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Realization of Quantum Receiver for M-Ary Signals

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SUMMARY In quantum communication theory, a realization of the optimum quantum receiver that minimizes the error probability is one of fundamental problems. A quantum receiver is described by detection operators. Therefore, it is very important to derive the optimum detection operators for a realization of the optimum quantum receiver. In general, it is difficult to derive the optimum detection operators, except for some simple cases. In addition, even if we could derive the optimum detection operators, it is not trivial what device corresponds to the operators. In this paper, we show a realization method of a quantum receiver which is described by a projection-valued measure (PVM) and apply the method to 3-ary phase-shift-keyed (3PSK) coherent-state signals.

key words: *quantum communication theory, quantum detection, quantum receiver, realization problem, error probability*

1. Introduction

Realization of the optimum quantum receiver that minimizes the error probability is one of fundamental problems in quantum communication theory [1]–[4]. A measurement process of a quantum receiver is described by detection operators which are defined in a Hilbert space. Therefore, it is very important to derive the optimum detection operators for realization of the optimum quantum receiver. In general, it is difficult to derive the optimum detection operators analytically. Before 1990's, optimum detection operators only for few signals, e.g. binary pure-state signals, were known. In addition, even if we could derive the optimum detection operators, it is not trivial what device corresponds to the operators. It is necessary to consider a physical correspondence of it. This is called a realization problem of the optimum quantum receiver.

Studies of realization of the optimum quantum receiver were started by Kennedy and Dolinar in 1970's [5], [6]. They did not find the realization method from the optimum detection operators but tried to construct physical system by means of a heuristic manner. So far realization methods have been proposed for the optimum quantum receiver of binary coherent-state signals [6] and quasi-optimum quantum receivers of binary coherent-state signals [5], 4-ary phase-shift-keyed (4PSK) coherent-state signals [7] and M-ary orthogonal coherent-state signals [8].

On the other hand, studies of constructing the optimum quantum receiver from the optimum detection operators were started in 1990's. So far realization algorithm has been proposed for binary signals [9]–[11], binary codes [12] and M-ary linearly dependent signals in two-state systems [13]. They are all based on two-state system or its extension. As for general M-ary linearly independent signals, a direction of realization was mentioned in Ref. [14]. However no construction method of the optimum quantum receiver for a specific physical system is clarified. Recently studies of deriving the optimum detection operators have been developed, and the optimum detection operators have been clarified for pure-state signals such as certain 3 or 4-ary signals [15], M-ary symmetric signals [16] and binary linear codes [17], [18] and for mixed-state signals such as signals in two-state system [13], [19], [20] and the other specific signals [21], [22]. Therefore, it is desired to clarify the construction method of the optimum quantum receiver from its optimum detection operators for general M-ary linearly independent signals.

In this paper, we develop Ref. [14] and generalize the approach for binary pure-state signals [11] to clarify a realization method of a quantum receiver which is described by a projection-valued measure (PVM). It is known that the set of the optimum detection operators for linearly independent signals is a PVM [23]. Therefore, we can realize the optimum quantum receiver for any M-ary linearly independent signal by using our realization method.

This paper is organized as follows. In Sect. 2, we explain the measurement process and the error probability in quantum communication theory. In Sect. 3, we show our realization method. In Sect. 4, we apply the method to 3PSK signals. And Sect. 5 is for concluding remarks.

2. Basis of Quantum Detection

In this section, we briefly survey quantum detection theory [1]–[4].

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We consider quantum communication for sending M-ary c lassical information $\{x_i|i=0,\cdots,M-1\}$. In this case, the outputs of the transmitter are quantum states $\{\hat{\rho}_i\}$ which correspond to classical information. A quantum state is described by a Hermitian operator on the Hilbert space of the quantum system and satisfies

$$\hat{\rho}_i \geq 0, \quad \mathrm{Tr}\hat{\rho}_i = 1. \tag{1}$$

where Tr denotes the trace, i.e., the sum of diagonal elements of a matrix. If a signal state $\hat{\rho}_i$ is pure, the state can be described by a unit vector $|\psi_i\rangle$ in the Hilbert space and

$$\hat{\rho}_i = |\psi_i\rangle\langle\psi_i|. \tag{2}$$

The pure-state signal contains no classical noise and $\mathrm{Tr}\hat{\rho}_i^2 = 1$ is satisfied. If a signal state is mixed, the state contains classical noises and $\mathrm{Tr}\hat{\rho}_i^2 < 1$ is satisfied.

