# Quantum field theory in a time-dependent gravitational field

Ikuo Ichinose
*Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo, 152-Japan*
(Received 6 October 1980)

A quantum field theory of a charged scalar field with self-interactions in a Robertson-Walker metric is considered. The structure of the Hilbert space of the state vectors in this theory is investigated; and taking account of the change of the Fock space, a method of calculating physical quantities is constructed and its diagrammatic version is given. By practical calculation to second order in the coupling constant, it is shown that the theory is renormalizable and renormalization constants are identical to those in the theory in Minkowski spacetime.

## I. INTRODUCTION

In recent years, quantum field theory in curved spacetime has been actively investigated and its importance has been recognized, especially in cosmology and astrophysics. Although the gravitational field should probably be treated as a quantized field in the final form, and many people have attempted to do this, several difficult problems are encountered. It has not been shown how to treat the nonlinearity of the gravitational field in quantum theory, and the linearized weak gravitational field is not renormalizable in the usual sense. It is expected that treating the gravitational field as a classical $c$-number field and studying quantum field theories in that curved spacetime is significant at least in the case that (the expectation value of) the curvature of spacetime is sufficiently small compared with the Planck length. Much of the work on quantum field theories in curved spacetime done so far has been restricted to the free field theories with no interactions between quantized fields; recently, there have been some investigations of interacting fields, in particular, the question of renormalizability.[1-3]

In the present paper, we shall consider a quantum field theory in a time-dependent classical gravitational field. Generally, in a nonstationary external field, quantum fields will lose the concept of "particle." Let us suppose that in the sufficiently distant past and future the gravitational field becomes stationary, that is,

$$g_{\mu\nu}(x) \to g_{\mu\nu}^{\text{in(out)}}(\vec{x}) , \quad \text{as } t \to -\infty\,(+\infty) ,$$

and interactions between quantized fields are adiabatically switched off. In that case one can construct the Fock spaces of "in" and "out" states. They represent the same Hilbert space and are connected by a unitary transformation if the degree of freedom of the system is finite; however, this is not the case if the degree of freedom is infinite, which is the case we shall consider in this paper. One must bear this fact in mind in constructing a method of calculation. It is rather difficult to obtain the Green's functions for a quantized field which is interacting nontrivially with a classical gravitational field, and a rigorous discussion of the Feynman-Dyson diagrammatic method, taking into account the change of the Fock space, has not yet been given. In this paper, we shall resolve the above problems, taking a charged scalar field in the spatially flat Robertson-Walker metric as an example. In this case the spatial components of momentum are conserved quantities, and it is easy to see in what situation the particle picture reappears.

This paper is organized as follows. In Sec. II we consider a given problem and present a prescription for treating the problem. We first construct a suitable Hamiltonian and then define creation and annihilation operators of the "instantaneous particle", which instantaneously diagonalize the quadratic part of the Hamiltonian. In Sec. III we consider a time-dependent quantum-mechanical system of two degrees of freedom and represent the transition matrix element by a functional integral in the coherent-state form. In Sec. IV the results obtained in Sec. III are extended to quantum field theory and we construct a diagrammatic method for perturbation expansion. Section V is devoted to a discussion of the ultraviolet divergences of a theory which is renormalizable in flat spacetime, and it is explicitly shown that, at least to second order in the coupling constant, the theory is renormalizable, although individual graphs contain divergences which cannot be eliminated by renormalization. Section VI is devoted to a conclusion and discussion.

## II. CONSIDERING THE PROBLEMS

Although our method is applicable to any theory in curved spacetime, we consider a charged scalar field in the spatially flat Robertson-Walker metric. In our convention

$$g_{\mu\nu} = c^2(t)\eta_{\mu\nu} , \qquad (2.1)$$
$$\operatorname{diag}\eta_{\mu\nu} = (+,-,-,-) ,$$

and the Riemann-Christoffel tensor is defined by

$$R^{\alpha}{}_{\beta,\gamma\delta}=\partial_\delta\Gamma^{\alpha}_{\beta\gamma}-\partial_\gamma\Gamma^{\alpha}_{\beta\delta}+\Gamma^{\alpha}_{\lambda\delta}\Gamma^{\lambda}_{\beta\gamma}-\Gamma^{\alpha}_{\lambda\gamma}\Gamma^{\lambda}_{\beta\delta}\,.$$

The Lagrangian density of the system is given by

$$\mathcal{L}(x)=g^{1/2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^\dagger-m^2\phi\phi^\dagger-\xi R\phi\phi^\dagger)+\mathcal{L}_I(g_{\mu\nu},\phi,\phi^\dagger)\,, \tag{2.2}$$

where $\mathcal{L}_I(g_{\mu\nu},\phi,\phi^\dagger)$ is the interaction part of the quantized field and generally may not be invariant under the conformal transformation ($\mathcal{L}_I$ also contains counterterms which are necessary for renormalization and $m^2$ and $\xi$ are renormalized quantities). As a matter of convenience, in the remainder of the discussion we take the following rescaled fields $\psi(x)$ and $\psi^\dagger(x)$ to be dynamical variables, instead of $\phi(x)$ and $\phi^\dagger(x)$,

$$\psi(x)=c(t)\phi(x)\,. \tag{2.3}$$

The Lagrangian density is written in terms of the rescaled field as follows:

$$\mathcal{L}(x)=\eta^{\mu\nu}\partial_\mu\psi\partial_\nu\psi^\dagger-m^2c^2\psi\psi^\dagger+(\tfrac{1}{6}-\xi)Rc^2\psi\psi^\dagger+\mathcal{L}_{\rm int}(c,\psi,\psi^\dagger)\,, \tag{2.4}$$

where an irrelevant total-divergent term has been discarded, and

$$\mathcal{L}_{\rm int}(c,\psi,\psi^\dagger)=\mathcal{L}_I(g_{\mu\nu},c^{-1}\psi,c^{-1}\psi^\dagger)\,. \tag{2.5}$$

From Eq. (2.4), in the case of the massless theory with conformal coupling, i.e., $m^2=0$ and $\xi=\frac{1}{6}$, the particle picture survives and is found to be the collective motion of the rescaled fields $\psi(x)$ and $\psi^\dagger(x)$.

The canonical conjugate momenta of $\psi(x)$ and $\psi^\dagger(x)$ are given as usual by

$$\pi(x)=\frac{\partial\mathcal{L}}{\partial\dot\psi(x)}=\dot\psi^\dagger(x)\,, \tag{2.6a}$$

$$\pi^\dagger(x)=\frac{\partial\mathcal{L}}{\partial\dot\psi(x)^\dagger}=\dot\psi(x)\,, \tag{2.6b}$$

and the canonical commutation relations are

$$[\psi(x),\pi(y)]_{x^0=y^0}=i\delta^3(\vec{x}-\vec{y})\,, \tag{2.7a}$$

$$[\psi^\dagger(x),\pi^\dagger(y)]_{x^0=y^0}=i\delta^3(\vec{x}-\vec{y}) \tag{2.7b}$$

and

all others commute at equal time. (2.7c)

These relations are equivalent to those obtained from the original Lagrangian (2.2), regarding $\phi(x)$ and $\phi^\dagger(x)$ as dynamical variables. From the Lagrangian (2.4), $\psi(x)$ is the charged scalar field which has a time-dependent mass $M^2(t)\equiv m^2c^2(t)+(\xi-\frac{1}{6})R(t)c^2(t)$ and exists in flat spacetime. If the effective mass $M^2(t)$ becomes negative, the Fock-space vacuum for $\psi(x)$ and $\psi^\dagger(x)$ becomes unstable and the state which contains particles and antiparticles coherently has lower energy than the vacuum, and the invariance under the global phase transformation,

$$\psi(x)\to e^{i\alpha}\psi(x)\,,$$

$$\psi^\dagger(x)\to e^{-i\alpha}\psi^\dagger(x)\,,$$

is spontaneously broken. This effect is very interesting from the point of view of elementary particle physics, especially in the case that $\psi(x)$ couples to gauge fields. However, in this paper we do not consider a situation such as this and assume $M^2(t)\geq 0$ for all $t$. In the case $(\xi-\frac{1}{6})R>0$, the scalar curvature plays a role of the cutoff of infrared divergences for massless theory.

From Eqs. (2.4), (2.6a), and (2.6b), we obtain the Hamiltonian in the Heisenberg picture,

$$\begin{aligned}H_H(t)&=\int d^3x\,\mathcal{H}_H(x)\\&=\int d^3x(\pi\dot\psi+\pi^\dagger\dot\psi^\dagger-\mathcal{L})\\&=\int d^3x\{\pi\pi^\dagger+\partial_k\psi\partial_k\psi^\dagger\\&\qquad+c^2[m^2+(\xi-\tfrac{1}{6})R]\psi\psi^\dagger\\&\qquad-\mathcal{L}_{\rm int}(c,\psi,\psi^\dagger)\}\,.\end{aligned} \tag{2.8}$$

It is easily verified that the Heisenberg equations

$$i\dot\psi(x)=[\psi(x),H_H(t)]\,, \tag{2.9a}$$

$$i\dot\pi(x)=[\pi(x),H_H(t)] \tag{2.9b}$$

are equivalent to the equations of motion which are obtained from the Lagrangian (2.4). As the system exists in the time-dependent external field, the Hamiltonian in the Schrödinger picture is different from that in the Heisenberg picture, and is given by

$$\begin{aligned}H_S(t)=\int d^3x\{&\pi_S(\vec{x})\pi_S^\dagger(\vec{x})+\partial_k\psi_S(\vec{x})\partial_k\psi_S^\dagger(\vec{x})\\&+c^2(t)[m^2+(\xi-\tfrac{1}{6})R(t)]\psi_S(\vec{x})\psi_S^\dagger(\vec{x})\\&-\mathcal{L}_{\rm int}(c(t),\psi_S(\vec{x}),\psi_S^\dagger(\vec{x}))\}\,,\end{aligned} \tag{2.10}$$

where

$$\psi_S(\vec{x})=U(t,t_0)\psi(x)U^\dagger(t,t_0)\,, \tag{2.11a}$$

$$\pi_S(\vec{x})=U(t,t_0)\pi(x)U^\dagger(t,t_0)\,, \tag{2.11b}$$

and $U(t,t_0)$ is defined as the solution of the following integral equation:

$$U(t,t_0)=1-i\int_{t_0}^{t}H_S(t')U(t',t_0)dt'\,. \tag{2.12}$$

Here we introduce the creation and annihilation operators, noticing that spatial momentum is conserved,

$$\psi_S(\vec{x}) \equiv \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{d^3k}{[2\omega_k(t_0)]^{1/2}} [a(\vec{k})e^{i\vec{k}\cdot\vec{x}} + b^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}], \tag{2.13a}$$

$$\pi_S(\vec{x}) \equiv \left(\frac{1}{2\pi}\right)^{3/2} \int d^3k\, i\left[\frac{\omega_k(t_0)}{2}\right]^{1/2} [a^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}} - b(\vec{k})e^{i\vec{k}\cdot\vec{x}}], \tag{2.13b}$$

where

$$\omega_k(t_0) = [\vec{k}^2 + m^2c^2(t_0) + (\xi - \tfrac{1}{6})R(t_0)c^2(t_0)]^{1/2} \tag{2.14}$$

