|) < \Lambda_0$, $-\infty < \omega < +\infty$, $-\infty < \Omega < +\infty$, and $\lambda_0(=1)$ is a parameter to indicate the effect of nonlinear terms in renormalization expansion.

A.2 Renormalization Procedure

Now the RNG method is to be started with the following scale separating assumption:

$$u_i(\hat{k}) = \begin{cases} u_i^<(\hat{k}) & : \quad \Lambda_e < k < \Lambda_0 \exp(-r) \\ u_i^>(\hat{k}) & : \quad \Lambda_0 \exp(-r) < k < \Lambda_0 \\ 0 & : \quad \text{otherwise} \end{cases} . \tag{A.34}$$

The parameter r is chosen to specify the mode $u_i^>(\hat{k})$ in the vicinity of the initial cutoff wave number Λ_0. According to this decomposition, the two Navier-Stokes equations:

$$\begin{aligned} u_i^<(\hat{k}) &= G_0^<(\hat{k})\, f_i^<(\hat{k}) - \frac{i\lambda_0}{2}\, G_0^<(\hat{k})\, P_{imn}^<(\boldsymbol{k}) \int_q u_m(\hat{q}) u_n(\hat{k}-\hat{q})\, \frac{d\hat{q}}{(2\pi)^{d+1}} \\ &= G_0^<(\hat{k})\, f_i^<(\hat{k}) - \frac{i\lambda_0}{2}\, G_0^<(\hat{k})\, P_{imn}^<(\boldsymbol{k}) \int_q \Big\{ u_m^<(\hat{q}) u_n^<(\hat{k}-\hat{q}) \\ &\quad +2u_m^>(\hat{q}) u_n^<(\hat{k}-\hat{q}) + u_m^>(\hat{q}) u_n^>(\hat{k}-\hat{q}) \Big\} \frac{d\hat{q}}{(2\pi)^{d+1}} \end{aligned} \tag{A.35}$$

$$\begin{aligned} u_i^>(\hat{k}) &= G_0^>(\hat{k})\, f_i^>(\hat{k}) - \frac{i\lambda_0}{2}\, G_0^>(\hat{k})\, P_{imn}^>(\boldsymbol{k}) \int_q u_m(\hat{q}) u_n(\hat{k}-\hat{q})\, \frac{d\hat{q}}{(2\pi)^{d+1}} \\ &= G_0^>(\hat{k})\, f_i^>(\hat{k}) - \frac{i\lambda_0}{2}\, G_0^>(\hat{k})\, P_{imn}^>(\boldsymbol{k}) \int_q \Big\{ u_m^<(\hat{q}) u_n^<(\hat{k}-\hat{q}) \\ &\quad +2u_m^>(\hat{q}) u_n^<(\hat{k}-\hat{q}) + u_m^>(\hat{q}) u_n^>(\hat{k}-\hat{q}) \Big\} \frac{d\hat{q}}{(2\pi)^{d+1}} \end{aligned} \tag{A.36}$$

are obtained. Thereafter, we seek to eliminate the higher wave-number mode $u_i^>(\hat{k})$ in the velocity field and to represent the field with the rest mode $u_i^<(\hat{k})$, which corresponds to the reduction of degree of freedom in the system. In the equation for the lower wave-number mode $u_i^<(\hat{k})$, the higher wave-number modes $\boldsymbol{u}^>$ in the nonlinear term are removed by using the renormalizing expansion for the Navier-Stokes equation, i.e., by replacing $\boldsymbol{u}^{>}$ in Eq. (A.35) with their transport equations on the basis of Eq. (A.36), which yields

$$u_i^{<}(\hat{k}) = O(\lambda_0^0) + O(\lambda_0^1) + O(\lambda_0^2) + \cdots \tag{A.37}$$

as an expansion of λ_0. The calculation is done as follows: the equations for the higher wave-number modes

$$\begin{aligned}
u_m^{>}(\hat{q}) &= G_0^{>}(\hat{q})\, f_m^{>}(\hat{q}) \\
&\quad -\frac{i\lambda_0}{2}\, G_0^{>}(\hat{q})\, P_{m\alpha\beta}^{>}(\boldsymbol{q}) \int_s \Big\{ u_\alpha^{<}(\hat{s}) u_\beta^{<}(\hat{q}-\hat{s}) + 2u_\alpha^{>}(\hat{s}) u_\beta^{<}(\hat{q}-\hat{s}) \\
&\quad + u_\alpha^{>}(\hat{s})\, u_\beta^{>}(\hat{q}-\hat{s}) \Big\} \frac{d\hat{s}}{(2\pi)^{d+1}} \\
&= G_0^{>}(\hat{q})\, f_m^{>}(\hat{q}) \\
&\quad -\frac{i\lambda_0}{2}\, G_0^{>}(\hat{q})\, P_{m\alpha\beta}^{>}(\boldsymbol{q}) \int_s \Big\{ u_\alpha^{<}(\hat{s}) u_\beta^{<}(\hat{q}-\hat{s}) + 2G_0^{>}(\hat{s})\, f_\alpha^{>}(\hat{s}) u_\beta^{<}(\hat{q}-\hat{s}) \\
&\quad + G_0^{>}(\hat{s})\, G_0^{>}(\hat{q}-\hat{s})\, f_\alpha^{>}(\hat{s})\, f_\beta^{>}(\hat{q}-\hat{s}) \Big\} \frac{d\hat{s}}{(2\pi)^{d+1}} \\
&\quad + O(\lambda_0^2)
\end{aligned} \tag{A.38}$$

$$\begin{aligned}
u_n^{>}(\hat{k}-\hat{q}) &= G_0^{>}(\hat{k}-\hat{q})\, f_n^{>}(\hat{k}-\hat{q}) \\
&\quad -\frac{i\lambda_0}{2}\, G_0^{>}(\hat{k}-\hat{q})\, P_{n\gamma\delta}^{>}(\boldsymbol{k}-\boldsymbol{q}) \int_r \Big\{ u_\gamma^{<}(\hat{r}) u_\delta^{<}(\hat{k}-\hat{q}-\hat{r}) \\
&\quad +2u_\gamma^{>}(\hat{r}) u_\delta^{<}(\hat{k}-\hat{q}-\hat{r}) + u_\gamma^{>}(\hat{r})\, u_\delta^{>}(\hat{k}-\hat{q}-\hat{r}) \Big\} \frac{d\hat{r}}{(2\pi)^{d+1}} \\
&= G_0^{>}(\hat{k}-\hat{q})\, f_n^{>}(\hat{k}-\hat{q}) \\
&\quad -\frac{i\lambda_0}{2}\, G_0^{>}(\hat{k}-\hat{q})\, P_{n\gamma\delta}^{>}(\boldsymbol{k}-\boldsymbol{q}) \int_r \Big\{ u_\gamma^{<}(\hat{r}) u_\delta^{<}(\hat{k}-\hat{q}-\hat{r}) \\
&\quad +2G_0^{>}(\hat{r})\, f_\gamma^{>}(\hat{r}) u_\delta^{<}(\hat{k}-\hat{q}-\hat{r}) \\
&\quad + G_0^{>}(\hat{r})\, G_0^{>}(\hat{k}-\hat{q}-\hat{r})\, f_\gamma^{>}(\hat{r})\, f_\delta^{>}(\hat{k}-\hat{q}-\hat{r}) \Big\} \frac{d\hat{r}}{(2\pi)^{d+1}}
\end{aligned}$$

$$+O(\lambda_0^2) \tag{A.39}$$

up to the order of λ_0^1 are obtained; and substituting $\boldsymbol{u}^>$ in Eq. (A.35) into Eqs. (A.38) and (A.39) gives

$$\begin{aligned}
(-i\omega+\nu_0 k^2)u_i^<(\hat{k}) \;=\;& f_i^<(\hat{k}) - \frac{i\lambda_0}{2} P_{imn}^<(\boldsymbol{k}) \int_q u_m^<(\hat{q})\, u_n^<(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\
& -i\lambda_0 P_{imn}^<(\boldsymbol{k}) \int_q \Big[G_0^>(\hat{q}) f_m^>(\hat{q}) \\
& -\frac{i\lambda_0}{2} G_0^>(\hat{q}) P_{m\alpha\beta}^>(\boldsymbol{q}) \int_s \big\{ u_\alpha^<(\hat{s})\, u_\beta^<(\hat{q}-\hat{s}) \\
& + 2G_0^>(\hat{s}) f_\alpha^>(\hat{s}) u_\beta^<(\hat{q}-\hat{s}) \\
& + G_0^>(\hat{s}) G_0^>(\hat{q}-\hat{s})\; f_\alpha^>(\hat{s}) f_\beta^>(\hat{q}-\hat{s}) \big\} \frac{d\hat{s}}{(2\pi)^{d+1}} \Big] \\
& \times u_n^<(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\
& -\frac{i\lambda_0}{2} P_{imn}^<(\boldsymbol{k}) \int_q \Big[G_0^>(\hat{q}) f_m^>(\hat{q}) \\
& -\frac{i\lambda_0}{2} G_0^>(\hat{q}) P_{m\alpha\beta}^>(\boldsymbol{q}) \int_s \big\{ u_\alpha^<(\hat{s}) u_\beta^<(\hat{q}-\hat{s}) \\
& + 2G_0^>(\hat{s}) f_\alpha^>(\hat{s}) u_\beta^<(\hat{q}-\hat{s}) \\
& + G_0^>(\hat{s}) G_0^>(\hat{q}-\hat{s}) f_\alpha^>(\hat{s}) f_\beta^>(\hat{q}-\hat{s}) \big\} \frac{d\hat{s}}{(2\pi)^{d+1}} \Big] \\
& \times \Big[G_0^>(\hat{k}-\hat{q}) f_n^>(\hat{k}-\hat{q}) \\
& -\frac{i\lambda_0}{2} G_0^>(\hat{k}-\hat{q}) P_{n\gamma\delta}^>(\boldsymbol{k}-\boldsymbol{q}) \int_r \big\{ u_\gamma^<(\hat{r}) u_\delta^<(\hat{k}-\hat{q}-\hat{r}) \\
& + 2G_0^>(\hat{r}) f_\gamma^>(\hat{r}) u_\delta^<(\hat{k}-\hat{q}-\hat{r}) \\
& + G_0^>(\hat{r}) G_0^>(\hat{k}-\hat{q}-\hat{r}) f_\gamma^>(\hat{r}) f_\delta^>(\hat{k}-\hat{q}-\hat{r}) \big\} \frac{d\hat{r}}{(2\pi)^{d+1}} \Big] \frac{d\hat{q}}{(2\pi)^{d+1}}
\end{aligned}$$

$$
\begin{aligned}
& + O(\lambda_0^3) \\
= \; & f_i^<(\hat{k}) - \frac{i\lambda_0}{2} P_{imn}^<(\boldsymbol{k}) \int_q u_m^<(\hat{q})\, u_n^<(\hat{k}-\hat{q}) \,\frac{d\hat{q}}{(2\pi)^{d+1}} \\
& - i\lambda_0 P_{imn}^<(\boldsymbol{k}) \int_q G_0^>(\hat{q}) f_m^>(\hat{q}) u_n^<(\hat{k}-\hat{q}) \,\frac{d\hat{q}}{(2\pi)^{d+1}} \\
& + 2\left(\frac{i\lambda_0}{2}\right)^2 P_{imn}^<(\boldsymbol{k}) \int_q\int_s G_0^>(\hat{q}) P_{m\alpha\beta}^>(\boldsymbol{q}) u_n^<(\hat{k}-\hat{q}) \\
& \times \Big\{ u_\alpha^<(\hat{s}) u_\beta^<(\hat{q}-\hat{s}) + 2G_0^>(\hat{s}) f_\alpha^>(\hat{s}) u_\beta^<(\hat{q}-\hat{s}) \\
& + G_0^>(\hat{s}) G_0^>(\hat{q}-\hat{s})\, f_\alpha^>(\hat{s}) f_\beta^>(\hat{q}-\hat{s}) \Big\} \frac{d\hat{s}d\hat{q}}{(2\pi)^{2d+2}} \\
& - \frac{i\lambda_0}{2} P_{imn}^<(\boldsymbol{k}) \int_q G_0^>(\hat{q}) G_0^>(\hat{k}-\hat{q}) f_m^>(\hat{q}) f_n^>(\hat{k}-\hat{q}) \,\frac{d\hat{q}}{(2\pi)^{d+1}} \\
& + 2\left(\frac{i\lambda_0}{2}\right)^2 P_{imn}^<(\boldsymbol{k}) \int_q\int_r G_0^>(\hat{q}) G_0^>(\hat{k}-\hat{q}) f_m^>(\hat{q}) P_{n\gamma\delta}^>(\boldsymbol{k}-\boldsymbol{q}) \\
& \times \Big\{ u_\gamma^<(\hat{r}) u_\delta^<(\hat{k}-\hat{q}-\hat{r}) + 2G_0^>(\hat{r}) f_\gamma^>(\hat{r}) u_\delta^<(\hat{k}-\hat{q}-\hat{r}) \\
& + G_0^>(\hat{r}) G_0^>(\hat{k}-\hat{q}-\hat{r})\, f_\gamma^>(\hat{r}) f_\delta^>(\hat{k}-\hat{q}-\hat{r}) \Big\} \frac{d\hat{r}d\hat{q}}{(2\pi)^{2d+2}} \\
& + 2\left(\frac{i\lambda_0}{2}\right)^2 P_{imn}^<(\boldsymbol{k}) \int_q\int_s G_0^>(\hat{q}) G_0^>(\hat{k}-\hat{q}) f_n^>(\hat{k}-\hat{q}) P_{m\alpha\beta}^>(\boldsymbol{q}) \\
& \times \Big\{ u_\alpha^<(\hat{s}) u_\beta^<(\hat{q}-\hat{s}) + 2G_0^>(\hat{s}) f_\alpha^>(\hat{s}) u_\beta^<(\hat{q}-\hat{s}) \\
& + G_0^>(\hat{s}) G_0^>(\hat{q}-\hat{s})\, f_\alpha^>(\hat{s}) f_\beta^>(\hat{q}-\hat{s}) \Big\} \frac{d\hat{s}d\hat{q}}{(2\pi)^{2d+2}} \\
& + O(\lambda_0^3). \qquad \text{(A.40)}
\end{aligned}
$$

