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PAPER

Time Series Analysis Based on Exponential Model Excited by *t*-Distribution Process and Its Algorithm

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SUMMARY In this paper, a new time series analysis method is proposed. The proposed method uses the exponential (EXP) model. The residual signal is assumed to be identically and independently distributed (IID). To achieve accurate and efficient estimates, the parameter of the system model is calculated by maximizing the logarithm of the likelihood of the residual signal which is assumed to be IID *t*-distribution. The EXP model theoretically assures the stability of the system. This model is appropriate for analyzing signals which have not only poles, but also poles and zeroes. The asymptotic efficiency of the EXP model is addressed. The optimal solution is calculated by the Newton-Raphson iteration method. Simulation results show that only a small number of iterations are necessary to reach stationary points which are always local minimum points. When the method is used to estimate the spectrum of synthetic signals, by using small α we can achieve a more accurate and efficient estimate than that with large α. To reduce the calculation burden an alternative algorithm is also proposed. In this algorithm, the estimated parameter is updated in every sampling instant using an imperfect, short-term, gradient method which is similar to the LMS algorithm.

key words: *exponential model, t-distribution, M-estimate, time series analysis, finite-length cepstrum*

1. Introduction

The estimation of parameters of a time series has been a widely addressed problem. In many cases and also in the conventional linear prediction (CLP) method, the autoregressive (AR) system model is used. In the CLP method, the coefficients of the models are calculated to minimize the sum of the square of the residuals. The autocorrelation and the covariance methods are usually utilized to calculate the coefficients of the predictor.[1] Unfortunately, the conventional method can achieve good estimation results only when the encountered driving source is a Gaussian process.[2]–[4] It is well known that in many cases the source is of a quasi-periodic nature with spiky excitation which is not a Gaussian process, such as voiced-speech signals, etc. For these kinds of processes the obtained results from the CLP is biased and inefficient.[2]–[5] The obtained estimates are very much affected by the strong signal parts. We cannot accurately and efficiently estimate the parameters, such as formant frequencies and bandwidths.

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To improve the accuracy and the efficiency of the estimator, we proposed recently the assumption that the residual signal has *t*-distribution with a small α degree of freedom and the use of the autoregressive (AR) system model.[2],[3] When $\alpha=\infty$, we get the Gaussian distribution. This case is equivalent to the CLP method. The *t*-distribution with small α has more probability on its tail than with large α. By using the small α assumption, we assume that the residual signal is more spiky than in the Gaussian assumption. By doing so, the optimal predictor is calculated by assigning a large weighting factor for the small amplitude residuals and a small weighting factor for the large amplitude residuals. In this way, the effect of the large amplitude signal on the obtained predictor is reduced, so that we can get less biased and more efficient estimates. The nature of the equation enables us to theoretically guarantee the stability of the inverse system in the conventional autocorrelation method only. In our previous method, the stability of the obtained inverse system can only be shown experimentally. Since in many cases, such as the speech analysis, the stability of the inverse system is one of the important factors,[1] extending our previous result, in this paper we propose an estimation method by assuming that the residual signal is a *t*-distribution process to achieve accurate and efficient estimates and use the theoretically stable exponential (EXP) system model. When $\alpha=\infty$, we get the similar method proposed by Imai et al.,[6] so this method can be seen as a generalization or an improvement of that method. The EXP model can represent poles and zeroes with equal weight. Thus we can use the EXP model instead of the ARMA model which has the stability problem.

Furthermore, to reduce the calculation burden, an alternative algorithm is proposed. In this approach, the estimation result is updated in every sampling instant using an imperfect, short-term, gradient which is similar to the LMS algorithm.[12] This algorithm is particularly useful in the implementation using real-time systems.

This paper is arranged as follows. The preliminary discussions and the derivation of the asymptotic efficiency are given in Sects. 2 and 3, respectively. Section 4 describes the method of calculating the solution and the basic properties of the proposed method.

The alternative algorithm is explained in Sect. 5. The simulation results are presented in Sect. 6, and this paper is concluded in Sect. 7.

2. Preliminary Discussions

We consider a zero mean stationary time series s_i which is generated by feeding a certain IID random process into a p-th order exponential model EXP(p). The signal is observed along a window $0 \leqq i \leqq M-1$. The number of samples of M is assumed to be large, $M \to \infty$. The signal outside the window is consider to be zero. The transfer function of the exponential model EXP(p) is

$$H(z) = KD(z), \tag{1}$$

$$D(z) = \exp\left\{\sum_{j=1}^{p} c_j z^{-j}\right\}, \quad K = \exp c_0. \tag{2}$$

The coefficients c_j, $1 \leqq j \leqq p$, represent a finite length cepstrum coefficient. The gain factor is K. The impulse responses of $D(z)$ and $1/D(z)$ have an infinite length. Therefore the EXP model is appropriate for modeling signals which have not only poles, but also both poles and zeroes. The EXP model in Eq. (1) can be represented by

$$H(z) = \frac{K}{A(z)}, \tag{3}$$

where

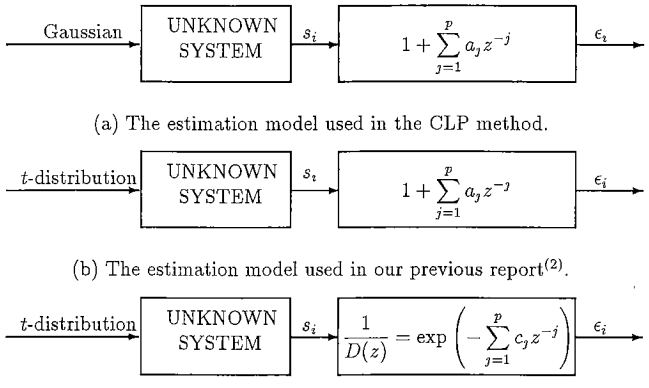
$$A(z) = \sum_{j=0}^{\infty} a_j z^{-j}, \quad a_0 = 1 \tag{4}$$

is the inverse system transfer function, so that the residual samples ε_i; $0 \leqq i \leqq M-1$; can be calculated from the signal samples s_i; $0 \leqq i \leqq M-1$; by

$$\varepsilon_i = a_i * s_i. \tag{5}$$

The sign "$*$" stands for convolution.

Similar to our previous result,[2],[3] in this paper we assume that the probability density function (PDF) of the residual signal ε_i is known. The models used in the CLP, in our previous paper,[2] and in this paper are shown in Figs. 1(a), 1(b) and 1(c), respectively.



(a) The estimation model used in the CLP method.

(b) The estimation model used in our previous report[2].

(c) The estimation model used in this paper.

Fig. 1 The block diagram of various estimation models.