[remember that we have assumed $\omega_k^2(t) \geq 0$ for all $t$]. From the commutation relations (2.7a)–(2.7c), we obtain

$$[a(\vec{k}), a^\dagger(\vec{p})] = \delta(\vec{k} - \vec{p}), \tag{2.15a}$$

$$[a(\vec{k}), a(\vec{p})] = 0, \tag{2.15b}$$

$$[b(\vec{k}), b^\dagger(\vec{p})] = \delta(\vec{k} - \vec{p}), \tag{2.15c}$$

$$[b(\vec{k}), b(\vec{p})] = 0, \tag{2.15d}$$

$$[a(\vec{k}), b(\vec{p})] = 0, \tag{2.15e}$$

$$[a(\vec{k}), b^\dagger(\vec{p})] = 0. \tag{2.15f}$$

The creation and annihilation operators in the Heisenberg picture,

$$a_H(\vec{k}, t) = U^{-1}(t, t_0)a(\vec{k})U(t, t_0), \tag{2.16a}$$

$$b_H(\vec{k}, t) = U^{-1}(t, t_0)b(\vec{k})U(t, t_0), \tag{2.16b}$$

also satisfy the commutation relations similar to (2.15a)–(2.15f), as the time translational operator $U(t, t_0)$ is unitary. The Hamiltonian in the Schrödinger picture is written in terms of $a(k), b(k)$ and their Hermitian conjugates as

$$\begin{aligned} H_S(t) = \int d^3k \Bigg\{ \frac{1}{2}\Bigg[ &\left(\omega_k(t_0) + \frac{\omega_k^2(t)}{\omega_k(t_0)}\right)[a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(-\vec{k})b(-\vec{k})] \\ &+ \left(\frac{\omega_k^2(t)}{\omega_k(t_0)} - \omega_k(t_0)\right)[a(\vec{k})b(-\vec{k}) + a^\dagger(\vec{k})b^\dagger(-\vec{k})]\Bigg]\Bigg\} + V[c(t), a(\vec{k}), \ldots, b^\dagger(\vec{p})] \\ = \int d^3k \big(\omega_k(t)\{&E_k(t)[a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(-\vec{k})b(-\vec{k})] + F_k(t)[a(\vec{k})b(-\vec{k}) + a^\dagger(\vec{k})b^\dagger(-\vec{k})]\}\big) \\ &+ V[c(t), a(\vec{k}), \ldots, b^\dagger(\vec{p})], \end{aligned} \tag{2.17}$$

where

$$\omega_k(t) = [\vec{k}^2 + M^2(t)]^{1/2}, \tag{2.18a}$$

$$E_k(t) = \frac{1}{2\omega_k(t)}\left[\frac{\omega_k^2(t)}{\omega_k(t_0)} + \omega_k(t_0)\right], \tag{2.18b}$$

$$F_k(t) = \frac{1}{2\omega_k(t)}\left[\frac{\omega_k^2(t)}{\omega_k(t_0)} - \omega_k(t_0)\right], \tag{2.18c}$$

and

$$V[c(t), a(k), \ldots, b^\dagger(p)] = -\int d^3x\, \mathcal{L}_{\rm int}(c(t), \psi_S(x), \psi_S^\dagger(x)). \tag{2.18d}$$

Here, instead of treating the self-interaction terms directly, we introduce source terms and use the same notation $H_S(t)$ for the Hamiltonian of the system with source terms, i.e.,

$$\mathcal{L} = \mathcal{L}_0 + \eta(x)\psi(x) + \bar{\eta}(x)\psi^\dagger(x), \tag{2.19}$$

$$\begin{aligned} H_S(t) &= H_{S0}(t) - \int d^3x[\eta(x)\psi_S(\vec{x}) + \bar{\eta}(x)\psi_S^\dagger(\vec{x})] \\ &= H_{S0}(t) - \int d^3k[\gamma(\vec{k}, t)a(\vec{k}) + \bar{\gamma}(\vec{k}, t)a^\dagger(\vec{k}) \\ &\qquad + \bar{\gamma}(-\vec{k}, t)b(\vec{k}) + \gamma(-\vec{k}, t)b^\dagger(\vec{k})], \end{aligned} \tag{2.20}$$

where $\mathcal{L}_0(x)$ and $H_{S0}(t)$ are the Lagrangian and Hamiltonian, respectively, of the free field with time-dependent mass and

$$\gamma(\vec{k}, t) \equiv \left(\frac{1}{2\pi}\right)^{3/2} \int d^3x \frac{1}{[2\omega_k(t_0)]^{1/2}} \eta(x) e^{i\vec{k}\cdot\vec{x}}. \tag{2.21}$$

We now define the creation and annihilation operators of the "instantaneous particle," which diagonalize the quadratic part of the Hamiltonian. This is performed by the Bogoliubov transformation

$$\alpha_S(\vec{k}, t) = a(\vec{k})\cosh\theta_k(t) - b^\dagger(-\vec{k})\sinh\theta_k(t), \tag{2.22a}$$

$$\beta_S(-\vec{k}, t) = b(-\vec{k})\cosh\theta_k(t) - a^\dagger(\vec{k})\sinh\theta_k(t), \tag{2.22b}$$

where

$$\begin{aligned} \cosh\theta_k(t) &= \frac{E_k(t)}{[E_k^2(t) - F_k^2(t)]^{1/2}}, \\ \sinh 2\theta_k(t) &= \frac{-F_k(t)}{[E_k^2(t) - F_k^2(t)]^{1/2}}. \end{aligned} \tag{2.23}$$

In terms of $\alpha_S(\vec{k}, t)$, $\beta_S(-\vec{k}, t)$, and their Hermitian conjugates,

$$H_S(t)=\int d^3k\,\mathcal{H}_S(x)$$
$$=\int d^3k\{\omega_k(t)[\alpha_S^\dagger(\vec{k},t)\alpha_S(\vec{k},t)+\beta_S^\dagger(-\vec{k},t)\beta_S(-\vec{k},t)]$$
$$-\gamma(\vec{k},t)e^{\theta_k(t)}[\alpha_S(\vec{k},t)+\beta_S^\dagger(-\vec{k},t)]$$
$$-\bar{\gamma}(\vec{k},t)e^{\theta_k(t)}[\alpha_S^\dagger(\vec{k},t)+\beta_S(-\vec{k},t)]\}\,, \tag{2.24}$$

where we have omitted irrelevant $c$-number terms. The operator which generates the Bogoliubov transformation (2.22a) and (2.22b) is explicitly given by

$$V(\vec{k},t)=\exp\{\theta_k(t)[b^\dagger(-\vec{k})a^\dagger(\vec{k})-a(\vec{k})b(-\vec{k})]\}$$
$$=\exp\{\theta_k(t)[\beta_S^\dagger(-\vec{k},t)\alpha_S^\dagger(\vec{k},t)-\alpha_S(\vec{k},t)\beta_S(-\vec{k},t)]\} \tag{2.25}$$

and

$$\alpha_S(\vec{k},t)=V(\vec{k},t)a(\vec{k})V^{-1}(\vec{k},t)\,,$$
$$\beta_S(-\vec{k},t)=V(\vec{k},t)b(-\vec{k})V^{-1}(\vec{k},t)\,. \tag{2.26}$$

Although each $V(k,t)$ is a unitary operator, in the case where the degree of freedom of the system is infinite, the operator $V(t)\equiv\prod_k V(k,t)$ becomes nonunitary and the Fock spaces of the instantaneous particle at different times are not connected by a unitary transformation. Suppose that the gravitational field becomes stationary in the sufficient past and future and interactions between quantized fields adiabatically switch on at an early time and switch off at a late time, then in and out Fock spaces, $\mathfrak{F}_{\text{in}}$ and $\mathfrak{F}_{\text{out}}$, are constructed by the creation and annihilation operators of "in particle" and "out particle," $\alpha^{\text{in}}(\vec{k})$, $\beta^{\text{in}}(\vec{k})$, $\alpha^{\text{out}}(\vec{k})$, and $\beta^{\text{out}}(\vec{k})$,

$$\psi^{\text{in}}(x)=\left(\frac{1}{2\pi}\right)^{3/2}\int d^3k\frac{1}{[2\omega_{\text{in}}(k)]^{1/2}}\{\alpha^{\text{in}}(\vec{k})\exp[-i\omega_{\text{in}}(k)x^0+i\vec{k}\cdot\vec{x}]+\beta^{\text{in}\dagger}(\vec{k})\exp[i\omega_{\text{in}}(k)x^0-i\vec{k}\cdot\vec{x}]\}\,, \tag{2.27a}$$

$$\psi^{\text{out}}(x)=\left(\frac{1}{2\pi}\right)^{3/2}\int d^3k\frac{1}{[2\omega_{\text{out}}(k)]^{1/2}}\{\alpha^{\text{out}}(\vec{k})\exp[-i\omega_{\text{out}}(k)x^0+i\vec{k}\cdot\vec{x}]+\beta^{\text{out}\dagger}(\vec{k})\exp[i\omega_{\text{out}}(k)x^0-i\vec{k}\cdot\vec{x}]\}\,, \tag{2.27b}$$

where $\psi^{\text{in}}(x)$ and $\psi^{\text{out}}(x)$ are the asymptotic in field and out field, respectively, and $\omega_{\text{in}}(k)=\omega_k(-\infty)$ and $\omega_{\text{out}}(k)=\omega_k(+\infty)$. In the subsequent sections, we shall discuss a prescription to compute the amplitude from $|f\rangle\in\mathfrak{F}_{\text{in}}$ to $|g\rangle\in\mathfrak{F}_{\text{out}}$.

## III. PATH INTEGRAL IN THE COHERENT-STATE FORM AND THE BOGOLIUBOV TRANSFORMATION

Just as we have constructed the problem into a suitable form in the previous section, we shall investigate it by using functional integral techniques. As a preliminary step, we consider in this section the quantum-mechanical system with two degrees of freedom. From the nature of this problem, it is both suitable and crucial to use the path-integral method in the coherent-state form, which was first contrived by Schweber.[4] For readers who are not familiar with this technique, we shall pursue the discussion in such a way that the reader need not refer to any literature.

Omitting the momentum suffix $k$, the Hamiltonian under consideration is

$$H_S(t)=\omega(t)[E(t)(a^\dagger a+b^\dagger b)+F(t)(ab+a^\dagger b^\dagger)]$$
$$-\gamma(t)(a+b^\dagger)-\bar{\gamma}(t)(a^\dagger+b)$$
$$=\omega(t)[\alpha_S^\dagger(t)\alpha_S(t)+\beta_S^\dagger(t)\beta_S(t)]$$
$$-\gamma(t)e^{\theta(t)}[\alpha_S(t)+\beta_S^\dagger(t)]$$
$$-\bar{\gamma}(t)e^{\theta(t)}[\alpha_S^\dagger(t)+\beta_S(t)]\,. \tag{3.1}$$

The creation and annihilation operators of the instantaneous particle in the "Heisenberg picture" are defined by

$$\alpha_H(t)=U^\dagger(t,t_0)\alpha_S(t)U(t,t_0)\,, \tag{3.2a}$$
$$\beta_H(t)=U^\dagger(t,t_0)\beta_S(t)U(t,t_0)\,, \tag{3.2b}$$

and they satisfy the following commutation relations:

$$[\alpha_H(t),\alpha_H^\dagger(t)]=1\,, \tag{3.3a}$$
$$[\beta_H(t),\beta_H^\dagger(t)]=1\,, \tag{3.3b}$$

and

$$\text{all the others commute}. \tag{3.3c}$$

Therefore, we can construct the "vacuum" and "$n$-particle" state as usual. The "no-particle" state at time $t$, $|0;t\rangle$, is defined by

$$\alpha_H(t)|0;t\rangle=0\,,$$
$$\beta_H(t)|0;t\rangle=0\,, \tag{3.4}$$

and the "$n$-particle and $m$-antiparticle" state is

$$|n,m;t\rangle=\frac{[\alpha_H^\dagger(t)]^n}{\sqrt{n!}}\frac{[\beta_H^\dagger(t)]^m}{\sqrt{m!}}|0;t\rangle\,. \tag{3.5}$$

The eigenstate of the annihilation operators, $\alpha_H(t)$ and $\beta_H(t)$, is called the (instantaneous) coherent state, that is,

$$\alpha_H(t)|Z_\alpha,Z_\beta;t\rangle=Z_\alpha|Z_\alpha,Z_\beta;t\rangle\,, \tag{3.6a}$$
$$\beta_H(t)|Z_\alpha,Z_\beta;t\rangle=Z_\beta|Z_\alpha,Z_\beta;t\rangle\,, \tag{3.6b}$$

where we note that $Z_\alpha$ and $Z_\beta$ take all complex values, and this state vector is explicitly given by

$$|Z_\alpha, Z_\beta; t\rangle = \exp[Z_\alpha \alpha_H^\dagger(t) + Z_\beta \beta_H^\dagger(t)]|0;t\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{Z_\alpha{}^n}{\sqrt{n!}}\frac{Z_\beta{}^m}{\sqrt{m!}}|n,m;t\rangle \tag{3.7}$$

and

$$\langle Z'_\alpha, Z'_\beta; t | Z_\alpha, Z_\beta; t\rangle = \exp(\bar{Z}'_\alpha Z_\alpha + \bar{Z}'_\beta Z_\beta). \tag{3.8}$$