Then, Eq. (A.4) for $\boldsymbol{f}^>$ is used to close this equation:

$$
(-i\omega + \nu_0 k^2) u_i^<(\hat{k}) = f_i^<(\hat{k}) - \frac{i\lambda_0}{2} P_{imn}^<(\boldsymbol{k}) \int_q u_m^<(\hat{q})\, u_n^<(\hat{k}-\hat{q}) \,\frac{d\hat{q}}{(2\pi)^{d+1}}
$$

$$
\begin{aligned}
& -i\lambda_0 P^{<}_{imn}(\boldsymbol{k}) \int_q G^{>}_0(\hat{q}) \langle f^{>}_m(\hat{q}) \rangle u^{<}_n(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\
& + 2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q\int_s G^{>}_0(\hat{q}) P^{>}_{m\alpha\beta}(\boldsymbol{q}) u^{<}_n(\hat{k}-\hat{q}) \\
& \times \Big\{ u^{<}_\alpha(\hat{s}) u^{<}_\beta(\hat{q}-\hat{s}) + 2G^{>}_0(\hat{s}) \langle f^{>}_\alpha(\hat{s}) \rangle u^{<}_\beta(\hat{q}-\hat{s}) \\
& + G^{>}_0(\hat{s}) G^{>}_0(\hat{q}-\hat{s}) \left\langle f^{>}_\alpha(\hat{s}) f^{>}_\beta(\hat{q}-\hat{s}) \right\rangle \Big\} \frac{d\hat{s}\,d\hat{q}}{(2\pi)^{2d+2}} \\
& -\frac{i\lambda_0}{2} P^{<}_{imn}(\boldsymbol{k}) \int_q G^{>}_0(\hat{q}) G^{>}_0(\hat{k}-\hat{q}) \left\langle f^{>}_m(\hat{q}) f^{>}_n(\hat{k}-\hat{q}) \right\rangle \frac{d\hat{q}}{(2\pi)^{d+1}} \\
& + 2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q\int_r G^{>}_0(\hat{q}) G^{>}_0(\hat{k}-\hat{q}) P^{>}_{n\gamma\delta}(\boldsymbol{k}-\boldsymbol{q}) \\
& \times \Big\{ \langle f^{>}_m(\hat{q}) \rangle u^{<}_\gamma(\hat{r}) u^{<}_\delta(\hat{k}-\hat{q}-\hat{r}) \\
& + 2G^{>}_0(\hat{r}) \left\langle f^{>}_\gamma(\hat{r}) f^{>}_m(\hat{q}) \right\rangle u^{<}_\delta(\hat{k}-\hat{q}-\hat{r}) \\
& + G^{>}_0(\hat{r}) G^{>}_0(\hat{k}-\hat{q}-\hat{r}) \left\langle f^{>}_\gamma(\hat{r}) f^{>}_\delta(\hat{k}-\hat{q}-\hat{r}) f^{>}_m(\hat{q}) \right\rangle \Big\} \frac{d\hat{r}\,d\hat{q}}{(2\pi)^{2d+2}} \\
& + 2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q\int_s G^{>}_0(\hat{q}) G^{>}_0(\hat{k}-\hat{q}) P^{>}_{m\alpha\beta}(\boldsymbol{q}) \\
& \times \Big\{ u^{<}_\alpha(\hat{s}) \left\langle f^{>}_n(\hat{k}-\hat{q}) \right\rangle u^{<}_\beta(\hat{q}-\hat{s}) \\
& + 2G^{>}_0(\hat{s}) \left\langle f^{>}_\alpha(\hat{s}) f^{>}_n(\hat{k}-\hat{q}) \right\rangle u^{<}_\beta(\hat{q}-\hat{s}) \\
& + G^{>}_0(\hat{s}) G^{>}_0(\hat{q}-\hat{s}) \left\langle f^{>}_\alpha(\hat{s}) f^{>}_\beta(\hat{q}-\hat{s}) f^{>}_n(\hat{k}-\hat{q}) \right\rangle \Big\} \frac{d\hat{s}\,d\hat{q}}{(2\pi)^{2d+2}} \\
& + O(\lambda_0^3) \\
= \;& f^{<}_i(\hat{k}) - \frac{i\lambda_0}{2} P^{<}_{imn}(\boldsymbol{k}) \int_q u^{<}_m(\hat{q})\, u^{<}_n(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}}
\end{aligned}
$$

$$
\begin{aligned}
&+2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q\int_s G_0^{>}(\hat{q}) P^{>}_{m\alpha\beta}(\boldsymbol{q}) \\
&\times u^{<}_n(\hat{k}-\hat{q}) \left\{ u^{<}_\alpha(\hat{s}) u^{<}_\beta(\hat{q}-\hat{s}) \right. \\
&\left. + G_0^{>}(\hat{s}) G_0^{>}(\hat{q}-\hat{s}) \left\langle f^{>}_\alpha(\hat{s})\, f^{>}_\beta(\hat{q}-\hat{s}) \right\rangle \right\} \frac{d\hat{s}d\hat{q}}{(2\pi)^{2d+2}} \\
&-\frac{i\lambda_0}{2} P^{<}_{imn}(\boldsymbol{k}) \int_q G_0^{>}(\hat{q}) G_0^{>}(\hat{k}-\hat{q}) \left\langle f^{>}_m(\hat{q}) f^{>}_n(\hat{k}-\hat{q}) \right\rangle \frac{d\hat{q}}{(2\pi)^{d+1}} \\
&+2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q\int_r G_0^{>}(\hat{q}) G_0^{>}(\hat{k}-\hat{q}) G_0^{>}(\hat{r})\, P^{>}_{n\gamma\delta}(\boldsymbol{k}-\boldsymbol{q}) \\
&\times \left\langle f^{>}_m(\hat{q}) f^{>}_\gamma(\hat{r}) \right\rangle u^{<}_\delta(\hat{k}-\hat{q}-\hat{r}) \frac{d\hat{r}d\hat{q}}{(2\pi)^{2d+2}} \\
&+2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q\int_s G_0^{>}(\hat{q}) G_0^{>}(\hat{k}-\hat{q}) G_0^{>}(\hat{s})\, P^{>}_{m\alpha\beta}(\boldsymbol{q}) \\
&\times \left\langle f^{>}_n(\hat{k}-\hat{q}) f^{>}_\alpha(\hat{s}) \right\rangle u^{<}_\beta(\hat{q}-\hat{s}) \frac{d\hat{s}d\hat{q}}{(2\pi)^{2d+2}} \\
&+O(\lambda_0^3), \qquad \text{(A.41)}
\end{aligned}
$$

where the third term of the r.h.s. is estimated as

$$
\begin{aligned}
& 2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q\int_s G_0^{>}(\hat{q}) P^{>}_{m\alpha\beta}(\boldsymbol{q})\, u^{<}_n(\hat{k}-\hat{q}) u^{<}_\alpha(\hat{s}) u^{<}_\beta(\hat{q}-\hat{s}) \frac{d\hat{s}d\hat{q}}{(2\pi)^{2d+2}} \\
=\ & O([u^{<}]^3), \qquad \text{(A.42)}
\end{aligned}
$$

the fourth term is

$$
\begin{aligned}
& 2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q\int_s G_0^{>}(\hat{q}) G_0^{>}(\hat{s}) G_0^{>}(\hat{q}-\hat{s}) P^{>}_{m\alpha\beta}(\boldsymbol{q}) \\
& \times \left\langle f^{>}_\alpha(\hat{s}) f^{>}_\beta(\hat{q}-\hat{s}) \right\rangle u^{<}_n(\hat{k}-\hat{q}) \frac{d\hat{s}d\hat{q}}{(2\pi)^{2d+2}} \\
=\ & 2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q\int_s G_0^{>}(\hat{q}) G_0^{>}(\hat{s}) G_0^{>}(\hat{q}-\hat{s}) P^{>}_{m\alpha\beta}(\boldsymbol{q})
\end{aligned}
$$

$$
\begin{aligned}
& \times \left\{ 2D_0\, s^{-y}\, P^{>}_{\alpha\beta}(\boldsymbol{s})\, (2\pi)^{d+1}\delta(\hat{q}-\hat{s}+\hat{s}) \right\} u^{<}_{n}(\hat{k}-\hat{q})\, \frac{d\hat{s}d\hat{q}}{(2\pi)^{2d+2}} \\
= \; & 2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q\int_s G^{>}_0(\hat{q})G^{>}_0(\hat{s})G^{>}_0(\hat{q}-\hat{s})P^{>}_{m\alpha\beta}(\boldsymbol{q}) \\
& \times \left\{ 2D_0\, s^{-y}\, P^{>}_{\alpha\beta}(\boldsymbol{s})\, (2\pi)^{d+1}\delta(\hat{q}) \right\} u^{<}_{n}(\hat{k}-\hat{q})\, \frac{d\hat{s}d\hat{q}}{(2\pi)^{2d+2}} \\
= \; & 0, \qquad \text{(A.43)}
\end{aligned}
$$

the fifth term is treated as

$$
\Delta f_i \equiv -\frac{i\lambda_0}{2}\, P^{<}_{imn}(\boldsymbol{k}) \int_q G^{>}_0(\hat{q})G^{>}_0(\hat{k}-\hat{q}) \left\langle f^{>}_{m}(\hat{q}) f^{>}_{n}(\hat{k}-\hat{q}) \right\rangle \frac{d\hat{q}}{(2\pi)^{d+1}}, \qquad \text{(A.44)}
$$

the sixth term is

$$
\begin{aligned}
& 2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q\int_r G^{>}_0(\hat{q})G^{>}_0(\hat{k}-\hat{q})G^{>}_0(\hat{r})\, P^{>}_{n\gamma\delta}(\boldsymbol{k}-\boldsymbol{q}) \\
& \times \left\langle f^{>}_{m}(\hat{q}) f^{>}_{\gamma}(\hat{r}) \right\rangle u^{<}_{\delta}(\hat{k}-\hat{q}-\hat{r})\, \frac{d\hat{r}d\hat{q}}{(2\pi)^{2d+2}} \\
= \; & 2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q\int_r G^{>}_0(\hat{q})G^{>}_0(\hat{k}-\hat{q})G^{>}_0(\hat{r})\, P^{>}_{n\gamma\delta}(\boldsymbol{k}-\boldsymbol{q}) \\
& \times \left\{ 2D_0\, q^{-y}\, P^{>}_{m\gamma}(\boldsymbol{q})\, (2\pi)^{d+1}\delta(\hat{q}+\hat{r}) \right\} u^{<}_{\delta}(\hat{k}-\hat{q}-\hat{r})\, \frac{d\hat{r}d\hat{q}}{(2\pi)^{2d+2}} \\
= \; & 2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q G^{>}_0(\hat{q})G^{>}_0(\hat{k}-\hat{q})G^{>}_0(-\hat{q})\, P^{>}_{n\gamma\delta}(\boldsymbol{k}-\boldsymbol{q}) \\
& \times 2D_0\, q^{-y}\, P^{>}_{m\gamma}(\boldsymbol{q}) u^{<}_{\delta}(\hat{k})\, \frac{d\hat{q}}{(2\pi)^{d+1}} \\
= \; & 2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q \left|G^{>}_0(\hat{q})\right|^2 G^{>}_0(\hat{k}-\hat{q})\, P^{>}_{n\gamma\delta}(\boldsymbol{k}-\boldsymbol{q}) \\
& \times 2D_0\, q^{-y}\, P^{>}_{m\gamma}(\boldsymbol{q})\, u^{<}_{\delta}(\hat{k})\, \frac{d\hat{q}}{(2\pi)^{d+1}}, \qquad \text{(A.45)}
\end{aligned}
$$

and the seventh term is

$$
2\left(\frac{i\lambda_0}{2}\right)^2 P^{<}_{imn}(\boldsymbol{k}) \int_q\int_s G^{>}_0(\hat{q})G^{>}_0(\hat{k}-\hat{q})G^{>}_0(\hat{s})\, P^{>}_{m\alpha\beta}(\boldsymbol{q})
$$

$$\times \left\langle f_n^>(\hat{k}-\hat{q}) f_\alpha^>(\hat{s}) \right\rangle u_\beta^<(\hat{q}-\hat{s}) \frac{d\hat{s}d\hat{q}}{(2\pi)^{2d+2}}$$

$$= 2\left(\frac{i\lambda_0}{2}\right)^2 P_{imn}^<(\boldsymbol{k}) \int_q\int_s G_0^>(\hat{q}) G_0^>(\hat{k}-\hat{q}) G_0^>(\hat{s})\, P_{m\alpha\beta}^>(\boldsymbol{q})$$

$$\times \left\{2D_0\, |\boldsymbol{k}-\boldsymbol{q}|^{-y}\, P_{n\alpha}^>(\boldsymbol{k}-\boldsymbol{q})\, (2\pi)^{d+1}\delta(\hat{k}-\hat{q}+\hat{s})\right\} u_\beta^<(\hat{q}-\hat{s}) \frac{d\hat{s}d\hat{q}}{(2\pi)^{2d+2}}$$

$$= 2\left(\frac{i\lambda_0}{2}\right)^2 P_{imn}^<(\boldsymbol{k}) \int_q G_0^>(\hat{q}) G_0^>(\hat{k}-\hat{q}) G_0^>(\hat{q}-\hat{k})\, P_{m\alpha\beta}^>(\boldsymbol{q})$$

$$\times 2D_0\, |\boldsymbol{k}-\boldsymbol{q}|^{-y}\, P_{n\alpha}^>(\boldsymbol{k}-\boldsymbol{q}) u_\beta^<(\hat{k}) \frac{d\hat{q}}{(2\pi)^{d+1}}$$

$$= 2\left(\frac{i\lambda_0}{2}\right)^2 P_{imn}^<(\boldsymbol{k}) \int_q \left|G_0^>(\hat{k}-\hat{q})\right|^2 G_0^>(\hat{q})\, P_{m\alpha\beta}^>(\boldsymbol{q})$$

$$\times 2D_0\, |\boldsymbol{k}-\boldsymbol{q}|^{-y}\, P_{n\alpha}^>(\boldsymbol{k}-\boldsymbol{q})\, u_\beta^<(\hat{k}) \frac{d\hat{q}}{(2\pi)^{d+1}}. \tag{A.46}$$

In particular, interchanging the indices $\alpha \to \gamma$ and $\beta \to \delta$ in the sixth and seventh terms can be done in association with the symmetric property $P_{imn}(\boldsymbol{k}) = P_{inm}(\boldsymbol{k})$. As a result, the renormalized Navier-Stokes equation is obtained as

$$\begin{aligned}
\left(-i\omega + \nu_0 k^2\right) u_i^<(\hat{k}) \;=\;& f_i^<(\hat{k}) + \Delta f_i \\
& -\frac{i\lambda_0}{2} P_{imn}^<(\boldsymbol{k}) \int_q u_m^<(\hat{q}) u_n^<(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\
& + 4D_0 \left(\frac{i\lambda_0}{2}\right)^2 P_{imn}^<(\boldsymbol{k}) \int_q |G_0^>(\hat{q})|^2\, G_0^>(\hat{k}-\hat{q})\, P_{n\alpha\beta}^>(\boldsymbol{k}-\boldsymbol{q}) \\
& \times P_{m\alpha}^>(\boldsymbol{q}) q^{-y} \frac{d\hat{q}}{(2\pi)^{d+1}} u_\beta^<(\hat{k}) \\
& + 4D_0 \left(\frac{i\lambda_0}{2}\right)^2 P_{imn}^<(\boldsymbol{k}) \int_q \left|G_0^>(\hat{k}-\hat{q})\right|^2 G_0^>(\hat{q})\, P_{n\alpha\beta}^>(\boldsymbol{q}) \\
& \times P_{m\alpha}^>(\boldsymbol{k}-\boldsymbol{q})\, |\boldsymbol{k}-\boldsymbol{q}|^{-y} \frac{d\hat{q}}{(2\pi)^{d+1}} u_\beta^<(\hat{k})
\end{aligned}$$

$$+ O[(u^<)^3]. \tag{A.47}$$

To carry out the integration of $\hat{q}$, the fourth and fifth terms in Eq. (A.47) are rewritten as

$$\begin{aligned} R_1 &\equiv -\lambda_0^2 \frac{D_0}{(2\pi)^{d+1}} P^<_{imn}(\boldsymbol{k})\, u^<_\beta(\hat{k}) \int_q |G^>_0(\hat{q})|^2\, G^>_0(\hat{k}-\hat{q}) \\ &\quad \times P^>_{n\alpha\beta}(\boldsymbol{k}-\boldsymbol{q})\, P^>_{m\alpha}(\boldsymbol{q})\, q^{-y}\, d\hat{q} \end{aligned} \tag{A.48}$$

$$\begin{aligned} R_2 &\equiv -\lambda_0^2 \frac{D_0}{(2\pi)^{d+1}} P^<_{imn}(\boldsymbol{k})\, u^<_\beta(\hat{k}) \int_q \left|G^>_0(\hat{k}-\hat{q})\right|^2 G^>_0(\hat{q}) \\ &\quad \times P^>_{n\alpha\beta}(\boldsymbol{q})\, P^>_{m\alpha}(\boldsymbol{k}-\boldsymbol{q})\, |\boldsymbol{k}-\boldsymbol{q}|^{-y}\, d\hat{q}. \end{aligned} \tag{A.49}$$