The joint PDF of the observed signal $\boldsymbol{s}$ can be expressed in the term of the joint PDF of the residual signal $\boldsymbol{\varepsilon}$, where

$$\boldsymbol{s} = [s_0 \ s_1 \ \cdots \ s_{M-1}]^T \text{ and } \boldsymbol{\varepsilon} = [\varepsilon_0 \ \varepsilon_1 \ \cdots \ \varepsilon_{M-1}]^T, \tag{6}$$

by the following relation:[7]

$$q\{\boldsymbol{s}\} = |\boldsymbol{J}| q\{\boldsymbol{\varepsilon}\}. \tag{7}$$

The joint PDF of the signal and the residual are denoted by $q\{\boldsymbol{s}\}$ and $q\{\boldsymbol{\varepsilon}\}$, respectively. The Jacobian of the transformation $\partial\boldsymbol{\varepsilon}/\partial\boldsymbol{s}$ is the matrix $\boldsymbol{J}$. The determinant of the matrix, $|\boldsymbol{J}|$, can be calculated using a way similar to that in the AR model.[7] By assuming that $\varepsilon_i = 0$ for $i < 0$, we get $|\boldsymbol{J}| = 1$, so that the joint PDF of the signal is

$$q\{\boldsymbol{s}\} = q\{\boldsymbol{\varepsilon}\} = \prod_{i=0}^{M-1} f(\varepsilon_i(\boldsymbol{c})), \tag{8}$$

where

$$\boldsymbol{c} = [c_1 \ c_2 \ c_3 \ \cdots \ c_p]^T. \tag{9}$$

The joint PDF is also known as the likelihood function of the samples.[8] In this paper, it is assumed that the probability density function (PDF) $f(\varepsilon_i(\boldsymbol{c}))$ of the residual signal is t-distribution with α degree of freedom $f_\alpha(x)$ defined as[8]

$$f_\alpha(x) = \frac{1}{\sqrt{\alpha\pi}} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{\left(1+\frac{x^2}{\alpha}\right)^{(\alpha+1)/2}}. \tag{10}$$

Please note that $f_1(x)$ is the Cauchy distribution and $f_\infty(x)$ is the Gaussian distribution with zero mean and unity standard deviation. When α is small, the PDF has more probability on its tail than when α is large.

The optimal coefficient $\boldsymbol{c}$ is sought with the maximum likelihood method (MLM) by maximizing the logarithm of the likelihood function $L(\boldsymbol{\varepsilon}|\boldsymbol{c})$:

$$\max L(\boldsymbol{\varepsilon}|\boldsymbol{c}) = \log \prod_{i=0}^{M-1} f_\alpha(\varepsilon_i(\boldsymbol{c})) = \sum_{i=0}^{M-1} \rho(\varepsilon_i(\boldsymbol{c})). \tag{11}$$

where

$$\rho(x) = \log f_\alpha(x). \tag{12}$$

For estimation purposes, the PDF has to have a finite second order moment.[8] Since the second order moment for $\alpha < 3$ is infinite, in this paper we use $\alpha \geqq 3$.

3. The Asymptotic Efficiency of the EXP Model

By using the MLM approach, the estimation results will have a PDF which is asymptotically Gaussian with the variance matrix defined as[5],[9]

$$\text{var}=\{E_G(\boldsymbol{d}^T\boldsymbol{d})\}\{E_G^2(\boldsymbol{H})\}^{-1},$$

$$E_G(y(x))=\int_x y(x)\,g(x)\,dx \tag{13}$$

The probability distribution function of the residual signal is denoted by $G(x)$ and its corresponding probability density function (PDF) is $g(x)$. Generally, the exact PDF of the residual signal $g(x)$ is unknown. Therefore, we assume that the PDF is $f(x)$ which might be different from $g(x)$. The gradient vector $\boldsymbol{d}$ is

$$\boldsymbol{d}=\left[\frac{\partial L(\boldsymbol{\varepsilon}|\boldsymbol{c})}{\partial c_1}\ \frac{\partial L(\boldsymbol{\varepsilon}|\boldsymbol{c})}{\partial c_2}\cdots\frac{\partial L(\boldsymbol{\varepsilon}|\boldsymbol{c})}{\partial c_p}\right]^T \tag{14}$$

where

$$\frac{\partial L(\boldsymbol{\varepsilon}|\boldsymbol{c})}{\partial c_l}=\sum_{i=0}^{M-1}\psi(\varepsilon_i)\,\varepsilon_{i-l},$$

$$\psi(x)=\frac{\partial \rho(x)}{\partial x},\quad 1\leqq l\leqq p. \tag{15}$$

The Hessian matrix $\boldsymbol{H}$ is

$$\boldsymbol{H}=\frac{\partial^2 L(\boldsymbol{\varepsilon}|\boldsymbol{c})}{\partial \boldsymbol{c}\partial \boldsymbol{c}^T}=\boldsymbol{H}^1+\boldsymbol{H}^2 \tag{16}$$

where the elements of $\boldsymbol{H}^1$ and $\boldsymbol{H}^2$ are given by

$$H^1_{r,s}=\sum_{i=0}^{M-1}\psi'(\varepsilon_i)\,\varepsilon_{i-r}\varepsilon_{i-s},\quad \psi'(x)=\frac{\partial\psi(x)}{\theta x},$$

$$1\leqq r\leqq p \text{ and } 1\leqq s\leqq p, \tag{17}$$

$$H^2_{r,s}=\sum_{i=0}^{M-1}\psi(\varepsilon_i)\,\varepsilon_{i-r-s},\quad 1\leqq r\leqq p \text{ and } 1\leqq s\leqq p. \tag{18}$$

By assuming that the number of samples M is large; $M\to\infty$; the first term in Eq. (13) can be written:

$$E_G(\boldsymbol{d}^T\boldsymbol{d})=\begin{bmatrix} n_{1,1}\cdots n_{1,p}\\ \vdots\quad\ \vdots\\ n_{p,1}\cdots n_{p,p}\end{bmatrix} \tag{19}$$

where

$$n_{r,s}=E_G\left\{\frac{\partial L(\boldsymbol{\varepsilon}|\boldsymbol{c})}{\partial c_r}\ \frac{\partial L(\boldsymbol{\varepsilon}|\boldsymbol{c})}{\partial c_s}\right\}$$

$$=\int_x \psi^2(x)\,g(x)\,dx\sum_{i=0}^{M-1}\varepsilon_{i-r}\varepsilon_{i-s}. \tag{20}$$

The signal s_i is assumed to be obtained by convolving the excitation signal ε_i with the impulse response of the system $H(z)$. Hence, the excitation at a certain time i determines the signal which occurs on that particular time or after and not before. Therefore ε_i and s_{i-r} for $r>0$ are independent and Eq. (19) can be simplified to become

$$E_G(\boldsymbol{d}^T\boldsymbol{d})=E_G(\psi^2(\varepsilon_i))\,\boldsymbol{V} \tag{21}$$

where

$$V_{r,s}=\sum_{i=0}^{M-1}\varepsilon_{i-r}\varepsilon_{i-s},\quad 1\leqq r\leqq p \text{ and } 1\leqq s\leqq p. \tag{22}$$

are the elements of matrix $\boldsymbol{V}$. The second term in Eq. (13) can be calculated in a similar way and the result is

$$E_G^2(\boldsymbol{H})=E_G^2(\psi'(\varepsilon_i))\,\boldsymbol{V}^T\boldsymbol{V}+2E_G(\psi(\varepsilon_i))E_G(\psi'(\varepsilon_i))\,\boldsymbol{V}^T\boldsymbol{V}^*+E_G^2(\psi(\varepsilon_i))\,\boldsymbol{V}^{*T}\boldsymbol{V}^* \tag{23}$$

where the elements of matrix $\boldsymbol{V}^*$ are

$$V^*_{r,s}=V_{0,r+s},\quad 1\leqq r\leqq p \text{ and } 1\leqq s\leqq p. \tag{24}$$