The coherent states are not orthonormal, but they span a complete set. Then for any state vectors $|f\rangle$ and $|g\rangle$,

$$\begin{aligned}\langle f|g\rangle &= \int dZ_\alpha d\bar{Z}_\alpha dZ_\beta d\bar{Z}_\beta \langle Z_\alpha, Z_\beta | Z_\alpha, Z_\beta\rangle^{-1} \\ &\quad\times \langle f|Z_\alpha, Z_\beta\rangle\langle Z_\alpha, Z_\beta|g\rangle \\ &= \int dZ_\alpha d\bar{Z}_\alpha dZ_\beta d\bar{Z}_\beta e^{-|Z_\alpha|^2-|Z_\beta|^2}\langle f|Z_\alpha, Z_\beta\rangle\langle Z_\alpha, Z_\beta|g\rangle \\ &= \int d\mu(Z_\alpha)d\mu(Z_\beta)\langle f|Z_\alpha, Z_\beta\rangle\langle Z_\alpha, Z_\beta|g\rangle,\end{aligned} \tag{3.9}$$

where

$$d\mu(Z) = \frac{dx\,dy}{\pi}e^{-|Z|^2}, \quad Z = x + iy.$$

The problem is reduced to computing the following transition matrix element:

$$F[Z'_\alpha, Z'_\beta, t''; \bar{Z}_\alpha, \bar{Z}_\beta, t'] \equiv \langle Z'_\alpha, Z'_\beta; t''|\bar{Z}_\alpha, \bar{Z}_\beta, t'\rangle, \tag{3.10}$$

which is also a functional of $\gamma(t)$. Here, we subdivide the time interval $(t', t'')$ into $N$ segments,

$$N\epsilon = t'' - t', \quad t_n = t' + n\epsilon.$$

From the completeness of the state vectors $|Z_\alpha, Z_\beta; t_n\rangle$, we have

$$\begin{aligned}F[Z'_\alpha, Z'_\beta, t''; \bar{Z}_\alpha, \bar{Z}_\beta, t'] = \int d\mu_1 \cdots \int d\mu_{N-1}\langle \bar{Z}'_\alpha, \bar{Z}'_\beta; t''|\bar{Z}_\alpha(N-1), \bar{Z}_\beta(N-1); t''-\epsilon\rangle \cdots \\ \times\langle \bar{Z}_\alpha(1), \bar{Z}_\beta(1); t'+\epsilon|\bar{Z}_\alpha, \bar{Z}_\beta, t'\rangle,\end{aligned} \tag{3.11}$$

where

$$\int d\mu_n = \int dZ_\alpha(n) d\bar{Z}_\alpha(n) dZ_\beta(n) d\bar{Z}_\beta(n)\exp[-|Z_\alpha(n)|^2 - |Z_\beta(n)|^2].$$

The transition amplitude between the coherent states which separate the infinitesimal time interval is obtained as follows. Neglecting the terms of order $\epsilon^2$,

$$\begin{aligned}F[\zeta_\alpha, \zeta_\beta, t+\epsilon; \bar{\xi}_\alpha, \bar{\xi}_\beta, t] &= \langle \bar{\zeta}_\alpha, \bar{\zeta}_\beta; t+\epsilon|\bar{\xi}_\alpha, \bar{\xi}_\beta; t\rangle \\ &= \langle \bar{\zeta}_\alpha, \bar{\zeta}_\beta; t+\epsilon|U(t+\epsilon, t)V^{-1}(t+\epsilon, t)|\bar{\xi}_\alpha, \bar{\xi}_\beta; t+\epsilon\rangle \\ &\cong \langle \bar{\zeta}_\alpha, \bar{\zeta}_\beta; t+\epsilon|[1 - i\epsilon H_H(t+\epsilon)] \\ &\quad\times\{1 + \epsilon\dot{\theta}(t)[\beta_H^\dagger(t+\epsilon)\alpha_H^\dagger(t+\epsilon) - \alpha_H(t+\epsilon)\beta_H(t+\epsilon)]\}|\bar{\xi}_\alpha, \bar{\xi}_\beta; t+\epsilon\rangle,\end{aligned} \tag{3.12}$$

where we have defined[5]

$$U(t_1, t_2) = U^{-1}(t_2, t_0)U(t_1, t_0)$$

and

$$V(t_1, t_2) = \exp\{[\theta(t_1) - \theta(t_2)][\beta_H^\dagger(t_1)\alpha_H^\dagger(t_1) - \alpha_H(t_1)\beta_H(t_1)]\}.$$

From Eq. (3.1) and the definition of the coherent states,

$$\begin{aligned}F[\zeta_\alpha, \zeta_\beta, t+\epsilon; \bar{\xi}_\alpha, \bar{\xi}_\beta, t] &\cong \exp\{-i\epsilon[\omega(t)(\zeta_\alpha\bar{\xi}_\alpha + \zeta_\beta\bar{\xi}_\beta) - \gamma(t)e^{\theta(t)}(\bar{\xi}_\alpha + \zeta_\beta) - \bar{\gamma}(t)e^{\theta(t)}(\zeta_\alpha + \bar{\xi}_\beta) \\ &\quad + i\dot{\theta}(t)(\zeta_\alpha\zeta_\beta - \bar{\xi}_\alpha\bar{\xi}_\beta)]\}\langle \bar{\zeta}_\alpha, \bar{\zeta}_\beta; t+\epsilon|\bar{\xi}_\alpha, \bar{\xi}_\beta; t+\epsilon\rangle \\ &\cong \exp[-i\epsilon h_{\mathrm{eff}}(\zeta_\alpha, \bar{\xi}_\alpha, \zeta_\beta, \bar{\xi}_\beta; t)]\langle \bar{\zeta}_\alpha, \bar{\zeta}_\beta; t+\epsilon|\bar{\xi}_\alpha, \bar{\xi}_\beta; t+\epsilon\rangle,\end{aligned} \tag{3.13}$$

where the effective Hamiltonian is defined by

$$h_{\mathrm{eff}}(\alpha^\dagger, \alpha, \beta^\dagger, \beta; t) = \omega(t)(\alpha^\dagger\alpha + \beta^\dagger\beta) - \gamma(t)e^{\theta(t)}(\alpha + \beta^\dagger) - \bar{\gamma}(t)e^{\theta(t)}(\alpha^\dagger + \beta) + i\dot{\theta}(t)(\alpha^\dagger\beta^\dagger - \alpha\beta). \tag{3.14}$$

We note that it is a Hermitian operator. Making use of Eqs. (3.11) and (3.13), the transition matrix is expressed in terms of the path integral,

$$F[Z_\alpha, Z_\beta, t''; \bar{Z}_\alpha, \bar{Z}_\beta, t'] = \int d\mu_1 \cdots d\mu_{N-1} \exp[Z'_\alpha \bar{Z}_\alpha(N-1) + Z'_\beta \bar{Z}_\beta(N-1) - i\epsilon h_{\text{eff}}(Z'_\alpha, \bar{Z}_\alpha(N-1), Z'_\beta, \bar{Z}_\beta(N-1); t''-\epsilon) + \cdots + Z_\alpha(1)\bar{Z}_\alpha + Z_\beta(1)\bar{Z}_\beta - i\epsilon h_{\text{eff}}(Z_\alpha(1), \bar{Z}_\alpha, Z_\beta(1), \bar{Z}_\beta; t')] + O(\epsilon)$$

$$= \int_{\alpha(t')=\bar{Z}_\alpha}^{\alpha^*(t'')=Z'_\alpha} [d\alpha^* d\alpha] \int_{\beta(t')=\bar{Z}_\beta}^{\beta^*(t'')=Z'_\beta} [d\beta^* d\beta] \exp\{\tfrac{1}{2}[\alpha^*(t'')\alpha(t'') + \alpha^*(t')\alpha(t') + \beta^*(t'')\beta(t'') + \beta^*(t')\beta(t')]\}$$
$$\times \exp\left\{ i \int_{t'}^{t''} dt \left[ \frac{1}{2i}[\dot{\alpha}^*(t)\alpha(t) - \alpha^*(t)\dot{\alpha}(t) + \dot{\beta}^*(t)\beta(t) - \beta^*(t)\dot{\beta}(t)] - h_{\text{eff}}(\alpha^*(t), \alpha(t), \beta^*(t), \beta(t); t) \right] \right\}, \quad (3.15)$$

where it must be remarked that the path integral $\int_{\alpha(t')=\bar{Z}_\alpha}^{\alpha^*(t'')=Z'_\alpha} [d\alpha^* d\alpha]$ should be evaluated over the paths $\alpha^*(t)$ and $\alpha(t)$ which satisfy the boundary condition $\alpha(t') = \bar{Z}_\alpha$ and $\alpha^*(t'') = Z'_\alpha$ [$\alpha^*(t)$ and $\alpha(t)$ are complex conjugate to each other, apart from the boundary values], and similarly for $\beta^*(t)$ and $\beta(t)$.

Since the effective Hamiltonian $h_{\text{eff}}(\alpha^*(t), \alpha(t), \beta^*(t), \beta(t); t)$ is a quadratic polynomial of $\alpha^*(t)$, $\alpha(t)\beta^*(t)$, and $\beta(t)$, the functional integral (3.15) can be evaluated if the solution of the stationary phase is known. However, it is difficult to solve this equation of motion which contains an arbitrary function $c(t)$, therefore we expand Eq. (3.15) in powers of $\dot{\theta}(t)$. Later it will become clear that this expansion brings out the vertices of particle production and annihilation. Adding the additional source terms for the expansion of the last two terms in Eq. (3.14), Eq. (3.15) is expressed as

$$F[Z'_\alpha, Z'_\beta, t''; \bar{Z}_\alpha, \bar{Z}_\beta, t'] = \exp\left\{ \int_{t'}^{t''} dt\dot{\theta}(t) \left[ \frac{\delta^2}{\delta\zeta(t)\delta\xi(t)} - \frac{\delta^2}{\delta\zeta^*(t)\delta\xi^*(t)} \right] \right\} G[Z'_\alpha, Z'_\beta, t''; \bar{Z}_\alpha, \bar{Z}_\beta, t'] , \quad (3.16)$$

where

$$G[Z'_\alpha, Z'_\beta, t''; \bar{Z}_\alpha, \bar{Z}_\beta, t'] = \int [d\alpha^* d\alpha] \int [d\beta^* d\beta] \exp\tfrac{1}{2}[\alpha^*(t'')\alpha(t'') + \alpha^*(t')\alpha(t') + \beta^*(t'')\beta(t'') + \beta^*(t')\beta(t')]$$
$$\times \exp\left\{ i \int_{t'}^{t''} dt \left[ \frac{1}{2i}(\dot{\alpha}^*\alpha - \alpha^*\dot{\alpha} + \dot{\beta}^*\beta - \beta^*\dot{\beta}) - \omega(t)(\alpha^*\alpha + \beta^*\beta) + [\gamma(t)e^{\theta(t)} - \zeta(t)]\alpha(t) + [\bar{\gamma}(t)e^{\theta(t)} - \zeta^*(t)]\alpha^*(t) + [\gamma(t)e^{\theta(t)} - \xi^*(t)]\beta^*(t) + [\bar{\gamma}(t)e^{\theta(t)} - \xi(t)]\beta(t) \right] \right\} . \quad (3.17)$$

We shall calculate $G[Z'_\alpha, Z'_\beta, t''; \bar{Z}_\alpha, \bar{Z}_\beta, t']$ by obtaining a set of paths $\alpha^*_{\text{sp}}(t)$, $\alpha_{\text{sp}}(t)$, $\beta^*_{\text{sp}}(t)$, and $\beta_{\text{sp}}(t)$ which give the stationary value of the integrand of Eq. (3.17), and they are a set of classical solutions of the following equations of motion:

$$\frac{d}{dt}\alpha(t) + i\omega(t)\alpha(t) - i[\bar{\gamma}(t)e^{\theta(t)} - \zeta^*(t)] = 0 , \quad (3.18a)$$

$$\frac{d}{dt}\alpha^*(t) - i\omega(t)\alpha^*(t) + i[\gamma(t)e^{\theta(t)} - \zeta(t)] = 0 , \quad (3.18b)$$

$$\frac{d}{dt}\beta(t) + i\omega(t)\beta(t) - i[\bar{\gamma}(t)e^{\theta(t)} - \xi^*(t)] = 0 , \quad (3.18c)$$

$$\frac{d}{dt}\beta^*(t) - i\omega(t)\beta^*(t) + i[\gamma(t)e^{\theta(t)} - \xi(t)] = 0 , \quad (3.18d)$$

with the boundary conditions

$$\begin{aligned}&\alpha(t')=\bar{Z}_\alpha, \quad \alpha^*(t'')=Z'_\alpha,\\&\beta(t')=\bar{Z}_\beta, \quad \beta^*(t'')=Z'_\beta.\end{aligned} \tag{3.19}$$

