Fluctuating Hydrodynamics for a Rarefied Gas Based on Extended Thermodynamics

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Abstract

We develop a theory of fluctuating hydrodynamics based on extended thermodynamics through studying the 13-variable theory for a monatomic rarefied gas as a representative case. After analyzing the relationship between the present theory and the Landau-Lifshitz theory, we discuss the hierarchy structure of the hydrodynamic fluctuations.

Keywords:
Fluctuating hydrodynamics, Extended thermodynamics, Landau-Lifshitz theory, Hierarchy structure
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1. Introduction

More than fifty years ago, Landau and Lifshitz [1, 2, 3] developed the theory of fluctuating hydrodynamics for viscous, heat-conducting fluids with constitutive equations of Navier-Stokes and Fourier type basing on thermodynamics of irreversible processes (TIP) [4, 5]. In order to incorporate thermal fluctuations into hydrodynamics, they introduced additional stochastic flux terms into the constitutive equations of the viscous stress and the heat flux by applying the fluctuation-dissipation theorem [6, 7, 8]. See also reviews [9, 10, 11].

Nowadays the Landau-Lifshitz (LL) theory attracts much attention, especially, as the basic theory for microflows and nanoflows, which may play an important role, for example, in the fields of nano-technology [12, 13] and molecular biology [14, 15]. Numerical analyses of the fluctuations by using the theory have been made extensively [16, 17, 18, 19, 20, 21, 22]. The fluctuating-hydrodynamic approach can also contribute to the study of fluctuations in nonequilibrium states [11, 23, 24].

It is well known that TIP rests on the local equilibrium assumption [4, 5]. Strictly speaking, it is highly probable that this assumption may no longer be valid, in particular, in the cases where nanoflows are involved, or in the cases where rarefied gases play a role. Actually, in small systems for example, physical quantities undergo evident changes in a spatio-temporal scale which is even smaller than the scale necessary for the local equilibrium assumption to be valid. As for the discussion on the validity criterion of the assumption, see, for example, Ref. [25]. Extended thermodynamics (ET) [26] is a generalized theory being applicable to such cases. ET for rarefied gases has a counterpart in the kinetic theory of gases. For example, ET of 13 variables (ET-13) coincides with the moment theory of the Boltzmann equation within the Grad's 13-moment approximation [27].

The purpose of this paper is to develop a theory of fluctuating hydrodynamics based on ET through studying the 13-variable theory as a representative case. After establishing the relationship between the present theory and the LL theory, the hierarchy structure of the hydrodynamic fluctuations will be discussed.

2. Theory of fluctuating hydrodynamics based on ET

The basic equations in the present study are the linearized equations of ET-13 for a monatomic rarefied gas [26] around an equilibrium state. The independent variables are the mass density ρ, velocity v_i, temperature T, shear stress $t_{\langle ij\rangle}$ (angular brackets stand for the symmetric traceless part with respect to the suffixes inside), and heat flux q_i, where $i = 1, 2, 3$. Note that the dynamic pressure vanishes identically in this case.

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial v_i}{\partial x_i} = 0, \\
&\frac{\partial v_i}{\partial t} + \frac{aT_0}{\rho_0}\frac{\partial \rho}{\partial x_i} + a\frac{\partial T}{\partial x_i} - \frac{1}{\rho_0}\frac{\partial t_{\langle ij\rangle}}{\partial x_j} = 0, \\
&a\frac{\partial T}{\partial t} + \frac{2}{3}aT_0\frac{\partial v_k}{\partial x_k} + \frac{2}{3\rho_0}\frac{\partial q_k}{\partial x_k} = 0, \\
&\frac{\partial t_{\langle ij\rangle}}{\partial t} - \frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} - 2a\rho_0 T_0 \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = -s_{\langle ij\rangle}, \\
&\frac{\partial q_i}{\partial t} - aT_0\frac{\partial t_{\langle ij\rangle}}{\partial x_j} + \frac{5}{2}a^2\rho_0 T_0\frac{\partial T}{\partial x_i} = \frac{s_{ppi}}{2},
\end{aligned} \tag{1}$$

where $a \equiv k_B/m$ with k_B being the Boltzmann constant and m the mass of a molecule, and $s_{\langle ij\rangle}$ and s_{ppi} are the source terms. The quantities with and without the suffix 0 are, respectively, the quantities at the equilibrium state and the deviations from the equilibrium state. The first three equations represent, respectively, the mass, momentum and energy conservation laws, and the last two are the equations of balance type for the irreversible fluxes $t_{\langle ij\rangle}$ and q_i. Owing to the existence of the second

