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A Parallel Algorithm for Constructing Strongly Convex Superhulls of Points*

Carla Denise CASTANHO†, **Wei CHEN**††, *and* **Koichi WADA**††, *Members*

SUMMARY Let S be a set of n points in the plane and $CH(S)$ be the convex hull of S. We consider the problem of constructing an approximate convex hull which contains $CH(S)$ with strong convexity. An *ϵ-convex δ-superhull* of S is a convex polygon P satisfying the following conditions: (1) P has at most $O(n)$ vertices, (2) P contains S, (3) no vertex of P lies farther than δ outside $CH(S)$, and (4) P remains convex even if its vertices are perturbed by as much as ϵ. The parameters ϵ and δ represent the strength of convexity of P and the degree of approximation of P to $CH(S)$, respectively. This paper presents the first parallel method for the problem. We show that an ϵ-convex $(8+4\sqrt{2})\epsilon$-superhull of S can be constructed in $O(\log n)$ time using $O(n)$ processors, or in $O(\log n)$ time using $O(n/\log n)$ processors if S is sorted, in the *EREW-PRAM* model. We implement the algorithm and find that the average performance is even much better: the results are more strongly convex and much more approximate to $CH(S)$ than the theoretical analysis shows.

key words: *computational geometry, convexity, strongly convex superhull, parallel algorithm, divide-and-conquer*

1. Introduction

In computational geometry the convex hull problem of points in the plane is one of the oldest and most-studied problems. The task is to determine the smallest convex polygon that contains all the given points. Over the past two decades a number of algorithms have been proposed [1]–[3], [5], [8], [9], [12]. Due to the nice properties of convexity, convex hulls have applications in a variety of problem domains including computer vision, computer graphics, and statistics [10]. Consequently, it is natural that we may desire that the solution has strong tolerance for convexity so that in the further computations, many properties from the convexity can be preserved in some fashion even if they are tested with imprecise computations.

Let S be a set of n points in the plane. The concept of strongly convex approximate hull of S first appeared in the problem of finding an *ϵ-convex δ-hull* of S [11] which is a simple polygon P satisfying the following conditions: (i) the vertices of P are taken from S, (ii) no point of S lies farther than δ ($\delta \geq 0$) outside P and (iii) P is convex and remains convex even if the vertices of P are perturbed by as much as ϵ ($\epsilon \geq 0$). The parameters ϵ and δ are used to describe the tolerance of P for the convexity and the approximation of P to $CH(S)$, respectively. According to the definition, the 0-convex 0-hull of S is the convex hull of S. Clearly, for a given ϵ, the smaller the value of δ is, the better an ϵ-convex δ-hull would be. Li and Milenkovic present the first algorithm for the problem which computes an ϵ-convex 12ϵ-hull in $O(n \log n)$ time [11]. Guibas, Salesin and Stolfi propose another algorithm which computes an ϵ-convex 6ϵ-hull in $O(n^3 \log n)$ time [9]. Recently, Chen, Wada and Kawaguchi have developed a parallel algorithm for constructing an ϵ-convex 6ϵ-hull which runs in $O(\log n)$ time using n processors in the *EREW PRAM* or in $O(n \log n)$ time if it is implemented sequentially [5]**.

In many applications, approximate hulls are required to contain all the points of S [12]. A major drawback of an ϵ-convex δ-hull is the fact that the points of S may lie outside of the hull. A recent work has been developed to solve this problem. Chen, Deng, Wada, and Kawaguchi have introduced a new concept, strongly convex approximate superhull of S [6]. A simple polygon P is an *ϵ-convex δ-superhull* of S if P satisfies the following conditions: (i) P has at most $O(n)$ vertices, (ii) P contains all the points of S, (iii) no vertex of P lies farther than δ outside the convex hull of S, and (iv) P is convex and remains convex even if its vertices are perturbed by as much as ϵ. Obviously, the 0-convex 0-superhull of S is the convex hull of S. Note that according to the definition, the vertices of a superhull are not necessary to be the points of S, in fact, it is impossible.

It is worth to notice the relation between strongly convex hulls and strongly convex superhulls. Let Q be an ϵ-convex δ_1-hull of S. An ϵ-convex δ_2-superhull of S can be constructed directly by expanding the boundary of Q as follows: draw lines outside Q such that the lines are parallel to the edges of Q with distance δ_1. The polygon Q' consisting of the intersections of the lines is an ϵ-convex δ_2-superhull of S, where δ_2 is the maximum distance from the intersections to Q (Fig. 1). However,

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†The author is with Nagoya Institute of Technology, Nagoya-shi, 466-8555 Japan.
††The authors are with the Department of Electrical and Computer Engineering, Nagoya Institute Techonology, Nagoya-shi, 466-8555 Japan.

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**These algorithms also consider imprecise computations. We only list the results relating to exact arithmetic.

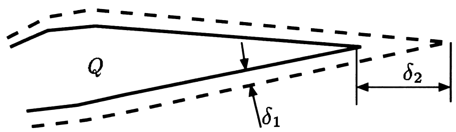

Fig. 1 δ_2 may be very large.

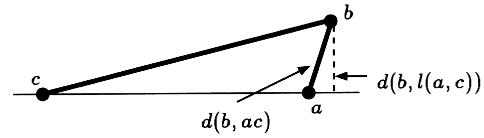

Fig. 2 An ϵ-convex vertex.

δ_2 may be much larger than δ_1, i.e., Q' may not be a good approximation of Q. Therefore, a strongly convex approximate hull cannot be used for constructing a strongly convex approximate superhull. Moreover, the algorithms developed for constructing Q depend on the technique of deleting the points of S, therefore, they can not be used to construct a strongly convex approximate superhull of S essentially.

Chen, et al. [6] present the first method to solve the convex superhull problem. They show a sequential algorithm that constructs an ϵ-convex $(2+4\sqrt{2})\epsilon$-superhull of S which has at most $n+1$ vertices in $O(n \log n)$ time, or in $O(n)$ time if S is sorted. They use a sweep technique which cannot be easily parallelized.

This paper presents, for the first time, a parallel algorithm for the strongly convex approximate superhull problem. We construct an ϵ-convex $(8+4\sqrt{2})\epsilon$-superhull of S in $O(\log n)$ time using $O(n)$ processors, or in $O(\log n)$ time using $O(n/\log n)$ processors if S is sorted. The main technique is the use of an almost ϵ-convex approximate superhull of S which controls the created vertices such that they do not lie farther and farther from $CH(S)$ in the process of construction. Throughout this paper, the model of parallel computation we use is the *EREW-PRAM*. It is a synchronous shared-memory model where no two processors can simultaneously read or write in the same memory location. In this paper, we also implement our algorithm and do some experiments and we find that (i) the average performance of the algorithm is much better than the theoretical analysis shows: the resulted superhulls are more strongly convex than the requested ϵ and much more approximate to $CH(S)$ than $(8+4\sqrt{2})\epsilon$, and (ii) our parallel method gives a stronger convexity than the sequential one, although its degree of approximation is a little bit worse than that of sequential one.

2. Definitions and Lemmas

Let S be a set of n points in the plane and P be a simple polygon.