A quantum receiver in which quantum measurement is performed is generally described by a positive operator-valued measure (POM) which is a set of nonnegative Hermitian operators $\{\hat{\Pi}_j\}$ satisfying the resolution of the identity:

$$\sum_{j=0}^{M-1} \hat{\Pi}_j = \hat{\mathrm{I}}, \quad \hat{\Pi}_j \geq 0, \tag{3}$$

where $\hat{\mathrm{I}}$ is the identity operator and $\hat{\Pi}_j$ is called a detection operator and expresses the measurement process which decides that a received state is $\hat{\rho}_j$. In general, the number of detection operators need not to equal the number of signals. But we suppose that they are equal since we only treat error probabilities as criteria. If $\{\hat{\Pi}_j\}$ satisfies not only Eq. (3) but also

$$\hat{\Pi}_j\hat{\Pi}_{j'} = \delta_{jj'}\hat{\Pi}_j, \tag{4}$$

$\{\hat{\Pi}_j\}$ is a projection-valued measure (PVM). Here $\delta_{jj'}$ is the Kronecker-delta which is defined as

$$\delta_{jj'} = \begin{cases} 0, & (j \neq j'), \\ 1, & (j = j'). \end{cases} \tag{5}$$

When the transmitted signal is $\hat{\rho}_i$, the probability that the received signal is decided to be $\hat{\rho}_j$ is represented by $\hat{\Pi}_j$ as

$$P(\hat{\rho}_j|\hat{\rho}_i) = \mathrm{Tr}\hat{\rho}_i\hat{\Pi}_j. \tag{6}$$

Then the error probability P_e is represented as

$$P_\mathrm{e} = 1 - \sum_{i=0}^{M-1} \xi_i \mathrm{Tr}\hat{\rho}_i\hat{\Pi}_i, \tag{7}$$

where ξ_i is *a priori* probability of $\hat{\rho}_i$. The operators $\{\hat{\Pi}_j^{(\mathrm{opt})}\}$ by which the P_e is minimized are called the optimum detection operators.

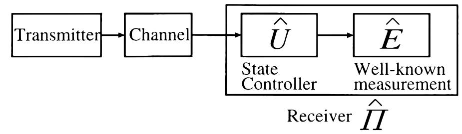

Fig. 1 Received quantum state control.

3. Realization of PVM

In this section, we show a realization method of a quantum receiver of which the detection operators are described by a PVM. Generally, the expression of detection operators $\{\hat{\Pi}_j\}$ of a quantum receiver is abstract and it is not obvious the physical correspondence of the operators. On the other hand, detection operators which correspond to measurement of signal observable are called standard detection operators [14], [24], [25], and its physical correspondence is obvious. Needless to say, physical correspondence of a measurement of the other well-known observable is also obvious. In the following, we represent standard detection operators or the other well-known measurement as $\{\hat{E}_j\}$.

It was shown in Ref. [10] that the optimum detection operators for any binary pure-state signals can be realized by a unitary operator $\hat{U}$ and well-known detection $\{\hat{E}_j\}$ as Fig. 1. These construction methods of receivers are called a received quantum state control [26]. Refs. [9] and [10] developed the idea of utilizing a unitary process as a received quantum state controller [27] and showed algorithm for binary pure-state signals, by which the unitary operator $\hat{U}$ is derived. Here $\hat{U}$ is related to the detection operator of the quantum receiver as

$$\hat{\Pi}_j = \hat{U}^\dagger\hat{E}_j\hat{U}. \tag{8}$$

This algorithm has been generalized for binary codes and M-ary linearly dependent signals in two-state system and the structure of the quantum circuit which plays the role of the unitary operator $\hat{U}$ was discussed [12], [13].