Since the residual or the excitation signal is independently and identically distributed (IID) and the number of samples M is large enough, $M\to\infty$, we have

$$V_{r,s}=\sum_{i=0}^{M-1}\varepsilon_{i-r}\varepsilon_{i-s}\approx E\{\varepsilon_{i-r}\varepsilon_{i-s}\}\quad\begin{cases}=0 \text{ for } r\neq s\\ \neq 0 \text{ for } r=s\end{cases} \tag{25}$$

Thus the second and the third terms in Eq. (23) are zero and by using Eq. (21), we can rewrite Eq. (13) to become

$$\text{var}=\frac{E_G(\psi^2(\varepsilon_i))}{E_G^2(\psi'(\varepsilon_i))}\,\boldsymbol{V}^{-1}. \tag{26}$$

The asymptotic efficiency is defined as[8]

$$\text{AEFF}=\{I_F\text{var}\}^{-1}. \tag{27}$$

The Fisher information matrix I_F, defined as[8]

$$I_F=E_G\left(\frac{-\dfrac{\partial g(\varepsilon_i)}{\partial \boldsymbol{c}}}{g(\varepsilon_i)}\right)^2, \tag{28}$$

is the minimum possible variance matrix, when $f(x)=g(x)$. Using the same approach to get Eq. (21), we can rewrite Eq. (28) to be

$$I_F=E_G\left(\frac{-g'(\varepsilon_i)}{g(\varepsilon_i)}\right)^2\boldsymbol{V},\quad g'(x)=\frac{\partial g(x)}{\partial x} \tag{29}$$

so that the asymptotic efficiency is

$$\text{AEFF}=\frac{\boldsymbol{I}}{v_{loc}E_G\left(\dfrac{-g'(\varepsilon_i)}{g(\varepsilon_i)}\right)^2},\quad v_{loc}=\frac{E_G(\psi^2(\varepsilon_i))}{E_G^2(\psi'(\varepsilon_i))}, \tag{30}$$

and $\boldsymbol{I}$ is the unit matrix. Equation (30) shows that the asymptotic efficiency of the EXP model is equal to the efficiency of the AR and ARMA models.[5]

The efficiency of an estimator based on the t-distribution assumption depends on the degree of freedom of α. Our previous investigation[2] indicates that the AR estimator with $\alpha=\infty$ assumption is efficient for the Gaussian process only. For heavy-tailed processes, this kind of estimator is inefficient. Since the efficiency of the EXP and AR models are equal, we can conclude that an EXP estimator with $\alpha=\infty$ is also efficient for the Gaussian process only. This estimator is the same with the method proposed by Imai et al.[6] On the other hand, our previous result also shows that the AR estimator with small α is efficient for both Gaussian and heavy-tailed processes. Therefore, to achieve an efficient EXP estimator, we have to use small α. Hence, we recommend using the small α t-distribution assumption to analyze signals, whose PDF are unknown and which might be heavy-tailed, to achieve efficient estimates, and to use the EXP model to get a theoretically stable system.

4. The Solution Method and the Basic Properties

The algorithm for solving the optimization problem and the basic properties of the proposed method are given in the following sections.

4.1 The Solution Method

By using Eq. (10), the logarithm of the likelihood function $L(\boldsymbol{\varepsilon}|\boldsymbol{c})$ can be rewritten to be

$$L(\boldsymbol{\varepsilon}|\boldsymbol{c})=N_\alpha-\bar{L}(\boldsymbol{c}), \tag{31}$$

where

$$\bar{L}(\boldsymbol{c})=\frac{\alpha+1}{2}\sum_{i=0}^{M-1}\log\left(1+\frac{(\varepsilon_i(\boldsymbol{c})/\hat{s})^2}{\alpha}\right) \tag{32}$$

$$N_\alpha=M\ \log\left(\frac{1}{\sqrt{\alpha\pi}}\frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\right) \tag{33}$$

To get a scale invariant estimate, ε_i is normalized with $\hat{s}$. Since the mean and the variance of the residual are very sensitive to outliers, we use the median which has been proved to be very robust[9] for the scale estimate. The outliers here are the innovation outliers (IO)[5] which are possibly due to the lack of knowledge about the PDF, so that the actual PDF of the residual signal deviates from the assumption. Thus, we use

$$\hat{s}=\text{median}\ |\varepsilon_i|,\quad 0\leqq i\leqq M-1 \tag{34}$$

as the robust scale estimate of the residual.

The residual signal can be calculated in two ways. The first method is by utilizing the convolution formula in Eq. (5). The second way is by calculating the inverse Fourier transform of the spectrum of the residual which is calculated by

$$E(e^{j\omega})=A(e^{j\omega})\,S(e^{j\omega}), \tag{35}$$

where $E(e^{j\omega})$, $A(e^{j\omega})$ and $S(e^{j\omega})$ are the Fourier transform of the residual, the inverse system transfer function and the Fourier transform of the input signal, respectively. Please note that in the second way, since the Fourier transform is calculated by FFT, we have to use the circular shift for the shifted signal.

Since N_α is a constant, maximizing $L(\boldsymbol{\varepsilon}|\boldsymbol{c})$ in Eq. (31) is equivalent to minimizing $\bar{L}(\boldsymbol{c})$ in Eq. (32). Equation (32) is minimized by the Newton-Raphson iteration procedure as follows:

$$\boldsymbol{G}\boldsymbol{c}^{(k+1)}=\boldsymbol{G}\boldsymbol{c}^{(k)}-\nabla \tag{36}$$

where k denote the iteration number, $\nabla=-\boldsymbol{d}$ and $\boldsymbol{d}$ is given in Eq. (14).

Since in practice the exact Hessian matrix is not always positive definite, in the iteration we use a positive definite matrix $\boldsymbol{G}$

$$\boldsymbol{G}=\begin{bmatrix} G_{1,1}\cdots G_{1,p} \\ \vdots\quad\quad\vdots \\ G_{p,1}\cdots G_{p,p}\end{bmatrix} \tag{37}$$

where

$$G_{r,s}=\frac{\alpha+1}{\alpha\hat{s}^2}\sum_{i=0}^{M-1}\varepsilon_{i-r}\varepsilon_{i-s}w_i,\quad 1\leqq r\leqq p \text{ and } 1\leqq s\leqq p. \tag{38}$$

$$w_i=\frac{1}{1+\dfrac{(\varepsilon_i/\hat{s})^2}{\alpha}} \tag{39}$$

Two criteria

$$\sqrt{\sum_{l=1}^{p}\left(\frac{\partial\bar{L}^{(k)}}{\partial c_l}\right)^2}\leqq 10^{-4} \text{ and } |\bar{L}^{(k)}-\bar{L}^{(k-1)}|\leqq 10^{-4} \tag{40}$$

are used to terminate the iteration. Simulation results show that 10^{-4} is a suitable value for stopping the iteration. No further significant improvement can be obtained even when a value lower than 10^{-4} is used. Only the number of iterations will increase.