Definition 1 (ϵ-convex polygon): P is ϵ-convex ($\epsilon \geq 0$), if P is convex and remains convex even after each vertex of P is perturbed as far as ϵ. □

Definition 2 (δ-hull of points): P is a δ-hull ($\delta \geq 0$) of S, if all vertices of P belong to S and no vertex of S lies farther than δ outside the polygon P. □

Definition 3 (ϵ-convex δ-hull of points): P is an ϵ-convex δ-hull of S, if P is a δ-hull of S and P is ϵ-convex. □

Definition 4 (δ-superhull of points): P is a δ-superhull ($\delta \geq 0$) of S, if P contains all the points of S, and no vertex of P lies farther than δ outside the convex hull of S. □

Definition 5 (ϵ-convex δ-superhull of points): P is an ϵ-convex δ-superhull of S, if P is a δ-superhull of S, P is ϵ-convex and P has at most $O(n)$ vertices. □

It is easily seen that an ϵ'-convex δ'-superhull of S is an ϵ-convex δ-superhull of S if $\epsilon' \geq \epsilon$ and $\delta' \leq \delta$. For any two points p and q, let $|pq|$ denote the length of line segment pq, $l(p,q)$ denote the straight line passing through p and q, and $int(l_1, l_2)$ denote the intersection of lines l_1 and l_2. Points p and q divide line $l(p,q)$ into three parts: segment pq and two half lines $l(p,q)_p$ and $l(p,q)_q$ which start at p and q, respectively. Let G be a polygon (or polygonal chain). A sub-chain H of G is contiguous if its vertices lie contiguously in G. Given polygonal chains H and H', where H is a *contiguous* sub-chain of G, $G(H \rightarrow H')$ is a polygon (or polygonal chain) obtained by replacing H with H' in G. Similarly, given k pairs of polygonal chains H_1 and H'_1, H_2 and H'_2, …, H_k and H'_k, where H_i $(1 \leq i \leq k)$ is a contiguous sub-chain in G, and for any i and j $(i \neq j)$ H_i and H_j share no common vertex, $G(H_1 \rightarrow H'_1, H_2 \rightarrow H'_2, \ldots, H_k \rightarrow H'_k)$ is a polygon (or polygonal chain) obtained by replacing H_i with H'_i for all i $(1 \leq i \leq k)$ in G.

Definition 6: (point-line distance, point-segment distance): Let a, b and c be three points in the plane. Define $d(b, l(a,c)) = \min\{|bd| \mid d \text{ is the point on } l(a,c)\}$ to be the distance from point b to line $l(a,c)$ and define $d(b, ac) = \min\{|bd| \mid d \text{ is the point on } ac\}$ to be the distance from point b to line segment ac (Fig. 2). □

Definition 7 (ϵ-convex and ϵ-flat vertices): Let P be a convex polygon (or convex polygonal chain), and a, b and c be three contiguous vertices of P listing in counter-clockwise. Vertex b is ϵ-convex in P if $d(b, l(a,c)) \geq 2\epsilon$ (Fig. 2), otherwise b is ϵ-flat. □

According to Definition 7, for three contiguous vertices a, b, c of P the convexity of b, i.e., the value of $d(b, l(a,c))$, is a local property which depends only on its two neighbors in P. If $d(b, ac) = d(b, l(a,c))$ then we say b is *normal* else b is *abnormal*. Obviously, if $\angle bac \leq 90°$ and $\angle bca \leq 90°$ then b is normal. If b is abnormal and $\angle bca > 90°$ then we say b is *left-abnormal*. If b is abnormal and $\angle bac > 90°$ then we say b is *right-abnormal*.

In a convex polygonal chain, the end vertices are neither ϵ-convex nor ϵ-flat. For any three contiguous vertices a, b and c of a simple polygon, if b is ϵ-convex, then b remains convex even after perturbing a, b and c as far as ϵ. Therefore, we have the following Lemma.

Lemma 1[11]**:** Let P be a simple polygon. If each vertex of P is ϵ-convex, then P is ϵ-convex. □

The distance from a point v to simple polygon P, denoted as $d(v, P)$, is defined as $d(v, P) = \min\{d(v, e)$, e is the edge of $P\}$. Therefore the point-line distance is used to measure the convexity of vertex and the point-segment distance is used to measure the distance from a point to a simple polygon.

Let a convex polygon (convex polygonal chain) be represented by a sequence of their vertices listing in counter clockwise. A convex polygonal chain G is monotonic in both x-axis and y-axis if both the x coordinates and y coordinates of its vertices increase or decrease monotonically, respectively. The following property holds obviously.

Property 1: Given a convex polygonal chain $G = (u_1, u_2, \ldots, u_n)$, if G is monotonic in both x-axis and y-axis, then $d(b, u_iu_j) = d(b, l(u_i, u_j))$ holds for any i and j $(1 \leq i \neq j \leq n)$, where b is the intersection of $l(u_i, u_{i+1})$ and $l(u_{j-1}, u_j)$.

Let $Q = (q_1, q_2, \ldots, q_m)$ be a convex polygon (or convex polygonal chain). Denote $Q(i : j)$ $(1 \leq i \leq j \leq m)$ to be the sequence of the contiguous vertices of Q from q_i to q_j ($Q(i : j)$ is empty if $i > j$). Given two polygonal chains U and V, notation $\bowtie$ denotes the operation of concatenating U and V, i.e., $U \bowtie V = (u_1, u_2, \ldots, u_f, v_1, v_2, \ldots, v_g)$ for $U = (u_1, u_2, \ldots, u_f)$ and $V = (v_1, v_2, \ldots, v_g)$. The following property holds obviously.

Property 2: Let $G = (u_1, u_2, \ldots, u_n)$ be a convex polygon and $H = (u_f, ..., u_g)$ be a sub-chain of G. Let I be the intersection of $l(u_{f-1}, u_f)$ and $l(u_g, u_{g+1})$. If $H' = (u)$, $H' = (v)$ or $H' = (u, v)$ such that u lies on segment u_fI and v lies on segment Iu_g, then $G(H \to H')$ is still convex and the convexities of the vertices of G other than those of H are not changed in $G(H \to H')$ (Fig. 3). □

It is not easy to revise vertices in a convex polygon locally, since it may change the convexities of their neighbors. In the following, we show that under some conditions, the vertices can be revised into ϵ-convex without changing the convexities of their neighbors.

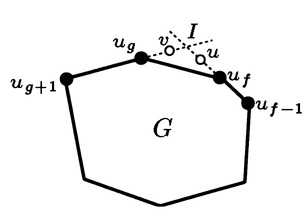

Fig. 3 $G(H \to H')$ is still convex.

Definition 8 ((ϵ, δ)-swap and (ϵ, δ)-subswap): Let G be a convex polygon (or a convex polygonal chain) and H be a contiguous polygonal sub-chain of G. An (ϵ, δ)-swap of H in G, denoted as $swp(G, H, \epsilon, \delta)$, is a convex polygonal chain H' satisfying the following conditions: (i) H' has at most $|H|$ vertices, (ii) $G(H \to H')$ is convex, and for the vertices of G other than those of H, their convexities in $G(H \to H')$ are the same as in G, (iii) all the vertices of H' are ϵ-convex in $G(H \to H')$ and (iv) the vertices of H' lie at most δ outside of H'. An (ϵ, δ)-subswap of H in G, denoted as $sswp(G, H, \epsilon, \delta)$, is a convex polygonal chain H' satisfying all the condition of a $swp(G, H, \epsilon, \delta)$ except that the last vertex of H' may not be ϵ-convex in $G(H \to H')$. □

The proof of Lemma 2 is given in Appendix, and the proof of Lemma 3 can be found in [6].

Lemma 2: Let $G = (u_1, \ldots, u_n)$ be a convex polygonal chain and $H = (u_{f+1}, \ldots, u_{g-1})$ be a contiguous sub-chain of G. If u_f and u_g are ϵ-convex in G, where u_f and u_g are the vertices of G lying directly before u_{f+1} and after u_{g-1}, respectively, then an $sswp(G, H, \epsilon, 2\epsilon)$ $(= H')$ can be found in $O(|H|)$ time using a single processor and if H has at least one ϵ-convex vertex, H' has also at least one ϵ-convex vertex. □

Lemma 3[6]**:** Let P be a convex polygonal chain and $H = (v, w, x)$ be a contiguous sub-chain of P. If w is ϵ-flat and u, v, x and y are $\sqrt{2}\epsilon$-convex, with u and y lying before and after v and x respectively, then H', a $swp(G, H, \epsilon, (2 + 2\sqrt{2})\epsilon)$ can be found in $O(|H|)$ time using a single processor. □

3. Algorithm

Let S be a set of n points in the plane. We construct a strongly convex approximate superhull of S as follows: find the convex hull of S, denoted as $CH(S)$, and then find an ϵ-convex $(8 + 4\sqrt{2})\epsilon$-superhull of $CH(S)$. Since $CH(S)$ can be found in $O(\log n)$ time using $O(n)$ processors, or in $O(\log n)$ time using $O(n/\log n)$ processors if S is sorted [1], [4], [8], in the rest of the paper, we only construct a strongly convex approximate superhull for a convex polygon.