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part, rapidly changing (deterministic) modes, which have been neglected in the traditional hydrodynamic analysis, can be taken into account. The specific entropy production Σ is obtained as follows:

$$\Sigma = \lambda_{\langle ij\rangle} s_{\langle ij\rangle} + \lambda_{ppi} s_{qqi} \geq 0, \tag{2}$$

where $\lambda_{\langle ij\rangle}$ and λ_{ppi} are so-called the Lagrange multipliers.

Within the linear constitutive equations, we have

$$s_{\langle ij\rangle} = b\lambda_{\langle ij\rangle}, \quad s_{ppi} = c\lambda_{ppi}, \tag{3}$$

where b and c are positive phenomenological coefficients. Furthermore we can prove the following relations [26]:

$$\lambda_{\langle ij\rangle} = \frac{1}{2a\rho_0 T_0^2} t_{\langle ij\rangle}, \quad \lambda_{ppi} = -\frac{1}{5a^2\rho_0 T_0^3} q_i. \tag{4}$$

For later analysis, we summarize briefly the fluctuation-dissipation theorem. In the generic case [28] where the specific entropy production is given by

$$\Sigma = -\dot{x}_a X_a \tag{5}$$

and where we assume the linear constitutive equation between $\dot{x}_a$ and X_a with the phenomenological coefficient C_{ab}, we can introduce the Gaussian white random force $\mathfrak{f}_a$ into the constitutive equation in such a way that

$$\dot{x}_a = -C_{ab} X_b + \mathfrak{f}_a, \tag{6}$$

where the mean of $\mathfrak{f}_a$ vanishes and its correlation is given by

$$\left\langle \mathfrak{f}_a(\boldsymbol{x},t)\mathfrak{f}_b(\boldsymbol{x}',t')\right\rangle = k_B(C_{ab}+C_{ba})\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'). \tag{7}$$

Now we easily notice the following correspondence relationship between the generic case above and the present case:

$$\dot{x}_a \to \{s_{\langle ij\rangle}, \quad s_{ppi}\}, \qquad X_a \to \{-\lambda_{\langle ij\rangle}, \quad -\lambda_{ppi}\}. \tag{8}$$

And we can introduce the Gaussian white random forces $\mathfrak{r}_{\langle ij\rangle}$ and $\mathfrak{s}_i$ into Eq. (3) as follows:

$$s_{\langle ij\rangle} = b\lambda_{\langle ij\rangle} + \mathfrak{r}_{\langle ij\rangle}, \quad s_{ppi} = c\lambda_{ppi} + \mathfrak{s}_i, \tag{9}$$

where the means of $\mathfrak{r}_{\langle ij\rangle}$ and $\mathfrak{s}_i$ vanish, and their correlations are given by

$$\begin{aligned}
&\langle \mathfrak{r}_{\langle ij\rangle}(\boldsymbol{x},t)\mathfrak{r}_{\langle mn\rangle}(\boldsymbol{x}',t')\rangle = k_B b \\
&\qquad \times(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}-\frac{2}{3}\delta_{ij}\delta_{mn})\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'), \\
&\langle \mathfrak{s}_i(\boldsymbol{x},t)\mathfrak{s}_j(\boldsymbol{x}',t')\rangle = 2k_B c\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'), \\
&\langle \mathfrak{r}_{\langle ij\rangle}(\boldsymbol{x},t)\mathfrak{s}_m(\boldsymbol{x}',t')\rangle = 0.
\end{aligned} \tag{10}$$