The initial coefficient $\boldsymbol{c}^{(0)}$ is calculated by using the recursive method[1] from its AR model which is obtained from the CLP method with a Hamming window. After the optimal coefficients are obtained, the gain factor K is calculated by taking the square root of the sum of the square of the residuals.

The analysis method explained in this section uses a block of M data samples, within a window $0\leqq i\leqq M-1$, to calculate the optimal solution, so that we call

this approach the block analysis algorithm.

4. 2 The Basic Properties

The basic properties of the proposed method are as follows:

1. In the iteration, instead of using the exact Hessian matrix which is not always positive definite, a positive definite matrix $\boldsymbol{G}$ is used. The simulation results show that by using this method, only a few iterations are needed to reach stationary points.
2. Since matrix $\boldsymbol{G}$ is always positive definite (see appendix A), the Cholesky decomposition method is used to calculate the new $\boldsymbol{c}$.
3. The simulation results on many signals show that at the stationary points, the Hessian matrix is always positive definite, so that we can be sure that we can always get a local minimum point.
4. The stability of the obtained system $D(z)$ is theoretically assured. However, it cannot be realized directly since $D(z)$ is not a rational form. Fortunately, we can use the LMA filter which can accurately approximate the EXP system.[10]
5. When $\alpha=\infty$, we get the same approach as the method proposed by Imai et al.[6]

5. The Alternative Algorithm

By utilizing the algorithm proposed in the previous section using small α we can get efficient estimates for many kinds of processes, including Gaussian and heavy-tailed processes (see the discussions in Sect. 3 and simulation results in Sect. 6). Since the calculation burden is very high, it is difficult to implement the algorithm in the real-time systems, using the general purpose DSP chip for example. To reduce the calculation burden, in this section we propose an alternative calculation algorithm. In this approach, instead of using a block of M data samples along a certain window, the estimate parameter is updated at every sampling instant.

In the algorithm, the coefficients of the predictor are calculated by using the imperfect, short-term, gradient as in the LMS algorithm[12] (see Fig. 2). At a certain time instant, $k+1$, the coefficients are calculated by

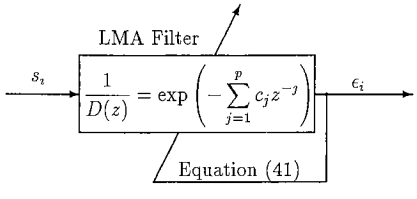


Fig. 2 The adaptation block diagram of the alternative algorithm.

$$\boldsymbol{c}^{(k+1)}=\boldsymbol{c}^{(k)}-\mu\boldsymbol{g}^{(k)} \tag{41}$$

where

$$\boldsymbol{g}^{(k)}=[g_1\ g_2\ g_3\ \cdots\ g_p]^T,\quad g_l=\frac{\varepsilon_k\varepsilon_{k-l}}{1+\frac{(\varepsilon_k/\hat{s})^2}{\alpha}}. \tag{42}$$

The convergence factor is μ and the scale $\hat{s}$ is calculated by

$$\hat{s}=\text{median}\ |\varepsilon_i|,\quad k-scl\leqq i\leqq k. \tag{43}$$

The calculation time to get the scale $\hat{s}$ is proportional to the value of scl. Experimental results show that the larger value of scl, the longer time is needed to reach the steady state, but we can get more efficient and accurate estimates. Compromising the above mentioned factors, the calculation and transient times on one side, the accuracy and the efficiency on the other side, we selected $scl=25$.

To further reduce the calculation burden, instead of using the convolution or the FFT approach which require many calculations to get ε_i, the EXP model is approximated by the log magnitude approximation (LMA) filter[10]

$$\frac{1}{D(z)}=\exp(-F(z))\approx R_L(-F(z))$$
$$=\frac{1+\sum_{l=1}^{L}B_{L,l}(-F(z))^l}{1+\sum_{l=1}^{L}B_{L,l}(F(z))^l} \tag{44}$$

where

$$F(z)=\sum_{j=1}^{p}c_jz^{-j}. \tag{45}$$

We use $L=4$ and the same $B_{L,l}$ coefficients as in the adaptive cepstral analysis.[11] Since, by cascading two stages of the LMA filter, we can obtain a more accurate approximation,[11] in this paper we use the cascade structure with

$$F_1(z)=c_1z^{-1}\ \text{and}\ F_2(z)=\sum_{q=2}^{p}c_qz^{-q}. \tag{46}$$

Readers who are interested in the implementation of the LMA filter are referred to the Ref. (11). The gain K in Eq. (1) is also updated in every sampling instant by

$$K=\sqrt{\zeta^{(k)}},\quad \zeta^{(k)}=\lambda\zeta^{(k-1)}+(1-\lambda)\,\varepsilon_k^2. \tag{47}$$

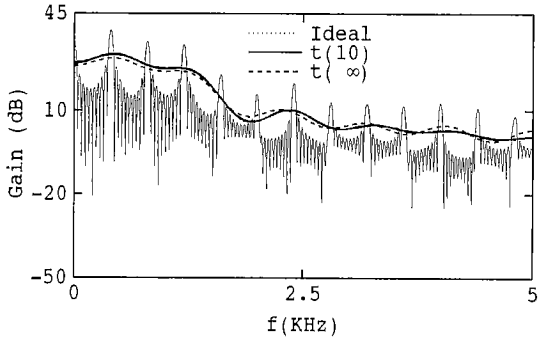
where $\zeta^{(k)}$ is the predicted power of the residual at the time k. The parameter λ is chosen experimentally and in all simulations we used $\lambda=0.98$. By using this approach, the calculation burden is very much reduced, so that it is possible to implement the algorithm in real-time systems.

6. The Simulation Results

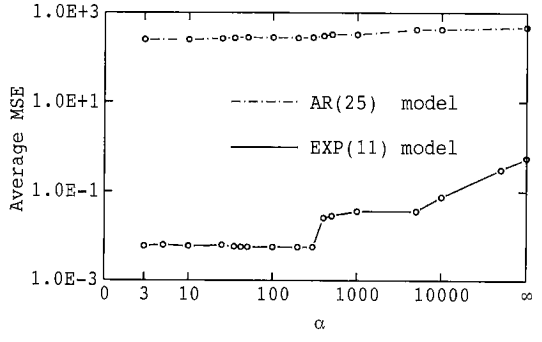
The two proposed methods, the block and the alternative algorithms, have been implemented and tested on several signals. The following are the reports of those testings.

6. 1 The Block Analysis Algorithm

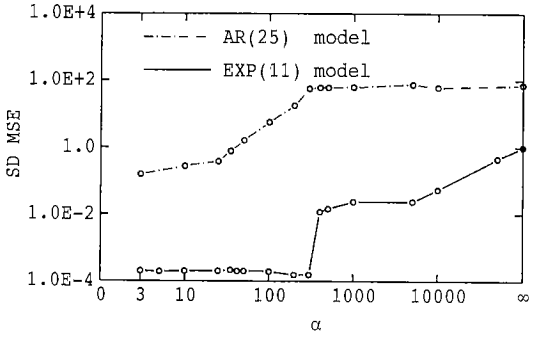
The output signals from an exponential system EXP (11) excited by various signals are used as the synthetic signals. The transfer function of the system is



(a) The ideal and estimated spectra.