3.1 Almost-ϵ-Convex-Ring and Almost-ϵ-Convex-Piece

Our parallel algorithm is based on divide-and-conquer. If we construct an ϵ-convex δ-superhull of a convex polygon P with n vertices, directly by divide-and-conquer technique, we would divide P into two sub-polygons P_1 and P_2 with the same size each, recursively

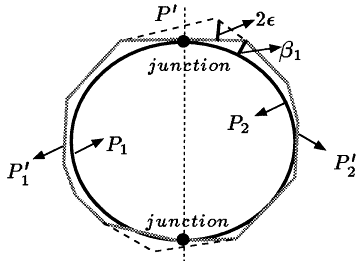

Fig. 4 Combining two ϵ-convex β_1-superhulls.

compute P_1' and P_2', ϵ-convex β_1-superhulls of P_1 and P_2 in parallel, and then combine P_1' and P_2' into P', an ϵ-convex β_2-superhull of P. In the combining step, the vertices at the junctions would be revised if they were not ϵ-convex (Fig. 4). It means that in the worst case (the case that the junctions and their two neighbors lie almost on a same line) the vertices of P' may lie 2ϵ outside P_1' and P_2', which means that P' may lie $2\epsilon + \beta_1$ outside P. On the other hand, P_1' and P_2' were constructed recursively with $\Theta(\log n)$ steps. It means that we would finally get an ϵ-convex $\Theta(\epsilon \log n)$-superhull of P, which is not a good approximation of P. Obviously, δ should not be a function of n but only be that of ϵ. Therefore, instead of constructing an ϵ-convex δ-superhull of P directly, we first find a δ-superhull of P in which only a proportion of the vertices are ϵ-convex, and then revise it to be ϵ-convex.

Definition 9 (Almost-ϵ-Convex-Ring): Let P be a convex polygon. Convex polygon P' is an Almost-ϵ-Convex-Ring of P, denoted as $a(P, \epsilon)$, if P' satisfies the following conditions: (i) P' has at most $|P|$ vertices, (ii) in every two contiguous vertices of P', at least one is ϵ-convex, and (iii) each vertex of P' lies at most 4ϵ outside P (Fig. 5(i)). □

Let F be a convex polygonal chain. By adding an edge between the endpoints of F we get a closed chain. Point v is said to be outside of F if v lies outside the closed F. The end-vertices of F are neither ϵ-convex nor ϵ-flat.

Definition 10 (Almost-ϵ-Convex-Piece): Let F be a convex polygonal chain. Convex polygonal chain F' is an Almost-ϵ-Convex-Piece of F, denoted as $a(F, \epsilon)$, if F' satisfies the following conditions: (i) F' has at most $|F|$ vertices, (ii) except the end-vertices, in any two contiguous vertices of F', at least one is ϵ-convex, (iii) the first and the last vertices of F' are the same as those of F, (iv) the second and the penultimate vertices of F' lie at most 2ϵ outside of F if they are ϵ-flat else they lie at most 4ϵ outside of F, and the other vertices of F' lie at most 4ϵ outside of F, and (v) the first and the last edges of F' contain those of F as line segments, respectively (Fig. 5(ii)). □

Given a convex polygon P, we construct a strongly convex approximate superhull of P in two steps: we find an Almost-ϵ-Convex-Ring of P, and then revise it into an ϵ-convex $(8+4\sqrt{2})\epsilon$-superhull of P. We construct an Almost-ϵ-Convex-Ring of convex polygon P as follows: (1) divide P into four contiguous polygonal chains P_1, P_2, P_3, P_4 at extreme vertices u, v, w and x which have the largest x coordinate, the largest y coordinate, the smallest x coordinate, and the smallest y coordinate, respectively, and then find an $a(P_i, \epsilon)$, for each i ($1 \leq i \leq 4$), and (2) concatenate them into a polygon $P' = a(P_1, \epsilon) \bowtie a(P_2, \epsilon) \bowtie a(P_3, \epsilon) \bowtie a(P_4, \epsilon)$ (note that P' is convex since for each i, $1 \leq i \leq 4$, the first and the last edges of $a(P_i, \epsilon)$ contains these of P_i), and then revise it into an Almost-ϵ-Convex-Ring of P.

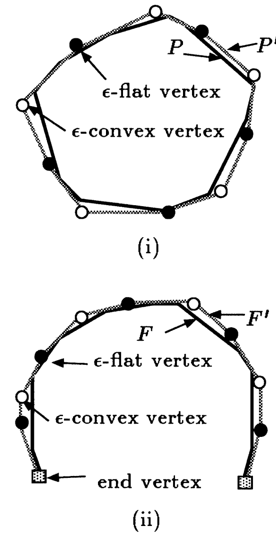

Fig. 5 (i) Almost-ϵ-convex-ring and (ii) Almost-ϵ-convex-piece.

3.2 Finding Almost-ϵ-Convex-Pieces

In this subsection, we find an $a(F, \epsilon)$ for a convex polygonal chain F which is monotonic in both x-axis and y-axis. It is easily seen that $a(F, \epsilon)$ is also monotonic in both x-axis and y-axis since its first and last edges contains these of F. From Property 1 point-line distance and point-segment distance are the same measure for F and $a(F, \epsilon)$. Thus, throughout this subsection we regard point-line distance and point-segment distance as the same distance.

Algorithm MakeAlmostConvexPiece(F)
Step 1 Let $F = (u_1, u_2, \ldots, u_m)$ and $w = u_{\lceil m/2 \rceil}$. Divide F into two sub-chains $F_1 = (u_1, u_2, \ldots, w)$ and $F_2 = (w, v_1, v_2, \ldots, v_{\lfloor m/2 \rfloor})$, where $v_i = u_{\lceil m/2 \rceil + i}$ ($1 \leq i \leq \lfloor m/2 \rfloor$), and recursively find an $a(F_1, \epsilon)$ and an $a(F_2, \epsilon)$ respectively, in parallel.

Step 2 Combine the $a(F_1, \epsilon)$ and the $a(F_2, \epsilon)$ into an $a(F, \epsilon)$ as follows.

Assume $a(F_1, \epsilon) = (z_1, z_2, \ldots, z_k, w')$ and $a(F_2, \epsilon) = (w'', z_{k+1}, \ldots, z_h)$. From Definition 10, $w' = w'' =$

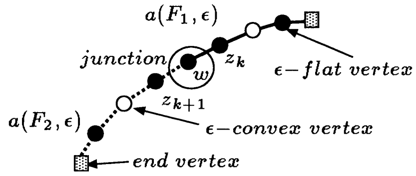

Fig. 6 Combining two almost-ϵ-convex-pieces.

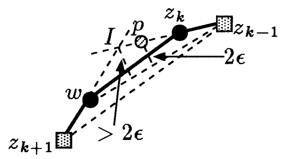

Fig. 7 Revising $H = (z_k, w)$.