Here we develop Ref. [14] in which possibility of applying a received quantum state control to realization of the optimum quantum receiver for general signals was mentioned and generalize the approach for binary pure-state signals [11] to clarify a realization method of a quantum receiver for M-ary signals which is described by a PVM.

The realization method is divided into two steps. The first step is to decompose each detection operator $\hat{\Pi}_j$ which describes the quantum receiver into a unitary operator $\hat{U}$ and a detection operator $\hat{E}_j$ corresponding to well-known measurement. The second step is to derive the quantum circuit which corresponds to the unitary operator $\hat{U}$.

3.1 Decomposition of Detection Operators

The Hilbert space spanned by signal quantum states is called the signal space. If the signals are M-ary and linearly independent, the dimensions of the signal space are M. The detection operators of the quantum receiver are defined on this M-dimensional space. In general, the dimensions d of the Hilbert space of the physical system are not always equal to M. For example, the dimensions of an optical system are infinity. Detection operators of well-known measurement are defined on d-dimensional space and projections of them onto the signal space are not generally a PVM. Therefore, it is impossible to derive the unitary operator which expresses the received quantum state controller on M-dimensional space.

Here we generalize the approach for binary pure-state signals [11] and introduce an extended space so that detection operators of well-known measurement become orthogonal on the extended space.

First, we apply the identity operator on d-dimensional space which is the sum of the detection operators of well-known measurement to each signal. Each of signal states $\{|\psi_i\rangle | i = 0, \cdots, M-1\}$ is projected into M states,

$$|\psi_i\rangle = \sum_{j=0}^{M-1} \hat{E}_j |\psi_i\rangle = \sum_{j=0}^{M-1} \sqrt{\epsilon_{ij}} |A_{ij}\rangle, \tag{9}$$

where

$$|A_{ij}\rangle = \frac{1}{\sqrt{\epsilon_{ij}}} \hat{E}_j |\psi_i\rangle. \tag{10}$$

Here ϵ_{ij} expresses the probability that the received signal is decided to be $\hat{\rho}_j$ by $\hat{E}_j$ when the transmitted signal is $\hat{\rho}_i$ and satisfies

$$\epsilon_{ij} = \langle\psi_i|\hat{E}_j|\psi_i\rangle, \tag{11}$$

$$\sum_{j=0}^{M-1} \epsilon_{ij} = 1. \tag{12}$$

In this way, M signals can be described by M^2 vectors $\{|A_{ij}\rangle | i, j = 0, \cdots, M-1\}$. We apply Schmidt orthonormalization to N linearly independent vectors of M^2 vectors $\{|A_{ij}\rangle | i, j = 0, \cdots, M-1\}$ and introduce N-dimensional extended space spanned by the orthonormal set which is derived by Schmidt orthonormalization.

$\{|A_{i0}\rangle, \cdots, |A_{i(M-1)}\rangle\}$ is orthogonal for any i. On the other hand, $\{|A_{0j}\rangle, \cdots, |A_{(M-1)j}\rangle\}$ is non-orthogonal and sometimes linearly dependent. So if n_j vectors of $\{|A_{0j}\rangle, \cdots, |A_{(M-1)j}\rangle\}$ are linearly independent, we orthonormalize these vectors and derive an orthonormal set $\{|r_{j'+l}\rangle | l = 0, \cdots, n_j - 1\}$($j' = \sum_{m=0}^{j-1} n_m$). This operation for all j derives $N(= \sum_{j=0}^{M-1} n_j)$ orthogonal vectors $\{|r_k\rangle | k = 0, \cdots, N-1\}$, where $M \le N \le M^2$. These N vectors construct a orthonormal set in N-dimensional space. $\hat{E}_j$ is related to this orthonormal set as follows:

$$\hat{E}_j = \sum_{k=j'}^{j'+n_j-1} |r_k\rangle\langle r_k|, \tag{13}$$

$$j' = \sum_{m=0}^{j-1} n_m. \tag{14}$$

Since the detection operators $\hat{\Pi}_j$ of the quantum receiver for M-ary signals are a PVM, it can be represented as