At last, taking the relations (4) into account and introducing the new phenomenological coefficients (relaxation times) α and β instead of b and c, whose relationships are evident from the equations below, we obtain the expressions for $s_{\langle ij\rangle}$ and s_{ppi} in terms of $t_{\langle ij\rangle}$, q_i and the random forces $\mathfrak{r}_{\langle ij\rangle}$, $\mathfrak{s}_i$:

$$\begin{aligned}
s_{\langle ij\rangle} &= \frac{1}{\alpha} t_{\langle ij\rangle} + \mathfrak{r}_{\langle ij\rangle}, \\
s_{ppi} &= -\frac{2}{\beta} q_i + \mathfrak{s}_i.
\end{aligned} \tag{11}$$

The correlations (10) are rewritten by

$$\begin{aligned}
&\langle \mathfrak{r}_{\langle ij\rangle}(\boldsymbol{x},t)\mathfrak{r}_{\langle mn\rangle}(\boldsymbol{x}',t')\rangle = k_B \frac{2a\rho_0 T_0^2}{\alpha} \\
&\qquad \times(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}-\frac{2}{3}\delta_{ij}\delta_{mn})\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'), \\
&\langle \mathfrak{s}_i(\boldsymbol{x},t)\mathfrak{s}_j(\boldsymbol{x}',t')\rangle = k_B \frac{20a^2\rho_0 T_0^3}{\beta}\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'), \\
&\langle \mathfrak{r}_{\langle ij\rangle}(\boldsymbol{x},t)\mathfrak{s}_m(\boldsymbol{x}',t')\rangle = 0.
\end{aligned} \tag{12}$$

Equations (1) with (11) and (12) constitute ***the basic system of equations for fluctuating hydrodynamics based on ET (ET-13)***.

The relaxation times α and β can be evaluated by experiments or kinetic-theoretical analysis. For gases with Maxwellian interatomic potential we have the relation $3\alpha = 2\beta$ [26]. Other realistic monatomic gases satisfy this relation approximately.

3. Two subsystems of the stochastic field equations

The basic system of equations obtained above may be decomposed into two uncoupled subsystems, that is, the subsystem composed of longitudinal modes (System-L) and the subsystem of transverse modes (System-T).

System-L: The relevant quantities of the system are given by

$$\begin{gathered}
\rho,\ T,\ \psi\left(\equiv \frac{\partial v_i}{\partial x_i}\right),\ \tau\left(\equiv \frac{\partial^2 t_{\langle ij\rangle}}{\partial x_i \partial x_j}\right),\ \varphi\left(\equiv \frac{\partial q_i}{\partial x_i}\right), \\
\mathfrak{v}\left(\equiv -\frac{\partial^2 \mathfrak{r}_{\langle ij\rangle}}{\partial x_i \partial x_j}\right) \text{ and } \mathfrak{w}\left(\equiv \frac{1}{2}\frac{\partial \mathfrak{s}_i}{\partial x_i}\right).
\end{gathered} \tag{13}$$

The spatial Fourier transform of the system is the system of the rate-type differential equations in the space of the wave number $\boldsymbol{k}$ and time t ($\boldsymbol{k}t$-representation) as follows:

$$\begin{aligned}
&\frac{\partial \rho(\boldsymbol{k},t)}{\partial t} + \rho_0 \psi(\boldsymbol{k},t) = 0, \\
&\frac{\partial \psi(\boldsymbol{k},t)}{\partial t} - \frac{aT_0 k^2}{\rho_0}\rho(\boldsymbol{k},t) - ak^2 T(\boldsymbol{k},t) - \frac{1}{\rho_0}\tau(\boldsymbol{k},t) = 0, \\
&a\frac{\partial T(\boldsymbol{k},t)}{\partial t} + \frac{2}{3}aT_0\psi(\boldsymbol{k},t) + \frac{2}{3\rho_0}\varphi(\boldsymbol{k},t) = 0, \\
&\frac{\partial \tau(\boldsymbol{k},t)}{\partial t} + \frac{8}{15}k^2\varphi(\boldsymbol{k},t) + \frac{4}{3}a\rho_0 T_0 k^2\psi(\boldsymbol{k},t) = -\frac{1}{\alpha}\tau(\boldsymbol{k},t) + \mathfrak{v}(\boldsymbol{k},t), \\
&\frac{\partial \varphi(\boldsymbol{k},t)}{\partial t} - aT_0\tau(\boldsymbol{k},t) - \frac{5}{2}a^2\rho_0 T_0 k^2 T(\boldsymbol{k},t) = -\frac{1}{\beta}\varphi(\boldsymbol{k},t) + \mathfrak{w}(\boldsymbol{k},t),
\end{aligned} \tag{14}$$