(b) The average of the mean square error.



(c) The SD of the mean square error.

Fig. 3 The estimation results of EXP model, excited by a 400 Hz pulse train and using a rectangular window.

$$D_{\text{true}}(z)=\exp\left\{\sum_{q=1}^{11} c_q^{\text{true}} z^{-q}\right\}. \tag{48}$$

Without losing generality, the gain factor K is chosen to be one. The true coefficient of the EXP model is denoted by c_q^{true}; $1 \leqq q \leqq 11$. The sampling frequency is 10 KHz. The coefficient is obtained by analyzing one frame of real human speech with the proposed method using $\alpha=\infty$. The first synthetic signal is generated by exciting the system with a 400 Hz pulse train.

We tried to estimate the spectrum of the system from the output signal. The signal was analyzed every msec using a 25.6 msec window. As many as 800 frames were analyzed. Various values of α and $p=11$ were applied. Figure 3(a) shows the ideal spectrum of the system, the obtained estimate spectrum from the proposed method with $\alpha=10$ and with $\alpha=\infty$, and the periodogram for a frame where the obtained estimate spectrum using $\alpha=\infty$ gives the largest error. The periodogram is plotted using a solid thin line. Although the ideal spectrum is actually plotted by a dotted line, it coincides with the estimated spectrum using $\alpha=10$, which is plotted using a solid line. Therefore we can see only one solid line. By denoting the ideal and estimated spectrum as $D_{\text{true}}(e^{j\omega})$ and $D(e^{j\omega})$, respectively, from each frame the mean square error (MSE) was calculated,

$$\begin{aligned} MSE &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\{20\log_{10}|D_{\text{true}}(e^{j\omega})| \\ &\quad -20\log_{10}|D(e^{j\omega})|\}^2 d\omega \\ &= \frac{N}{2}\sum_{j=1}^{p}(c_j^{\text{true}}-c_j)^2, \quad N=\left(\frac{20}{\log_{10} e}\right)^2, \end{aligned} \tag{49}$$

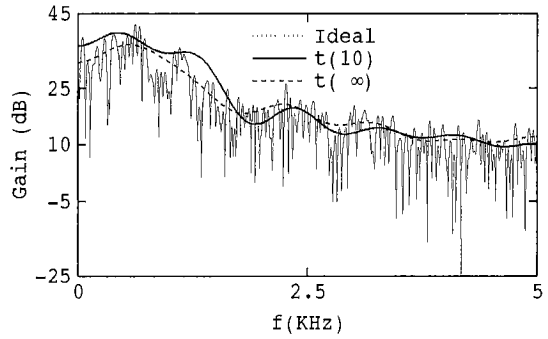
where c_j is the estimated EXP coefficients. From those 800 MSE values obtained from the corresponding 800 frames, the average and the SD of the MSE were calculated and plotted as a function of α in Figs. 3(b) and 3(c), respectively. Figures 3(b) and 3(c) show that by using small α we can get a smaller error and more efficient estimates than that by using large α. This is because we analyzed a heavy-tailed signal and it is consistent with the efficiency discussion in Sect. 3.

To simulate more complex heavy-tailed processes, we generate the second synthetic signal by exciting the same system as above with a random binary signal. The signal is generated by using a non-linear operation

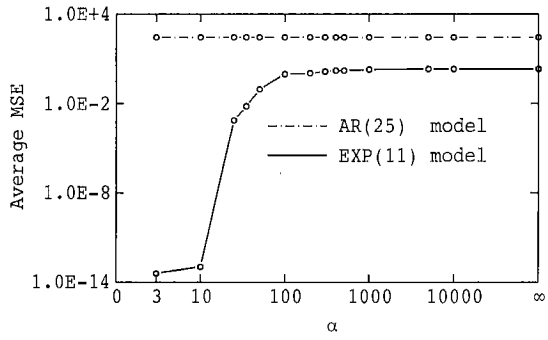
$$rbs_i=\begin{cases} -1 \text{ if } r_i \leqq -0.75 \\ 1 \text{ if } r_i \geqq 0.75 \\ 0 \quad \text{otherwise,} \end{cases} \tag{50}$$

where rbs_i is the random binary signal and r_i is the random noise which has a flat PDF between -1 and 1.

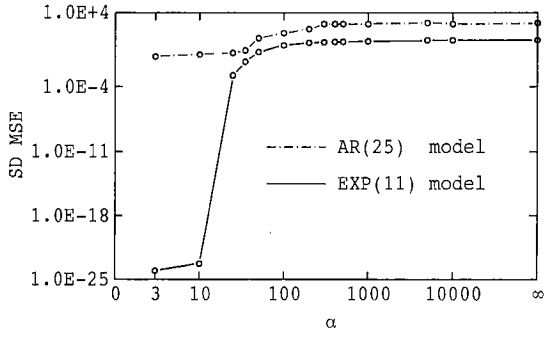
We carried out the same experiment as above. The obtained estimate and ideal spectra, again for a frame where the resulting estimated spectrum using $\alpha=\infty$



(a) The ideal and estimated spectra.

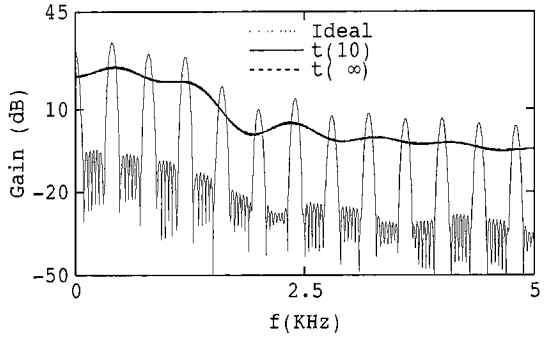


(b) The average of the mean square error.

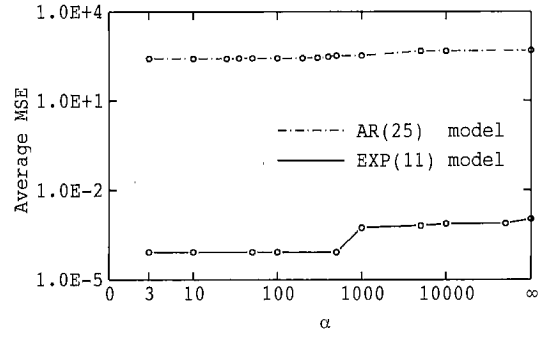


(c) The SD of the mean square error.