$$\hat{\Pi}_j = |\omega_j\rangle\langle\omega_j|, \tag{15}$$

where $\{|\omega_j\rangle | j = 0, \cdots, M-1\}$ are called measurement basis vectors. We extend these M basis vectors to basis vectors in N-dimensional extended space. We make an orthonormal set $\{|\omega'_k\rangle | k = 0, \cdots, N-1\}$ on N-dimensional extended space by adding $N - M$ basis vectors obtained by Schmidt orthonormalization, where $|\omega_j\rangle = |\omega'_k\rangle$ if $j = k$.

In this way, both $\{\hat{\Pi}_j\}$ and $\{\hat{E}_j\}$ become PVM on N-dimensional extended space. Thus realization of the detection operators will be in principle possible by combining a quantum circuit which plays the role of $\hat{U}$ and the detection $\{\hat{E}_j\}$ if we define the unitary operator as

$$|\omega'_k\rangle = \hat{U}^\dagger |r_k\rangle. \tag{16}$$

3.2 Decomposition of Unitary Operator

In order to construct the quantum receiver we have to derive the Hamiltonian which corresponds to $\hat{U}$ and clarify the structure of a quantum circuit which corresponds to $\hat{U}$. The unitary operator defined in Eq. (16) is the operator on N-dimensional space. Therefore, it is difficult to derive the Hamiltonian which corresponds to $\hat{U}$ and to construct a quantum circuit. We can not directly apply the derivation method in Ref. [11] in which the realization of the optimum quantum detection for binary signals was shown and it was studied how to construct the quantum circuit which corresponds to a unitary operator on 1 or 2-dimensional space. However, if we can decompose a unitary operator on N-dimensional space into 1 or 2-dimensional operations, it will be possible to construct the quantum circuit. Here we consider the spectral decomposition of $\hat{U}$. According to Stone's theorem, $\hat{U}$ on N-dimensional space can be decomposed as

$$\hat{U} = \sum_{t=0}^{N-1} \mathrm{e}^{\mathrm{i}\theta_t} |x_t\rangle\langle x_t|, \tag{17}$$

where $e^{i\theta_t}$ and $|x_t\rangle$ are an eigenvalue and an eigenvector of $\hat{U}$, respectively and $\{|x_t\rangle\}$ is an orthonormal set in N-dimensional space. Then $\hat{U}$ is expressed as

$$\hat{U} = \prod_{t=0}^{N-1} \hat{R}_t(\theta_t), \tag{18}$$

$$\hat{R}_t(\theta_t) = \exp[i\theta_t |x_t\rangle\langle x_t|]. \tag{19}$$

In this way $\hat{U}$ can be decomposed into N 1-dimensional rotations. Thus it will be possible to construct the quantum circuit if we derive the Hamiltonians for 1-dimensional rotations in Eq. (19). Derivation of Hamiltonians should be considered for respective signals based on a certain physical system. Here we show the Hamiltonian for an optical system. Because the optical system is most important system in quantum communication. In the optical system, the Hamiltonian for $\hat{R}_t(\theta_t)$ in Eq. (19) is generally derived as follows:

$$\hat{H}_{\hat{R}_t} = -\hbar g \hat{X}, \tag{20}$$

$$\begin{aligned}\hat{X} &= |x_t\rangle\langle x_t| \\ &= \sum_{m=0}^{\infty} \frac{C_{t,m}}{\sqrt{m!}} (\hat{a}^\dagger)^m \sum_{l=0}^{\infty} \frac{(-\hat{a}^\dagger)^l (\hat{a})^l}{l!} \\ &\quad \cdot \sum_{n=0}^{\infty} \frac{C_{t,n}^*}{\sqrt{n!}} (\hat{a})^n,\end{aligned} \tag{21}$$

where $\hat{a}$ ($\hat{a}^\dagger$) is the photon annihilation (creation) operator,

$$C_{t,n} = \langle n|x_t\rangle, \tag{22}$$

$$\sum_n \sum_m C_{t,n} C_{t',m} = \begin{cases} 1, & k = k', \\ 0, & k \neq k', \end{cases} \tag{23}$$

and $g = \theta_t/\tau$ is the coupling constant of the physical process represented by the Hamiltonian $\hat{H}_{\hat{R}_t}$ and τ is the interaction time, and $|n\rangle$ is a photon number state with photon number n. The Hamiltonian corresponds to nonlinear multiphoton processes.