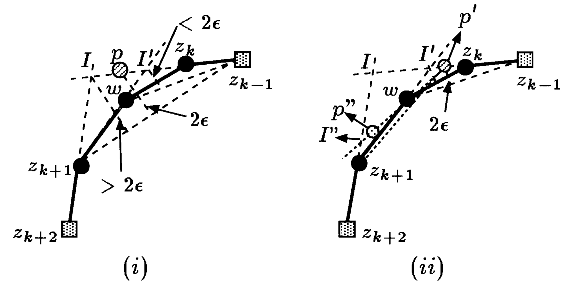

Fig. 8 Revising $H = (z_k, w, z_{k+1})$.

w, $z_1 = u_1$ and $z_h = v_{\lfloor m/2 \rfloor}$. Let chain $G = a(F_1, \epsilon) \bowtie a(F_2, \epsilon) = (z_1, z_2, \ldots, z_k, w, z_{k+1}, \ldots, z_h)$. It is easily seen that G satisfies all the conditions as an $a(F, \epsilon)$ except there may be more than one ϵ-flat vertex appearing at the junction w (Fig. 6). At most three contiguous vertices, z_k, w and z_{k+1}, may be ϵ-flat at the junction, since if z_k is ϵ-flat, z_{k-1} is ϵ-convex or it is the first vertex of G according to the definition of $a(F_1, \epsilon)$, and similarly if z_{k+1} is ϵ-flat in $a(F_2, \epsilon)$, z_{k+2} is ϵ-convex or it is the last vertex of G. Therefore, assuming that H is the longest contiguous sequence consisting of the ϵ-flat vertices at the junction w, $|H| \le 3$ holds. If H contains only one vertex then G is already an $a(F, \epsilon)$. If H contains more than two vertices, use the following procedure REVISION(G, H) to revise H into H' such that $G(H \to H')$ is an $a(F, \epsilon)$. □

Procedure REVISION(G,H)

(Case 1) $|H| = 2$. In this case, $H = (z_k, w)$ or $H = (w, z_{k+1})$.

(Subcase 1) $H = (z_k, w)$ (Fig. 7).
Find the intersection I of $l(z_{k-1}, z_k)$ and $l(w, z_{k+1})$. If $d(I, z_{k-1}w) \le 2\epsilon$, let (i) $H' = (I)$, else find a point p on segment $z_k I$ such that $d(p, l(w, z_{k-1})) = 2\epsilon$ (such p must exist since $d(z_k, l(z_{k-1}, w)) < 2\epsilon$ and $d(I, l(z_{k-1}, w)) > 2\epsilon$), and let (ii) $H' = (p, w)$. Revise H into H'.
(It is easily seen that only the last vertex (I or w) of H' may not be ϵ-convex in $G(H \to H')$.)
(Subcase 2) $H = (w, z_{k+1})$.
It is the symmetry case of Subcase 1.

(Case 2) $|H| = 3$. In this case, $H = (z_k, w, z_{k+1})$ (Fig. 8).

Let I be the intersection of $l(z_{k-1}, z_k)$ and $l(z_{k+1}, z_{k+2})$, and let I' be the intersection of $l(z_{k-1}, z_k)$ and $l(w, z_{k+1})$. Revise H as follows.

(Subcase 1) $d(I, z_{k-1}z_{k+1}) \le 2\epsilon$.
Let (iii) $H' = (I)$ and revise H into H'.
(Subcase 2) $d(I, z_{k-1}z_{k+1}) > 2\epsilon$ and $d(I', z_{k-1}w) \le 2\epsilon$.
If $d(I', z_{k-1}z_{k+1}) \ge 2\epsilon$, let (iv) $H' = (I', z_{k+1})$. Else, find a point p on segment $I'I$ such that $d(p, l(z_{k-1}, z_{k+1})) = 2\epsilon$ (Fig. 8(i)) and let (v) $H' = (p, z_{k+1})$. Revise H into H'.
(It is easily seen that only the last vertex z_{k+1} of H' may not be ϵ-convex in $G(H \to H')$.)
(Subcase 3) $d(I', z_{k-1}w) > 2\epsilon$.
Find a point p' on segment $z_k I'$ such that $d(p', l(z_{k-1}, w)) = 2\epsilon$ (Fig. 8(ii)). Let I'' be the intersection of $l(p', w)$ and $l(z_{k+1}, z_{k+2})$. If $d(I'', p'z_{k+1}) \le 2\epsilon$, let (vi) $H' = (p', I'')$, else find a point p'' on segment wI'' such that $d(p'', l(p', z_{k+1})) = 2\epsilon$ (such p'' must exist since $d(w, l(p', z_{k+1})) \le d(w, l(z_k, z_{k+1})) < 2\epsilon$ and $d(I'', l(p', z_{k+1})) > 2\epsilon$), and let (vii) $H' = (p', p'', z_{k+1})$. Revise H into H'.
(It is easily seen that only the last vertex of H' may not be ϵ-convex in $G(H \to H')$.) □

Let u, v and w be three contiguous vertices of a convex polygonal chain F listing in counter-clockwise order. We call edges (u, v) and (v, w) to be *the right and the left edges* of v, respectively. In the following, we prove that algorithm MakeAlmostConvexPiece(F) finds a correct $a(F, \epsilon)$.

Property 3: Let F be a convex polygonal chain and F' be the output of algorithm MakeAlmostConvexPiece(F). If v is the first ϵ-convex vertex of F', the right edge (u, v) of v must contain some edge of F.

Proof: Let $F'_1 = a(F_1, \epsilon)$, $F'_2 = a(F_2, \epsilon)$, and $G = F'_1 \bowtie F'_2$. From the algorithm, $F = G(H \to H')$, where H is the longest contiguous sequence of the ϵ-flat vertices at the junction, $F = F_1 \bowtie F_2$, and H' is computed in Procedure REVISION(G,H). F'_1 and F'_2 are computed recursively by the algorithm. Assuming that this property has held for both F'_1 and F'_2 already, we show that the property holds for F'. If F'_1 contains ϵ-convex vertices, since its first ϵ-convex vertex v is the first ϵ-convex vertex of F' and its vertices in the right of v (including v) are never changed in procedure REVISION. From the assumption the right edge of v contains some edge of F_1.

Now let F'_1 contains no ϵ-convex vertices. If H'

contains no ϵ-convex vertices, the first ϵ-convex vertex v of F_2' ($v = z_{k+1}$ in Case 1 of procedure REVISION and $v = z_{k+2}$ in Case 2) is the first ϵ-convex vertex of F'. From the construction of H', the right edge of v in F' always contains the right edge of v in F_2'. According to the assumption the right edge of v in F_2' contains some edge of F_2. If H' contains ϵ-convex vertices, from the construction of H', the first ϵ-convex vertex of H' always lies on the lengthening line of the first edge of F_1'. From Definition 10, the first edge of F_1' contains the first edge of F_1. □

Lemma 4: Let F be a convex polygonal chain with m vertices. If F is monotonic in both x-axis and y-axis, then an Almost-ϵ-Convex-Piece of F can be computed in $O(\log m)$ time using $O(m/\log m)$ processors.

Proof: Assuming that $a(F_1, \epsilon)$ and $a(F_2, \epsilon)$ are computed correctly, we prove $G(H \to H')$ is a correct $a(F, \epsilon)$.

It is easily seen that $|H'| \leq |H|$, and only the last vertex of H' may be not ϵ-convex in G. From the construction of H' and Property 2, $G(H \to H')$ is convex. Therefore, $G(H \to H')$ satisfies the first two conditions of Definition 10 as an $a(F, \epsilon)$. The vertices of H are either the end-vertices or the second vertex or the penultimate vertex of the $a(F_1, \epsilon)$ and the $a(F_2, \epsilon)$, therefore, they lie at most 2ϵ from F according to the definitions of $a(F_1, \epsilon)$ or $a(F_2, \epsilon)$. It means that the vertices of H' lie at most 4ϵ from F since from the construction of H' the vertices lie at most 2ϵ from H. The end-vertices of $G(H \to H')$ are the same as those of G. Thus, we only need to show that $G(H \to H')$ satisfies the remaining conditions of Definition 10: (a) the second vertex and the penultimate vertex of $G(H \to H')$ lie at most 2ϵ outside of F if they are ϵ-flat and (b) the first and the last edges of $G(H \to H')$ contain those of F, respectively. Note that when G is changed into $G(H \to H')$, only the vertices of H, i.e., the edges chain $E = f \bowtie H \bowtie g$ can be changed, where f and g are the vertices of G lying before the first vertex of H and after the last vertex of H, respectively.