Equations (3.18a)–(3.18d) and (3.19) are easily solved and the solutions are

$$\alpha_{sp}(t)=\exp\left[-i\int_{t'}^{t}\omega(\tau)d\tau\right]\left[\bar{Z}_\alpha+i\int_{t'}^{t}du[\bar{\gamma}(u)e^{\theta(u)}-\zeta^*(u)]\exp\left(i\int_{t'}^{u}\omega(s)ds\right)\right], \tag{3.20a}$$

$$\alpha^*_{sp}(t)=\exp\left[-i\int_{t}^{t''}\omega(\tau)d\tau\right]\left[Z'_\alpha+i\int_{t}^{t''}du[\gamma(u)e^{\theta(u)}-\zeta(u)]\exp\left(i\int_{u}^{t''}\omega(s)ds\right)\right], \tag{3.20b}$$

$$\beta_{sp}(t)=\exp\left[-i\int_{t'}^{t}\omega(\tau)d\tau\right]\left[\bar{Z}_\beta+i\int_{t'}^{t}du[\gamma(u)e^{\theta(u)}-\xi^*(u)]\exp\left(i\int_{t'}^{u}\omega(s)ds\right)\right], \tag{3.20c}$$

$$\beta^*_{sp}(t)=\exp\left[-i\int_{t}^{t''}\omega(\tau)d\tau\right]\left[Z'_\beta+i\int_{t}^{t'}du[\gamma(u)e^{\theta(u)}-\xi(u)]\exp\left(i\int_{u}^{t''}\omega(s)ds\right)\right]. \tag{3.20d}$$

Substituting these solutions into Eq. (3.17), we obtain

$$\begin{aligned}G[Z'_\alpha,Z'_\beta,t'';\bar{Z}_\alpha,\bar{Z}_\beta,t']=\exp\Bigg[&(Z'_\alpha\bar{Z}_\alpha+Z'_\beta\bar{Z}_\beta)\exp\left(-i\int_{t'}^{t''}\omega(\tau)d\tau\right)\\&+iZ'_\alpha\int_{t'}^{t''}du[\bar{\gamma}(u)e^{\theta(u)}-\zeta^*(u)]\exp\left(-i\int_{u}^{t''}\omega(s)ds\right)\\&+i\bar{Z}_\alpha\int_{t'}^{t''}du[\gamma(u)e^{\theta(u)}-\zeta(u)]\exp\left(-i\int_{t'}^{u}\omega(s)ds\right)\\&+iZ'_\beta\int_{t'}^{t''}du[\gamma(u)e^{\theta(u)}-\xi^*(u)]\exp\left(-i\int_{u}^{t''}\omega(s)ds\right)\\&+i\bar{Z}_\beta\int_{t'}^{t''}du[\bar{\gamma}(u)e^{\theta(u)}-\xi(u)]\exp\left(-i\int_{t'}^{u}\omega(s)ds\right)\\&-\int_{t'}^{t''}dt\int_{t'}^{t}du\exp\left(-i\int_{u}^{t}\omega(s)ds\right)[\gamma(t)e^{\theta(t)}-\zeta(t)][\bar{\gamma}(u)e^{\theta(u)}-\zeta^*(u)]\\&-\int_{t'}^{t''}dt\int_{t'}^{t}du\exp\left(-i\int_{u}^{t}\omega(s)ds\right)[\bar{\gamma}(t)e^{\theta(t)}-\xi(t)][\gamma(u)e^{\theta(u)}-\xi^*(u)]\Bigg].\end{aligned} \tag{3.21}$$

We denote the interaction of this system by the potential $V[a^\dagger, a, b^\dagger, b]$ which, we assume, is invariant under the following two discrete transformations:

$$a\leftrightarrow b^\dagger \tag{3.22a}$$

and

$$a^\dagger\leftrightarrow b. \tag{3.22b}$$

Then the final form of the transition matrix is

$$\begin{aligned}U[Z'_\alpha,Z'_\beta,t'';\bar{Z}_\alpha,\bar{Z}_\beta,t']=&\exp\left\{-i\int_{t'}^{t''}dt\,V\left[\frac{1}{i}\frac{\delta}{\delta\gamma(t)},\frac{1}{i}\frac{\delta}{\delta\bar{\gamma}(t)}\right]\right\}\\&\times\exp\left\{\int_{t'}^{t''}dt\,\dot{\theta}(t)\left[\frac{\delta^2}{\delta\zeta(t)\delta\xi(t)}-\frac{\delta^2}{\delta\zeta^*(t)\delta\xi^*(t)}\right]\right\}\\&\times G[Z'_\alpha,Z'_\beta,t'';\bar{Z}_\alpha,\bar{Z}_\beta,t']\ \text{all sources set to zero.}\end{aligned} \tag{3.23}$$

As stated in Sec. II, if

$$c(t)\to\begin{cases}c_- & \text{as } t\to-\infty,\\ c_+ & \text{as } t\to+\infty,\end{cases} \tag{3.24}$$

in and out Fock spaces can be constructed, and the $S$-matrix element is defined as usual by

$$|f;\text{ in}\rangle\in\mathfrak{F}_{\text{in}},\quad |g;\text{ out}\rangle\in\mathfrak{F}_{\text{out}}, \qquad S_{gf}=\langle g;\text{ out}|f;\text{ in}\rangle\,. \tag{3.25}$$

Making use of $U[Z'_\alpha, Z'_\beta, t''; \bar{Z}_\alpha, \bar{Z}_\beta, t']$, the $S$-matrix element $S_{gf}$ is expressed as follows:

$$S_{gf}=\lim_{\substack{t'\to-\infty\\ t''\to+\infty}}\int d\mu(Z')\int d\mu(Z)\langle g;\text{ out}\,|\,\bar{Z}'_\alpha,\bar{Z}'_\beta;t''\rangle\langle \bar{Z}'_\alpha,\bar{Z}'_\beta,t''\,|\,\bar{Z}_\alpha,\bar{Z}_\beta;t'\rangle\langle\bar{Z}_\alpha,\bar{Z}_\beta;t'|f;\text{ in}\rangle\,, \tag{3.26}$$

and the behavior of the wave functions $\langle\bar{Z}_\alpha,\bar{Z}_\beta;t'|f;\text{ in}\rangle$ and $\langle\bar{Z}'_\alpha,\bar{Z}'_\beta,t''|g;\text{ out}\rangle$ is very simple in the limit $t'\to-\infty$ and $t''\to+\infty$, respectively. This is because, from the discussion given above, the operators $\alpha_H(t)$, $\beta_H(t)$ and their Hermitian conjugates diagonalize the Hamiltonian in the Heisenberg picture $H_H(t)$, and their equations of motion become

$$\frac{d}{dt}\alpha_H(\vec{k},t)=-i\omega_k(t)\alpha_H(\vec{k},t)\,,\qquad \frac{d}{dt}\beta_H(\vec{k},t)=-i\omega_k(t)\beta_H(\vec{k},t)\,.\qquad \text{as } t\to-\infty(+\infty) \tag{3.27}$$

Therefore we obtain

$$\alpha_H(\vec{k},t)\to\alpha^{\text{in(out)}}(\vec{k})\exp[-i\omega_{\text{in(out)}}(k)t]\,,\qquad \beta_H(\vec{k},t)\to\beta^{\text{in(out)}}(\vec{k})\exp[-i\omega_{\text{in(out)}}(k)t]\,,\qquad \text{as } t\to-\infty(+\infty) \tag{3.28}$$

up to an irrelevant constant phase factor (and the factor coming from wave-function renormalization).

## IV. QUANTUM FIELD THEORY AND THE DIAGRAMMATIC METHOD

The transition matrix in quantum field theory is derived by extending the manipulation in Sec. III to the system having many degrees of freedom. From Eqs. (2.17) and (2.20), the Hamiltonian of the system under consideration is

$$\begin{aligned}H_S(t)=\int d^3k\big(&\omega_k(t)\{E_k(t)[a^\dagger(\vec{k})a(\vec{k})+b^\dagger(-\vec{k})b(-\vec{k})]+F_k(t)[a(\vec{k})b(-\vec{k})+a^\dagger(\vec{k})b^\dagger(-\vec{k})]\}\\
&-[\gamma(\vec{k},t)a(\vec{k})+\bar{\gamma}(\vec{k},t)a^\dagger(\vec{k})+\bar{\gamma}(-\vec{k},t)b(\vec{k})+\gamma(-\vec{k},t)b^\dagger(\vec{k})]\big)\\
=\int d^3k\,\{&\omega_k(t)[\alpha^\dagger_S(\vec{k},t)\alpha_S(\vec{k},t)+\beta^\dagger_S(\vec{k},t)\beta_S(\vec{k},t)]-\gamma(\vec{k},t)e^{\theta_k(t)}[\alpha_S(\vec{k},t)+\beta^\dagger_S(-\vec{k},t)]\\
&-\bar{\gamma}(\vec{k},t)e^{\theta_k(t)}[\alpha^\dagger_S(\vec{k},t)+\beta_S(-\vec{k},t)]\}\,.\end{aligned} \tag{4.1}$$

From Eqs. (2.25) and (4.1), and using (3.21) with the formula (3.16), the transition matrix of the system with the external source is given as

$$\begin{aligned}F[Z'_\alpha(\vec{k}),Z_\beta(\vec{k}),t'';Z_\alpha(\vec{k}),Z_\beta(\vec{k}),t']=\exp\Bigg\{\bigg(\int d^3k\int_{t'}^{t''}dt\,\dot{\theta}_k(t)\bigg[&\frac{\delta^2}{\delta\xi(-\vec{k},t)\delta\zeta(\vec{k},t)}-\frac{\delta^2}{\delta\xi^*(-\vec{k},t)\delta\zeta^*(\vec{k},t)}\bigg]\\
\times\exp\Bigg\{\,d^3k\bigg[-\int_{t'}^{t''}dt\int_{t'}^{t''}du\,&\bar{\gamma}(\vec{k},t)e^{\theta_k(t)}\gamma(\vec{k},u)e^{\theta_k(u)}\exp\bigg(-i\bigg|\int_u^t\omega_k(\tau)d\tau\bigg|\\
&+\int_{t'}^{t''}dt\int_t^{t'}du\,\gamma(\vec{k},u)e^{\theta_k(u)}\zeta^*(\vec{k},t)\exp\bigg(-i\int_t^u\omega_k(\tau)d\tau\bigg)\\
&+\int_{t'}^{t''}dt\int_{t'}^{t}du\,\bar{\gamma}(\vec{k},u)e^{\theta_k(u)}\zeta(\vec{k},t)\exp\bigg(-i\int_u^t\omega_k(\tau)d\tau\bigg)\\
&+\int_{t'}^{t''}dt\int_t^{t''}du\,\bar{\gamma}(\vec{k},u)e^{\theta_k(u)}\xi^*(-\vec{k},t)\exp\bigg(-i\int_t^u\omega_k(\tau)d\tau\bigg)\end{aligned}$$

$$+\int_{t'}^{t''} dt \int_{t'}^{t} du\, \gamma(\vec{k},u) e^{\theta_k(u)} \xi(-\vec{k},t) \exp\left(-i\int_u^t \omega_k(\tau)d\tau\right)$$
$$-\int_{t'}^{t''} dt \int_{t'}^{t} du\, \zeta(\vec{k},t)\zeta^*(\vec{k},u) \exp\left(-i\int_u^t \omega_k(\tau)d\tau\right)$$
$$-\int_{t'}^{t''} dt \int_{t'}^{t} du\, \xi(-\vec{k},t)\xi^*(-\vec{k},u) \exp\left(-i\int_u^t \omega_k(\tau)d\tau\right)$$
$$+Z'_\alpha(\vec{k})Z_\alpha(\vec{k}) \exp\left(-i\int_{t'}^{t''} \omega_k(\tau)d\tau\right)$$
$$+iZ'_\alpha(\vec{k}) \int_{t'}^{t''} du[\bar{\gamma}(\vec{k},u)e^{\theta_k(u)} - \zeta^*(\vec{k},u)] \exp\left(-i\int_u^{t''} \omega_k(\tau)d\tau\right)$$
$$+i\bar{Z}_\alpha(\vec{k}) \int_{t'}^{t''} du[\gamma(\vec{k},u)e^{\theta_k(u)} - \zeta(\vec{k},u)] \exp\left(-i\int_{t'}^{u} \omega_k(\tau)d\tau\right)$$
$$+Z'_\beta(-\vec{k})Z_\beta(-\vec{k}) \exp\left(-i\int_{t'}^{t''} \omega_k(\tau)d\tau\right)$$
$$+iZ'_\beta(-\vec{k}) \int_{t'}^{t''} du[\gamma(\vec{k},u)e^{\theta_k(u)} - \xi^*(-\vec{k},u)] \exp\left(-i\int_u^{t''} \omega_k(\tau)d\tau\right)$$
$$+i\bar{Z}_\beta(-\vec{k}) \int_{t'}^{t''} du[\bar{\gamma}(\vec{k},u)e^{\theta_k(u)} - \xi(-\vec{k},u)] \exp\left(-i\int_{t'}^{u} \omega_k(\tau)d\tau\right)\Bigg]\Bigg\}. \quad (4.2)$$

In Eq. (4.2), line 2 represents what we will call the propagator, lines 3–8 represent pair production and annihilation, and lines 9–14 represent the external line.