where $\rho(\boldsymbol{k},t)$ is the spatial Fourier transform of $\rho(\boldsymbol{x},t)$ defined as

$$\rho(\boldsymbol{k},t) \equiv \frac{1}{(2\pi)^3}\int \rho(\boldsymbol{x},t)\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{15}$$

and so on.

From Eq. (12), the quantities $\mathfrak{v}(\boldsymbol{k},t)$ and $\mathfrak{w}(\boldsymbol{k},t)$ are the Gaussian white random forces with null means and correlations:

$$\begin{aligned}
&\langle \mathfrak{v}(\boldsymbol{k},t)\mathfrak{v}(\boldsymbol{k}',t')\rangle = k_B\frac{a\rho_0 T_0^2}{3\pi^3\alpha}k^4\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'), \\
&\langle \mathfrak{w}(\boldsymbol{k},t)\mathfrak{w}(\boldsymbol{k}',t')\rangle = k_B\frac{5a^2\rho_0 T_0^3}{8\pi^3\beta}k^2\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'), \\
&\langle \mathfrak{v}(\boldsymbol{k},t)\mathfrak{w}(\boldsymbol{k}',t')\rangle = 0.
\end{aligned} \tag{16}$$

System-T: The relevant quantities of the system are given by

$$\omega_i\,(\equiv(\mathrm{curl}\boldsymbol{v})_i)\,,\quad \sigma_i\left(\equiv\varepsilon_{ijk}\frac{\partial^2 t_{\langle kn\rangle}}{\partial x_j\partial x_n}\right),\quad \pi_i(\equiv(\mathrm{curl}\boldsymbol{q})_i),$$
$$\mathfrak{x}_i\left(\equiv-\varepsilon_{ijk}\frac{\partial^2\mathfrak{r}_{\langle kn\rangle}}{\partial x_j\partial x_n}\right)\ \text{ and }\ \mathfrak{y}_i\left(\equiv\frac{1}{2}(\mathrm{curl}\mathfrak{s})_i\right). \tag{17}$$

The field equations in the $\boldsymbol{k}t$-representation are as follows:

$$\begin{aligned}
&\frac{\partial\omega_i(\boldsymbol{k},t)}{\partial t}-\frac{1}{\rho_0}\sigma_i(\boldsymbol{k},t)=0,\\
&\frac{\partial\sigma_i(\boldsymbol{k},t)}{\partial t}+\frac{2}{5}k^2\pi_i(\boldsymbol{k},t)+a\rho_0T_0k^2\omega_i(\boldsymbol{k},t)=-\frac{1}{\alpha}\sigma_i(\boldsymbol{k},t)+\mathfrak{x}_i(\boldsymbol{k},t),\\
&\frac{\partial\pi_i(\boldsymbol{k},t)}{\partial t}-aT_0\sigma_i(\boldsymbol{k},t)=-\frac{1}{\beta}\pi_i(\boldsymbol{k},t)+\mathfrak{y}_i(\boldsymbol{k},t).
\end{aligned} \tag{18}$$