Situation 1: neither f is the first vertex of G nor g is the last vertex of G.

In this situation, the first and the last edges of G are not contained in E, therefore, they are not changed in $G(H \to H')$. Thus, conditions (a) and (b) hold automatically.

Situation 2: f is the first vertex of G or g is the last vertex of G.

In this case, the first or the last edges of G may be contained in E. We divide the proof into two parts: (1) if f is the first vertex of G, we prove that (a) the second vertex of $G(H \to H')$ lies at most 2ϵ outside of F if it is ϵ-flat and (b) the first edge of $G(H \to H')$ contains that of F; and (2) if g is the last vertex of G, we prove that (a) the penultimate of $G(H \to H')$ lies at most 2ϵ outside of F if it is ϵ-flat and (b) the last edge of $G(H \to H')$ contains that of F.

Sub-Situation 1: f is the first vertex of G.

The first and the second vertices of $G(H \to H')$ are f and the first vertex of H', say p respectively. From the construction of H', the first vertex of H' must lie on the lengthening line of the first edge of G and from the assumption that $a(F_1, \epsilon)$ is computed correctly, the first edge of the $a(F_1, \epsilon)$ (i.e., the first edge of G) contains the first edge of F_1. Therefore, edge (f, p) contains the first edge of G, i.e., condition (b) holds.

If the first vertex of H', i.e., the second vertex of $G(H \to H')$, is ϵ-convex in $G(H \to H')$, condition (a) holds automatically. If the first vertex of H' is ϵ-flat in $G(H \to H')$, H' contains only one vertex I such that $d(I, fg) \leq 2\epsilon$ (Subcase 1 (i) of Case 1, where $f = z_{k-1}$ and $g = z_{k+1}$, and Subcase 1 (iii) of Case 2, where $f = z_{k-1}$ and $g = z_{k+2}$, in procedure REVISION). Vertex g is either the first ϵ-convex vertex of $a(F_2, \epsilon)$ or the last vertex of G. In both cases, the right edge of g contains some edge of F from Property 3 and from Definition 10, respectively. It means that F begins from its first vertex f and passes through the a part of the right edge of g. Therefore, F must enter the region bounded by chain E and segment fg. Thus, $d(I, F) \leq d(I, fg) \leq 2\epsilon$, i.e., I lies at most 2ϵ outside F. Therefore, condition (a) holds.

Sub-Situation 2: g is the last vertex of G. This is the symmetry case of Sub-Situation 1.

The algorithm can be executed in $O(\log m)$ time using $O(m)$ processors since it contains $O(\log m)$ recursive steps and each step can be executed in $O(1)$ time using $O(m)$ processors. It can be easily modified to run in $O(\log m)$ time using $O(m/\log m)$ processors: dividing F into $m/\log m$ sub-chains with $O(\log m)$ vertices each, computing an Almost-ϵ-Convex-Piece using the sequential method [6] for each chain in $O(\log m)$ time, and combining these pieces using the method of our above algorithm. □

3.3 Finding Almost-ϵ-Convex-Rings and Making Them ϵ-Convex

Now we are ready to make an Almost-ϵ-Convex-Ring of a convex polygon.

Lemma 5: Given a convex polygon $P = (p_1, p_2, \ldots, p_n)$, either an Almost-$\epsilon$-Convex-Ring or an ϵ-convex $(4+4\sqrt{2})\epsilon$-superhull of P can be constructed in $O(\log n)$ time using $O(n/\log n)$ processors.

Proof: We divide P in four parts, $F_1 = (u, \ldots, v)$, $F_2 = (v, \ldots, w)$, $F_3 = (w, \ldots, x)$ and $F_4 = (x, \ldots, u)$ according to the extreme vertices u, v, w and x which have the largest x coordinate, the largest y coordinate, the smallest x coordinate, and the smallest y coordinate, respectively. Each F_i $(1 \leq i \leq 4)$ is monotonic in both x-

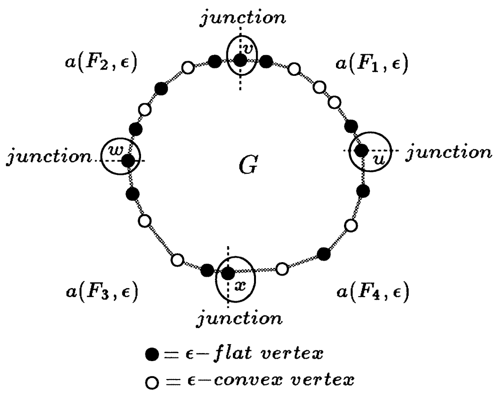

Fig. 9 Concatenating four almost-ϵ-convex-pieces.

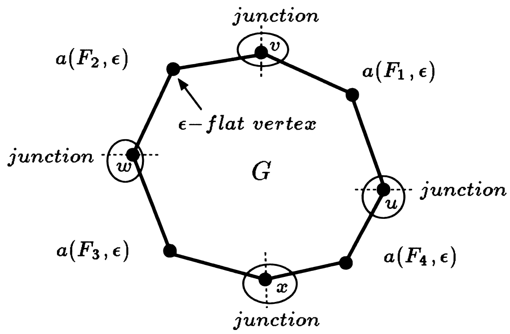

Fig. 10 G has no ϵ-convex-vertex.

axis and y-axis. We construct Almost-ϵ-Convex-Pieces $a(F_1,\epsilon)$, $a(F_2,\epsilon)$, $a(F_3,\epsilon)$ and $a(F_4,\epsilon)$, by Lemma 4 in parallel in $O(\log n)$ time using $O(n/\log n)$ processors. Let G be the polygon obtained by concatenating $a(F_1,\epsilon)$, $a(F_2,\epsilon)$, $a(F_3,\epsilon)$ and $a(F_4,\epsilon)$ (Fig. 9). Notice that the last vertex of $a(F_i,\epsilon)$ and the first vertex of $a(F_{i+1},\epsilon)$ (the index is taken with modulo 4) are the same. Polygon G is convex, since P and $a(F_i,\epsilon)$ $(1 \le i \le 4)$ are convex and the first and the last edges of $a(F_i,\epsilon)$ contain those of F_i. Convex polygon G satisfies all conditions as an $a(P,\epsilon)$ except that more than one contiguous ϵ-flat vertices may appear at the junctions u, v, w and x. In the following, we show how to revise G into either an $a(P,\epsilon)$ or an ϵ-convex $(4+4\sqrt{2})\epsilon$-superhull in $O(1)$ time using $O(n)$ processors as follows. Note that the method in Sect. 3.2 can not be used here since point-line distance and point-segment distance may be different.

(Case 1) No ϵ-convex vertex exists in G.

If G contains no ϵ-convex vertex, there exists no ϵ-convex vertex in $a(F_i,\epsilon)$ $(1 \le i \le 4)$. It means that for each i, F_i consists of at most three vertices. Therefore, G contains at most eight vertices (Fig. 10). In this case, we construct G', an ϵ-convex $(2+4\sqrt{2})\epsilon$-superhull of G in $O(1)$ time using the sequential method [6]. Since the vertices of G are ϵ-flat, according to the definition of Almost-ϵ-Convex-Piece, they lie at most 2ϵ outside P. Therefore, G' is an ϵ-convex $(4+4\sqrt{2})\epsilon$-superhull of P.

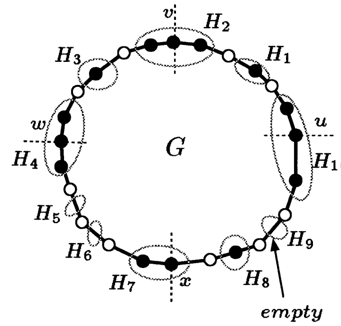

Fig. 11 Constructing an almost ϵ-convex-ring.

(Case 2) At least one ϵ-convex vertex exists in G.