Here the integral operator indicates

$$\int_q d\hat{q} \equiv \int_{\Lambda_0 \exp(-r)<q<\Lambda_0} d\boldsymbol{q} \int_{-\infty}^{+\infty} d\Omega. \tag{A.50}$$

Firstly, the integration of the frequency Ω is easily carried out by using the residue theorem:

$$\int_{-\infty}^{+\infty} |G^>_0(\hat{q})|^2\, G^>_0(\hat{k}-\hat{q})\, d\Omega \tag{A.51}$$

$$\begin{aligned}
&= \int_{-\infty}^{+\infty} \frac{d\Omega}{(-i\Omega+\nu_0 q^2)(i\Omega+\nu_0 q^2)\left\{-i(\omega-\Omega)+\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2\right\}} \\
&= 2\pi i \left[\frac{\Omega - i\nu_0 q^2}{(-i\Omega+\nu_0 q^2)(i\Omega+\nu_0 q^2)\left\{-i(\omega-\Omega)+\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2\right\}}\right]_{\Omega=+i\nu_0 q^2} \\
&\quad + 2\pi i \left[\frac{\Omega - \omega - i\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2}{(-i\Omega+\nu_0 q^2)(i\Omega+\nu_0 q^2)\left\{-i(\omega-\Omega)+\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2\right\}}\right]_{\Omega=\omega+i\nu_0|k-q|^2} \\
&= \frac{2\pi}{2\nu_0 q^2\left(-i\omega-\nu_0 q^2+\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2\right)} \\
&\quad + \frac{2\pi}{\left(-i\omega+\nu_0 q^2+\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2\right)\left(i\omega+\nu_0 q^2-\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2\right)} \\
&= \frac{\pi}{\nu_0 q^2\left(-i\omega+\nu_0 q^2+\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2\right)}
\end{aligned} \tag{A.52}$$

and

$$\int_{-\infty}^{+\infty} G_0^>(\hat{q}) \left|G_0^>(\hat{k}-\hat{q})\right|^2 d\Omega \tag{A.53}$$

$$= \int_{-\infty}^{+\infty} \frac{d\Omega}{\left\{-i(\omega-\Omega)+\nu_0 |\boldsymbol{k}-\boldsymbol{q}|^2\right\}\left\{i(\omega-\Omega)+\nu_0 |\boldsymbol{k}-\boldsymbol{q}|^2\right\}(-i\Omega+\nu_0 q^2)}$$

$$= 2\pi i \left[\frac{\Omega-\omega-i\nu_0 |\boldsymbol{k}-\boldsymbol{q}|^2}{\left\{-i(\omega-\Omega)+\nu_0 |\boldsymbol{k}-\boldsymbol{q}|^2\right\}\left\{i(\omega-\Omega)+\nu_0 |\boldsymbol{k}-\boldsymbol{q}|^2\right\}}\right.$$

$$\left.\frac{}{(-i\Omega+\nu_0 q^2)}\right]_{\Omega=\omega+i\nu_0|k-q|^2}$$

$$= \frac{\pi}{\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2(-i\omega+\nu_0 q^2+\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2)}, \tag{A.54}$$

so that R_1 and R_2 become

$$R_1 = -\lambda_0^2 \frac{\pi D_0}{\nu_0(2\pi)^{d+1}} P_{imn}^<(\boldsymbol{k}) u_\beta^<(\hat{k}) \int_q \frac{P_{n\alpha\beta}^>(\boldsymbol{k}-\boldsymbol{q}) P_{m\alpha}^>(\boldsymbol{q}) q^{-y-2} d\boldsymbol{q}}{-i\omega+\nu_0 q^2+\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2} \tag{A.55}$$

$$R_2 = -\lambda_0^2 \frac{\pi D_0}{\nu_0(2\pi)^{d+1}} P_{imn}^<(\boldsymbol{k}) u_\beta^<(\hat{k}) \int_q \frac{P_{n\alpha\beta}^>(\boldsymbol{q}) P_{m\alpha}^>(\boldsymbol{k}-\boldsymbol{q})|\boldsymbol{k}-\boldsymbol{q}|^{-y-2} d\boldsymbol{q}}{-i\omega+\nu_0 q^2+\nu_0|\boldsymbol{k}-\boldsymbol{q}|^2}. \tag{A.56}$$

Under the assumption $|\boldsymbol{k}| \ll |\boldsymbol{q}|$ and $|i\omega| \ll \nu_0 q^2$ the interchange $\boldsymbol{q} \to \boldsymbol{q} + \frac{1}{2}\boldsymbol{k}$ is valid, so that

$$R_1 = -\lambda_0^2 \frac{D_0 P_{imn}^<(\boldsymbol{k})}{2\nu_0(2\pi)^d} u_\beta^<(\hat{k}) \int_q \frac{P_{n\alpha\beta}^>\left(\frac{1}{2}\boldsymbol{k}-\boldsymbol{q}\right) P_{m\alpha}^>\left(\boldsymbol{q}+\frac{1}{2}\boldsymbol{k}\right)\left|\boldsymbol{q}+\frac{1}{2}\boldsymbol{k}\right|^{-y-2} d\boldsymbol{q}}{\nu_0\left|\boldsymbol{q}+\frac{1}{2}\boldsymbol{k}\right|^2+\nu_0\left|\frac{1}{2}\boldsymbol{k}-\boldsymbol{q}\right|^2} \tag{A.57}$$

$$R_2 = -\lambda_0^2 \frac{D_0 P_{imn}^<(\boldsymbol{k})}{2\nu_0(2\pi)^d} u_\beta^<(\hat{k}) \int_q \frac{P_{n\alpha\beta}^>(\boldsymbol{q}+\frac{1}{2}\boldsymbol{k}) P_{m\alpha}^>(\frac{1}{2}\boldsymbol{k}-\boldsymbol{q})\left|\frac{1}{2}\boldsymbol{k}-\boldsymbol{q}\right|^{-y-2} d\boldsymbol{q}}{\nu_0\left|\boldsymbol{q}+\frac{1}{2}\boldsymbol{k}\right|^2+\nu_0\left|\frac{1}{2}\boldsymbol{k}-\boldsymbol{q}\right|^2} \tag{A.58}$$

and

$$\left|\boldsymbol{q}+\frac{1}{2}\boldsymbol{k}\right|^2+\left|\frac{1}{2}\boldsymbol{k}-\boldsymbol{q}\right|^2 = 2q^2+\frac{1}{2}k^2 \simeq 2q^2. \tag{A.59}$$

Thus, by making the use of the following relations:

$$\left(q_\alpha+\frac{1}{2}k_\alpha\right) P_{m\alpha}\left(\boldsymbol{q}+\frac{1}{2}\boldsymbol{k}\right)$$

$$= \left(q_\alpha + \frac{1}{2}\,k_\alpha\right)\left\{\delta_{m\alpha} - \frac{\left(q_\alpha + \frac{1}{2}\,k_\alpha\right)\left(q_m + \frac{1}{2}\,k_m\right)}{\left|\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right|^2}\right\}$$

$$= \left(q_\alpha + \frac{1}{2}\,k_\alpha\right)\,\delta_{m\alpha} - \left(q_m + \frac{1}{2}\,k_m\right)\frac{\left(q_\alpha + \frac{1}{2}\,k_\alpha\right)^2}{\left|\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right|^2}$$

$$= \left(q_m + \frac{1}{2}\,k_m\right) - \left(q_m + \frac{1}{2}\,k_m\right)$$

$$= 0 \tag{A.60}$$

$$\left(q_\alpha - \frac{1}{2}\,k_\alpha\right)P_{m\alpha}\left(\boldsymbol{q} - \frac{1}{2}\,\boldsymbol{k}\right) = 0 \tag{A.61}$$

$$k_\beta u_\beta(\hat{k}) = 0, \tag{A.62}$$

the compound projection operator is reduced as follows:

$$P_{n\alpha\beta}\left(\frac{1}{2}\,\boldsymbol{k} - \boldsymbol{q}\right)P_{m\alpha}\left(\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right)$$

$$= \left\{\left(\frac{1}{2}\,k_\alpha - q_\alpha\right)P_{n\beta}\left(\frac{1}{2}\,\boldsymbol{k} - \boldsymbol{q}\right) + \left(\frac{1}{2}\,k_\beta - q_\beta\right)P_{n\alpha}\left(\frac{1}{2}\,\boldsymbol{k} - \boldsymbol{q}\right)\right\}P_{m\alpha}\left(\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right)$$

$$= \left\{k_\alpha P_{n\beta}\left(\frac{1}{2}\,\boldsymbol{k} - \boldsymbol{q}\right) - q_\beta P_{n\alpha}\left(\frac{1}{2}\,\boldsymbol{k} - \boldsymbol{q}\right)\right\}P_{m\alpha}\left(\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right)$$

$$= \left\{k_\alpha P_{n\beta}\left(\boldsymbol{q} - \frac{1}{2}\,\boldsymbol{k}\right) - q_\beta P_{n\alpha}\left(\boldsymbol{q} - \frac{1}{2}\,\boldsymbol{k}\right)\right\}P_{m\alpha}\left(\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right) \tag{A.63}$$

$$P_{n\alpha\beta}\left(\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right)P_{m\alpha}\left(\frac{1}{2}\,\boldsymbol{k} - \boldsymbol{q}\right)$$

$$= \left\{\left(q_\alpha + \frac{1}{2}\,k_\alpha\right)P_{n\beta}\left(\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right) + \left(q_\beta + \frac{1}{2}\,k_\beta\right)P_{n\alpha}\left(\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right)\right\}P_{m\alpha}\left(\frac{1}{2}\,\boldsymbol{k} - \boldsymbol{q}\right)$$

$$= \left\{k_\alpha P_{n\beta}\left(\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right) + q_\beta P_{n\alpha}\left(\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right)\right\}P_{m\alpha}\left(\boldsymbol{q} - \frac{1}{2}\,\boldsymbol{k}\right), \tag{A.64}$$

where the approximations

$$P_{n\alpha}\left(\boldsymbol{q} - \frac{1}{2}\,\boldsymbol{k}\right)P_{m\alpha}\left(\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right)$$

$$= \left\{ \delta_{n\alpha} - \frac{\left(q_n - \frac{1}{2}\,k_n\right)\left(q_\alpha - \frac{1}{2}\,k_\alpha\right)}{\left|\boldsymbol{q} - \frac{1}{2}\,\boldsymbol{k}\right|^2} \right\} \left\{ \delta_{m\alpha} - \frac{\left(q_m + \frac{1}{2}\,k_m\right)\left(q_\alpha + \frac{1}{2}\,k_\alpha\right)}{\left|\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right|^2} \right\}$$

$$= \delta_{mn} - \frac{\left(q_n - \frac{1}{2}\,k_n\right)\left(q_m - \frac{1}{2}\,k_m\right)}{\left|\boldsymbol{q} - \frac{1}{2}\,\boldsymbol{k}\right|^2} - \frac{\left(q_n + \frac{1}{2}\,k_n\right)\left(q_m + \frac{1}{2}\,k_m\right)}{\left|\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right|^2}$$

$$+ \frac{\left(q_n - \frac{1}{2}\,k_n\right)\left(q_m + \frac{1}{2}\,k_m\right)\left(q_\alpha^2 - \frac{1}{4}\,k_\alpha^2\right)}{\left|\boldsymbol{q} - \frac{1}{2}\,\boldsymbol{k}\right|^2\left|\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right|^2} \tag{A.65}$$

and

$$\left|\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right|^{-2} = q^{-2}\left(1 - \frac{k_j q_j}{q^2} + \cdots\right) \tag{A.66}$$

$$\left|\boldsymbol{q} - \frac{1}{2}\,\boldsymbol{k}\right|^{-2} = q^{-2}\left(1 + \frac{k_j q_j}{q^2} + \cdots\right) \tag{A.67}$$

yield

$$\begin{aligned}
& P_{n\alpha}\left(\boldsymbol{q} - \frac{1}{2}\,\boldsymbol{k}\right) P_{m\alpha}\left(\boldsymbol{q} + \frac{1}{2}\,\boldsymbol{k}\right) \\
= \;& \delta_{mn} - q^{-2}\left(1 + \frac{k_j q_j}{q^2}\right)\left(q_m - \frac{1}{2}\,k_m\right)\left(q_n - \frac{1}{2}\,k_n\right) \\
& -q^{-2}\left(1 - \frac{k_j q_j}{q^2}\right)\left(q_m + \frac{1}{2}\,k_m\right)\left(q_n + \frac{1}{2}\,k_n\right) \\
& + q^{-2}\left(q_m + \frac{1}{2}\,k_m\right)\left(q_n - \frac{1}{2}\,k_n\right) \\
& + O(k^2) \\
= \;& \delta_{mn} - \frac{1}{q^2}\left(1 + \frac{k_j q_j}{q^2}\right)\left(q_m q_n - \frac{1}{2}\,k_m q_n - \frac{1}{2}\,k_n q_m\right) \\
& -\frac{1}{q^2}\left(1 - \frac{k_j q_j}{q^2}\right)\left(q_m q_n + \frac{1}{2}\,k_m q_n + \frac{1}{2}\,k_n q_m\right) + \frac{1}{q^2}\left(q_m q_n + \frac{1}{2}\,k_m q_n - \frac{1}{2}\,k_n q_m\right)
\end{aligned}$$

$$
\begin{aligned}
& + O(k^2) \\
= \;& \delta_{mn} - \frac{q_m q_n}{q^2} + \frac{1}{2}\frac{k_m q_n}{q^2} - \frac{1}{2}\frac{k_n q_m}{q^2} + O(k^2) \\
= \;& P_{mn}(\boldsymbol{q}) + O(k^2).
\end{aligned} \tag{A.68}
$$