Note that, for given $\mathfrak{x}_i$ and $\mathfrak{y}_i$, the equations for the set of variables $(\omega_i,\sigma_i,\pi_i)$ with the same suffix i can be solved separately from those with the different suffix $j(\neq i)$. In view of Eq. (12), $\mathfrak{x}_i$ and $\mathfrak{y}_i$ are the Gaussian white random forces with null means and correlations:

$$\begin{aligned}
\langle\mathfrak{x}_i(\boldsymbol{k},t)\mathfrak{x}_m(\boldsymbol{k}',t')\rangle&=k_B\frac{a\rho_0T_0^2}{4\pi^3\alpha}k^4\left(\delta_{im}-\frac{k_ik_m}{k^2}\right)\\
&\qquad\times\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'),\\
\langle\mathfrak{y}_i(\boldsymbol{k},t)\mathfrak{y}_m(\boldsymbol{k}',t')\rangle&=k_B\frac{5a^2\rho_0T_0^3}{8\pi^3\beta}k^2\left(\delta_{im}-\frac{k_ik_m}{k^2}\right)\\
&\qquad\times\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'),\\
\langle\mathfrak{x}_i(\boldsymbol{k},t)\mathfrak{y}_m(\boldsymbol{k}',t')\rangle&=0.
\end{aligned} \tag{19}$$

4. Relationship to the Landau-Lifshitz theory

In the successive approximation explained below, it is necessary to express the shear stress and the heat flux in terms of the other quantities. We therefore solve the last two equations of (14) and (18) with respect to (τ, φ) and (σ_i, π_i), respectively, assuming, for the moment, that the other 3 variables (or 1 variable) are some given functions of $(\boldsymbol{k},t)$. The solutions can be expressed in a generic way because the last two equations of both systems can be written in the following matrix form:

$$\frac{d\boldsymbol{y}(\boldsymbol{k},t)}{dt}+\boldsymbol{M}(\boldsymbol{k})\cdot\boldsymbol{y}(\boldsymbol{k},t)=\boldsymbol{d}(\boldsymbol{k},t)+\mathfrak{a}(\boldsymbol{k},t), \tag{20}$$

where $\boldsymbol{y}(\boldsymbol{k},t)$, $\boldsymbol{M}(\boldsymbol{k})$, $\boldsymbol{d}(\boldsymbol{k},t)$ and $\mathfrak{a}(\boldsymbol{k},t)$ are given explicitly in Eq. (30) or (34) below. $\mathfrak{a}(\boldsymbol{k},t)$ is a Gaussian white random force vector with 2 components. As is well known, the general solution for the variable $\boldsymbol{y}$ can be easily obtained explicitly by using the eigenvalues λ_1 and λ_2 of the matrix $\boldsymbol{M}$, and the corresponding eigenvectors $[u_1,1]^T$ and $[u_2,1]^T$:

$$\boldsymbol{y}(\boldsymbol{k},t)=\tilde{\boldsymbol{y}}(\boldsymbol{k},t)+\boldsymbol{y}^*(\boldsymbol{k},t), \tag{21}$$

where

$$\tilde{\boldsymbol{y}}(\boldsymbol{k},t)=\begin{bmatrix}u_1&u_2\\1&1\end{bmatrix}\cdot\begin{bmatrix}C_1\mathrm{e}^{-\lambda_1t}\\C_2\mathrm{e}^{-\lambda_2t}\end{bmatrix} \tag{22}$$

with integration constants C_1 and C_2 that are determined by the initial condition at $t=0$, and

$$\boldsymbol{y}^*(\boldsymbol{k},t)=\begin{bmatrix}u_1&u_2\\1&1\end{bmatrix}\cdot\begin{bmatrix}\int_0^t\mathrm{e}^{-\lambda_1(t-t')}f_1(\boldsymbol{k},t')dt'\\\int_0^t\mathrm{e}^{-\lambda_2(t-t')}f_2(\boldsymbol{k},t')dt'\end{bmatrix}, \tag{23}$$