4. Example

In this section, we show realization of the optimum quantum receiver for 3-ary phase-shift-keyed (3PSK) coherent-state signals as an example. The quantum system in this case is an optical system and the dimensions are infinity. Therefore, the dimensions of the quantum system are much higher than the signal space ($M \ll d$).

Signal states of 3PSK are expressed as

$$|\psi_0\rangle = |\alpha\rangle, |\psi_1\rangle = |\alpha e^{\frac{2\pi i}{3}}\rangle, |\psi_2\rangle = |\alpha e^{-\frac{2\pi i}{3}}\rangle. \tag{24}$$

We assumed that α is real since no generality is lost by this assumption. An orthonormal set $\{|e_j\rangle\}$ on 3-dimensional Hilbert space spanned by these signal states can be chosen as follows [28]

$$|e_0\rangle = \frac{1}{\sqrt{3h_0}}(|\psi_0\rangle + e^{-\frac{2\pi i}{3}}|\psi_1\rangle + e^{\frac{2\pi i}{3}}|\psi_2\rangle), \tag{25}$$

$$|e_1\rangle = \frac{1}{\sqrt{3h_1}}(|\psi_0\rangle + e^{\frac{2\pi i}{3}}|\psi_1\rangle + e^{-\frac{2\pi i}{3}}|\psi_2\rangle), \tag{26}$$

$$|e_2\rangle = \frac{1}{\sqrt{3h_2}}(|\psi_0\rangle + |\psi_1\rangle + |\psi_2\rangle), \tag{27}$$

where h_j denotes an eigenvalue of the Gram matrix

$$h_0 = 1 - K_c + \sqrt{3}K_s, \tag{28}$$

$$h_1 = 1 - K_c - \sqrt{3}K_s, \tag{29}$$

$$h_2 = 1 + 2K_c, \tag{30}$$

$$K_c = e^{-\frac{3}{2}|\alpha|^2} \cos\frac{\sqrt{3}}{2}|\alpha|^2, \tag{31}$$

$$K_s = e^{-\frac{3}{2}|\alpha|^2} \sin\frac{\sqrt{3}}{2}|\alpha|^2. \tag{32}$$

It was shown in Ref. [16] that the optimum detection operators for 3PSK signal with equal *a priori* probability correspond to a measurement process which is called a Square-root measurement(SRM) [29]–[31]. The measurement basis vectors $\{|\omega_i\rangle\}$ of SRM are expressed as

$$|\omega_0\rangle = \frac{1}{\sqrt{3}}(|e_0\rangle + |e_1\rangle + |e_2\rangle), \tag{33}$$

$$|\omega_1\rangle = \frac{1}{\sqrt{3}}(e^{\frac{2\pi i}{3}}|e_0\rangle + e^{-\frac{2\pi i}{3}}|e_1\rangle + |e_2\rangle), \tag{34}$$

$$|\omega_2\rangle = \frac{1}{\sqrt{3}}(e^{-\frac{2\pi i}{3}}|e_0\rangle + e^{\frac{2\pi i}{3}}|e_1\rangle + |e_2\rangle). \tag{35}$$

Now each of the signal states is projected into three states,

$$|\psi_i\rangle = \sum_{j=0}^{2} \hat{E}_j|\psi_i\rangle = \sum_{j=0}^{2} \sqrt{\epsilon_{ij}}|A_{ij}\rangle, \quad i,j = 0,1,2, \tag{36}$$

where we assumed the detection of well-known observable is a photon counting and $\hat{E}_0 = |0\rangle\langle 0|, \hat{E}_1 = |1\rangle\langle 1|, \hat{E}_2 = \hat{I} - |0\rangle\langle 0| - |1\rangle\langle 1|$. $|0\rangle$ and $|1\rangle$ are vacuum state and photon number state with photon number 1, respectively. Ordinary, standard detection operators express the optimum classical receiver. But in this case, the optimum classical receiver is a heterodyne receiver which corresponds to simultaneous measurement of two observables and its detection operators are not a PVM. So we used the photon counting as the detection process though it is not the optimum classical receiver.