Then the reduction

$$
P_{n\alpha}\left(\boldsymbol{q} - \frac{1}{2}\boldsymbol{k}\right) P_{m\alpha}\left(\boldsymbol{q} + \frac{1}{2}\boldsymbol{k}\right) \simeq P_{mn}(\boldsymbol{q}) \tag{A.69}
$$

$$
P_{n\alpha}\left(\boldsymbol{q} + \frac{1}{2}\boldsymbol{k}\right) P_{m\alpha}\left(\boldsymbol{q} - \frac{1}{2}\boldsymbol{k}\right) \simeq P_{mn}(\boldsymbol{q}) \tag{A.70}
$$

is applicable under the condition $k \to 0$, and R_1 and R_2 are rewritten as

$$
\begin{aligned}
R_1 = & -\frac{\lambda_0^2 D_0 P^<_{imn}(\boldsymbol{k})}{4\nu_0^2(2\pi)^d} u^<_\beta(\hat{k}) \int_q q^{-2} \left|\boldsymbol{q} + \frac{1}{2}\boldsymbol{k}\right|^{-y-2} \\
& \times \left\{ k_\alpha P^>_{n\beta}\left(\boldsymbol{q} - \frac{1}{2}\boldsymbol{k}\right) P^>_{m\alpha}\left(\boldsymbol{q} + \frac{1}{2}\boldsymbol{k}\right) - q_\beta P^>_{mn}(\boldsymbol{q}) \right\} d\boldsymbol{q}
\end{aligned} \tag{A.71}
$$

$$
\begin{aligned}
R_2 = & -\frac{\lambda_0^2 D_0 P^<_{imn}(\boldsymbol{k})}{4\nu_0^2(2\pi)^d} u^<_\beta(\hat{k}) \int_q q^{-2} \left|\boldsymbol{q} - \frac{1}{2}\boldsymbol{k}\right|^{-y-2} \\
& \times \left\{ k_\alpha P^>_{n\beta}\left(\boldsymbol{q} + \frac{1}{2}\boldsymbol{k}\right) P^>_{m\alpha}\left(\boldsymbol{q} - \frac{1}{2}\boldsymbol{k}\right) + q_\beta P^>_{mn}(\boldsymbol{q}) \right\} d\boldsymbol{q}
\end{aligned} \tag{A.72}
$$

or

$$
\begin{aligned}
& R_1 + R_2 \\
= & -\frac{\lambda_0^2 D_0 P^<_{imn}(\boldsymbol{k})}{4\nu_0^2(2\pi)^d} u^<_\beta(\hat{k}) \int_q q^{-y-4} \left[\left(1 - \frac{y+2}{2}\frac{k_j q_j}{q^2}\right) \right. \\
& \times \left\{ k_\alpha P^>_{n\beta}\left(\boldsymbol{q} - \frac{1}{2}\boldsymbol{k}\right) P^>_{m\alpha}\left(\boldsymbol{q} + \frac{1}{2}\boldsymbol{k}\right) - q_\beta P^>_{mn}(\boldsymbol{q}) \right\} \\
& \left. + \left(1 + \frac{y+2}{2}\frac{k_j q_j}{q^2}\right) \left\{ k_\alpha P^>_{n\beta}\left(\boldsymbol{q} + \frac{1}{2}\boldsymbol{k}\right) P^>_{m\alpha}\left(\boldsymbol{q} - \frac{1}{2}\boldsymbol{k}\right) + q_\beta P^>_{mn}(\boldsymbol{q}) \right\} \right] d\boldsymbol{q}.
\end{aligned} \tag{A.73}
$$

Actually, the $O(k^2)$ term is neglected because of $k \to 0$; thus,

$$
\begin{aligned}
& k_\alpha P_{n\beta}\left(\boldsymbol{q}-\frac{1}{2}\boldsymbol{k}\right) P_{m\alpha}\left(\boldsymbol{q}+\frac{1}{2}\boldsymbol{k}\right) \\
= \;& k_\alpha \left\{ \delta_{n\beta} - \frac{\left(q_n - \frac{1}{2}k_n\right)\left(q_\beta - \frac{1}{2}k_\beta\right)}{\left|\boldsymbol{q}-\frac{1}{2}\boldsymbol{k}\right|^2} \right\} \left\{ \delta_{m\alpha} - \frac{\left(q_m + \frac{1}{2}k_m\right)\left(q_\alpha + \frac{1}{2}k_\alpha\right)}{\left|\boldsymbol{q}+\frac{1}{2}\boldsymbol{k}\right|^2} \right\} \\
= \;& k_\alpha \left\{ \delta_{n\beta} - \frac{1}{q^2}\left(1 + \frac{k_j q_j}{q^2} + \cdots\right)\left(q_n - \frac{1}{2}k_n\right)\left(q_\beta - \frac{1}{2}k_\beta\right) \right\} \\
& \times \left\{ \delta_{m\alpha} - \frac{1}{q^2}\left(1 - \frac{k_j q_j}{q^2} + \cdots\right)\left(q_m + \frac{1}{2}k_m\right)\left(q_\alpha + \frac{1}{2}k_\alpha\right) \right\} \\
= \;& k_\alpha \left\{ \delta_{n\beta} - \frac{q_n q_\beta}{q^2} + O(k) \right\} \left\{ \delta_{m\alpha} - \frac{q_m q_\alpha}{q^2} + O(k) \right\} \\
= \;& k_\alpha \left(\delta_{n\beta} - \frac{q_n q_\beta}{q^2} \right) \left(\delta_{m\alpha} - \frac{q_m q_\alpha}{q^2} \right) + O(k^2) \\
\simeq \;& k_\alpha P_{n\beta}(\boldsymbol{q}) P_{m\alpha}(\boldsymbol{q})
\end{aligned}
\tag{A.74}
$$

and

$$
\begin{aligned}
& R_1 + R_2 \\
= \;& -\frac{\lambda_0^2 D_0 P^<_{imn}(\boldsymbol{k})}{4\nu_0^2 (2\pi)^d} u^<_\beta(\hat{k}) \int_q q^{-y-4} \left[\left(1 - \frac{y+2}{2}\frac{k_j q_j}{q^2}\right) \left\{ k_\alpha P^>_{n\beta}(\boldsymbol{q}) P^>_{m\alpha}(\boldsymbol{q}) \right. \right. \\
& \left. \left. - q_\beta P^>_{mn}(\boldsymbol{q}) \right\} + \left(1 + \frac{y+2}{2}\frac{k_j q_j}{q^2}\right) \left\{ k_\alpha P^>_{n\beta}(\boldsymbol{q}) P^>_{m\alpha}(\boldsymbol{q}) + q_\beta P^>_{mn}(\boldsymbol{q}) \right\} \right] d\boldsymbol{q} \\
= \;& -\frac{\lambda_0^2 D_0 P^<_{imn}(\boldsymbol{k})}{2\nu_0^2 (2\pi)^d} u^<_\beta(\hat{k}) \int_q q^{-y-4} \left\{ k_\alpha P^>_{n\beta}(\boldsymbol{q}) P^>_{m\alpha}(\boldsymbol{q}) + \frac{y+2}{2}\frac{k_j q_j}{q^2} q_\beta P^>_{mn}(\boldsymbol{q}) \right\} d\boldsymbol{q}.
\end{aligned}
\tag{A.75}
$$

The integration of the wave vector $\boldsymbol{q}$ is done by the following relations:

$$
\int d\boldsymbol{q} = \int d^d q = S_d \int q^{d-1} dq, \tag{A.76}
$$

$$\int q_\alpha q_\beta d^d q = \frac{S_d}{d}\,\delta_{\alpha\beta}\int q^{d+1}dq, \tag{A.77}$$

and

$$\int q_\alpha q_\beta q_\gamma q_\delta d^d q = \frac{S_d}{d(d+2)}\left(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}\right)\int q^{d+3}dq, \tag{A.78}$$

where S_d is the area of a d-dimensional unit sphere (i.e., $S_3 = 4\pi$). The first term of the r.h.s. in Eq. (A.75) becomes

$$\begin{aligned}
& P^{<}_{imn}(\boldsymbol{k})k_\alpha u^{<}_\beta(\hat{k})\int_q q^{-y-4}P^{>}_{n\beta}(\boldsymbol{q})P^{>}_{m\alpha}(\boldsymbol{q})d\boldsymbol{q} \\
= {} & P^{<}_{imn}(\boldsymbol{k})k_\alpha u^{<}_\beta(\hat{k})\int_q q^{-y-4}\left(\delta_{n\beta}-\frac{q_nq_\beta}{q^2}\right)\left(\delta_{m\alpha}-\frac{q_mq_\alpha}{q^2}\right)d\boldsymbol{q} \\
= {} & P^{<}_{imn}(\boldsymbol{k})k_\alpha u^{<}_\beta(\hat{k})\int_q q^{-y-4}\left(\delta_{n\beta}\delta_{m\alpha}-\frac{q_nq_\beta}{q^2}\,\delta_{m\alpha}-\frac{q_mq_\alpha}{q^2}\,\delta_{n\beta}+\frac{q_mq_nq_\alpha q_\beta}{q^4}\right)d\boldsymbol{q} \\
= {} & P^{<}_{imn}(\boldsymbol{k})k_\alpha u^{<}_\beta(\hat{k})S_d\left\{\delta_{m\alpha}\delta_{n\beta}-\frac{1}{d}\,\delta_{m\alpha}\delta_{n\beta}-\frac{1}{d}\,\delta_{m\alpha}\delta_{n\beta}\right. \\
& \left.+\frac{1}{d(d+2)}\left(\delta_{\alpha\beta}\delta_{mn}+\delta_{m\alpha}\delta_{n\beta}+\delta_{n\alpha}\delta_{m\beta}\right)\right\}\int_{\Lambda_0\exp(-r)}^{\Lambda_0} q^{-y+d-5}dq \\
= {} & P^{<}_{imn}(\boldsymbol{k})k_m u^{<}_n(\hat{k})S_d\left\{1-\frac{2}{d}+\frac{1}{d(d+2)}\right\}\int_{\Lambda_0\exp(-r)}^{\Lambda_0} q^{-y+d-5}dq \\
& +P^{<}_{imn}(\boldsymbol{k})\underbrace{k_\alpha u^{<}_\alpha(\hat{k})}_{0}\frac{S_d}{d(d+2)}\int_{\Lambda_0\exp(-r)}^{\Lambda_0} q^{-y+d-5}dq \\
& +P^{<}_{imn}(\boldsymbol{k})k_n u^{<}_m(\hat{k})\frac{S_d}{d(d+2)}\int_{\Lambda_0\exp(-r)}^{\Lambda_0} q^{-y+d-5}dq \\
= {} & P^{<}_{imn}(\boldsymbol{k})k_m u^{<}_n(\hat{k})S_d\left\{1-\frac{2}{d}+\frac{2}{d(d+2)}\right\}\int_{\Lambda_0\exp(-r)}^{\Lambda_0} q^{-y+d-5}dq,
\end{aligned} \tag{A.79}$$

where

$$\begin{aligned}
P_{imn}(\boldsymbol{k})k_m u_n(\hat{k}) &= k_m\left\{k_mP_{in}(\boldsymbol{k})+k_nP_{im}(\boldsymbol{k})\right\}u_n(\hat{k}) \\
&= k^2P_{in}(\boldsymbol{k})u_n(\hat{k})+\underbrace{k_mP_{im}(\boldsymbol{k})k_nu_n(\hat{k})}_{0}
\end{aligned}$$

$$\begin{aligned}
&= k^2\left(\delta_{in} - \frac{k_i k_n}{k^2}\right) u_n(\hat{k}) \\
&= k^2\delta_{in}u_n(\hat{k}) - \underbrace{k_i k_n u_n(\hat{k})}_{0} \\
&= k^2 u_i(\hat{k}),
\end{aligned} \tag{A.80}$$

$$\begin{aligned}
\int_{\Lambda_0\exp(-r)}^{\Lambda_0} q^{-y+d-5}dq &= \left[\frac{1}{-y+d-4}\,q^{-y+d-4}\right]_{\Lambda_0\exp(-r)}^{\Lambda_0} \\
&= \left[-\frac{1}{\epsilon}\,q^{-\epsilon}\right]_{\Lambda_0\exp(-r)}^{\Lambda_0} = \frac{1}{\epsilon\Lambda_0^\epsilon}\left\{\exp(\epsilon r)-1\right\}
\end{aligned} \tag{A.81}$$

and

$$\epsilon \equiv 4 + y - d. \tag{A.82}$$

The result is

$$\begin{aligned}
&P^<_{imn}(\boldsymbol{k})k_\alpha u^<_\beta(\hat{k}) \int_q q^{-y-4} P^>_{n\beta}(\boldsymbol{q})P^>_{m\alpha}(\boldsymbol{q})d\boldsymbol{q} \\
=\;& \frac{d^2-2}{d(d+2)}\,\frac{S_d\left\{\exp(\epsilon r)-1\right\}}{\epsilon\Lambda_0^\epsilon}\,k^2 u^<_i(\hat{k}).
\end{aligned} \tag{A.83}$$

Similarly, the second term is reduced to

$$\begin{aligned}
&P^<_{imn}(\boldsymbol{k})u^<_\beta(\hat{k}) \int_q \frac{y+2}{2}\,q^{-y-4}\,\frac{k_j q_j}{q^2} q_\beta P^>_{mn}(\boldsymbol{q})d\boldsymbol{q} \\
=\;& P^<_{imn}(\boldsymbol{k})u^<_\beta(\hat{k})\,\frac{y+2}{2}\,k_j \int_q q_j q_\beta q^{-y-6}\left(\delta_{mn} - \frac{q_m q_n}{q^2}\right)d\boldsymbol{q} \\
=\;& \frac{y+2}{2}\,P^<_{imn}(\boldsymbol{k})k_j u^<_\beta(\hat{k})\left\{\frac{1}{d}\,\delta_{mn}\delta_{j\beta} - \frac{1}{d(d+2)}\right. \\
&\left.\times\left(\delta_{mn}\delta_{j\beta} + \delta_{jm}\delta_{n\beta} + \delta_{jn}\delta_{m\beta}\right)\right\} S_d \int_{\Lambda_0\exp(-r)}^{\Lambda_0} q^{-y+d-5}dq \\
=\;& \frac{y+2}{2}\left\{\frac{1}{d} - \frac{1}{d(d+2)}\right\} P^<_{imn}(\boldsymbol{k})\,\underbrace{k_j u^<_j(\hat{k})}_{0}\,S_d \int_{\Lambda_0\exp(-r)}^{\Lambda_0} q^{-y+d-5}dq
\end{aligned}$$

$$
\begin{aligned}
& -\frac{y+2}{2}\frac{S_d}{d(d+2)}P^<_{imn}(\boldsymbol{k})k_m u^<_n(\hat{k})\int_{\Lambda_0\exp(-r)}^{\Lambda_0} q^{-y+d-5}dq \\
& -\frac{y+2}{2}\frac{S_d}{d(d+2)}P^<_{imn}(\boldsymbol{k})k_n u^<_m(\hat{k})\int_{\Lambda_0\exp(-r)}^{\Lambda_0} q^{-y+d-5}dq \\
= & -\frac{(y+2)S_d}{d(d+2)}P^<_{imn}(\boldsymbol{k})k_n u^<_m(\hat{k})\int_{\Lambda_0\exp(-r)}^{\Lambda_0} q^{-y+d-5}dq \\
= & -\frac{(y+2)S_d}{d(d+2)}\frac{\exp(\epsilon r)-1}{\epsilon\Lambda_0^\epsilon}k^2u^<_i(\hat{k}). \qquad \text{(A.84)}
\end{aligned}
$$

Finally, Eqs. (A.48) and (A.49) are formed as follows:

$$
\begin{aligned}
R_1+R_2 &= -\frac{\lambda_0^2 D_0}{2\nu_0^2(2\pi)^d}\frac{d^2-y-4}{d(d+2)}\frac{S_d\{\exp(\epsilon r)-1\}}{\epsilon\Lambda_0^\epsilon}k^2u^<_i(\hat{k}) \\
&= -\frac{\lambda_0^2 D_0}{2\nu_0^2(2\pi)^d}\frac{d^2-d-\epsilon}{d(d+2)}\frac{S_d\{\exp(\epsilon r)-1\}}{\epsilon\Lambda_0^\epsilon}k^2u^<_i(\hat{k}), \qquad \text{(A.85)}
\end{aligned}
$$

$$
\begin{aligned}
R_1+R_2 &\equiv -\lim_{\substack{k\to 0\\ \omega\to 0}}\Delta\nu(k)k^2u^<_i(\hat{k}) \\
&= -\Delta\nu(0)k^2u^<_i(\hat{k}), \qquad \text{(A.86)}
\end{aligned}
$$

$$\tilde{A}_d = \frac{1}{2}\frac{d^2-d-\epsilon}{d(d+2)}, \qquad \text{(A.87)}$$

$$A_d = \tilde{A}_d\frac{S_d}{(2\pi)^d}, \qquad \text{(A.88)}$$

and

$$\Delta\nu(0) = A_d\frac{\lambda_0^2 D_0}{\nu_0^2\Lambda_0^\epsilon}\frac{\exp(\epsilon r)-1}{\epsilon}. \qquad \text{(A.89)}$$

This term represents the effect of the eliminated components $\boldsymbol{u}^>$ and is added to the molecular (bare) viscosity as an increment.

$$
\begin{aligned}
\nu_r &\equiv \nu_0+\Delta\nu(0) \\
&= \nu_0 + A_d\frac{\lambda_0^2 D_0}{\nu_0^2\Lambda_0^\epsilon}\frac{\exp(\epsilon r)-1}{\epsilon} \\
&= \nu_0\left\{1+A_d\bar{\lambda}_0^2\frac{\exp(\epsilon r)-1}{\epsilon}\right\} \qquad \text{(A.90)}
\end{aligned}
$$

$$\bar{\lambda}_0^2 = \frac{\lambda_0^2 D_0}{\nu_0^3 \Lambda_0^\epsilon} \tag{A.91}$$

Then the renormalized Navier-Stokes equation becomes

$$\begin{aligned} u_i^<(\hat{k}) &= G_r(\hat{k})\left\{f_i^<(\hat{k}) + \Delta f_i\right\} \\ &\quad -\frac{i\lambda_0}{2} G_r(\hat{k}) P_{imn}^<(\boldsymbol{k}) \int_{q<\Lambda_0 \exp(-r)} u_m^<(\hat{q}) u_n^<(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\ &\quad + O[(u^<)^3], \end{aligned} \tag{A.92}$$

where

$$G_r(\hat{k}) \equiv \left(-i\omega + \nu_r k^2\right)^{-1} \tag{A.93}$$

is the renormalized Green function. The wave number k is defined in the range $\Lambda_e < k < \Lambda_0 \exp(-r)$.