Assume that G has k ϵ-convex vertices. We divide the ϵ-flat vertices of G into k seperated sub-chains H_1, H_2, …, H_k such that H_i $(1 \le i \le k)$ consists of all the ϵ-flat vertices lying between the ith and the $(i+1)$th ϵ-convex vertices of G (Fig. 11). From the construction of G, $|H_i| \le 9$ holds (when G contains only one ϵ-convex vertex, H_1 may have nine vertices). H_i $(1 \le i \le k)$ is contiguous sub-chain of G and they share no common vertices. For each i, if H_i has more than one vertex we revise H_i into H'_i, a $sswp(G,H_i,\epsilon,2\epsilon)$ of H_i, in $O(1)$ time by Lemma 2, in parallel. If $|H_i| \le 1$, let $H'_i = H_i$. We get a chain $G' = G(H_1 \to H'_1, H_2 \to H'_2, \ldots, H_k \to H'_k)$. According to Definition 8, (i) $|G'| \le |G|$ and G' is convex, (ii) in every two contiguous vertices of G' at least one is ϵ-convex since H'_i contains at most one ϵ-flat vertex and the convexities of the vertices other than H'_i are not changed, and (iii) the vertices of H'_i lies at most 2ϵ outside from H_i, therefore, they lie at almost 4ϵ outside from P from the fact that the vertices H_i lie almost 2ϵ outside from P. Thus, G' is an Almost-ϵ-Convex-Ring of P. □

Finally, we use an Almost-ϵ-Convex-Ring of P to compute an ϵ-convex $(2+8\sqrt{2})$-superhull of P.

Lemma 6: Let P' be a convex polygon with n vertices and there is at least one $\sqrt{2}$ ϵ-convex vertex in every two contiguous vertices of P'. An ϵ-convex $(2+4\sqrt{2})\epsilon$-superhull of P' can be constructed in $O(1)$ time using $O(n)$ processors.

Proof: First, we revise P' such that in every three contiguous vertices, at least two are $\sqrt{2}\epsilon$-convex, then we use Lemma 3 to change all the $\sqrt{2}\epsilon$-flat vertices into ϵ-convex.

Assume that $P' = (u_1, u_2, \ldots, u_n)$ has $\sqrt{2}\epsilon$-flat vertices, else it is already $\sqrt{2}\epsilon$-convex. We divide P' into k contiguous sub-chains $\overline{H_1}, \ldots, \overline{H_k}$ such that $\overline{H_i}$ contains four $\sqrt{2}\epsilon$-convex vertices of P' (only $\overline{H_k}$ may contain six $\sqrt{2}\epsilon$-convex vertices), the first and the last vertices of $\overline{H_i}$ are ϵ-convex, and $\overline{H_i}$ and $\overline{H_{i+1}}$ share a same endpoint, i.e., the last vertex of $\overline{H_i}$ is the same as the first vertex of $\overline{H_{i+1}}$ ($\overline{H_1}$ when $i = k$). Let H_i consists of the vertices of $\overline{H_i}$ by deleting the first and the last ones (Fig. 12). H_i $(1 \le i \le k)$ is contiguous sub-chain of G

and they share no common vertices. From the condition of P', H_i $(1 \leq i \leq k-1)$ contains at most five vertices (only H_k may contain at most nine vertices). For every i $(1 \leq i \leq k)$, if H_i contains any $\sqrt{2}\epsilon$-flat vertex, we revise H_i into H'_i, an $sswp(P', H_i, \sqrt{2}\epsilon, 2\sqrt{2}\epsilon)$ in $O(1)$ time, by Lemma 2, in parallel. If H_i contains no $\sqrt{2}\epsilon$-flat vertex, we just let $H'_i = H_i$. That is, we get a chain $R = P'(H_1 \to H'_1, H_2 \to H'_2, \ldots, H_k \to H'_k)$. Similar to the proof of Lemma 5, we can easily prove that (i) R is convex, (ii) $|R| \leq |P'|$, (iii) R is a $2\sqrt{2}\epsilon$-superhull of P', since the revised vertices lie at most $2\sqrt{2}\epsilon$ from P'. (iv) Before and after every sub-chain H_i in P' there is an $\sqrt{2}\epsilon$-convex vertex. According to the properties of a $sswp(P', H_i, \sqrt{2}\epsilon, 2\sqrt{2}\epsilon)$ the vertices of P' other than those of H_i are not changed in $P(H_i \to H'_i)$. Therefore, the $\sqrt{2}\epsilon$-convex vertices located immediately before and after every sub-chain H'_i are still $\sqrt{2}\epsilon$-convex in R. Moreover, according to Lemma 2 H'_i contains at most one $\sqrt{2}\epsilon$-flat vertices and at least one $\sqrt{2}\epsilon$-convex vertices, therefore in every three contiguous vertices of R at least two are $\sqrt{2}\epsilon$-convex.

Convex polygon R satisfies all the conditions as an ϵ-convex $(2\sqrt{2})$ ϵ-superhull of P' except in every three contiguous vertices there may be one ϵ-flat vertex. We revise the ϵ-flat vertices of P' by Lemma 3 as follows. Suppose there are k ϵ-flat vertices $w_1, w_2, \ldots, w_k$ in R. For every w_i $(1 \leq i \leq k)$, there must be two $\sqrt{2}\epsilon$-convex vertices u_i and v_i lying before and two $\sqrt{2}\epsilon$-convex vertices x_i and y_i lying after w_i, respectively. We use Lemma 3 to revise chain $H_i = (v_i, w_i, x_i)$ into H'_i, an $swp(R, H_i, \epsilon, (2+2\sqrt{2})\epsilon)$ of H_i, for each i, in parallel. It is easily seen that H_i $(1 \leq i \leq k)$ are contiguous sub-chains of G and they share no common vertices. That is, we get a chain $R' = R(H_1 \to H'_1, H_2 \to H'_2, \ldots, H_k \to H'_k)$. According to Definition 8, (i) $|R'| \leq |R|$ and R' is convex, (ii) the vertices of H'_i lie at most $(2+2\sqrt{2})\epsilon$ outside of H_i, (iii) each vertex of H'_i is ϵ-convex in R', and (v) the convexities of the vertices of R other than those of H_i are not changed. Therefore, R' is an ϵ-convex $(2+2\sqrt{2})\epsilon$-superhull of R. Since R is a $2\sqrt{2}\epsilon$-superhull of P', R' is an ϵ-convex $(2+4\sqrt{2}\epsilon)$-superhull of P'. □

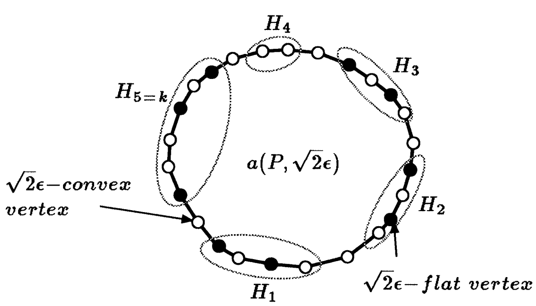

Fig. 12 Dividing the ring into k sub-chains.

Theorem 1: Let P be a convex polygon with n vertices, an ϵ-convex $(8+4\sqrt{2})\epsilon$-superhull of P can be found in $O(\log n)$ time using n processors.

Proof: Set $\epsilon' = \sqrt{2}\epsilon$ and construct an ϵ-convex $(4+4\sqrt{2})\epsilon'$-superhull of P or an Almost-ϵ'-Convex-Ring of P in $O(\log n)$ time using $O(n)$ processors by applying Lemma 5. An ϵ'-convex $(4+4\sqrt{2})\epsilon'$-superhull of P is $(8+4\sqrt{2})\epsilon$-superhull of P. If we find P', an Almost-ϵ'-Convex-Ring of P we can find R, an ϵ-convex $(2+4\sqrt{2})\epsilon$-superhull of P' by Lemma 6. According to the definition of Almost-ϵ'-Convex-Ring, the vertices of P' lie at most $4\epsilon'$ outside of P. Therefore, R is an ϵ-convex $(2+8\sqrt{2})\epsilon$-superhull of P. □

4. Implementation and Experiment

Let the size of the input (a set of points in the plane) be 10,000. The points are taken from three distributions: uniformly distributed inside a square, uniformly distributed inside a disk, and uniformly distributed on the boundary of a disk. We implement our algorithm and do the experiments for the inputs of the above three distributions. The results of the experiments are listed in Table 1, where the data of (I), (II) and (III) correspond to the first, the second and the third distributions, respectively.