The subspace $\mathcal{H}_0$ spanned by $\{|A_{i0}\rangle\}$, the subspace $\mathcal{H}_1$ spanned by $\{|A_{i1}\rangle\}$ and the subspace $\mathcal{H}_2$ spanned by $\{|A_{i2}\rangle\}$ are orthogonal each other. But $|A_{0j}\rangle, |A_{1j}\rangle$ and $|A_{2j}\rangle$ are not. Both $\{|A_{i0}\rangle\}$ and $\{|A_{i1}\rangle\}$ are linearly dependent set as

$$|A_{00}\rangle = |A_{10}\rangle = |A_{20}\rangle = |0\rangle, \tag{37}$$

$$|A_{01}\rangle = \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}}|A_{11}\rangle = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}|A_{21}\rangle = |1\rangle. \tag{38}$$

So we determine $|r_0\rangle$ and $|r_1\rangle$ as

$$|r_0\rangle = |A_{00}\rangle = |0\rangle, \tag{39}$$

$$|r_1\rangle = |A_{01}\rangle = |1\rangle. \tag{40}$$

On the other hand, $\{|A_{i2}\rangle\}$ is linearly independent set. So we apply Schmidt orthonormalization to $\{|A_{i2}\rangle\}$ and determine the orthonormal vectors $|r_2\rangle, |r_3\rangle$ and $|r_4\rangle$:

$$|r_2\rangle = |A_{02}\rangle, \tag{41}$$

$$|r_3\rangle = \frac{|A_{12}\rangle - \langle r_2|A_{12}\rangle|r_2\rangle}{\sqrt{1-|\langle r_2|A_{12}\rangle|^2}}, \tag{42}$$

$$|r_4\rangle = \frac{|A_{22}\rangle - \langle r_2|A_{22}\rangle|r_2\rangle - \langle r_3|A_{22}\rangle|r_3\rangle}{\sqrt{1-|\langle r_2|A_{22}\rangle|^2-|\langle r_3|A_{22}\rangle|^2}}, \tag{43}$$

where

$$|A_{m2}\rangle = \frac{|\alpha \mathrm{e}^{\frac{2m\pi \mathrm{i}}{3}}\rangle - \mathrm{e}^{-\frac{|\alpha|^2}{2}}|0\rangle - \alpha \mathrm{e}^{\frac{2m\pi \mathrm{i}}{3}-\frac{|\alpha|^2}{2}}|1\rangle}{\sqrt{1-\mathrm{e}^{-|\alpha|^2}-|\alpha|^2\mathrm{e}^{-|\alpha|^2}}}, \tag{44}$$

$$\langle A_{02}|A_{12}\rangle = \langle A_{12}|A_{22}\rangle = \langle A_{02}|A_{22}\rangle^* \tag{45}$$

$$= \frac{\mathrm{e}^{\frac{-3+\sqrt{3}\mathrm{i}}{2}|\alpha|^2} - \mathrm{e}^{-|\alpha|^2}(1+\frac{1-\sqrt{3}\mathrm{i}}{2}|\alpha|^2)}{1-\mathrm{e}^{-|\alpha|^2}(1+|\alpha|^2)}. \tag{46}$$