A.3 Application to Scalar Field

The transport equation for a passive scalar T is

$$\frac{\partial T}{\partial \tau} + \lambda_0 u_i \frac{\partial T}{\partial x_i} = \chi_0 \frac{\partial^2 T}{\partial x_j \partial x_j}, \tag{A.94}$$

where $\lambda_0 (= 1)$ is an indicator to describe the effect of the nonlinear term in renormalization expansion. By using the Fourier integral representation:

$$T(\boldsymbol{x}, \tau) = \int_{k<\Lambda_0} \int_{-\infty}^{+\infty} T(\boldsymbol{k}, \omega) \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau) \frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}, \tag{A.95}$$

Equation (A.94) is transferred to Fourier space as follows:

$$\begin{aligned} \frac{\partial T}{\partial \tau} &= \frac{\partial}{\partial \tau} \int_{k<\Lambda_0} \int_{-\infty}^{+\infty} T(\boldsymbol{k}, \omega) \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau) \frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}} \\ &= \int_{k<\Lambda_0} \int_{-\infty}^{+\infty} \{-i\omega T(\boldsymbol{k}, \omega) \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\tau)\} \frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}, \end{aligned} \tag{A.96}$$

$$\lambda_0 u_i \frac{\partial T}{\partial x_i} = \lambda_0 \int_{q<\Lambda_0} \int_{-\infty}^{+\infty} u_i(\boldsymbol{q}, \Omega) \exp(i\boldsymbol{q}\cdot\boldsymbol{x} - i\Omega\tau) \frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}}$$

$$
\begin{aligned}
& \times\frac{\partial}{\partial x_i}\int_{k'<\Lambda_0}\int_{-\infty}^{+\infty} T(\boldsymbol{k}',\omega')\exp(i\boldsymbol{k}'\cdot\boldsymbol{x}-i\omega'\tau)\frac{d\omega' d\boldsymbol{k}'}{(2\pi)^{d+1}} \\
= {} & \lambda_0\int_{k'<\Lambda_0}\int_{-\infty}^{+\infty}\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} ik_i' u_i(\boldsymbol{q},\Omega)T(\boldsymbol{k}',\omega') \\
& \times\exp\left\{i(\boldsymbol{k}'+\boldsymbol{q})\cdot\boldsymbol{x}-i(\omega'+\Omega)\tau\right\}\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}}\frac{d\omega' d\boldsymbol{k}'}{(2\pi)^{d+1}} \\
= {} & \lambda_0\int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} i(k_i-q_i)u_i(\boldsymbol{q},\Omega)T(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega) \\
& \times\exp(i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega\tau)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}}\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}} \\
= {} & \lambda_0\int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\left\{\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} ik_i u_i(\boldsymbol{q},\Omega)T(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}}\right\} \\
& \times\exp(i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}},
\end{aligned}
\tag{A.97}
$$

$$
\begin{aligned}
\chi_0\frac{\partial^2 T}{\partial x_j\partial x_j} &= \frac{\partial^2}{\partial x_j\partial x_j}\int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\chi_0 T(\boldsymbol{k},\omega)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}} \\
&= -\int_{k<\Lambda_0}\int_{-\infty}^{+\infty}\chi_0 k^2 T(\boldsymbol{k},\omega)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega\tau)\frac{d\omega d\boldsymbol{k}}{(2\pi)^{d+1}}.
\end{aligned}
\tag{A.98}
$$

Then, the transport equation for the passive scalar T in Fourier space becomes

$$
\begin{aligned}
& -i\omega T(\boldsymbol{k},\omega)+i\lambda_0 k_i\int_{q<\Lambda_0}\int_{-\infty}^{+\infty} u_i(\boldsymbol{q},\Omega)T(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega)\frac{d\Omega d\boldsymbol{q}}{(2\pi)^{d+1}} \\
= {} & -\chi_0 k^2 T(\boldsymbol{k},\omega)
\end{aligned}
\tag{A.99}
$$

or

$$
T(\hat{k}) = -i\lambda_0 k_i g_0(\hat{k})\int_{q<\Lambda_0} u_i(\hat{q})T(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^{d+1}} \tag{A.100}
$$

with

$$
g_0(\boldsymbol{k},\omega)\equiv g_0(\hat{k}) = (-i\omega+\chi_0 k^2)^{-1}. \tag{A.101}
$$

Renormalization for the passive scalar is done in the same manner as for the velocity field. Firstly, the component T is also divided into two parts:

$$T(\hat{k}) = \begin{cases} T^{<}(\hat{k}) & : \ \Lambda_e < k < \Lambda_0 \exp(-r) \\ T^{>}(\hat{k}) & : \ \Lambda_0 \exp(-r) < k < \Lambda_0 \\ 0 & : \ \text{otherwise} \end{cases} . \tag{A.102}$$

The nonlinear term in Eq. (A.100) can be expanded as

$$\begin{aligned} \int_q u_i(\hat{q}) T(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} &= \int_q u_i^{<}(\hat{q}) T^{<}(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\ &+ \int_q u_i^{>}(\hat{q}) T^{<}(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\ &+ \int_q u_i^{<}(\hat{q}) T^{>}(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\ &+ \int_q u_i^{>}(\hat{q}) T^{>}(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}}. \end{aligned} \tag{A.103}$$

To eliminate the components defined in the range $\Lambda_0 \exp(-r) < k < \Lambda_0$, $\boldsymbol{u}^{>}$ and $T^{>}$ in Eq. (A.103) are substituted for their transport equations based on Eqs. (A.36) and (A.100). In the equation for $T^{<}$:

$$\begin{aligned} (-i\omega + \chi_0 k^2) T^{<}(\hat{k}) &= -i\lambda_0 k_i \int_q u_i^{<}(\hat{q}) T^{<}(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\ &\quad -i\lambda_0 k_i \int_q u_i^{>}(\hat{q}) T^{<}(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\ &\quad -i\lambda_0 k_i \int_q u_i^{<}(\hat{q}) T^{>}(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\ &\quad -i\lambda_0 k_i \int_q u_i^{>}(\hat{q}) T^{>}(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}}, \end{aligned} \tag{A.104}$$

the effect of the second and third terms in the r.h.s. is negligible compared to the other terms if the wave number of $\boldsymbol{u}^{<}$ and $T^{<}$ is quite a distance from that of $\boldsymbol{u}^{>}$ and $T^{>}$. Then, the fourth term is expanded as

$$-i\lambda_0 k_i \int_q u_i^{>}(\hat{q}) T^{>}(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}}$$

$$
\begin{aligned}
&= -i\lambda_0 k_i \int_q \left[G_0^>(\hat{q}) f_i^>(\hat{q}) - \frac{i\lambda_0}{2} G_0^>(\hat{q}) P_{imn}^>(\boldsymbol{q}) \int_s u_m(\hat{s}) u_n(\hat{q}-\hat{s}) \frac{d\hat{s}}{(2\pi)^{d+1}} \right] \\
&\times \left[-i\lambda_0 (k_l - q_l) g_0^>(\hat{k}-\hat{q}) \int_r u_l(\hat{r}) T(\hat{k}-\hat{q}-\hat{r}) \frac{d\hat{r}}{(2\pi)^{d+1}} \right] \frac{d\hat{q}}{(2\pi)^{d+1}} \\
&= -\lambda_0^2 k_i \int_q \int_r (k_l - q_l) G_0^>(\hat{q}) g_0^>(\hat{k}-\hat{q}) f_i^>(\hat{q}) \left\{ u_l^<(\hat{r}) T^<(\hat{k}-\hat{q}-\hat{r}) \right. \\
&+ G_0^>(\hat{r}) f_l^>(\hat{r}) T^<(\hat{k}-\hat{q}-\hat{r}) + u_l^<(\hat{r}) T^>(\hat{k}-\hat{q}-\hat{r}) \\
&+ \left. G_0^>(\hat{r})\ f_l^>(\hat{r}) T^>(\hat{k}-\hat{q}-\hat{r}) \right\} \frac{d\hat{r}\, d\hat{q}}{(2\pi)^{2d+2}} \\
&+ O(\lambda_0^3).
\end{aligned}
\tag{A.105}
$$

Averaging over the range $\Lambda_0 \exp(-r) < k < \Lambda_0$ is applied to the random force $\boldsymbol{f}^>$, and Eq. (A.105) becomes

$$
\begin{aligned}
& -\lambda_0^2 k_i \int_q \int_r (k_l - q_l) G_0^>(\hat{q}) g_0^>(\hat{k}-\hat{q}) \left\{ \langle f_i^>(\hat{q}) \rangle u_l^<(\hat{r}) T^<(\hat{k}-\hat{q}-\hat{r}) \right. \\
&+ G_0^>(\hat{r}) \langle f_i^>(\hat{q}) f_l^>(\hat{r}) \rangle T^<(\hat{k}-\hat{q}-\hat{r}) + \langle f_i^>(\hat{q}) \rangle u_l^<(\hat{r}) T^>(\hat{k}-\hat{q}-\hat{r}) \\
&+ \left. G_0^>(\hat{r}) \langle f_i^>(\hat{q}) f_l^>(\hat{r}) \rangle T^>(\hat{k}-\hat{q}-\hat{r}) \right\} \frac{d\hat{r}\, d\hat{q}}{(2\pi)^{2d+2}} \\
&+ O(\lambda_0^3) \\
&= -\lambda_0^2 k_i \int_q \int_r (k_l - q_l) G_0^>(\hat{q}) G_0^>(\hat{r}) g_0^>(\hat{k}-\hat{q}) \langle f_i^>(\hat{q}) f_l^>(\hat{r}) \rangle \\
&\times T^<(\hat{k}-\hat{q}-\hat{r}) \frac{d\hat{r}\, d\hat{q}}{(2\pi)^{2d+2}} + O(\lambda_0^3) \\
&= -\lambda_0^2 k_i \int_q \int_r (k_l - q_l) G_0^>(\hat{q}) G_0^>(\hat{r}) g_0^>(\hat{k}-\hat{q}) 2 D_0 q^{-y} P_{il}^>(\boldsymbol{q}) (2\pi)^{d+1} \\
&\times \delta(\hat{q}+\hat{r}) T^<(\hat{k}-\hat{q}-\hat{r}) \frac{d\hat{r}\, d\hat{q}}{(2\pi)^{2d+2}} \\
&+ O(\lambda_0^3)
\end{aligned}
$$

$$
\begin{aligned}
&= -\lambda_0^2 k_i \int_q (k_l - q_l)|G_0^>(\hat{q})|^2 g_0^>(\hat{k}-\hat{q}) 2D_0 q^{-y} P_{il}^>(\boldsymbol{q}) T^<(\hat{k}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\
&\quad + O(\lambda_0^3) \\
&= -2\lambda_0^2 D_0 k_i k_l T^<(\hat{k}) \int_q |G_0^>(\hat{q})|^2 g_0^>(\hat{k}-\hat{q}) q^{-y} P_{il}^>(\boldsymbol{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\
&\quad + O(\lambda_0^3). \qquad \text{(A.106)}
\end{aligned}
$$

Then, the transport equation for $T^<$ becomes

$$
\begin{aligned}
(-i\omega + \chi_0 k^2) T^<(\hat{k}) &= -i\lambda_0 k_i \int_q u_i^<(\hat{q}) T^<(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\
&\quad -2\lambda_0^2 D_0 k_i k_l T^<(\hat{k}) \int_q |G_0^>(\hat{q})|^2 g_0^>(\hat{k}-\hat{q}) q^{-y} P_{il}^>(\boldsymbol{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\
&\quad + O(\lambda_0^3). \qquad \text{(A.107)}
\end{aligned}
$$

If we define

$$
k^2 \Delta\chi(k) T^<(\hat{k}) \equiv 2\lambda_0^2 D_0 k_i k_l T^<(\hat{k}) \int_q |G_0^>(\hat{q})|^2 g_0^>(\hat{k}-\hat{q}) q^{-y} P_{il}^>(\boldsymbol{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \qquad \text{(A.108)}
$$

or

$$
k^2 \Delta\chi(k) = 2\lambda_0^2 D_0 k_i k_l \int_q |G_0^>(\hat{q})|^2 g_0^>(\hat{k}-\hat{q}) q^{-y} P_{il}^>(\boldsymbol{q}) \frac{d\hat{q}}{(2\pi)^{d+1}}, \qquad \text{(A.109)}
$$

the renormalized equation for the passive scalar becomes

$$
\left\{-i\omega + (\chi_0 + \Delta\chi) k^2\right\} T^<(\hat{k}) = -i\lambda_0 k_i \int_q u_i^<(\hat{q}) T^<(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}}. \qquad \text{(A.110)}
$$