From Table 1, we see that for a given (requested) ϵ, our parallel algorithm computes an ϵ'-convex δ'-superhull of S, where ϵ' is larger than ϵ and δ' is much smaller than the theoretical value $(8+4\sqrt{2})\epsilon$. In distribution (I), note that due to the large size of the input set and the shape of that distribution, the results showed to be very convex. As a comparison, we also list the results of the experiments of the sequential algorithm [6] in Table 2. We can observe that the parallel algorithm has a fairly good average performance

Table 1 Average performance of our parallel algorithm.

ϵ				δ			
requested	actual			theoretical	actual		
	(I)	(II)	(III)		(I)	(II)	(III)
0.0200	0.0295	0.0283	0.0283	0.2663	0.1066	0.1527	0.0638
0.0800	0.1353	0.2298	0.2346	1.0651	0.1844	0.5038	0.2179
0.3240	0.4694	0.5313	1.0339	4.3136	0.9941	0.9770	0.7218
0.5600	7.5956	0.8403	1.0184	7.4557	0.9610	1.9172	0.7112
0.9700	7.9824	1.4832	1.3718	12.914	1.9739	3.2725	3.3924
1.6300	20.9204	3.9363	4.5167	21.701	1.9057	3.2830	2.6691
2.0460	21.1435	4.0635	4.5095	27.239	0.9224	3.2828	2.6796

Table 2 Average performance of the sequential algorithm.

ϵ				δ			
requested	actual			theoretical	actual		
	(I)	(II)	(III)		(I)	(II)	(III)
0.0200	0.0210	0.0200	0.0833	0.1520	0.0321	0.0469	0.0249
0.0800	0.0839	0.0811	0.1177	0.6080	0.1319	0.1891	0.1267
0.3240	0.4416	0.3602	0.4841	2.4624	0.9153	0.4928	0.5313
0.5600	0.6009	0.6558	0.8358	4.2560	0.7905	0.7695	1.0266
0.9700	0.9773	1.1931	1.4534	7.3720	1.3783	1.4291	1.1378
1.6300	1.6728	1.9788	2.2468	12.388	2.3988	2.2223	1.6717
2.0460	14.6911	2.6134	2.0876	15.549	3.4117	3.3905	2.1086

as the sequential algorithm does. Moreover, it can be easily seen that the parallel algorithm gives a stronger convexity than the sequential one. This is due to the fact that in the parallel method some vertices may be revised more than once, consequently becoming more convex. While in the sequential algorithm the vertices are revised only once by the sweeping process.

5. Conclusion

We introduced the first parallel algorithm that computes an ϵ-convex $(8+4\sqrt{2})\epsilon$-superhull of a set S of n points in the plane in $O(\log n)$ time using n processors. The computational model we used is the $EREW$ $PRAM$. We did experiments and found that (i) the practical data are much better than the theoretical analysis shows, and (ii) our parallel method gives a stronger convexity than the sequential one.

References

[1] A. Aggarwal, B. Chazelle, L. Guibas, C. O'Dunlaing, and C. Yap, "Parallel computational geometry," Algorithmica, vol.3, pp.293–327, 1988.

[2] M.J. Atallah and M.T. Goodrich, "Efficient parallel solutions to some geometric problems," Parallel and Distributed Computing, vol.3, pp.492–507, 1986.

[3] M.J. Atallah and M.T. Goodrich, "Parallel algorithms for some functions of two convex polygons," Algorithmica, vol.3, pp.535–548, 1988.

[4] W. Chen, K. Nakano, T. Masuzawa, and N. Tokura, "A parallel method for the prefix convex hulls problem," IEICE Trans. Fundamentals, vol.E77-A, no.10, pp.1675–1683, Oct. 1994.

[5] W. Chen, K. Wada, and K. Kawaguchi, "Parallel robust algorithms for constructing strongly convex hulls," Proc. 12th Annual ACM Symposium on Computational Geometry, pp.133–140, 1996.

[6] W. Chen, X.W. Deng, K. Wada, and K. Kawaguchi, "Constructing a strongly convex superhull of points," COCOON'97, Lecture Notes in Computer Science 1276, pp.4–51, 1997.

[7] R. Cole and M.T. Goodrich, "Optimal parallel algorithms for point-set polygon problems," Algorithmica, vol.7, pp.3–23, 1992.

[8] P.-O. Fjällström, J. Katajainen, C. Levcopoulos, and O. Petersson, "A sublogarithmic convex hull algorithm," Bit, vol.30, pp.378–384, 1990.

[9] L. Guibas, D. Salesin, and J. Stolfi, "Constructing strongly convex approximate hulls with inaccurate primitives," Algorithmica, vol.9, pp.534–560, 1993.

[10] D.T. Lee and F.P. Preparata, "Computational geometry—A survey," IEEE Trans. Comput., vol.C-33, no.12, pp.1072–1101, 1984.

[11] Z. Li and V.J. Milenkovic, "Constructing strongly convex hulls using exact or rounded arithmetic," Algorithmica, vol.8, pp.345–364, 1992.

[12] F.P. Preparata and M.L. Shamos, Computational Geometry: An Introduction, Springer-Verlag, 1985.

Appendix A: Proof of Lemma 2

Lemma 2 Let $G = (u_1, \ldots, u_n)$ be a convex polygonal chain and $H = (u_{f+1}, \ldots, u_{g-1})$ be a contiguous subchain of G. If the vertices u_f and u_g of G which lie directly before u_{f+1} and after u_{g-1} are ϵ-convex, then an $H' = sswp(G, H, \epsilon, 2\epsilon)$ can be found in $O(|H|)$ time using a single processor, and if H has at least one ϵ-convex vertex H' has also at least one ϵ-convex vertex.

Proof: We revise H into H' from u_{f+1} to u_{g-1}. Let $i = f+1$ and while $i < g$ apply the following scanning-and-revising process to H, repeatedly.

If u_i is ϵ-convex, set $i = i+1$, else scan the vertices of $H(i : g-1)$ from u_i one by one in counterclockwise to find k such that $d(I_j, u_{i-1}u_j) < 2\epsilon$ for all j $(i \le j \le k)$ and $d(I_{k+1}, u_{i-1}u_{k+1}) \ge 2\epsilon$ where $I_j = int(l(u_{i-1}, u_i), l(u_j, u_{j+1}))$ (k must exist since $d(I_i, u_{i-1}u_i) = 0$). We revise $H(i : k)$ as follows.

(Case 1) I_k is normal in triangle $\triangle u_{i-1}I_{k+1}u_{k+1}$, i.e., $d(I_{k+1}, u_{i-1}u_{k+1}) = d(I_{k+1}, l(u_{i-1}, u_{k+1}))$.

Note that while scanning, whenever an ϵ-convex vertex is found, say u_s $(i < s < g-1)$, it must be selected as u_k since it is easily seen that $d(I_{s+1}, u_{i-1}u_{s+1}) > d(u_s, l(u_{s-1}, u_{s+1})) \ge 2\epsilon$. Therefore u_{s+1} can not be selected as u_k since u_s is ϵ-convex. That is, the scanning always stops at u_s if such vertex exists. It is easily seen that $H(i : k)$ contains at most one ϵ-convex vertex (u_s). In the following we revise $H(i : k)$ into F such that if $H(i : k)$ contains one ϵ-convex vertex, then in F there is at least one ϵ-convex vertex.