From the above calculation, we obtain an orthonormal set $\{|r_k\rangle|k=0,\cdots,4\}$ in 5-dimensional space. $\{\hat{E}_j\}$ is expressed by this orthonormal set as follows:

$$\hat{E}_0 = |r_0\rangle\langle r_0|, \tag{47}$$

$$\hat{E}_1 = |r_1\rangle\langle r_1|, \tag{48}$$

$$\hat{E}_2 = \sum_{k=2}^{4}|r_k\rangle\langle r_k|. \tag{49}$$

The measurement basis vectors of the optimum detection operators are expressed as

$$|\omega_j\rangle = \sum_{k=0}^{4} c_{jk}|r_k\rangle, \quad j=0,1,2, \tag{50}$$

where $c_{jk} = \langle r_k|\omega_j\rangle$ which can be easily calculated from Eqs. (24)–(35) and Eqs. (39)–(46). Then we add two basis vectors $|\omega_3'\rangle$ and $|\omega_4'\rangle$ to these three basis vectors to construct an orthonormal set $\{|\omega_k'\rangle\}$. Here $\{\hat{\Pi}_j^{(\mathrm{opt})}\}$ is expressed by $\{|\omega_k'\rangle\}$ as follows:

$$\hat{\Pi}_0^{(\mathrm{opt})} = |\omega_0'\rangle\langle\omega_0'|, \tag{51}$$

$$\hat{\Pi}_1^{(\mathrm{opt})} = |\omega_1'\rangle\langle\omega_1'|, \tag{52}$$

$$\hat{\Pi}_2^{(\mathrm{opt})} = \sum_{k=2}^{4}|\omega_k'\rangle\langle\omega_k'|. \tag{53}$$

where we determined $|\omega_3'\rangle$ and $|\omega_4'\rangle$ as follows:

$$|\omega_k'\rangle = \frac{|r_{k-3}\rangle - \sum_{m=0}^{k-1}\langle\omega_m|r_{k-3}\rangle|\omega_m\rangle}{\sqrt{1-\sum_{m=0}^{k-1}|\langle\omega_m|r_{k-3}\rangle|^2}}, \quad k=3,4. \tag{54}$$

These two basis vectors do not affect the measurement result since they are orthogonal to signal states.

We define the unitary operator $\hat{U}$ which connects $\{\hat{\Pi}_j^{(\mathrm{opt})}\}$ and $\{\hat{E}_j\}$ as

$$\hat{\Pi}_j^{(\mathrm{opt})} = \hat{U}^\dagger \hat{E}_j \hat{U}. \tag{55}$$

Here $\hat{U}$ is expressed as

$$\hat{U} = [c_{jk}^*], \quad j,k=0,1,\cdots,4 \tag{56}$$

where $c_{jk} = \langle r_k|\omega_j'\rangle$.

Next, $\hat{U}$ should be decomposed into some 1-dimensional rotations in order to derive a Hamiltonian. According to Stone's theorem, $\hat{U}$ on N-dimensional space can be decomposed as

$$\hat{U} = \sum_{t=0}^{4}\mathrm{e}^{\mathrm{i}\theta_t}|x_t\rangle\langle x_t|. \tag{57}$$

Then $\hat{U}$ can be decomposed into five 1-dimensional rotations as follows

$$\hat{U} = \prod_{t=0}^{4}\hat{R}_t(\theta_t), \tag{58}$$

$$\hat{R}_t(\theta_t) = \exp[\mathrm{i}\theta_t|x_t\rangle\langle x_t|]. \tag{59}$$

We must derive eigenvalues of $\hat{U}$ in order to derive five rotation angles $\{\theta_t\}$. It is not easy to derive these angles analytically. However it is comparatively easy to derive them numerically. Figure 2 shows numerical values of θ_t with respect to $N_\mathrm{s} = |\alpha|^2$. We can see from Fig. 2 that each angle θ_t converges a certain value when N_s is large.

Next, we have to describe the eigenvectors $\{|x_t\rangle\}$ by annihilation (creation) operators $\hat{a}(\hat{a}^\dagger)$ in order to derive Hamiltonians of $\hat{R}_t(\theta_t)$. It is enough to derive the description of $\{|x_t\rangle\}$ by $\{|r_k\rangle\}$ and the description of $\{|r_k\rangle\}$ by $\hat{a}$ and $\hat{a}^\dagger$ for this purpose. Let $r_{tk} = \langle r_k|x_t\rangle$ then