The term $\Delta\chi(k)$ arises from eliminating the modes in the range $\Lambda_0 \exp(-r) < k < \Lambda_0$ and is added as an increment to the molecular diffusivity χ_0. Thereafter, the integration of $\hat{q}$ in Eq. (A.110) is carried out. The integration of the frequency Ω is

$$
\begin{aligned}
\int_{-\infty}^{+\infty} |G_0^>(\hat{q})|^2 g_0^>(\hat{k}-\hat{q}) d\Omega &= \int_{-\infty}^{+\infty} \frac{d\Omega}{(-i\Omega + \nu_0 q^2)(i\Omega + \nu_0 q^2)\left\{-i(\omega - \Omega) + \chi_0 |\boldsymbol{k} - \boldsymbol{q}|^2\right\}} \\
&= 2\pi i \left[\frac{\Omega - i\nu_0 q^2}{(-i\Omega + \nu_0 q^2)(i\Omega + \nu_0 q^2)\left\{-i(\omega - \Omega) + \chi_0 |\boldsymbol{k} - \boldsymbol{q}|^2\right\}}\right]_{\Omega = +i\nu_0 q^2}
\end{aligned}
$$

$$+\,2\pi i\left[\frac{\Omega-\omega-i\chi_0|\boldsymbol{k}-\boldsymbol{q}|^2}{(-i\Omega+\nu_0q^2)(i\Omega+\nu_0q^2)\left\{-i(\omega-\Omega)+\chi_0|\boldsymbol{k}-\boldsymbol{q}|^2\right\}}\right]_{\Omega=\omega+i\chi_0|k-q|^2}$$

$$= \frac{\pi}{\nu_0q^2(-i\omega-\nu_0q^2+\chi_0|\boldsymbol{k}-\boldsymbol{q}|^2)}$$

$$+\frac{2\pi}{(-i\omega+\nu_0q^2+\chi_0|\boldsymbol{k}-\boldsymbol{q}|^2)(i\omega+\nu_0q^2-\chi_0|\boldsymbol{k}-\boldsymbol{q}|^2)}$$

$$= \frac{\pi}{\nu_0q^2(-i\omega+\nu_0q^2+\chi_0|\boldsymbol{k}-\boldsymbol{q}|^2)}$$

$$\simeq \frac{\pi}{\nu_0q^2(\nu_0q^2+\chi_0|\boldsymbol{k}-\boldsymbol{q}|^2)}, \tag{A.111}$$

which yields

$$\Delta\chi(k)=\frac{\lambda_0^2D_0k_ik_l}{\nu_0(2\pi)^dk^2}\int_{\Lambda_0\exp(-r)<q<\Lambda_0}\frac{q^{-y-2}P_{il}^{>}(\boldsymbol{q})d\boldsymbol{q}}{\nu_0q^2+\chi_0|\boldsymbol{k}-\boldsymbol{q}|^2} \tag{A.112}$$

and

$$\Delta\chi(k) \simeq \frac{\lambda_0^2D_0k_ik_l}{\nu_0(2\pi)^dk^2}\int_{\Lambda_0\exp(-r)<q<\Lambda_0}\frac{q^{-y-2}P_{il}^{>}(\boldsymbol{q})d\boldsymbol{q}}{\nu_0q^2+\chi_0q^2}$$

$$= \frac{\lambda_0^2D_0}{\nu_0(2\pi)^dk^2}\frac{k_ik_l}{\nu_0+\chi_0}\int_{\Lambda_0\exp(-r)<q<\Lambda_0}q^{-y-4}P_{il}^{>}(\boldsymbol{q})d\boldsymbol{q} \tag{A.113}$$

in the limit $k\longrightarrow 0$, and the integration of the wave vector $\boldsymbol{q}$ is calculated as

$$\int_{\Lambda_0\exp(-r)<q<\Lambda_0}q^{-y-4}P_{il}^{>}(\boldsymbol{q})d\boldsymbol{q} = \int_{\Lambda_0\exp(-r)<q<\Lambda_0}q^{-y-4}\left(\delta_{il}-\frac{q_iq_l}{q^2}\right)d\boldsymbol{q}$$

$$= \delta_{il}\int_{\Lambda_0\exp(-r)<q<\Lambda_0}q^{-y-4}d\boldsymbol{q}-\int_{\Lambda_0\exp(-r)<q<\Lambda_0}q_iq_lq^{-y-6}d\boldsymbol{q}$$

$$= \left(1-\frac{1}{d}\right)S_d\delta_{il}\int_{\Lambda_0\exp(-r)}^{\Lambda_0}q^{-y+d-5}dq$$

$$= \left(1-\frac{1}{d}\right)S_d\delta_{il}\left[\frac{1}{-y+d-4}q^{-y+d-4}\right]_{\Lambda_0\exp(-r)}^{\Lambda_0}$$

$$= \left(1-\frac{1}{d}\right)S_d\delta_{il}\frac{\exp(\epsilon r)-1}{\epsilon\Lambda_0^{\epsilon}}, \tag{A.114}$$

where $\epsilon=4+y-d$. As a result, the increment to the molecular diffusivity becomes

$$\Delta\chi(k) = \frac{\lambda_0^2D_0}{\nu_0(2\pi)^d}\frac{(d-1)S_d}{d(\chi_0+\nu_0)}\frac{\exp(\epsilon r)-1}{\epsilon\Lambda_0^{\epsilon}}$$

$$= \frac{d-1}{d} K_d \frac{\bar{\lambda}_0^2 \nu_0^2}{\chi_0 + \nu_0} \frac{\exp(\epsilon r) - 1}{\epsilon}, \tag{A.115}$$

where

$$\bar{\lambda_0}^2 = \frac{D_0 \lambda_0^2}{\nu_0^3 \Lambda_0^\epsilon}, \tag{A.116}$$

$$K_d = \frac{S_d}{(2\pi)^d}. \tag{A.117}$$

The resultant renormalized equation for $T^<$ is

$$\left\{-i\omega + \chi(r)k^2\right\} T^<(\hat{k}) = -i\lambda_0 k_i \int_q u_i^<(\hat{q}) T^<(\hat{k} - \hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \tag{A.118}$$

with

$$\chi(r) = \chi_0 \left\{1 + \frac{d-1}{d} K_d \frac{\bar{\lambda}_0^2 \nu_0^2}{\chi_0 + \nu_0} \frac{\exp(\epsilon r) - 1}{\epsilon}\right\}. \tag{A.119}$$

The effective diffusivity $\chi(r)$ at each renormalization has the relation:

$$\begin{aligned}
\chi(\Lambda - \Delta\Lambda) &= \chi(\Lambda) + \Delta\chi(\Lambda) \\
&= \chi(\Lambda) \\
&\quad + \frac{d-1}{d} \frac{S_d}{(2\pi)^d} \frac{D_0 \lambda_0^2}{\nu(\Lambda)\{\chi(\Lambda) + \nu(\Lambda)\}\epsilon} \left\{\frac{1}{(\Lambda - \Delta\Lambda)^\epsilon} - \frac{1}{\Lambda^\epsilon}\right\}
\end{aligned} \tag{A.120}$$

so that the differential relation for $\chi(r)$ becomes

$$\begin{aligned}
\frac{d\chi(\Lambda)}{d\Lambda} &= \lim_{\Delta\Lambda \to 0} \frac{\chi(\Lambda - \Delta\Lambda) - \chi(\Lambda)}{(\Lambda - \Delta\Lambda) - \Lambda} \\
&= \lim_{\Delta\Lambda \to 0} \frac{\frac{d-1}{d} \frac{S_d}{(2\pi)^d} \frac{D_0 \lambda_0^2}{\nu(\Lambda)\{\nu(\Lambda) + \chi(\Lambda)\}\epsilon} \left\{\frac{1}{(\Lambda - \Delta\Lambda)^\epsilon} - \frac{1}{\Lambda^\epsilon}\right\}}{-\Delta\Lambda} \\
&= -\lim_{\Delta\Lambda \to 0} \frac{d-1}{d} \frac{S_d}{(2\pi)^d} \frac{D_0 \lambda_0^2}{\nu(\Lambda)\{\nu(\Lambda) + \chi(\Lambda)\}\epsilon} \frac{(\Lambda - \Delta\Lambda)^{-\epsilon} - \Lambda^{-\epsilon}}{\Delta\Lambda} \\
&= -\frac{d-1}{d} \frac{S_d}{(2\pi)^d} \frac{D_0 \lambda_0^2}{\nu(\Lambda)\{\nu(\Lambda) + \chi(\Lambda)\}\epsilon} \left[-\frac{d\Lambda^{-\epsilon}}{d\Lambda}\right] \\
&= -\frac{d-1}{d} \frac{S_d}{(2\pi)^d} \frac{D_0 \lambda_0^2}{\nu(\Lambda)\{\nu(\Lambda) + \chi(\Lambda)\}\Lambda^{\epsilon+1}},
\end{aligned} \tag{A.121}$$

and replacing Λ with $\Lambda_0 \exp(-r)$ gives

$$\frac{d\chi(r)}{dr} = \frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{D_0}{\nu(r)\Lambda_0^\epsilon}\frac{1}{\chi(r)+\nu(r)}\exp(\epsilon r) \tag{A.122}$$

or

$$\frac{d\chi(r)}{dr} = \frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{\bar{\lambda}^2(r)\nu^2(r)}{\chi(r)+\nu(r)}, \tag{A.123}$$

where

$$\bar{\lambda}^2(r) = \frac{\lambda_0^2 D_0}{\nu^3(r)\Lambda_0^\epsilon}\exp(\epsilon r). \tag{A.124}$$

Then, the new parameter α is defined as

$$\alpha(r) = \frac{\chi(r)}{\nu(r)} \tag{A.125}$$

with its differential equation:

$$\frac{d\alpha(r)}{dr} = \frac{1}{\nu(r)}\frac{d\chi(r)}{dr} - \frac{\chi(r)}{\nu^2(r)}\frac{d\nu(r)}{dr}. \tag{A.126}$$

Replacing $d\chi(r)/dr$ and $d\nu(r)/dr$ with their equations gives

$$\begin{aligned}
\frac{d\alpha(r)}{dr} &= \frac{1}{\nu(r)}\frac{d-1}{d}K_d\frac{\bar{\lambda}^2(r)\nu^2(r)}{\chi(r)+\nu(r)} - \alpha(r)A_d\bar{\lambda}^2(r) \\
&= \frac{d-1}{d}K_d\frac{S_d}{(2\pi)^d}\frac{\bar{\lambda}^2(r)}{1+\alpha(r)} - \alpha(r)A_d\bar{\lambda}^2(r) \\
&= A_d\bar{\lambda}^2(r)\left\{\frac{d-1}{d\tilde{A}_d}\frac{1}{1+\alpha(r)} - \alpha(r)\right\} \\
&= A_d\bar{\lambda}_0^2\exp(\epsilon r)\left\{\frac{d-1}{d\tilde{A}_d}\frac{1}{1+\alpha(r)} - \alpha(r)\right\}\left[1+\frac{3}{\epsilon}A_d\bar{\lambda}_0^2\left\{\exp(\epsilon r)-1\right\}\right]^{-1} \\
&= A_d\bar{\lambda}_0^2\exp(\epsilon r)\left[1+\frac{3}{\epsilon}A_d\bar{\lambda}_0^2\left\{\exp(\epsilon r)-1\right\}\right]^{-1}\frac{1}{d\tilde{A}_d\{1+\alpha(r)\}} \\
&\quad \times\left[d-1-\alpha(r)\{1+\alpha(r)\}d\tilde{A}_d\right]
\end{aligned} \tag{A.127}$$

and

$$\frac{d\tilde{A}_d(1+\alpha)}{d-1-d\tilde{A}_d\alpha(1+\alpha)}d\alpha = \frac{A_d\bar{\lambda}_0^2\exp(\epsilon r)}{1+\frac{3}{\epsilon}A_d\bar{\lambda}_0^2\left\{\exp(\epsilon r)-1\right\}}dr. \tag{A.128}$$

The integration with the initial condition $\alpha(0) = \chi_0/\nu_0$ becomes

$$\int_{\alpha_0}^{\alpha(r)} \frac{d\tilde{A}_d(1+\alpha)}{d-1-d\tilde{A}_d(\alpha^2+\alpha)} d\alpha = \int_0^r \frac{A_d\bar{\lambda}_0^2 \exp(\epsilon r)}{1+\frac{3}{\epsilon}A_d\bar{\lambda}_0^2\{\exp(\epsilon r)-1\}} dr \tag{A.129}$$

or

$$-\int_{\alpha_0}^{\alpha(r)} \frac{1}{a+b}\left(\frac{a+1}{\alpha-a}+\frac{b-1}{\alpha+b}\right) d\alpha = \int_0^r \frac{A_d\bar{\lambda}_0^2 \exp(\epsilon r)}{1+\frac{3}{\epsilon}A_d\bar{\lambda}_0^2\{\exp(\epsilon r)-1\}} dr \tag{A.130}$$

with

$$a = \frac{1}{2}\left(-1+\sqrt{1+4\frac{d-1}{d\tilde{A}_d}}\right) \tag{A.131}$$

$$b = \frac{1}{2}\left(1+\sqrt{1+4\frac{d-1}{d\tilde{A}_d}}\right). \tag{A.132}$$

The result is

$$\begin{aligned} & -\frac{1}{a+b}\left\{(a+1)\ln\left|\frac{\alpha-a}{\alpha_0-a}\right| + (b-1)\ln\left|\frac{\alpha+b}{\alpha_0+b}\right|\right\} \\ = \; & \frac{1}{3}\ln\left[1+\frac{3}{\epsilon}A_d\bar{\lambda}_0^2\{\exp(\epsilon r)-1\}\right] \end{aligned} \tag{A.133}$$

or

$$\begin{aligned} \left|\frac{\alpha-a}{\alpha_0-a}\right|^{\frac{a+1}{a+b}} \left|\frac{\alpha+b}{\alpha_0+b}\right|^{\frac{b-1}{a+b}} &= \frac{1}{\left[1+\frac{3}{\epsilon}A_d\bar{\lambda}_0^2\{\exp(\epsilon r)-1\}\right]^{\frac{1}{3}}} \\ &= \frac{\nu_0}{\nu(r)}. \end{aligned} \tag{A.134}$$

Appendix B

CALCULATIONS ON ITERATIVE AVERAGING

B.1 Fourier Transformation of the Navier-Stokes Equ tion

Here we begin by transforming the equation of motion to Fourier space. Using Eqs. (3.3) and (3.4), we can represent Eq. (3.11) as the Fourier series:

$$\sum_{k \geq \Lambda_e} \Bigg\{ \frac{\partial}{\partial \tau} u_i(\boldsymbol{k}, \tau) + ik_j \sum_q \overline{U}_i(\boldsymbol{q}, \tau) u_j(\boldsymbol{k}-\boldsymbol{q}, \tau)$$

$$+ik_j \sum_q \overline{U}_j(\boldsymbol{q}, \tau) u_i(\boldsymbol{k}-\boldsymbol{q}, \tau) + ik_j \sum_q u_i(\boldsymbol{q}, \tau) u_j(\boldsymbol{k}-\boldsymbol{q}, \tau)$$

$$+ \frac{ik_i}{\rho} p(\boldsymbol{k}, \tau) + \nu_0 k^2 u_i(\boldsymbol{k}, \tau) \Bigg\} \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) = 0. \tag{B.1}$$

Then, in order to eliminate the pressure $p(\boldsymbol{k}, \tau)$, we make use of the Poisson equation:

$$\frac{\partial u_m}{\partial x_n} \frac{\partial \overline{U}_n}{\partial x_m} + \frac{\partial u_n}{\partial x_m} \frac{\partial \overline{U}_m}{\partial x_n} = -\frac{1}{\rho} \frac{\partial^2 p}{\partial x_m \partial x_m} - \frac{\partial^2}{\partial x_m \partial x_n} \left(u_m u_n - \overline{u_m u_n} \right), \tag{B.2}$$

which can be transformed to Fourier space as follows:

$$\frac{1}{\rho} p(\boldsymbol{k}, \tau) = -\frac{k_m k_n}{k^2} \Bigg[\sum_q \Big\{ \overline{U}_n(\boldsymbol{q}, \tau) u_m(\boldsymbol{k}-\boldsymbol{q}, \tau) + \overline{U}_m(\boldsymbol{q}, \tau) u_n(\boldsymbol{k}-\boldsymbol{q}, \tau)$$

$$+u_m(\boldsymbol{q},\tau)u_n(\boldsymbol{k}-\boldsymbol{q},\tau)\}\Big]. \tag{B.3}$$