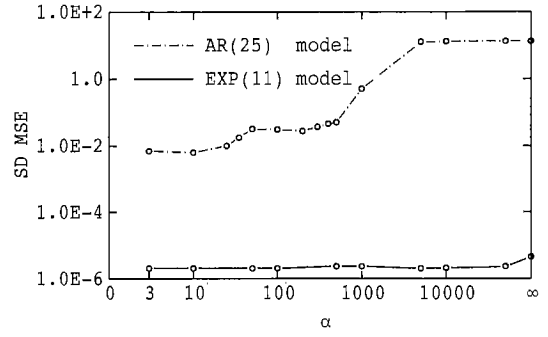
Fig. 4 The estimation results of the EXP model, excited by a random binary signal and using a rectangular window.



(a) The ideal and estimated spectra.



(b) The average of the mean square error.



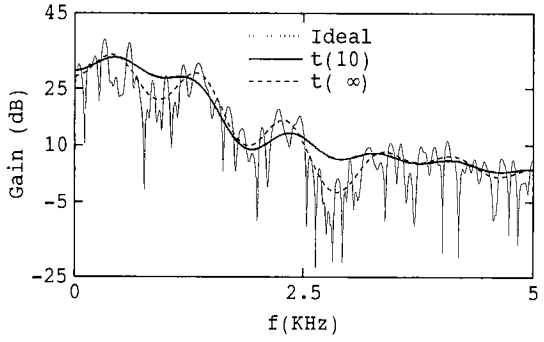
(c) The SD of the mean square error.

Fig. 5 The estimation results of EXP model, excited by a 400 Hz pulse train and using a Hamming window.

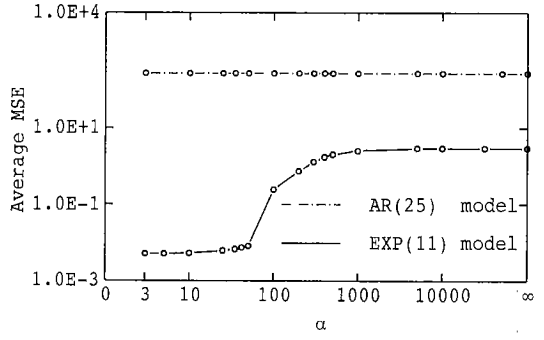
produces the largest error, are shown in Fig. 4(a). The average of the MSE as a function of α is depicted in Fig. 4(b). Figure 4(c) shows the SD of the MSE. From Fig. 4(a) we can see that by using small α we can get a more precise spectral estimate than that with large α. Figures 4(b) and 4(c) also show that the accuracy and the efficiency of the estimator can be improved by using small α.

In addition to the above simulations, we have also applied a 25.6 msec Hamming window in the analysis and the results are shown in Figs. 5 and 6 for the pulse train and the random binary inputs, respectively. These figures show that the Hamming window improves the accuracy and the efficiency of the estimator with $\alpha=\infty$. For a simple impulsive input, the pulse train input, the improvement by using a small α is insignificant. The difference between the obtained estimate spectrum with $\alpha=10$ and $\alpha=\infty$ is small, so that in Fig. 5(a) we can only see one solid line. For a more complex impulsive input such as the random binary signal, the improvement by using a small α remains significant.

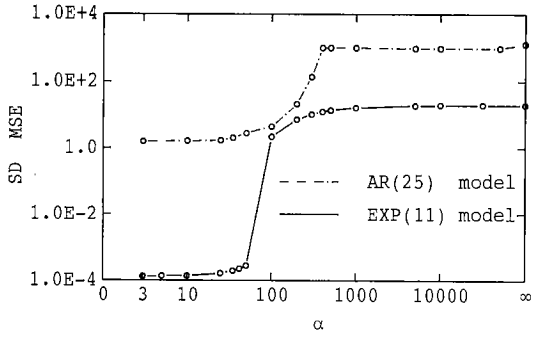
Those simulations indicate that the accuracy and the efficiency of the proposed estimator depend on α. By using small α, we can achieve accurate and efficient estimates for many kinds of processes, the Gaussian



(a) The ideal and estimated spectra.

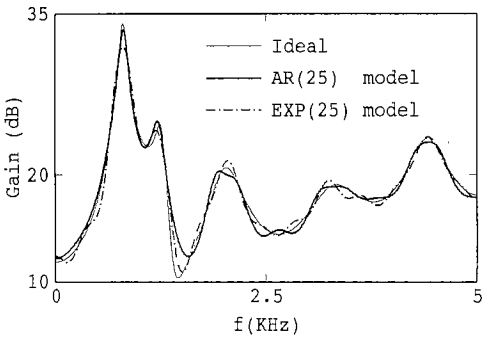


(b) The average of the mean square error.

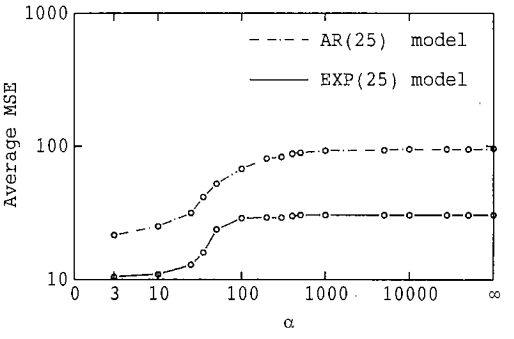


(c) The SD of the mean square error.

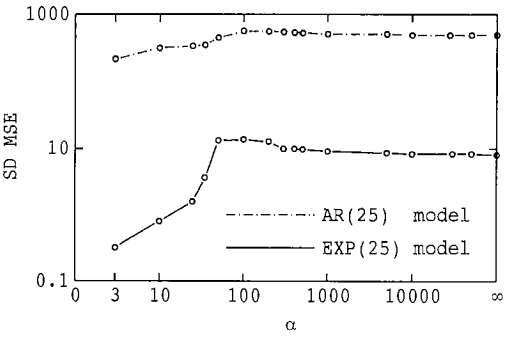
Fig. 6 The estimation results of the EXP model, excited by a random binary signal and using a Hamming window.



(a) The ideal and estimated spectra.



(b) The average of the mean square error.



(c) The SD of the mean square error.

Fig. 7 The estimation results of the ARMA model excited by a random binary signal, using AR and EXP models with a rectangular window.

and the heavy-tailed processes. All the plotted results in Figs. 3 to 6 are calculated by using the FFT approach. Since the results from the convolution approach are the same, those results are omitted.

To compare the performance of the proposed method with our previous proposal which uses the AR model and assumption that the residual signal is t-distribution with α degree of freedom,[2] we carried out the same experiments as above using an AR model estimator. The obtained average and SD by using the rectangular and Hamming windows for both synthetic signals are plotted in Figs. 3 to 6. In the experiment, we used 25th-order AR estimator. When a smaller order AR model is used, the average and SD of the MSE will be higher. Those figures show that although a high order AR model has been used, the average MSE is still high. This mean that the AR model cannot accurately estimate the EXP time series. From those figures, we can also conclude that for analyzing an EXP signal, we can achieve higher efficiency by using the EXP model than by using the AR model.