$$|x_t\rangle = \sum_{k=0}^{4} r_{tk}|r_k\rangle. \tag{60}$$

The analytical description of $\{|x_t\rangle\}$ by $\{|r_k\rangle\}$ is very

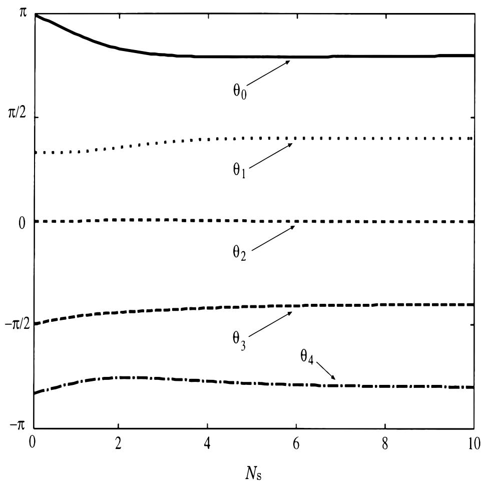

Fig. 2 Angles of rotations.

complicated. Here we show numerical values for the case of $N_\mathrm{s} = 3$ as an example.

$$|x_0\rangle = \begin{bmatrix} -0.289624 + 0.340012\mathrm{i} \\ -0.168546 - 0.402603\mathrm{i} \\ 0.183842 - 0.410997\mathrm{i} \\ 0.538176 \\ 0.0241692 + 0.342168\mathrm{i} \end{bmatrix}, \tag{61}$$

$$|x_1\rangle = \begin{bmatrix} -0.271052 - 0.147831\mathrm{i} \\ 0.0925722 + 0.203245\mathrm{i} \\ 0.0686031 - 0.395389\mathrm{i} \\ -0.230179 + 0.298503\mathrm{i} \\ 0.742747 \end{bmatrix}, \tag{62}$$

$$|x_2\rangle = \begin{bmatrix} 0.663316 \\ 0.379212 - 0.0696386\mathrm{i} \\ 0.408074 + 0.044467\mathrm{i} \\ 0.365357 - 0.00786251\mathrm{i} \\ 0.316223 - 0.0965159\mathrm{i} \end{bmatrix}, \tag{63}$$

$$|x_3\rangle = \begin{bmatrix} 0.062142 + 0.32745\mathrm{i} \\ -0.478264 - 0.158766\mathrm{i} \\ 0.560824 \\ -0.367122 + 0.103642\mathrm{i} \\ -0.0163204 - 0.417925\mathrm{i} \end{bmatrix}, \tag{64}$$

$$|x_4\rangle = \begin{bmatrix} -0.0445443 + 0.390037\mathrm{i} \\ 0.59752 \\ 0.0938877 - 0.380006\mathrm{i} \\ -0.413167 - 0.344246\mathrm{i} \\ -0.213751 + 0.0270972\mathrm{i} \end{bmatrix}. \tag{65}$$

The rotation $\hat{R}_t(\theta_t)$ and corresponding Hamiltonian $\hat{H}_{\hat{R}_t}$ can be expressed by $\hat{a}$ and $\hat{a}^\dagger$ as in Eqs. (20)–(23). Thus the physical correspondence of the optimum quantum receiver for 3PSK signals, which is expressed by Eqs. (33)–(35) was shown.

5. Conclusion

In this paper we showed a realization method of the optimum quantum receiver for M-ary linearly independent signals and applied the method to 3PSK as an example. This is the first example of realization of the optimum quantum receiver for M-ary linearly independent signals which are not based on two-state system. This method is applicable to any case that detection operators are expressed as a PVM on a signal space. Therefore, it is also applicable to mixed-state signals defined in Ref. [22] and to the case that a quasi-optimum quantum receiver is described by a PVM [32].

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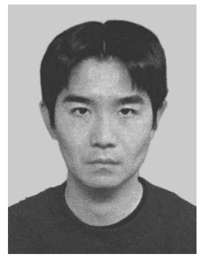

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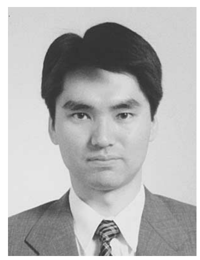

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