Note that $\overline{u_m u_n}$ does not appear in Eq. (B.3) because the wave number k is defined in the range $k \geq \Lambda_e$. Hence, Eq. (B.1) is reformed as follows:

$$\sum_{k\geq\Lambda_e}\Bigg[\frac{\partial}{\partial\tau}u_i(\boldsymbol{k},\tau)+\underbrace{ik_j\sum_q\Big\{\overline{U}_i(\boldsymbol{q},\tau)u_j(\boldsymbol{k}-\boldsymbol{q},\tau)+\overline{U}_j(\boldsymbol{q},\tau)u_i(\boldsymbol{k}-\boldsymbol{q},\tau)}_{*}$$

$$\underbrace{+u_i(\boldsymbol{q},\tau)u_j(\boldsymbol{k}-\boldsymbol{q},\tau)\}+\frac{ik_i}{\rho}\,p(\boldsymbol{k},\tau)}_{*}+\nu_0k^2u_i(\boldsymbol{k},\tau)\Bigg]\exp(\boldsymbol{k}\cdot\boldsymbol{x})=0, \tag{B.4}$$

where the terms marked with * become

$$ik_j\sum_q\Big\{\overline{U}_i(\boldsymbol{q},\tau)u_j(\boldsymbol{k}-\boldsymbol{q},\tau)+\overline{U}_j(\boldsymbol{q},\tau)u_i(\boldsymbol{k}-\boldsymbol{q},\tau)+u_i(\boldsymbol{q},\tau)u_j(\boldsymbol{k}-\boldsymbol{q},\tau)\Big\}$$

$$+ik_i\Bigg[-\frac{k_mk_n}{k^2}\sum_q\Big\{\overline{U}_m(\boldsymbol{q},\tau)u_n(\boldsymbol{k}-\boldsymbol{q},\tau)+\overline{U}_n(\boldsymbol{q},\tau)u_m(\boldsymbol{k}-\boldsymbol{q},\tau)$$

$$+\,u_m(\boldsymbol{q},\tau)u_n(\boldsymbol{k}-\boldsymbol{q},\tau)\}\Bigg]$$

$$=\ i\sum_q\left\{k_j\overline{U}_i(\boldsymbol{q},\tau)u_j(\boldsymbol{k}-\boldsymbol{q},\tau)-\frac{k_ik_mk_n}{k^2}\overline{U}_m(\boldsymbol{q},\tau)u_n(\boldsymbol{k}-\boldsymbol{q},\tau)\right\}$$

$$+i\sum_q\left\{k_j\overline{U}_j(\boldsymbol{q},\tau)u_i(\boldsymbol{k}-\boldsymbol{q},\tau)-\frac{k_ik_mk_n}{k^2}\overline{U}_n(\boldsymbol{q},\tau)u_m(\boldsymbol{k}-\boldsymbol{q},\tau)\right\}$$

$$+i\sum_q\left\{k_ju_i(\boldsymbol{q},\tau)u_j(\boldsymbol{k}-\boldsymbol{q},\tau)-\frac{k_ik_mk_n}{k^2}u_m(\boldsymbol{q},\tau)u_n(\boldsymbol{k}-\boldsymbol{q},\tau)\right\}. \tag{B.5}$$

Then, each term is reformed by turns:

$$k_j\overline{U}_i(\boldsymbol{q},\tau)u_j(\boldsymbol{k}-\boldsymbol{q},\tau)-\frac{k_ik_mk_n}{k^2}\overline{U}_m(\boldsymbol{q},\tau)u_n(\boldsymbol{k}-\boldsymbol{q},\tau)$$

$$=\ k_n\delta_{im}\overline{U}_m(\boldsymbol{q},\tau)u_n(\boldsymbol{k}-\boldsymbol{q},\tau)-\frac{k_ik_mk_n}{k^2}\overline{U}_m(\boldsymbol{q},\tau)u_n(\boldsymbol{k}-\boldsymbol{q},\tau)$$

$$=\ k_nP_{im}(\boldsymbol{k})\overline{U}_m(\boldsymbol{q},\tau)u_n(\boldsymbol{k}-\boldsymbol{q},\tau) \tag{B.6}$$

$$
\begin{aligned}
& k_j \overline{U}_j(\boldsymbol{q},\tau) u_i(\boldsymbol{k}-\boldsymbol{q},\tau) - \frac{k_i k_m k_n}{k^2} \overline{U}_n(\boldsymbol{q},\tau) u_m(\boldsymbol{k}-\boldsymbol{q},\tau) \\
= \; & k_n P_{im}(\boldsymbol{k}) \overline{U}_n(\boldsymbol{q},\tau) u_m(\boldsymbol{k}-\boldsymbol{q},\tau) \\
= \; & k_m P_{in}(\boldsymbol{k}) \overline{U}_m(\boldsymbol{q},\tau) u_n(\boldsymbol{k}-\boldsymbol{q},\tau)
\end{aligned} \tag{B.7}
$$

$$
\begin{aligned}
& k_j u_i(\boldsymbol{q},\tau) u_j(\boldsymbol{k}-\boldsymbol{q},\tau) - \frac{k_i k_m k_n}{k^2} u_n(\boldsymbol{q},\tau) u_m(\boldsymbol{k}-\boldsymbol{q},\tau) \\
= \; & k_n P_{im}(\boldsymbol{k}) u_m(\boldsymbol{q},\tau) u_n(\boldsymbol{k}-\boldsymbol{q},\tau) \\
= \; & \frac{1}{2} P_{imn}(\boldsymbol{k}) u_m(\boldsymbol{q},\tau) u_n(\boldsymbol{k}-\boldsymbol{q},\tau).
\end{aligned} \tag{B.8}
$$

As a result, transforming Eq. (3.11) to Fourier space yields

$$
\begin{aligned}
\left(\frac{\partial}{\partial\tau} + \nu_0 k^2\right) u_i(\boldsymbol{k},\tau) \; = \; & -\frac{i}{2} P_{imn}(\boldsymbol{k}) \sum_q u_m(\boldsymbol{k},\tau) u_n(\boldsymbol{k}-\boldsymbol{q},\tau) \\
& - i P_{imn}(\boldsymbol{k}) \sum_q \overline{U}_m(\boldsymbol{q},\tau) u_n(\boldsymbol{k}-\boldsymbol{q},\tau),
\end{aligned} \tag{B.9}
$$

for $k \geq \Lambda_e$.

B.2 Iterative Averaging for Velocity Field

By replacing the term $\langle \boldsymbol{u}^> \boldsymbol{u}^> \rangle_c$ with its transport equation, we expand the fourth term of the r.h.s. in Eq. (3.23) up to $O\left(\overline{U}^1\right)$. As a result, many terms appear, most of which are neglected under conditional averaging in the band $\Lambda_1 \leq q < \Lambda_0$.

First, after replacing $\boldsymbol{u}^> \boldsymbol{u}^>$ with its transport equation, the term $\langle \boldsymbol{u}^> \boldsymbol{u}^> \rangle_c$ becomes

$$
\begin{aligned}
& -i \sum_q \sum_r \left(\frac{\partial}{\partial\tau} + \nu_0 q^2 + \nu_0 \left|\boldsymbol{k}-\boldsymbol{q}\right|^2 \right)^{-1} \\
& \times \left\{ P^>_{jmn}(\boldsymbol{k}-\boldsymbol{q}) \overline{U}_m(\boldsymbol{r},\tau) \left\langle u_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau) u^>_i(\boldsymbol{q},\tau) \right\rangle_c \right. \\
& + P^>_{imn}(\boldsymbol{q}) \overline{U}_m(\boldsymbol{r},\tau) \left\langle u_n(\boldsymbol{q}-\boldsymbol{r},\tau) u^>_j(\boldsymbol{k}-\boldsymbol{q},\tau) \right\rangle_c
\end{aligned}
$$

$$+\frac{1}{2}\,P^{>}_{jmn}(\boldsymbol{k}-\boldsymbol{q})\,\langle u_m(\boldsymbol{r},\tau)u_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_i(\boldsymbol{q},\tau)\rangle_c$$

$$+\,\frac{1}{2}\,P^{>}_{imn}(\boldsymbol{q})\left\langle u_m(\boldsymbol{r},\tau)u_n(\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_j(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c\Big\}. \tag{B.10}$$

Here, by applying the decomposition, Eq. (3.21), to the components $\boldsymbol{u}$ in Eq. (B.10), the terms in the bracket $\{\quad\}$ yield

$$P^{>}_{jmn}(\boldsymbol{k}-\boldsymbol{q})\,\{\langle u^{<}_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_i(\boldsymbol{q},\tau)\rangle_c$$

$$+\,\langle u^{>}_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_i(\boldsymbol{q},\tau)\rangle_c\}\,\overline{U}_m(\boldsymbol{r},\tau)$$

$$+P^{>}_{imn}(\boldsymbol{q})\left\{\left\langle u^{<}_n(\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_j(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c\right.$$

$$\left.+\,\left\langle u^{>}_n(\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_j(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c\right\}\overline{U}_m(\boldsymbol{r},\tau)$$

$$+\frac{1}{2}\,P^{>}_{jmn}(\boldsymbol{k}-\boldsymbol{q})\,\{\langle u^{<}_m(\boldsymbol{r},\tau)u^{<}_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_i(\boldsymbol{q},\tau)\rangle_c$$

$$+2\,\langle u^{<}_m(\boldsymbol{r},\tau)u^{>}_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_i(\boldsymbol{q},\tau)\rangle_c$$

$$+\,\langle u^{>}_m(\boldsymbol{r},\tau)u^{>}_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_i(\boldsymbol{q},\tau)\rangle_c\}$$

$$+\frac{1}{2}\,P^{>}_{imn}(\boldsymbol{q})\left\{\left\langle u^{<}_m(\boldsymbol{r},\tau)u^{<}_n(\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_j(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c\right.$$

$$+2\left\langle u^{<}_m(\boldsymbol{r},\tau)u^{>}_n(\boldsymbol{q}-\boldsymbol{r},\tau)u^{>}_j(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c$$

$$\left.+\left\langle u^{>}_m(\boldsymbol{r},\tau)u^{>}_n(\boldsymbol{q}-\boldsymbol{r},\tau)\;u^{>}_j(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c\right\}. \tag{B.11}$$

From the results of Eqs. (3.30) and (3.31), the term $\langle \boldsymbol{u}^{>}\boldsymbol{u}^{<}\rangle_c$ can be neglected. We also apply the relation given by Eq. (3.25) to the estimation of the terms $\langle \boldsymbol{u}^{<}\boldsymbol{u}^{<}\boldsymbol{u}^{>}\rangle_c$ and $\langle \boldsymbol{u}^{>}\boldsymbol{u}^{>}\boldsymbol{u}^{<}\rangle_c$; thus, these terms are assumed to be

$$\langle \boldsymbol{u}^{<}\boldsymbol{u}^{<}\boldsymbol{u}^{>}\rangle_c \simeq \boldsymbol{u}^{<}\boldsymbol{u}^{<}\,\langle \boldsymbol{v}^{>}\rangle_c \tag{B.12}$$

and

$$\langle \boldsymbol{u}^{>}\boldsymbol{u}^{>}\boldsymbol{u}^{<}\rangle_c \simeq \langle \boldsymbol{v}^{>}\boldsymbol{v}^{>}\rangle_c\,\boldsymbol{u}^{<} \tag{B.13}$$

under the condition that the effect of the coupling function $\Delta^>$ is small. Eventually, by applying the relation (3.32) to the term $\langle \boldsymbol{v}^> \boldsymbol{v}^> \rangle_c$, Eq. (B.10) becomes

$$
\begin{aligned}
&-i\sum_q \left(\frac{2\pi}{L}\right)^3 \left(\frac{\partial}{\partial \tau} + \nu_0 q^2 + \nu_0 |\boldsymbol{k}-\boldsymbol{q}|^2\right)^{-1} \\
&\times \left\{ P^>_{jmn}(\boldsymbol{k}-\boldsymbol{q}) \langle v^>_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau) v^>_i(\boldsymbol{q},\tau)\rangle_c \overline{U}_m(\boldsymbol{r},\tau) \right. \\
&+P^>_{imn}(\boldsymbol{q}) \left\langle v^>_n(\boldsymbol{q}-\boldsymbol{r},\tau) v^>_j(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c \overline{U}_m(\boldsymbol{r},\tau) \\
&+P^>_{jmn}(\boldsymbol{k}-\boldsymbol{q}) \langle v^>_n(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau) v^>_i(\boldsymbol{q},\tau)\rangle_c u^<_m(\boldsymbol{r},\tau) \\
&\left. +P^>_{imn}(\boldsymbol{q}) \left\langle v^>_n(\boldsymbol{q}-\boldsymbol{r},\tau) v^>_j(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c u^<_m(\boldsymbol{r},\tau) \right\} \\
&= -i\sum_q \left(\frac{2\pi}{L}\right)^3 \left(\frac{\partial}{\partial \tau} + \nu_0 q^2 + \nu_0 |\boldsymbol{k}-\boldsymbol{q}|^2\right)^{-1} \\
&\times \left\{ P^>_{jmn}(\boldsymbol{k}-\boldsymbol{q}) P^>_{in}(\boldsymbol{q}) Q^>_v(q) \right. \\
&\left. +P^>_{imn}(\boldsymbol{q}) P^>_{jn}(\boldsymbol{k}-\boldsymbol{q}) Q^>_v(|\boldsymbol{k}-\boldsymbol{q}|) \right\} \overline{U}_m(\boldsymbol{k},\tau) \\
&-i\sum_q \left(\frac{2\pi}{L}\right)^3 \left(\frac{\partial}{\partial \tau} + \nu_0 q^2 + \nu_0 |\boldsymbol{k}-\boldsymbol{q}|^2\right)^{-1} \\
&\times \left\{ P^>_{jmn}(\boldsymbol{k}-\boldsymbol{q}) P^>_{in}(\boldsymbol{q}) Q^>_v(q) \right. \\
&\left. +P^>_{imn}(\boldsymbol{q}) P^>_{jn}(\boldsymbol{k}-\boldsymbol{q}) Q^>_v(|\boldsymbol{k}-\boldsymbol{q}|) \right\} u^<_m(\boldsymbol{k},\tau)
\end{aligned}
\tag{B.14}
$$

Note that, in ordinary turbulent shear flows, the mean velocity $\overline{U}_m$ is much larger than the corresponding fluctuating component $u^<_m$. Thus, it is the $\overline{U}_m$-related term that governs Eq. (B.14).