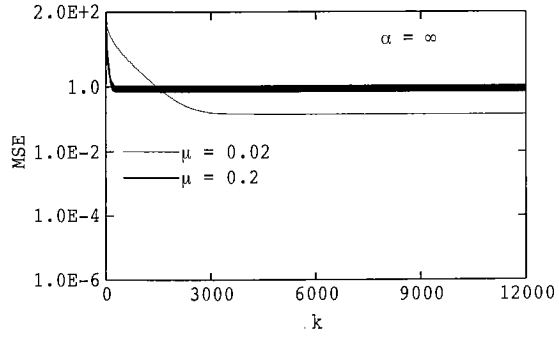
Besides that, we have tried to estimate the spectrum of a complex spiky ARMA signal. The signal was generated by exciting an ARMA system which has

10 poles and 2 zeroes with a random binary signal. In the analysis, we used a rectangular window, and 25th-order AR and EXP models. The ideal spectrum, the obtained estimate spectra by using AR and EXP models with $\alpha=10$ for a frame where the obtained spectrum estimate using the AR model produces the largest spectrum error, are shown in Fig. 7(a). We calculated the average and SD of the MSE using the same method as above and plot them as a function of α in Figs. 7(b) and 7(c), respectively. Figure 7(b) shows that the average of the MSE from the EXP model is smaller than from the AR model. These results indicate that the EXP model can produce a closer spectrum estimate for the ARMA signal than the AR model. The obtained SD from the EXP model is also smaller than from the AR model, so that it can be concluded that for analyzing the ARMA signal we can achieve a higher efficiency by using the EXP model. When the Hamming window or the 400 Hz pulse train excitation is used, we get almost the same results. Thus, those results are omitted. Finally, therefore we recommend using the EXP model to analyze unknown signals which in general may contain poles and zeroes, and also to use the small α to achieve accurate and efficient estimates.

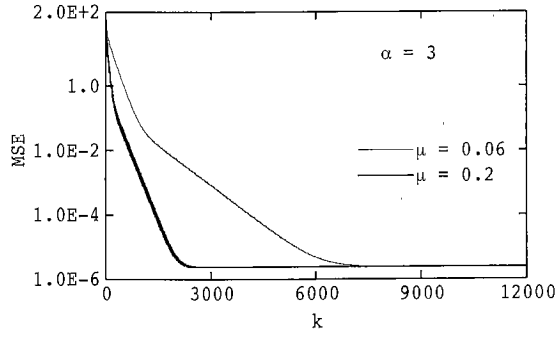
All simulations show that, indeed, only a small number of iterations are needed to get stationary points. At stationary points, the Hessian matrix is always positive definite. Furthermore, the simulations show that it is always possible to get accurate and efficient estimates by using small α. Hence, in practice we recommend to use small α; i.e. $\alpha \leqq 20$.

6.2 The Alternative Algorithm

In the alternative algorithm, the estimated parameter is updated in every sampling instant. Therefore, similar to the LMS algorithm, there are two important factors which determine the performance of the algorithm, the transient and the steady state. Both of those factors were examined and reported in this subsection. The transient is investigated by measuring the progression of the MSE between the ideal and the estimated spectra as a function of the iteration number k after setting $\boldsymbol{c}^{(0)}=\mathbf{0}$. The obtained MSE as a function of the k for the 400 Hz pulse-train excitation with $\mu=0.02$ and $\mu=0.2$, both using $\alpha=\infty$, are shown in Fig. 8(a). Those two curves show that the usage of larger μ will speed up the transient. The two curves for $\mu=0.06$ and $\mu=0.2$, both using $\alpha=3$, which are depicted in Fig. 8

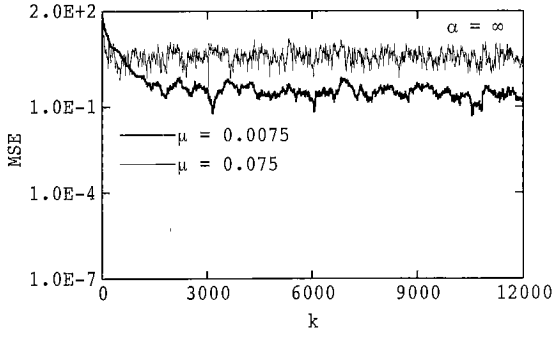


(a) For $\mu = 0.02$ and $\mu = 0.2$, both using $\alpha = \infty$.

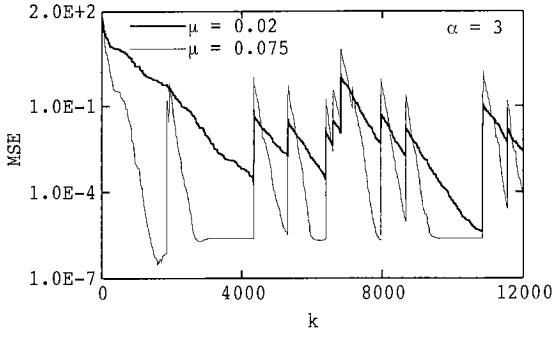


(b) For $\mu = 0.06$ and $\mu = 0.2$, both using $\alpha = 3$.

Fig. 8 The progression of the MSE of the alternative algorithm as a function of iteration number k for the EXP model excited by a 400 Hz pulse-train.



(a)For $\mu = 0.0075$ and $\mu = 0.075$, both using $\alpha = \infty$.



(b) For $\mu = 0.02$ and $\mu = 0.075$, both using $\alpha = 3$.

Fig. 9 The progression of the MSE of the alternative algorithm as a function of iteration number k for the EXP model excited by a random binary signal.

(b), behave in the same way.

By comparing Figs. 8(a) and 8(b), we can conclude that we get a slower transient when small α is used; please compare the curves for $\mu=0.2$ using $\alpha=3$ and $\alpha=\infty$. To encounter this problem, when we use small α, we have to use larger μ than when $\alpha=\infty$ is applied. Please compare the case for $\alpha=\infty$ and $\mu=0.02$ with the case using $\alpha=3$ and $\mu=0.2$. From Figs. 8(a) and 8(b), the merit of using small α can be clearly understood. By using small α, after the transient has deceased, we can obtain a smaller MSE than when large α is used.

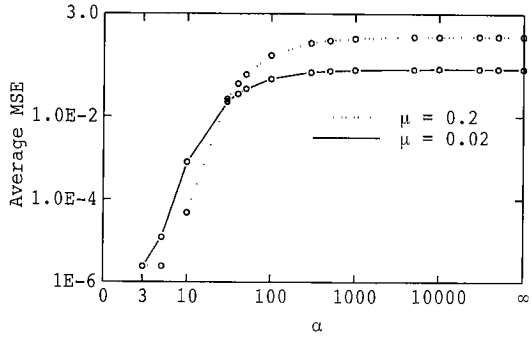
The same investigation was conducted for the random binary excitation. The obtained results for $\alpha=\infty$ using $\mu=0.0075$ and $\mu=0.075$ are shown in Fig. 9(a). In Fig. 9(b), the progression of the MSE for $\alpha=3$, with $\mu=0.02$ and $\mu=0.075$, are depicted. Those results behave in the same way as in the 400 Hz pulse-train excitation.

In the second simulation, we investigate the behavior of the MSE after the transient has ended. We used various values of α and μ. The MSE was computed at every 100 μsec or each sample. We calculated the average and the SD of the MSE from 800 samples. The results for the pulse-train excitation are shown in Fig. 10 for $\mu=0.02$ and $\mu=0.2$. When μ is assumed to be constant, by using small α we can achieve more accurate and more efficient estimates than with large α.