In a subsequent discussion, we consider the theory with the following interaction as an example (besides counterterms),

$$\mathcal{L}_I(g_{\mu\nu}(x), \phi(x), \phi^\dagger(x)) = -\frac{\lambda}{4} g^{1/2}(x)[\phi(x)\phi^\dagger(x)]^2, \quad (4.3)$$

which is renormalizable in flat spacetime. The interaction (4.3) is invariant (at the classical level) under the conformal transformation

$$g_{\mu\nu}(x) \to \Omega^2(x)\tilde{g}_{\mu\nu}(x),$$
$$\phi(x) \to \Omega^{-1}(x)\tilde{\phi}(x),$$

and

$$\mathcal{L}_{\text{int}}(c(x), \psi(x), \psi^\dagger(x)) = -\frac{\lambda}{4}[\psi(x)\psi^\dagger(x)]^2. \quad (4.4)$$

In this case, the transition matrix $U[Z'_\alpha(k), Z'_\beta(k), t''; Z_\alpha(k), Z_\beta(k), t']$ is given by

$$U[Z'_\alpha(\vec{k}), Z'_\beta(\vec{k}), t''; Z_\alpha(\vec{k}), Z_\beta(\vec{k}), t'] = \exp\left\{-i\int_{t'}^{t''} dt\, V\left[\frac{1}{i}\frac{\delta}{\delta\gamma(\vec{k},t)}, \frac{1}{i}\frac{\delta}{\delta\bar{\gamma}(\vec{p},t)}\right]\right\}$$
$$\times F[Z'_\alpha(\vec{k}), Z'_\beta(\vec{k}), t''; Z_\alpha(\vec{k}), Z_\beta(\vec{k}), t'], \quad (4.5)$$

where

$$V\left[\frac{1}{i}\frac{\delta}{\delta\gamma(\vec{k},t)}, \frac{1}{i}\frac{\delta}{\delta\bar{\gamma}(\vec{p},t)}\right] = \frac{\lambda}{4}\frac{1}{(2\pi)^3}\int d^3k_1 \int d^3k_2 \int d^3k_3 \int d^3k_4\, \delta^3(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4)$$
$$\times \frac{1}{i[2\omega_{k_1}(t_0)]^{1/2}} \frac{1}{i[2\omega_{k_2}(t_0)]^{1/2}} \frac{1}{i[2\omega_{k_3}(t_0)]^{1/2}} \frac{1}{i[2\omega_{k_4}(t_0)]^{1/2}}$$
$$\times \frac{\delta^4}{\delta\gamma(\vec{k}_1,t)\delta\gamma(\vec{k}_2,t)\delta\bar{\gamma}(\vec{k}_3,t)\delta\bar{\gamma}(\vec{k}_4,t)}, \quad (4.6)$$

and we set $\gamma(\vec{k},t) = \bar{\gamma}(\vec{p},t) = 0$ in Eq. (4.5) after the calculation. Since we have obtained the prescription to compute the transition matrix, we finally give the $S$ matrix. As we stated following Eq. (3.26), the wave function $\langle \bar{\kappa}_\alpha(\vec{k}), \bar{\kappa}_\beta(\vec{k}), t' | f\rangle$ of in state $|f\rangle \in \mathfrak{F}_{\text{in}}$ is simple in the limit $t' \to -\infty$ and is given as

$$\langle \bar{\kappa}_\alpha(\vec{k}), \bar{\kappa}_\beta(\vec{k}); t' | f \rangle \underset{t' \to -\infty}{\sim} \int d\mu_Z \langle \bar{\kappa}_\alpha(\vec{k}), \bar{\kappa}_\beta(\vec{k}); \text{in} | e^{-iH^{\text{in}}t'} | Z_\alpha(\vec{k}), Z_\beta(\vec{k}); \text{in} \rangle$$
$$\times \langle Z_\alpha(\vec{k}), Z_\beta(\vec{k}); \text{in} | f \rangle , \tag{4.7}$$

where

$$H^{\text{in}} = \int d^3k\, \omega_k(-\infty)(\alpha_k^{\text{in}\dagger}\alpha_k^{\text{in}} + \beta_k^{\text{in}\dagger}\beta_k^{\text{in}}) , \tag{4.8}$$

and

$$\alpha_k^{\text{in}} | Z_\alpha(\vec{k}), Z_\beta(\vec{k}); \text{in} \rangle = Z_\alpha(\vec{k}) | Z_\beta(\vec{k}), Z_\beta(\vec{k}); \text{in} \rangle , \tag{4.9a}$$

$$\beta_k^{\text{in}} | Z_\alpha(\vec{k}), Z_\beta(\vec{k}); \text{in} \rangle = Z_\beta(\vec{k}) | Z_\alpha(\vec{k}), Z_\beta(\vec{k}); \text{in} \rangle , \tag{4.9b}$$

and similarly for out state $|g\rangle \in \mathfrak{F}_{\text{out}}$. Then the $S$-matrix element $\langle Z'_\alpha(\vec{k}), Z_\beta(\vec{k}); \text{out} | Z_\alpha(\vec{k}), Z_\beta(\vec{k}); \text{in} \rangle$ is given by

$$S[Z'_\alpha(\vec{k}), Z'_\beta(\vec{k}); Z_\alpha(\vec{k}), Z_\beta(\vec{k})]$$
$$= \lim_{\substack{t'' \to +\infty \\ t' \to -\infty}} \int d\mu(\eta) \int d\mu(\kappa) \langle Z'_\alpha(\vec{k}), Z_\beta(\vec{k}); \text{out} | e^{iH^{\text{out}}t''} | \bar{\eta}_\alpha(\vec{k}), \bar{\eta}_\beta(\vec{k}); \text{out} \rangle$$
$$\times \langle \bar{\eta}_\alpha(\vec{k}), \bar{\eta}_\beta(\vec{k}); t'' | \bar{\kappa}_\alpha(\vec{k}), \bar{\kappa}_\beta(\vec{k}); t' \rangle$$
$$\times \langle \bar{\kappa}_\alpha(\vec{k}), \bar{\kappa}_\beta(\vec{k}); \text{in} | e^{-iH^{\text{in}}t'} | Z_\alpha(\vec{k}), Z_\beta(\vec{k}); \text{in} \rangle . \tag{4.10}$$

The matrix element of the evolution operator of the free oscillator system in the coherent states is easily obtained, and

$$\langle Z' | \exp[(-i\omega a^\dagger a)t] | Z \rangle = \exp(Z'Ze^{-i\omega t}) ,$$

therefore, $S[Z'_\alpha, Z'_\beta; Z_\alpha, Z_\beta]$ is different from $U[Z'_\alpha, Z'_\beta, t''; Z_\alpha, Z_\beta; t']$ only in the external line part, i.e., lines 9–14 in Eq. (4.2), and

$$S[Z'_\alpha, Z'_\beta; Z_\alpha, Z_\beta] = \lim_{\substack{t'' \to +\infty \\ t' \to -\infty}} U[Z'_\alpha(\vec{k})e^{i\omega_k(+\infty)t''}, Z'_\beta(\vec{k})e^{i\omega_k(+\infty)t''}, t''; Z_\alpha(\vec{k})e^{-i\omega_k(-\infty)t'}, Z_\beta(\vec{k})e^{-i\omega_k(-\infty)t'}, t'] . \tag{4.11}$$

Making use of Eqs. (4.2), (4.4), (4.5), and (4.10), one can calculate $S$-matrix elements, the two-point function, etc., but it is rather complicated to use directly these equations. Fortunately these equations can be compiled into diagrammatic rules, noticing that

$$e^{\theta_k(t)} = \left[\frac{\omega_k(t_0)}{\omega_k(t)}\right]^{1/2} , \tag{4.12}$$

$$\dot{\theta}_k = -\frac{\dot{M}^2(t)}{4\omega_k^2(t)} , \tag{4.13}$$

and they are given in Fig. 1. In Figs. 1(c) and 1(d) are typical diagrams of pair production and pair annihilation, and thus from Eq. (4.13) a particle with smaller momentum has more of a chance to be produced and annihilated. From Eq. (4.13) and the Feynman rules in Fig. 1, we note that if the gravitational field becomes asymptotically static at sufficiently distant past and future, such as Eq. (3.24) (this must be satisfied for the asymptotic field to be introduced), the vertices of particle production and annihilation vanish in these regions. However, if, for example, we take $c^2(t)$ as an expansion factor, the interaction between the particle and the external field remains nonvanishing in the regions where the gravitational field becomes static. This means that the concept of in (or out) particle defined by expanding the theory in terms of $c^2(t)$ is not a suitable one.

## V. THE RENORMALIZATION

In Secs. II–IV we constructed the method of calculation of perturbation expansion. In this section, making use of this method, we shall investigate the ultraviolet divergences and the renormalizability of the theory which is renormalizable in Minkowski spacetime. By simple power counting, it is easily verified that the ultraviolet divergences will appear only in the diagrams of the two- and four-point functions, besides the amplitude from the in vacuum to the out vacuum. We shall investigate in a practical manner the divergences to second order in the coupling constant $\lambda$.

### A. One-loop level: renormalization of mass and the coupling constant

The two-point function (the Feynman propagator) is defined by

$$G_F(t,s;\vec{k})=\frac{\langle \text{out; vac}|\, T[\tilde{\psi}^\dagger(t,\vec{k})\tilde{\psi}(s,\vec{k})]\,|\text{in; vac}\rangle}{\langle \text{out; vac}|\text{in; vac}\rangle}$$
$$=\left[\frac{1}{i[2\omega_k(t_0)]^{1/2}}\right]^2\frac{\delta^2}{\delta\gamma(\vec{k},s)\delta\bar{\gamma}(\vec{k},t)}$$
$$\times \ln S[Z'_\alpha,Z'_\beta;Z_\alpha,Z_\beta]\Big|_{\substack{\gamma=\bar{\gamma}=0\\ Z'_\alpha=Z'_\beta=\bar{Z}_\alpha=\bar{Z}_\beta=0}}\,, \tag{5.1}$$

where

$$\tilde{\psi}(t,\vec{k})=\left(\frac{1}{2\pi}\right)^{3/2}\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\psi(x)\,. \tag{5.2}$$

In the zeroth order in the coupling constant $\lambda$, diagrams which contribute to the propagator are readily obtained from the Feynman rules in Fig. 1 and Eq. (5.1), and they are given in Fig. 2. It is easily proved that they do not involve any divergences. For example, the contribution from the graphs of Figs. 2(b) and 2(c) is

$$[\text{graph }2(\text{b})+2(\text{c})]=\frac{1}{2[\omega_k(t)\omega_k(s)]^{1/2}}$$
$$\times\int du\left[\dot{\theta}_k(u)\theta(t-u)\theta(s-u)\exp\left(-i\int_u^t\omega_k(\tau)d\tau-i\int_u^s\omega_k(\tau)d\tau\right)\right.$$
$$\left.-\dot{\theta}_k(u)\theta(u-t)\theta(u-s)\exp\left(-i\int_t^u\omega_k(\tau)d\tau-i\int_s^u\omega_k(\tau)d\tau\right)\right]\,. \tag{5.3}$$

One can expect that the integral over time $u$ in the above equation exists if

$$\dot{\theta}_k(u)=-\frac{\dot{M}^2(u)}{4\omega_k^{\,2}(u)}$$
$$=-\frac{\dot{M}^2(u)}{4[\vec{k}^2+M^2(u)]}$$

is a square-integrable function.