If $k = g-1$ then let $F = (I_k)$. Note that in this case I_k may be ϵ-flat. In the following assume that $k < g-1$. From the definition of k, $d(I_k, u_{i-1}u_k) < 2\epsilon$. If $d(I_k, l(u_{i-1}, u_{k+1})) \ge 2\epsilon$, then let $F = (I_k)$. Else find point z on segment I_kI_{k+1} such that $d(z, l(u_{i-1}, u_{k+1})) = 2\epsilon$ (such z must exist since

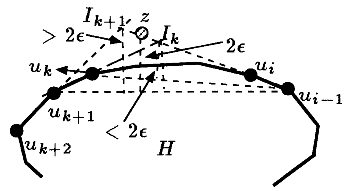

Fig. A·1 Finding z.

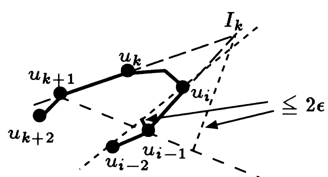

Fig. A·2 Finding F.

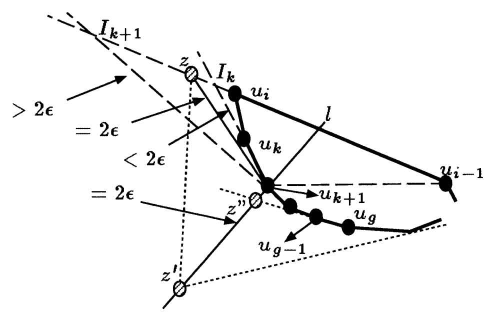

Fig. A·3 Finding z and z' or z''.

$d(I_k, l(u_{i-1}, u_{k+1})) < 2\epsilon$ and $d(I_{k+1}, l(u_{i-1}, u_{k+1})) \geq 2\epsilon$), and let $F = (z)$ (Fig. A·1). Here, obviously I_k and z are ϵ-convex. Revise $H(i : k)$ into F. Let $i = k + 1$.

(Case 2) I_k is abnormal in triangle $\triangle u_{i-1} I_{k+1} u_{k+1}$, i.e., $d(I_{k+1}, u_{k+1}u_{i-1}) \neq d(I_{k+1}, l(u_{k+1}u_{i-1}))$

Subcase 1 Angle $\angle u_i u_{i-1} u_{k+1} \geq \pi/2$ (Fig. A·2).

Let $F = (I_k)$ and revise the sub-sequence $H(i : k)$ into a new sequence F. Set $i = k + 1$.

(It is easily seen that I_k is ϵ-convex since $d(I_k, l(u_{i-1}u_{k+1})) \geq 2\epsilon$ from the fact that u_{i-1} is ϵ-convex in G.)

Subcase 2 Angle $\angle u_i u_{i-1} u_{k+1} < \pi/2$.

If $d(I_k, u_{i-1}u_{k+1}) \geq 2\epsilon$, let $F = (I_k)$, revise $H(i : k)$ into F and set $i = k + 1$. Else do the following (Fig. A·3).

Find a point z on the segment $I_k I_{k+1}$ such that $d(z, u_{i-1}u_{k+1}) = 2\epsilon$ (Such z must exist since $d(I_{k+1}, u_{i-1}u_{k+1}) \geq 2\epsilon$ and $d(I_k, u_{i-1}u_{k+1}) \leq 2\epsilon$). Draw a line l vertical to line $l(I_{k+1}, u_{k+1})$, and find a point z' on line l such that $|z' u_{k+1}| = 2\epsilon$. Determine whether line $l(z', u_j)$is the tangent from point z' to $H(k + 1 : g)$ for each j from $j = k + 1$ to g.

(a) If there is some j $(k+1 \leq j \leq g)$ such that line $l(z', u_j)$ is the tangent, let $F = (z, z')$, revise $H(i : j-1)$ into F and set $i = j$. Obviously z and z' are ϵ-convex from their definition.

(b) If $l(z', u_j)$ is not the tangent for any j $(k+1 \leq j \leq g)$, then let z'' be the intersection of lines $l(u_{g-1}, u_g)$ and $l(u_{k+1}, z')$. Let $F = (z, z'')$, revise $H(i : g - 1)$ into F and set $i = g$. Obviously, z is ϵ-convex from its definition and vertex z'' may be ϵ-flat.

After executing the above scanning-and-revising process repeatedly, $H(f+1 : g-1)$ is finally revised into H'. We prove that the following conditions hold: (1) H' is a $sswp(G, H, \epsilon, 2\epsilon)$ and (2) if H has at least one ϵ-convex vertex then H' also has at least one ϵ-convex vertex.

Now we prove that H' is a $sswp(G, H, \epsilon, 2\epsilon)$. In the above scanning-and-revising process, $H(i : k)$ is changed into F. It is easily seen that $|F| \leq |H(i : k)|$, $G(H(i : k) \rightarrow F)$ is convex, and the convexities of the vertices of G other than those of $H(i : k)$ are not changed. On the other hand, from the construction of F, its vertices lie at most 2ϵ outside of $H(i : k)$, and they are ϵ-convex except in two situations: Situation (i), in Case 1 when no ϵ-convex vertex exists in $H(i : k)$ and $k = g - 1$. In such case $F = (I_k)$ and I_k may be ϵ-flat. Since $k = g - 1$ then I_k is the last vertex of H'. Situation (ii), in Subcase 2 (b) of Case 2, where the last vertex u_j of F may be not ϵ-convex. If this case happens, i is set to g and no more scanning-and-revising process will be done, then u_j is the last vertex of H'. Therefore, $|G| \leq |H(i : k)|$, $G(H \rightarrow H')$ is convex, and the convexities of its vertices other than those of H are not changed. The vertices of H' lie at most 2ϵ outside of $H(i : k)$, and they are ϵ-convex except the last one. Therefore, H' is a $sswp(G, H, \epsilon, 2\epsilon)$.

Now we show that if $H(i : k)$ contains at least one ϵ-convex vertex then F contains at least one ϵ-convex vertex too. It is clear that $H(i : k)$ is revised into F such that all its vertices are ϵ-convex except in the two situations (i) and (ii) shown above. However, in Situation (i) if u_k is ϵ-convex then I_k is also ϵ-convex since $d(I_k, u_{i-1}u_{k+1}) > d(u_k, u_{k-1}u_{k+1}) \geq 2\epsilon$. And in Situation (ii) although the last vertex of F (u_j) may be ϵ-flat, according to Subcase 2(b) of Case 2, the first vertex of F (z) is ϵ-convex. Therefore, condition (2) holds.

H' can be computed in $O(g - 1)$ time since each vertex of H is scanned only once. □

Carla Denise Castanho received the B.S. degree in Computer Science from Passo Fundo University, Brazil, in 1994. She received the M.E. degree in Computer Science from Nagoya Institute of Technology, Japan, in 1998. She is currently a Ph.D. candidate in the Department of Electrical and Computer Engineering at Nagoya Institute of Technology. Her research interests include parallel algorithms and computational geometry.

Wei Chen received the B.A. degree in mathematics from Shanghai Marine University in 1982, and received M.E., and Ph.D. degrees from the Department of Information Engineering, Faculty of Engineering Science, Osaka University in 1991 and 1994. Since 1994 she has been working at the Department of Electrical and Computer Engineering, Nagoya Institute of Technology. She is now an associate professor of that university. Her research interests include parallel and distributed algorithms, computational geometry and graph theory. She is a member of ACM, IEEE, LA Symposium and IPSJ.

Koichi Wada graduated in 1978 from the Department of Information Engineering, Faculty of Engineering Science, Osaka University, and received his M.S. and Ph.D. degrees both from the same university in 1980 and 1983, respectively. He was a research associate at Osaka University during 1983–1984. In 1984 he joined Nagoya Institute of Technology, where he is currently a professor in the Department of Electrical and Computer Engineering. He was a visiting associate professor at University of Minnesota, Duluth and University of Wisconsin, Milwaukee during 1987–1988. His research interests include graph theory, parallel/distributed algorithms and VLSI theory. Dr. Wada is a member of IEEE, ACM, LA Symposium, Japan SIAM and IPSJ.