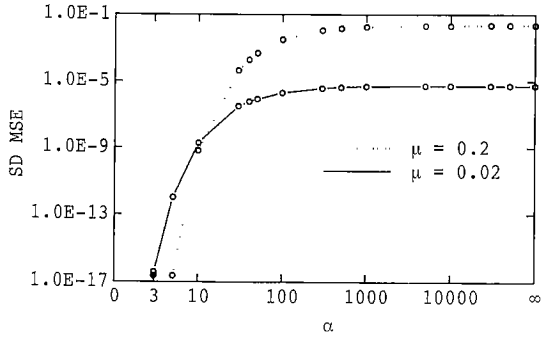
Figures 8 and 10 show that by using $\alpha=3$ and $\mu=0.2$, we can settle the transient faster and also get a more accurate and more efficient estimate than by applying $\alpha=\infty$ and $\mu=0.02$. The obtained average and SD of the MSE for the random binary excitation are shown in Fig. 11. By comparing Figs. 9 and 11, we can conclude that those results behave in the same way.

From Figs. 3(b) and 10(a), which show the average of the MSE, we can conclude that for a simple excitation (400 Hz pulse train in this case) the usage of the alternative algorithm does not affect the accuracy of the estimator. From Figs. 3(c) and 10(b), we can see that the SD is also not affected.

On the other hand, by comparing Figs. 4(b) and 11(a), and Figs. 4(c) and 11(b) we can clearly see that, although the usage of the alternative algorithm reduces the performance of the proposed method, the merit of using small α is still significant. The efficiency and the accuracy of the alternative algorithm using $\alpha=3$ and $\mu=0.075$ is still much better than that of the block analysis using $\alpha=\infty$. Thus, by using the alternative algorithm with small α we can implement the algo-

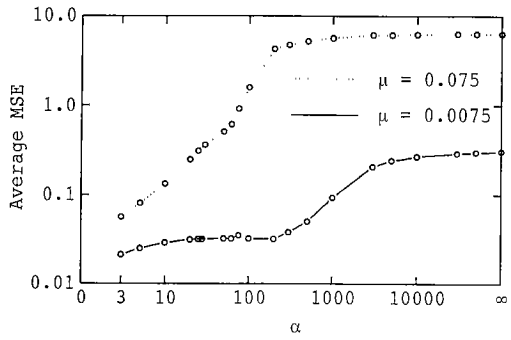


(a) The average of the mean square of the spectra.

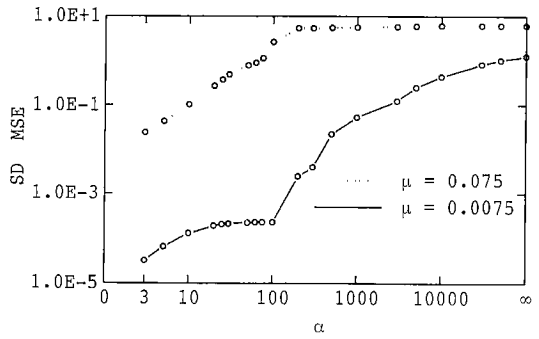


(b) The SD of the mean square error of the spectra.

Fig. 10 The estimation results of the alternative algorithm for the EXP model excited by a 400 Hz pulse train, using the convergence factor $\mu=0.02$ and $\mu=0.2$.



(a) The average of the mean square error of the spectra.



(b) The SD of the mean square error of the spectra.

Fig. 11 The estimation results of the alternative algorithm for the EXP model, excited by a random binary signal, using a convergence factor $\mu=0.0075$ and $\mu=0.075$.

rithm using real-time systems and achieve accurate and efficient estimates.

An adaptive analysis approach using $\alpha=\infty$ has been developed and reported to be fast enough to be used for analyzing time variant signals, such as the speech signal.[11] From the above simulation results, we can conclude that by using small α and a carefully selected μ, we can get an adaptive analysis method which is comparably fast with $\alpha=\infty$, and additional merit accurate and efficient estimates.

7. Conclusions

A new time series analysis method based on the exponential model and assumption that the excitation signal has an IID t-distribution has been proposed. Theoretically, the estimator is efficient for many kinds of processes, the Gaussian and heavy-tailed. The simulation results show that when the signal is spiky or heavy-tailed, we can achieve accurate and efficient estimates by using small α. When we use the small degree of freedom of α, the proposed EXP estimator can achieve more accurate and more efficient estimates for EXP and ARMA signals than the AR estimator. Since the EXP estimator is appropriate for analyzing signals which have poles and zeroes, it can be used to replace the ARMA model which has a stability problem.

To reduce the calculation burden, an alternative algorithm has also been proposed. By using small α and a carefully selected μ, we can get a fast convergence algorithm. The accuracy and efficiency of the estimator is comparable with the block analysis method.

Future research efforts are directed toward the application of the proposed method, the way of selecting the proper p, and calculating optimal α.

Acknowledgment

The authors would like to deeply thank Associate Professor Mineo Kaneko of Tokyo Institute of Technology for his helpful advice and discussion and all members of the Onoda, Kuneida and Kaneko laboratory who have provided a pleasant working and studying environment.

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Appendix A: The Proof of the Positive Definite Property of G

We can write for any b_k, $k=0, 1, \cdots, p$,

$$\sum_{i=0}^{M-1+p}\left\{\left(\sum_{j=0}^{p} b_{p-j}\varepsilon_{i-p+j}\right)^2 w_i\right\}\geqq 0. \qquad (A\cdot 1)$$

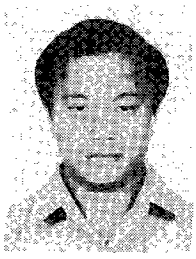
Equation (A·1) can be manipulated to become

$$\sum_{r=0}^{p} b_r \sum_{t=0}^{p} b_t \sum_{i=0}^{ba} \varepsilon_{i-r}\varepsilon_{i-t} w_i \geqq 0. \qquad (A\cdot 2)$$

The residual is non-zero in the range of $0\leqq i\leqq M-1$ when the convolution approach is used, so that $ba=M-1$. When the FFT approach is used, the residual is non-zero in the range of $0\leqq i\leqq M+q$; $q=\max\{r, t\}$, so that $ba=M+q$. Equation (A·1) can be manipulated to become

$$\sum_{r=0}^{p} b_r \sum_{t=0}^{p} b_t G_{r,t} \geqq 0 \qquad (A\cdot 3)$$

for any real b_r, $0\leqq r\leqq p$, which proves that matrix $\boldsymbol{G}$ is a positive definite matrix.



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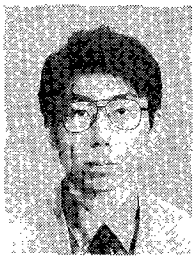

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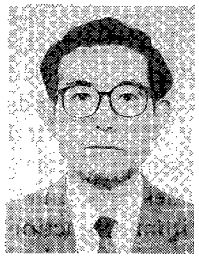

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