B.3 Integration of Wave Vector

Now we begin by integrating Eq. (3.38) over the wave vector $\boldsymbol{q}$ in the band $\Lambda_1 \leq q < \Lambda_0$. After taking the limit $L \to \infty$, Eq. (3.38) becomes

$$
\begin{aligned}
& -i \lim_{L\to\infty} \sum_{q} \left(\frac{2\pi}{L}\right)^3 \frac{1}{2\nu_0 q^2} \left\{P^{>}_{jmn}(\boldsymbol{k}-\boldsymbol{q})P^{>}_{in}(\boldsymbol{q}) + P^{>}_{imn}(\boldsymbol{q})P^{>}_{jn}(\boldsymbol{k}-\boldsymbol{q})\right\} \\
& \times Q^{>}(q)\,\overline{U}_m(\boldsymbol{k},\tau) \\
= & -\frac{i}{2\nu_0}\int_q \frac{E(q)}{4\pi q^4}\left\{P^{>}_{jmn}(\boldsymbol{k}-\boldsymbol{q})P^{>}_{in}(\boldsymbol{q}) + P^{>}_{imn}(\boldsymbol{q})P^{>}_{jn}(\boldsymbol{k}-\boldsymbol{q})\right\} d\boldsymbol{q}\,\overline{U}_m(\boldsymbol{k},\tau) \\
= & -\frac{i}{2\nu_0}\int_q \frac{E(q)}{4\pi q^4}\left\{k_i\delta_{jm} + k_j\delta_{im} - \frac{k_i q_j q_m}{q^2} - \frac{k_j q_i q_m}{q^2}\right. \\
& \left. -\frac{k_n q_i q_n}{q^2}\delta_{im} - \frac{k_n q_j q_n}{q^2}\delta_{jm} + 2\frac{k_n q_i q_j q_m q_n}{q^4} + O(k^2)\right\} d\boldsymbol{q}\,\overline{U}_m(\boldsymbol{k},\tau),
\end{aligned}
\tag{B.15}
$$

where $E(q)[= 4\pi q^2 Q^{>}(q)]$ denotes the energy spectrum for $\Lambda_1 \leq q < \Lambda_0$. Next, we introduce the following standard identities for transforming the integration from the wave vector $\boldsymbol{q}$ to the wave number q:

$$\int d\boldsymbol{q} = S_d \int q^{d-1} dq \tag{B.16}$$

$$\int q_\alpha q_\beta d\boldsymbol{q} = \frac{S_d}{d}\,\delta_{\alpha\beta} \int q^{d+1} dq \tag{B.17}$$

$$\int q_\alpha q_\beta q_\gamma q_\delta d\boldsymbol{q} = \frac{S_d}{d(d+2)}\,(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \int q^{d+3} dq, \tag{B.18}$$

where S_d is the area of a d-dimensional unit sphere ($S_3 = 4\pi$). Hence, the result of the integration of Eq. (B.15) with respect to the wave vector $\boldsymbol{q}$ becomes

$$
\begin{aligned}
& -\frac{i}{2\nu_0}\int_{\Lambda_1}^{\Lambda_0} \frac{E(q)}{q^2}\left\{1 - \frac{2}{d} + \frac{2}{d(d+2)}\right\}(k_i\delta_{jm} + k_j\delta_{im})\,dq\,\overline{U}_m(\boldsymbol{k},\tau) \\
& = -\frac{7}{30\nu_0}\int_{\Lambda_1}^{\Lambda_0} \frac{E(q)}{q^2}(ik_i\delta_{jm} + ik_j\delta_{im})\,dq\,\overline{U}_m(\boldsymbol{k},\tau) \\
& = -\nu_1\left\{ik_i\overline{U}_j(\boldsymbol{k},\tau) + ik_j\overline{U}_i(\boldsymbol{k},\tau)\right\}.
\end{aligned}
\tag{B.19}
$$

B.4 Iterative Averaging for Thermal Field

For the second and third terms of the r.h.s. in Eq. (3.81), there is an inevitable problem in carrying out the conditional average: the $\boldsymbol{u}^{>}$ mode in real turbulent flow is always not independent of the $\boldsymbol{u}^{<}$ mode because of the nonlinear term in the Navier-Stokes equation, i.e., performing the partial average of the $\boldsymbol{u}^{>}$ mode does affect at least the property of $\boldsymbol{u}^{<}$ mode. Hence, to avoid this problem, we begin by dividing the $\boldsymbol{u}^{>}$ mode into two components with respect to another velocity field $\boldsymbol{v}^{>}$:

$$u_i^{>}(\boldsymbol{q},\tau) = v_i^{>}(\boldsymbol{q},\tau) + \Delta_i^{>}(\boldsymbol{q},\tau), \tag{B.20}$$

where $v_i^{>}$ is a velocity field of the same type as $\boldsymbol{u}^{>}$ except that it is never coupled to the $\boldsymbol{u}^{>}$ modes. Thus, the properties of $\boldsymbol{v}^{>}$ under total averaging, denoted by $\langle \quad \rangle$, are the same as those of $\boldsymbol{u}^{>}$:

$$\langle v_i^{>}(\boldsymbol{q},\tau)\rangle = \langle u_i^{>}(\boldsymbol{q},\tau)\rangle \tag{B.21}$$

and

$$\left\langle v_i^{>}(\boldsymbol{q},\tau)v_j^{>}(\boldsymbol{q}',\tau)\right\rangle = \left\langle u_i^{>}(\boldsymbol{q},\tau)u_j^{>}(\boldsymbol{q}',\tau)\right\rangle. \tag{B.22}$$

On the other hand, the function $\Delta^{>}$ represents the part coupled to the $\boldsymbol{u}^{<}$ modes in the $\boldsymbol{u}^{>}$ modes, the properties of which under total averaging are

$$\langle \Delta_i^{>}(\boldsymbol{q},\tau)\rangle = 0 \tag{B.23}$$

and

$$\left\langle \Delta_i^{>}(\boldsymbol{q},\tau)\Delta_j^{>}(\boldsymbol{q}',\tau)\right\rangle = 0. \tag{B.24}$$

Then, the second term can be evaluated as

$$\begin{aligned}
-\sum_q \langle u_i^{>}(\boldsymbol{q},\tau)t^{<}(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c &= -\sum_q \underbrace{\langle v_i^{>}(\boldsymbol{q},\tau)\rangle_c}_{0} t^{<}(\boldsymbol{k}-\boldsymbol{q},\tau) - \sum_q \langle \Delta_i^{>}(\boldsymbol{q},\tau)\rangle_c t^{<}(\boldsymbol{k}-\boldsymbol{q},\tau) \\
&= -\sum_q \langle \Delta_i^{>}(\boldsymbol{q},\tau)\rangle_c t^{<}(\boldsymbol{k}-\boldsymbol{q},\tau) \\
&\simeq O(\lambda^m) \quad (m \geq 1),
\end{aligned} \tag{B.25}$$

which is considered to be negligiby small as compared to the first and fourth terms in Eq. (3.81) under the condition that the band $\Lambda_1 \le q < \Lambda_0$, or the band width parameter $\lambda[= (\Lambda_0 - \Lambda_1)/\Lambda_0]$, is sufficiently small (Itazu & Nagano 1997b; Nagano & Itazu 1997b).

The higher wave-number mode $t^>$ in the third term is also divided into two counterparts under the statistical scale-separation assumption:

$$t^>(\boldsymbol{k}-\boldsymbol{q},\tau) = \theta^>(\boldsymbol{k}-\boldsymbol{q},\tau) + \Delta_t^>(\boldsymbol{k}-\boldsymbol{q},\tau), \tag{B.26}$$

so that the third term becomes

$$\begin{aligned}
-\sum_q \langle u_i^<(\boldsymbol{q},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c &= -\sum_q u_i^<(\boldsymbol{q},\tau)\underbrace{\langle\theta^>(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c}_{0} - \sum_q u_i^<(\boldsymbol{q},\tau)\langle\Delta_t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c \\
&= -\sum_q u_i^<(\boldsymbol{q},\tau)\langle\Delta_t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c \\
&\simeq O(\lambda^m)\ \ (m \ge 1),
\end{aligned} \tag{B.27}$$

which may be the same order of the second term and is considered to be negligibly small.

B.5 Renormalization Expansion

For the second term of the r.h.s. in Eq. (3.83), we replace the correlation $u_i^>(\boldsymbol{q},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)$ with its transport equation [Eq. (3.87)]:

$$\begin{aligned}
-\sum_q \langle u_i^>(\boldsymbol{q},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c &= i\sum_q\sum_r \left(\frac{\partial}{\partial\tau} + \nu_0 q^2 + \alpha_0|\boldsymbol{k}-\boldsymbol{q}|^2\right)^{-1} \\
&\times \Big[(k_j - q_j)\Big\{\langle u_i^>(\boldsymbol{q},\tau)u_j(\boldsymbol{r},\tau)\rangle_c \overline{T}(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau) \\
&+ \overline{U}_j(\boldsymbol{r},\tau)\langle t(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u_i^>(\boldsymbol{q},\tau)\rangle_c \\
&+ \langle u_j(\boldsymbol{r},\tau)t(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u_i^>(\boldsymbol{q},\tau)\rangle_c\Big\} \\
&+ P_{imn}^>(\boldsymbol{q})\Big\{\frac{1}{2}\langle u_m(\boldsymbol{r},\tau)u_n(\boldsymbol{q}-\boldsymbol{r},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c \\
&+ \overline{U}_m(\boldsymbol{r},\tau)\langle u_n(\boldsymbol{q}-\boldsymbol{r},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c\Big\}\Big].
\end{aligned} \tag{B.28}$$

Applying Eqs. (3.79) and (3.80) to the fluctuations ($\boldsymbol{u}$ and t) in the r.h.s. of Eq. (B.28) yields

$$
\begin{aligned}
& i\sum_{q}\sum_{r}\left(\frac{\partial}{\partial\tau}+\nu_0 q^2+\alpha_0|\boldsymbol{k}-\boldsymbol{q}|^2\right)^{-1}\Big[(k_j-q_j)\Big\{\left\langle u_i^>(\boldsymbol{q},\tau)u_j^<(\boldsymbol{r},\tau)\right\rangle_c\overline{T}(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)\\
&+\left\langle u_i^>(\boldsymbol{q},\tau)u_j^>(\boldsymbol{r},\tau)\right\rangle_c\overline{T}(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)+\overline{U}_j(\boldsymbol{r},\tau)\left\langle t^<(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u_i^>(\boldsymbol{q},\tau)\right\rangle_c\\
&+\overline{U}_j(\boldsymbol{r},\tau)\left\langle t^>(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u_i^>(\boldsymbol{q},\tau)\right\rangle_c+\left\langle u_j^<(\boldsymbol{r},\tau)t^<(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u_i^>(\boldsymbol{q},\tau)\right\rangle_c\\
&+\left\langle u_j^<(\boldsymbol{r},\tau)t^>(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u_i^>(\boldsymbol{q},\tau)\right\rangle_c+\left\langle u_j^>(\boldsymbol{r},\tau)t^<(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u_i^>(\boldsymbol{q},\tau)\right\rangle_c\\
&+\left\langle u_j^>(\boldsymbol{r},\tau)t^>(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)u_i^>(\boldsymbol{q},\tau)\right\rangle_c\Big\}\\
&+P_{imn}^>(\boldsymbol{q})\left\{\frac{1}{2}\left\langle u_m^<(\boldsymbol{r},\tau)u_n^<(\boldsymbol{q}-\boldsymbol{r},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c+\left\langle u_m^>(\boldsymbol{r},\tau)u_n^<(\boldsymbol{q}-\boldsymbol{r},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c\right.\\
&+\frac{1}{2}\left\langle u_m^>(\boldsymbol{r},\tau)u_n^>(\boldsymbol{q}-\boldsymbol{r},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c+\overline{U}_m(\boldsymbol{r},\tau)\left\langle u_n^<(\boldsymbol{q}-\boldsymbol{r},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c\\
&\left.+\overline{U}_m(\boldsymbol{r},\tau)\left\langle u_n^>(\boldsymbol{q}-\boldsymbol{r},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c\right\}\Big]\,. \qquad \text{(B.29)}
\end{aligned}
$$

Here the $\langle \boldsymbol{u}^> t^>\rangle_c$-related terms are expanded in sequence by replacing the correlation $\langle \boldsymbol{u}^> t^>\rangle_c$ with its transport equation, the quantitative contribution of which to the turbulent heat flux is considered to be small compared to the $\overline{T}^1$-related term for simple shear flow at high Reynolds number limit. The triple correlation consisting of only the higher wave-number modes should be neglected. Then, the conditional average of the joint correlations between the higher wave-number modes ($\boldsymbol{u}^>$ and $t^>$) and the lower wave-number modes ($\boldsymbol{u}^<$ and $t^<$) is performed by means of the scale-separation assumption [Eqs. (B.20), (B.25)-(B.27)]; thus,

$$
\begin{aligned}
& i\sum_{q}\sum_{r}\left(\frac{\partial}{\partial\tau}+\nu_0 q^2+\alpha_0|\boldsymbol{k}-\boldsymbol{q}|^2\right)^{-1}\Big[(k_j-q_j)\Big\{\left\langle v_i^>(\boldsymbol{q},\tau)v_j^>(\boldsymbol{r},\tau)\right\rangle_c\overline{T}(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)\\
&+u_j^<(\boldsymbol{r},\tau)\left\langle \theta^>(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)v_i^>(\boldsymbol{q},\tau)\right\rangle_c+\left\langle v_j^>(\boldsymbol{r},\tau)v_i^>(\boldsymbol{q},\tau)\right\rangle_c t^<(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)\Big\}\\
&+P_{imn}^>(\boldsymbol{q})\left\langle v_m^>(\boldsymbol{r},\tau)\theta^>(\boldsymbol{k}-\boldsymbol{q},\tau)\right\rangle_c u_n^<(\boldsymbol{q}-\boldsymbol{r},\tau)\Big]
\end{aligned}
$$

$$\simeq \; i\sum_q\sum_r\left(\frac{\partial}{\partial\tau}+\nu_0 q^2+\alpha_0|\boldsymbol{k}-\boldsymbol{q}|^2\right)^{-1}(k_j-q_j)\left\{\left\langle v_i^>(\boldsymbol{q},\tau)v_j^>(\boldsymbol{r},\tau)\right\rangle_c\overline{T}(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)\right.$$

$$\left.+\left\langle v_j^>(\boldsymbol{r},\tau)v_i^>(\boldsymbol{q},\tau)\right\rangle_c t^<(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{r},\tau)\right\}, \tag{B.30}$$

where all the $\Delta_i^>$ and $\Delta_t^>$-related terms are neglected. Finally, we pay attention to the $\overline{T}^1$-related term as a leading one in Eq. (B.30), and apply the second-order moment for isotropic turbulence:

$$\left\langle v_\alpha^>(\boldsymbol{q},\tau)v_\beta^>(\boldsymbol{q}',\tau)\right\rangle_c=\left(\frac{2\pi}{L}\right)^3 P_{\alpha\beta}^>(\boldsymbol{q})Q_v^>(q,\tau)\delta_{q+q',0}, \tag{B.31}$$

$$\delta_{q+q',0}=\left(\frac{1}{L}\right)^3\int_L\int_L\int_L \exp\left\{-i(\boldsymbol{q}+\boldsymbol{q}')\cdot\boldsymbol{x}\right\}dx_1dx_2dx_3 \tag{B.32}$$

to the fundamental velocity field $\boldsymbol{v}^>$ as a moment closure hypothesis in our theory. Consequently, the second term of the r.h.s. in Eq. (3.83) can be modeled as

$$-\sum_q\langle u_i^>(\boldsymbol{q},\tau)t^>(\boldsymbol{k}-\boldsymbol{q},\tau)\rangle_c \;\simeq\; i\sum_q\left(\frac{2\pi}{L}\right)^3\left(\frac{\partial}{\partial\tau}+\nu_0q^2+\alpha_0|\boldsymbol{k}-\boldsymbol{q}|^2\right)^{-1}$$

$$\times(k_j-q_j)P_{ij}^>(\boldsymbol{q})Q_v^>(q,\tau)\overline{T}(\boldsymbol{k},\tau) \tag{B.33}$$

relevant to the order of $\overline{T}^1$ in the renormalized expansion.