Making use of the Feynman rules, we estimate the divergence in the one-loop diagram given in Fig. 3. To regularize the integral over the momentum, we use the dimensional method. Introducing a parameter $\mu$ of mass dimension, the interaction part of the Lagrangian in $D$-dimensional

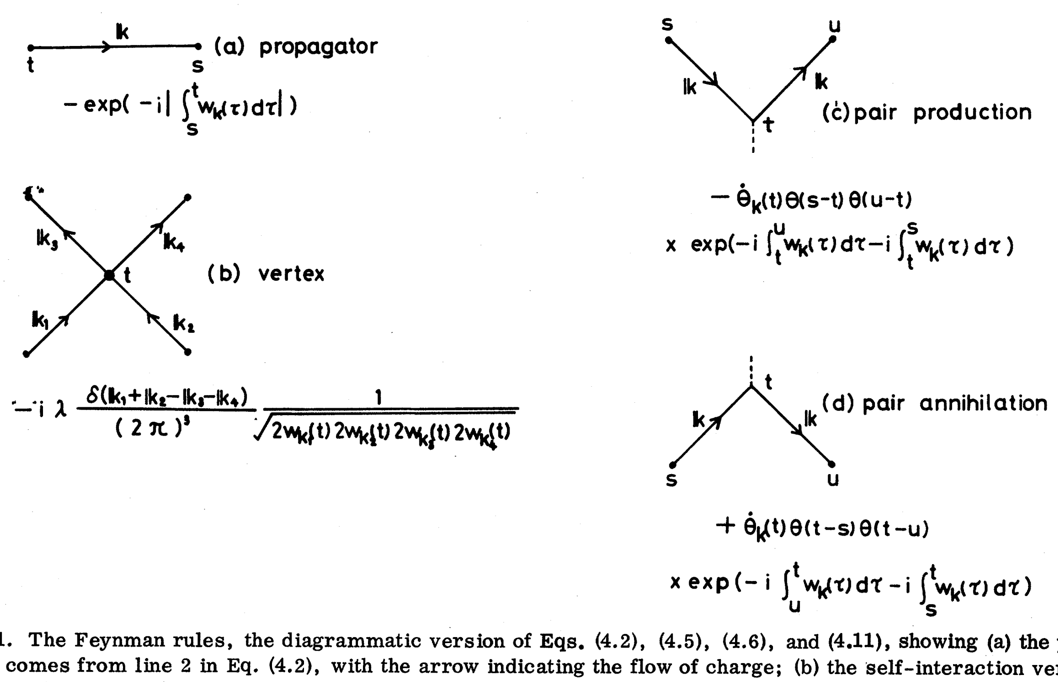



FIG. 1. The Feynman rules, the diagrammatic version of Eqs. (4.2), (4.5), (4.6), and (4.11), showing (a) the propagator that comes from line 2 in Eq. (4.2), with the arrow indicating the flow of charge; (b) the self-interaction vertex which comes from Eq. (4.6); and (c) and (d) the vertices of pair production and annihilation. The dashed line merely indicates that time $t$ is earlier (or later) than $u$ and $s$, and it does not mean directly an external field. That comes from lines 3–8 in Eq. (4.2). The factor of external lines is easily calculated from Eqs. (4.2) and (4.11).

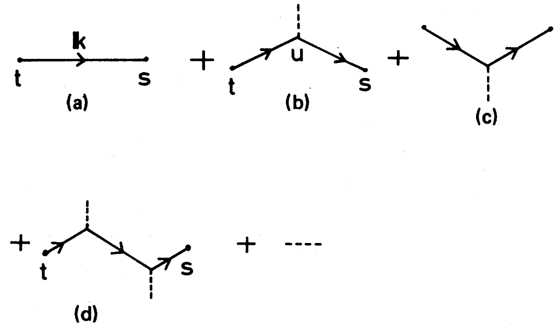



FIG. 2. Graphs which contribute to the two-point function in zeroth order in the coupling constant.

spacetime is

$$\mathcal{L}_{\rm int} = -\frac{\lambda\mu^\epsilon}{4}[\psi(x)\psi^\dagger(x)]^2\,, \tag{5.4}$$

where $\epsilon = 4 - D$, and divergences will appear in the form of poles as $\epsilon \to 0$. The contribution from the graph of Fig. 3 is

$$\text{graph } 3 = i\lambda\mu^\epsilon \int du \int \frac{d^{D-1}p}{(2\pi)^{D-1}} G_F^0(t,u;\vec{k}) G_F^0(u,u;\vec{p}) \times G_F^0(u,s;\vec{k})\,, \tag{5.5}$$

where $G_F^0(t,s;\vec{k})$ is a "free" propagator

$$G_F^0(t,s;\vec{k}) = -i\frac{\exp[-i|\int_s^t \omega_k(\tau)d\tau|]}{2i[\omega_k(t)\omega_k(s)]^{1/2}}\,. \tag{5.6}$$

In Eq. (5.5), we must estimate the following integral:

$$\begin{aligned} I_1 &= i\lambda\mu^\epsilon \int \frac{d^{D-1}p}{(2\pi)^{D-1}} G_F^0(u,u;\vec{p}) \\ &= \lambda\mu^\epsilon \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{1}{2i\omega_p(u)} \\ &= \lambda\mu^\epsilon \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 - M^2(u) + i\delta} \\ &= \frac{i\lambda}{8\pi^2} M^2(u)\left[\frac{1}{\epsilon} + \frac{1}{2}\ln\frac{4\pi\mu^2}{M^2(u)} - \frac{1}{2}\gamma\right] + O(\epsilon)\,, \end{aligned} \tag{5.7}$$

where $\gamma$ is Euler's constant. In our discussion so far, mass parameter $m^2$ was considered to be the

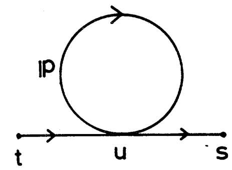



FIG. 3. The one-loop correction for the two-point function.

physical mass. However, for convenience, in the discussion of the ultraviolet divergences, we use the minimal-subtraction scheme. The renormalization constants of mass, the coupling constant, and the wave function are introduced as usual, that is,

$$\begin{aligned} \mathcal{L}_{\rm tot} &= \eta^{\mu\nu}\partial_\mu\psi_0\partial_\nu\psi_0^\dagger - M_0{}^2\psi_0\psi_0^\dagger - \frac{\lambda_0}{4}(\psi_0\psi_0^\dagger)^2 \\ &= \eta^{\mu\nu}Z_2\partial_\mu\psi_{re}\partial_\nu\psi_{re}^\dagger - Z_M M^2 Z_2\psi_{re}\psi_{re}^\dagger \\ &\quad - \frac{\lambda\mu^\epsilon}{4}Z_\lambda Z_2{}^2(\psi_{re}\psi_{re}^\dagger)^2\,, \end{aligned} \tag{5.8}$$

where the subscript zero on any quantity denotes that it is a bare quantity, and we expand the renormalization constants in the power of the renormalized coupling constant $\lambda$ as

$$Z_j = 1 + \sum_{n=1} Z_j^{(n)}\lambda^n\,, \quad j = M,\ \lambda,\ \text{and } 2\,. \tag{5.9}$$

The counterterm of the mass renormalization in first order in the coupling constant is given from Eq. (5.7),

$$\begin{aligned} \mathcal{L}_{\rm ct}^{(M)} &= -\frac{\lambda}{8\pi^2}\frac{1}{\epsilon}M^2(t)\psi\psi^\dagger \\ &= -\lambda Z_M^{(1)} M^2(t)\psi\psi^\dagger\,. \end{aligned} \tag{5.10}$$

We also calculate the graphs for the exact propagator with the vertices of pair creation and (or) pair annihilation in first order in the coupling constant. Some of these graphs are given in Fig. 4, and it is proved that they are all finite quantities without ultraviolet divergences. For example, the contribution from the graphs of Figs. 4(a) and 4(b)

$$\begin{aligned} [\text{graphs } 4(\text{a}) + 4(\text{b})] = i\lambda \int du \int dv \int \frac{d^3p}{(2\pi)^3} G_F^0(t,u;\vec{k}) G_F^0(u,s;\vec{k}) \Big[&\theta(v-u)\exp\Big(-2i\int_u^v \omega_p(\tau)d\tau\Big) \\ &- \theta(u-v)\exp\Big(-2i\int_v^u \omega_p(\tau)d\tau\Big)\Big]\frac{\dot{\theta}_p(v)}{2\omega_p(u)}\,. \end{aligned} \tag{5.11}$$

From Eq. (4.12) the following integral in the above equation,

$$J = -\int \frac{d^3p}{(2\pi)^3}\Big[\theta(v-u)\exp\Big(-2i\int_u^v \omega_p(\tau)d\tau\Big) - \theta(u-v)\exp\Big(-2i\int_v^u \omega_p(\tau)d\tau\Big)\Big]\frac{\dot{M}^2(v)}{8\omega_p(u)\omega_p{}^2(v)}\,,$$

is finite. It is also verified that the graphs with more creations and annihilations of pairs, e.g., Fig. 4(c), are all finite. This fact is very important and the renormalization of mass in this theory is identical with that in Minkowski space-time.

We turn to the investigation of the four-point function. In one-loop order, only the graphs given in Fig. 5 have divergences, and we first evaluate the graph of Fig. 5(a),

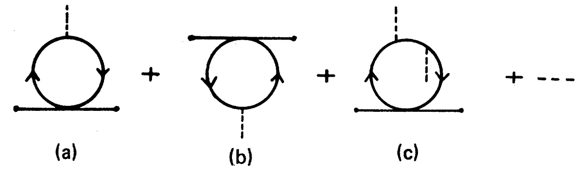



FIG. 4. One-loop corrections for the two-point function with pair creation and/or annihilation.

$$\text{graph 5(a)}=\frac{-i\lambda\mu^{\epsilon}}{(2\pi)^{D-1}}(-i\lambda\mu^{\epsilon})\int du\int dv\int\frac{d^{D-1}k}{(2\pi)^{D-1}}G_F^0(t,u;\vec{p}_1)G_F^0(u,s;\vec{p}_2)G_F^0(u,v;\vec{k})G_F^0(v,u;\vec{k}+\vec{p})$$
$$\times G_F^0(t',v;\vec{p}_3)G_F^0(v,s';\vec{p}_4)$$
$$=\frac{-i\lambda\mu^{\epsilon}}{(2\pi)^{D-1}}i\lambda\mu^{\epsilon}\int du\int dv G_F^0(t,u;\vec{p}_1)G_F^0(u,s;\vec{p}_2)G_F^0(t',v;\vec{p}_3)G_F^0(v,s';\vec{p}_4)$$
$$\times\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{\exp[-i|\int_u^v\omega_k(\tau)d\tau|]}{2i[\omega_k(u)\omega_k(v)]^{1/2}}\frac{\exp(-i|\int_u^v\omega_{k+p}(\tau)d\tau|)}{2i[\omega_{k+p}(u)\omega_{k+p}(v)]^{1/2}}\,, \tag{5.12}$$

where $\vec{p}=\vec{p}_2-\vec{p}_1=\vec{p}_3-\vec{p}_4$. From Eq. (5.12) we estimate the divergent part of the integral,

$$I_2(u,v;p)=i\lambda\mu^{\epsilon}\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{\exp[-i|\int_u^v\omega_k(\tau)d\tau|]}{2i[\omega_k(u)\omega_k(v)]^{1/2}}\frac{\exp[-i|\int_u^v\omega_{k+p}(\tau)d\tau|]}{2i[\omega_{k+p}(u)\omega_{k+p}(v)]^{1/2}}$$
$$=i\lambda\mu^{\epsilon}\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{\exp[-i\omega_k(u)|v-u|]}{2i\omega_k(u)}\frac{\exp[-i\omega_{k+p}(u)|v-u|]}{2i\omega_{k+p}(u)}+\text{finite}$$
$$=i\lambda\mu^{\epsilon}\int\frac{d^Dk}{(2\pi)^D}\int\frac{dp_0}{2\pi}\frac{e^{-ik_0(v-u)}}{k^2-M^2(u)+i\delta}\frac{e^{ip_0(v-u)}}{p_0{}^2-(\mathbf{k}+\vec{p})^2-M^2(u)+i\delta}+\text{finite}$$
$$=i\lambda\mu^{\epsilon}\int\frac{dp_0}{2\pi}e^{-ip_0(v-u)}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{[k^2-M^2(u)+i\delta][(k+p)^2-M^2(u)+i\delta]}+\text{finite}$$
$$=\int\frac{dp_0}{2\pi}e^{-ip_0(v-u)}\left\{\frac{-\lambda}{\delta\pi^2\epsilon}+\frac{\lambda}{16\pi^2}\left[\gamma+\int_0^1dx\ln\frac{M^2(u)+x(x-1)p^2}{4\pi\mu^2}\right]\right\}+\text{finite}$$
$$=-\frac{\lambda}{8\pi^2\epsilon}\delta(v-u)+\text{finite}\,. \tag{5.13}$$

Calculating the divergences in the crossed diagrams 5(b) and 5(c) in the same manner, we obtain the counterterm for the coupling-constant renormalization,

$$\mathcal{L}_{\text{ct}}^{(\lambda)}=-\lambda\mu^{\epsilon}\frac{5\lambda}{16\pi^2\epsilon}\frac{1}{4}(\psi\psi^{\dagger})^2$$
$$=-Z_\lambda^{(1)}\lambda\frac{\lambda\mu^{\epsilon}}{4}(\psi\psi^{\dagger})^2\,. \tag{5.14}$$

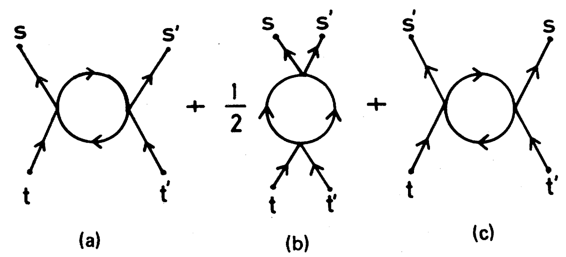



FIG. 5. One-loop corrections for the four-point function.

### B. Two-loop level: renormalization of wave function

In one-loop order, the renormalizations of mass and coupling constant have been demonstrated, and in two-loop order we meet another type of divergence, which is concerned with wave-function renormalization. For convenience, we divide the free propagator into two parts,

$$G_F^0(t,s;\vec{k})=-i\frac{\exp[-i|\int_s^t\omega_k(\tau)d\tau|]}{2i[\omega_k(t)\omega_k(s)]^{1/2}}$$
$$=D_1(t,s;\vec{k})+D_2(t,s;\vec{k})\,, \tag{5.15}$$

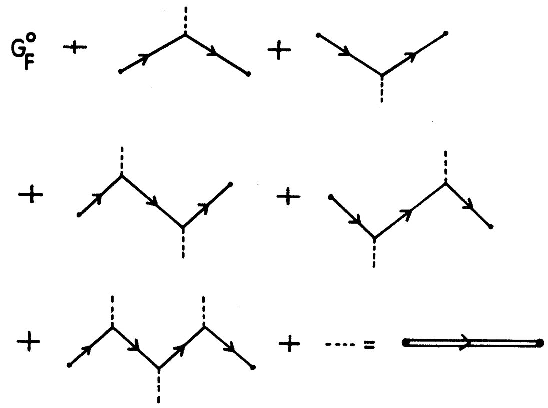
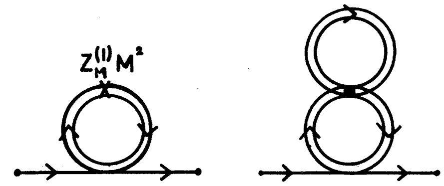


FIG. 6. The corrected two-point function in zeroth order in the coupling constant.




(a) (b)

FIG. 7. Second-order corrections for the two-point function.

where

$$D_1(t,s;\vec{k}) = -i\int \frac{dk_0}{2\pi}\frac{e^{-ik_0(s-t)}}{k_0^2-\omega_k^2(t)+i\delta}, \tag{5.16}$$

and $D_2(t,s;\vec{k})$ is a regular function and especially satisfies

$$D_2(t,t;\vec{k}) = 0. \tag{5.17}$$

Adding all contributions of pair-creation and pair-annihilation vertices, we denote the exact propagator in zeroth order in the coupling constant $\lambda$ to be $D_F(t,s;\vec{k})$, that is,

$$D_F(t,s;\vec{k}) = D_1(t,s;\vec{k}) + D_R(t,s;\vec{k}), \tag{5.18}$$

where $D_R(t,s;\vec{k})$ consists of $D_2(t,s;\vec{k})$ in Eq. (5.15) and the contributions from the graphs given in Fig. 6. From Eq. (5.17) and the discussion in Sec. V A, it is proved that

$$D_R(t) = \int \frac{d^3k}{(2\pi)^3} D_R(t,t;\vec{k}) = \text{finite}. \tag{5.19}$$

As indicated in Fig. 6, we use a double line for $D_F(t,s;\vec{k})$ in diagrammatic rules.

The graphs which contribute to the propagator in two-loop order are given in Figs. 7–9, and thus

$$\text{graph 7(a)} = -(i\lambda\mu^{\epsilon})iZ_M^{(1)}\lambda\int du\int dv\, M^2(v)\int\frac{d^{D-1}p}{(2\pi)^{D-1}}G_F^0(t,u;\vec{k})G_F^0(u,s;\vec{k})D_F(u,v;\vec{p})D_F(v,u;\vec{p}), \tag{5.20}$$

$$\begin{aligned}\text{graph 7(b)} &= -(i\lambda\mu^{\epsilon})^2\int du\int dv\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\int\frac{d^{D-1}q}{(2\pi)^{D-1}}G_F^0(t,u;\vec{k})G_F^0(u,s;\vec{k})D_F(u,v;\vec{p})D_F(v,u;\vec{p})\,G_F^0(v,v;\vec{q})\\ &= -(-i\lambda\mu^{\epsilon})^2\int du\int dv\int\frac{d^{D-1}}{(2\pi)^{D-1}}G_F^0(t,u;\vec{k})G_F^0(u,s;\vec{k})\\ &\qquad\times D_F(u,v;\vec{p})D_F(v,u;\vec{p})\left\{\mu^{-\epsilon}M^2(v)\left[Z_M^{(1)}+\frac{1}{2^4\pi^2}\left(\ln\frac{4\pi\mu^2}{M^2(v)}-\gamma\right)\right]+D_R(v)\right\},\end{aligned} \tag{5.21}$$

$$\begin{aligned}\text{graph 8} &= -(i\lambda\mu^{\epsilon})Z_\lambda^{(1)}\lambda\int du\int\frac{d^{D-1}p}{(2\pi)^{D-1}}G_F^0(t,u;\vec{k})G_F^0(u,s;\vec{k})D_F(u,u;\vec{p})\\ &= -(-i\lambda\mu^{\epsilon})Z_\lambda^{(1)}\lambda\int du\, G_F^0(t,u;\vec{k})G_F^0(u,s;\vec{k})\left\{\mu^{-\epsilon}M^2(u)\left[Z_M^{(1)}+\frac{1}{2^4\pi^2}\left(\ln\frac{4\pi\mu^2}{M^2(u)}-\gamma\right)\right]+D_R(u)\right\},\end{aligned} \tag{5.22}$$

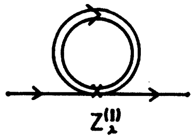



FIG. 8. Second-order corrections for the two-point function.

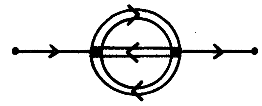


FIG. 9. Second-order corrections for the two-point function.

$$\text{graph } 9 = -\tfrac{1}{2}(-i\lambda\mu^{\epsilon})^2\int du\int dv\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\int\frac{d^{D-1}q}{(2\pi)^{D-1}}G_F^0(t,u;\vec{k})G_F^0(v,s;\vec{k})D_F(u,v;\vec{p})\times D_F(v,u;\vec{q})D_F(v,u;\vec{p}-\vec{q}-\vec{k}),\tag{5.23}$$

where we have used Eqs. (5.7), (5.18), and (5.19). Adding Eqs. (5.20) and (5.21),

$$\begin{aligned}\text{graph } 7(a)+7(b) &= -(-i\lambda\mu^{\epsilon})^2\int du\int dv\, G_F^0(t,u;\vec{k})G_F^0(u,s;\vec{k})\left[D_R(v)+\frac{\mu^{-\epsilon}}{2^4\pi^2}\left(\ln\frac{4\pi\mu^2}{M^2(v)}-\gamma\right)\right]\\&\quad\times\int\frac{d^{D-1}p}{(2\pi)^{D-1}}D_F(u,v;\vec{p})D_F(v,u;\vec{p})\\&=(-i\lambda\mu^{\epsilon})^2\int du\int dv\, G_F^0(t,u;\vec{k})G_F^0(u,s;\vec{k})\left[D_R(v)+\frac{\mu^{-\epsilon}}{2^4\pi^2}\left(\ln\frac{4\pi\mu^2}{M^2(v)}-\gamma\right)\right]\int\frac{dq_0}{2\pi}e^{-iq_0(u-v)}\\&\quad\times\int\frac{d^Dp}{(2\pi)^D}\frac{1}{[p^2-M^2(u)+i\delta][(p_0-q_0)^2-\vec{p}^2-M^2(u)+i\delta]}+\text{finite}\\&=-i\lambda\mu^{\epsilon}\frac{\lambda}{2^3\pi^2}\frac{1}{\epsilon}\int du\, G_F^0(t,u;\vec{k})G_F^0(u,s;\vec{k})\left[D_R(u)+\frac{\mu^{-\epsilon}M^2(u)}{2^4\pi^2}\left(\ln\frac{4\pi\mu^2}{M^2(u)}-\gamma\right)\right]+\text{finite}.\end{aligned}\tag{5.24}$$

In Eq. (5.23), we estimate the integral

$$\begin{aligned}I_3(u,v)&=-\tfrac{1}{2}(-i\lambda\mu^{\epsilon})^2\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\int\frac{d^{D-1}q}{(2\pi)^{D-1}}D_F(u,v;\vec{p})D_F(v,u;\vec{q})D_F(v,u;\vec{p}-\vec{q}-\vec{k})\\&=-\tfrac{1}{2}(-i\lambda\mu^{\epsilon})^2\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\int\frac{d^{D-1}q}{(2\pi)^{D-1}}[D_1(u,v;\vec{p})D_1(u,v;\vec{q})D_1(u,v;\vec{p}-\vec{q}-\vec{k})\\&\quad+3D_1(u,v;\vec{p})D_1(u,v;\vec{q})D_R(u,v;\vec{p}-\vec{q}-\vec{k})\\&\quad+3D_1(u,v;\vec{p})D_R(u,v;\vec{q})D_R(u,v;\vec{p}-\vec{q}-\vec{k})\\&\quad+D_R(u,v;\vec{p})D_R(u,v;\vec{q})D_R(u,v;\vec{p}-\vec{q}-\vec{k})],\end{aligned}\tag{5.25}$$

and from Eq. (5.16), only the first and second terms in the above square bracket are divergent,

$$\begin{aligned}I_3^{(1)}&=\mu^{2\epsilon}\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\int\frac{d^{D-1}q}{(2\pi)^{D-1}}D_1(u,v;\vec{p})D_1(u,v;\vec{q})D_1(u,v;\vec{p}-\vec{q}-\vec{k})\\&=(-i)^3\mu^{2\epsilon}\int\frac{d^Dp}{(2\pi)^D}\int\frac{d^Dq}{(2\pi)^D}\int\frac{dl_0}{2\pi}\frac{e^{-ip_0(u-v)}}{p^2-M^2(u)+i\delta}\frac{e^{-iq_0(u-v)}}{q^2-M^2(u)+i\delta}\frac{e^{-il_0(u-v)}}{l_0^2-(\vec{p}-\vec{q}-\vec{k})^2-M^2(u)+i\delta}\\&=i\mu^{2\epsilon}\int\frac{dk_0}{2\pi}e^{-ik_0(u-v)}\int\frac{d^Dp}{(2\pi)^D}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{p^2-M^2(u)+i\delta}\frac{1}{q^2-M^2(u)+i\delta}\frac{1}{(p-q-k)^2-M^2(u)+i\delta}\\&=i\mu^{2\epsilon}\int\frac{dk_0}{2\pi}e^{-ik_0(u-v)}\frac{1}{2^{2D}\pi^D}\frac{\Gamma(\epsilon)}{3-D}\left[6\frac{(M^2(u))^{D-3}}{\epsilon}-\tfrac{1}{2}k^2+3M^2(u)+O(\epsilon)\right]\\&=i\left\{\frac{-3M^2(u)}{2^7\pi^4\epsilon^2}-\frac{3M^2(u)}{2^7\pi^4\epsilon}\left[\ln\frac{4\pi\mu^2}{M^2(u)}-\gamma\right]-\frac{1}{2^8\pi^4\epsilon}[\tfrac{1}{2}(\partial_u^2+\vec{k}^2)+3M^2(u)]\right\}\delta(u-v)+\text{finite},\end{aligned}\tag{5.26}$$

where we have used the formula which was derived by Collins,[6]

$$\begin{aligned}I_3^{(2)}&=\mu^{\epsilon}\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\int\frac{d^{D-1}q}{(2\pi)^{D-1}}D_1(u,v;\vec{p})D_1(u,v;\vec{q})D_R(u,v;\vec{p}-\vec{q}-\vec{k})\\&=\mu^{\epsilon}\int\frac{d^{D-1}x}{(2\pi)^{D-1}}\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\int\frac{d^{D-1}q}{(2\pi)^{D-1}}\int\frac{d^{D-1}l}{(2\pi)^{D-1}}D_1(u,v;\vec{p})D_1(u,v;\vec{q})D_R(u,v;\vec{l})e^{-i\vec{x}\cdot(\vec{l}-\vec{p}+\vec{q}+\vec{k})}\\&=-\mu^{\epsilon}\int\frac{d^{D-1}x}{(2\pi)^{D-1}}\int\frac{d^{D-1}l}{(2\pi)^{D-1}}e^{-i\vec{x}\cdot(\vec{l}+\vec{k})}D_R(u,v;\vec{l})\int\frac{d^Dp}{(2\pi)^D}\int\frac{d^Dq}{(2\pi)^D}\frac{e^{-ip_0(u-v)}}{p^2-M^2(u)+i\delta}\frac{e^{iq_0(u-v)}}{q^2-M^2(u)+i\delta}e^{-i\vec{x}\cdot(\vec{q}-\vec{p})}\\&=-\mu^{\epsilon}\int\frac{d^{D-1}x}{(2\pi)^{D-1}}\int\frac{d^{D-1}l}{(2\pi)^{D-1}}e^{-i\vec{x}\cdot(\vec{l}+\vec{k})}D_R(u,v;\vec{l})\int\frac{d^Dp}{(2\pi)^D}e^{-ip_0(u-v)+i\vec{p}\cdot\vec{x}}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{[q^2-M^2(u)+i\delta][(q+p)^2-M^2(u)+i\delta]}\\&=-\int\frac{d^{D-1}x}{(2\pi)^{D-1}}\int\frac{d^{D-1}l}{(2\pi)^{D-1}}e^{-i\vec{x}\cdot(\vec{l}+\vec{k})}D_R(u,v;\vec{l})\int\frac{d^Dp}{(2\pi)^D}e^{-ip_0(u-v)+i\vec{p}\cdot\vec{x}}\frac{i}{2^3\pi^2}\frac{1}{\epsilon}+\text{finite}\\&=-\frac{i}{2^3\pi^2}\frac{1}{\epsilon}\delta(u-v)D_R(u)+\text{finite}.\end{aligned}\tag{5.27}$$

Adding the contributions from the graphs of Figs. 7(a), 7(b), 8, and 9, the divergent part of the Feynman propagator in two-loop order is

$$\lambda^2 \int du \int dv\, G_F^0(t,u;\vec{k})G_F^0(v,s;\vec{k})\left[\frac{7i}{2^8\pi^4\epsilon^2} - \frac{3i}{2^8\pi^4\epsilon} M^2(u)\delta(u-v) - \frac{i}{2^9\pi^4\epsilon}(\partial_u{}^2 + \vec{k}^2)\delta(u-v)\right]. \tag{5.28}$$

In the contributions from the individual graphs there exist terms of the form $(1/\epsilon)D_R(t)$ which cannot be eliminated by renormalization. However, these terms cancel each other in the final result (5.28). Therefore in two-loop order, the renormalization constants of mass and wave function are

$$Z_M^{(2)} = \frac{7}{2^8\pi^4}\frac{1}{\epsilon^2} - \frac{3}{2^8\pi^4}\frac{1}{\epsilon}, \tag{5.29}$$

$$Z_2^{(2)} = -\frac{1}{2^9\pi^4}\frac{1}{\epsilon}. \tag{5.30}$$

In a similar way, one can investigate the ultraviolet divergences of the four-point function in two-loop order, and verify that the theory under consideration is renormalizable and the renormalization constants are identical with those in the theory in Minkowski spacetime.

If we discuss the renormalization, starting with the original Lagrangian (2.2), the essential feature of the renormalization is unchanged, but the parameter $\xi$ suffers another renormalization which comes from the wave-function renormalization, as can be easily seen, and the finite terms are slightly altered, that is, the physical meanings of the renormalized mass, $\xi$, and the coupling constant are slightly modified. However, there is one thing to be mentioned. If we start with the Lagrangian (2.2) in $D$ dimensions, the interaction term of the rescaled fields becomes

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4}(\mu c(t))^\epsilon(\psi(x)\psi^\dagger(x))^2 \tag{5.31}$$

instead of Eq. (5.4). Singular parts in radiative corrections are absorbed by renormalization with the same renormalization constants obtained in this section. After the coupling-constant renormalization, in addition to the ordinary terms, there remain terms of order $\epsilon$ having the form $\ln[\mu^2c^2(t)/p^2]$. When they combine with other singular parts of order $1/\epsilon$, there appear time-dependent terms of order $\epsilon^0$ and particle production takes place even in the theory of massless and conformed coupling $[m^2=0$ and $\xi=\frac{1}{4}(D-2)/(D-1)]$, as suggested by Birrell and Davies.[7]

## VI. CONCLUSION AND DISCUSSION

In this paper, we have considered the charged scalar field with self-interaction in the spatially flat Robertson-Walker metric and constructed the method of calculation for perturbation expansion. In this discussion, the functional integral technique in the coherent-state form plays an essential role, and we have introduced creation and annihilation operators of the instantaneous particle. Although it may be considered that there is some physical entity corresponding to it, it seems correct that the concept of instantaneous particle is introduced merely as a calculational device. [In fact if we can solve exactly the field equations in curved spacetime or evaluate the functional integral (3.15) without expanding it in powers of $\dot{\theta}(t)$, the description, in terms of the instantaneous particle and the vertices of pair production and pair annihilation, does not appear in the final results.] In the general case, the in Fock space and out Fock space are two different Hilbert spaces, and it would seem that this awkward point did not surface superficially in our discussion. However, this is not true. For example, quantities such as the $S$-matrix element $\langle g;\text{out}|f;\text{in}\rangle$ and the Green's function $\langle\text{vac; out}|T(\psi(x)\psi^\dagger(y))|\text{vac; in}\rangle$ have meaning only when they are divided, for example, by the amplitude from the in vacuum to the out vacuum, $\langle\text{vac; out}|\text{vac; in}\rangle$, since one need not consider the disconnected diagrams. To see this more clearly, we calculate the vacuum-to-vacuum amplitude in the zeroth order in the coupling constant. The graphs which contribute to this amplitude are given in Fig. 10 and they do not contain ultraviolet divergences. However,

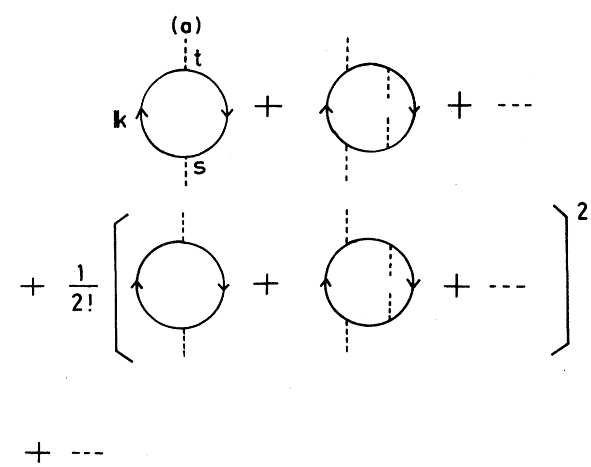



FIG. 10. The graphs which contribute to the vacuum-to-vacuum amplitude.

$$\text{graph 10(a)} = -\int d^3k \int dt \int d^3p \int ds\, \dot\theta_k(t)\dot\theta_p(s) \frac{\delta^2}{\delta\xi(-\vec{k},t)\delta\zeta(\vec{k},t)} \frac{\delta^2}{\delta\xi^*(-\vec{p},s)\delta\zeta^*(\vec{p},s)}$$
$$\times \exp\left\{-\int d^3q \left[\int_{-\infty}^{+\infty} du \int_{-\infty}^{u} dv\, \zeta(\vec{q},u)\zeta^*(\vec{q},v) \exp\left(-i\int_v^u \omega_q(\tau)d\tau\right)\right.\right.$$
$$+\int_{-\infty}^{+\infty} du \int_{-\infty}^{u} dv\, \xi(-\vec{q},u)\xi^*(-\vec{q},v)$$
$$\left.\left.\times \exp\left(-i\int_v^u \omega_q(\tau)dt\right)\right]\right\}\Bigg|_{\xi=\zeta=\xi^*=\zeta^*=0}$$
$$= -V\left[\int d^3k \int dt \int ds\, \theta(t-s)\dot\theta_k(t)\dot\theta_k(s) \exp\left(-2i\int_s^t \omega_k(\tau)d\tau\right)\right], \tag{6.1}$$

where $V$ is the volume of the space in which we consider the quantum field theory (or in other words, the space in which the external field has sensible time dependence, in a general case). Adding all contributions in zeroth order in the coupling constant, the amplitude under consideration is

$$\langle \text{vac; out} \mid \text{vac; in}\rangle \propto e^{-VA}, \tag{6.2}$$

where finite quantity $A$ has the dimension $(\text{length})^{-3}$, and generally $A$ has a positive real part. Similarly, it is easily verified that after the renormalization, for any $|f\rangle \in \mathfrak{F}_{\text{in}}$ and $|g\rangle \in \mathfrak{F}_{\text{out}}$,

$$\langle g;\ \text{out}|f;\ \text{in}\rangle = \text{finite} \times \langle \text{vac; out} \mid \text{vac; in}\rangle\,. \tag{6.3}$$

Notice that the Fock space is constructed to be separable, that is, it contains only a countable number of basis vectors. This is in the context that in and out Fock spaces become two different representations of the field operators in the limit $V \to \infty$. If we think naively, the above fact suggests to us to cease adhering to the Fock space and to consider a wider space, such as the space of the coherent states for all annihilation operators of the asymptotic fields. If we do so, it of course produces other difficulties. Although quantities, such as the expectation value of the number of produced particles, have meaning, at present we have no suitable solution for this problem.

Making use of the calculational method obtained here, we have discussed the ultraviolet divergences and the renormalizability of the theory, and it has been proved that the essential feature of the renormalization does not depend on the structure of the spacetime. It is interesting to compare our method with that given by Bunch, Panangaden, and Parker.[1] They approximately obtained the Green's function in the in Fock space and, using this Green's function, considered ultraviolet divergences.


## ACKNOWLEDGMENTS

It is a pleasure to thank Y. Yamagata for several valuable discussions and for reading the manuscript.



[1] T. S. Bunch and L. Parker, Phys. Rev. D 20, 2499 (1979); T. S. Bunch, P. Panangaden, and L. Parker, J. Phys. A 13, 901 (1980); T. S. Bunch and P. Panangaden, *ibid.* 13, 919 (1980).

[2] N. D. Birrell and L. H. Ford, Phys. Rev. D 22, 330 (1980).

[3] N. B. Birrell and J. G. Taylor, J. Math. Phys. 21, 1740 (1980).

[4] S. S. Schweber, J. Math. Phys. 3, 831 (1962); L. D. Faddeev, in *Methods in Field Theory*, 1975 Les Houches Lectures, edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976), p. 1.


[5] It is easily verified that $U(t_1, t_2)$ satisfies the following equations:

$$\frac{\partial}{\partial t_1}U(t_1,t_2) = -iU(t_1,t_2)H_H(t_1),$$
$$\frac{\partial}{\partial t_2}U(t_1,t_2) = iH_H(t_2)U(t_1,t_2),$$

and

$$U(t_1,t_1) = 1.$$

Using Eqs. (2.16a), (2.22a), (2.25), and (3.2a), it is proved that

$$V(t_1,t_2)U^{-1}(t_1,t_2)\alpha_H(t_2)U(t_1,t_2)V^{-1}(t_1,t_2)$$
$$= V(t_1,t_2)[a_H(t_1)\cosh\theta(t_2) - b_H^\dagger(t_1)\sinh\theta(t_2)]V^{-1}(t_1,t_2)$$
$$= a_H(t_1)\cosh\theta(t_1) - b_H^\dagger(t_1)\sinh\theta(t_1)$$
$$= \alpha_H(t_1),$$

and similarly for $\beta_H(t)$. Making use of the above identities, we arrive at the last line of Eq. (3.12).


[6] J. C. Collins, Phys. Rev. D 10, 1213 (1974).

[7] N. D. Birrell and P. C. W. Davies, Phys. Rev. D 22, 322 (1980).