where $f_1(\boldsymbol{k},t)$ and $f_2(\boldsymbol{k},t)$ are given by

$$\begin{bmatrix}f_1(\boldsymbol{k},t)\\f_2(\boldsymbol{k},t)\end{bmatrix}=\frac{1}{u_1-u_2}\begin{bmatrix}1&-u_2\\-1&u_1\end{bmatrix}\cdot\begin{bmatrix}d_1(\boldsymbol{k},t)+\mathfrak{a}_1(\boldsymbol{k},t)\\d_2(\boldsymbol{k},t)+\mathfrak{a}_2(\boldsymbol{k},t)\end{bmatrix} \tag{24}$$

with $d_1(\boldsymbol{k},t)$, $d_2(\boldsymbol{k},t)$ and $\mathfrak{a}_1(\boldsymbol{k},t)$, $\mathfrak{a}_2(\boldsymbol{k},t)$ being the components of the vectors $\boldsymbol{d}(\boldsymbol{k},t)$ and $\mathfrak{a}(\boldsymbol{k},t)$, respectively.

Let us now adopt the coarse graining approximation by eliminating the rapidly changing modes in ET expressed by Eqs. (14) and (18). In comparison with the conserved quatities (mass, momentum and energy), the rapidly changing modes usually have much smaller relaxation times, and decay quickly. Therefore the elimination can be done by retaining one or a few terms in the expansion of the general solution with respect to the characteristic times of the rapidly changing modes, λ_1^{-1} and λ_2^{-1} [29, 30, 31]. More explicitly, we apply to (23) the following kind of approximation obtained by successive integrations by parts and by neglecting transient terms:

$$\int_0^t\mathrm{e}^{-\lambda(t-t')}f(t')dt'\sim\frac{1}{\lambda}f(t)-\frac{1}{\lambda^2}\frac{df(t)}{dt}+\frac{1}{\lambda^3}\frac{d^2f(t)}{dt^2}-\cdots. \tag{25}$$

If we retain only the leading term in the approximation (25) and discard also the transient term $\tilde{\boldsymbol{y}}(\boldsymbol{k},t)$, we obtain the following contracted solution from Eq. (21):

$$\boldsymbol{y}(\boldsymbol{k},t)=\boldsymbol{Y}(\boldsymbol{k})\boldsymbol{\cdot}\boldsymbol{d}(\boldsymbol{k},t)+\mathfrak{b}(\boldsymbol{k},t), \tag{26}$$

where

$$\boldsymbol{Y}=\frac{1}{u_1-u_2}\begin{bmatrix}\dfrac{u_1}{\lambda_1}-\dfrac{u_2}{\lambda_2}&-\dfrac{u_1u_2}{\lambda_1}+\dfrac{u_1u_2}{\lambda_2}\\\dfrac{1}{\lambda_1}-\dfrac{1}{\lambda_2}&-\dfrac{u_2}{\lambda_1}+\dfrac{u_1}{\lambda_2}\end{bmatrix}. \tag{27}$$

The quantity $\mathfrak{b}(\boldsymbol{k},t)$ is the random force vector introduced by the relation:

$$\mathfrak{b}(\boldsymbol{k},t)=\boldsymbol{Y}(\boldsymbol{k})\cdot\mathfrak{a}(\boldsymbol{k},t). \tag{28}$$

Then $\mathfrak{b}$ is again Gaussian and white with null mean and correlation:

$$\left\langle\mathfrak{b}(\boldsymbol{k},t)\mathfrak{b}^T(\boldsymbol{k}',t')\right\rangle=\boldsymbol{Y}(\boldsymbol{k})\cdot\left\langle\mathfrak{a}(\boldsymbol{k},t)\mathfrak{a}^T(\boldsymbol{k}',t')\right\rangle\cdot\boldsymbol{Y}^T(\boldsymbol{k}'). \tag{29}$$

In what follows, we show explicitly the contracted solutions for the System-L and System-T:

System-L: The quantities in Eq. (20) are given by

$$\boldsymbol{y}(\boldsymbol{k},t)=\begin{bmatrix}\tau(\boldsymbol{k},t)\\\varphi(\boldsymbol{k},t)\end{bmatrix},\quad\boldsymbol{M}(\boldsymbol{k})=\begin{bmatrix}\dfrac{1}{\alpha}&\dfrac{8}{15}k^2\\-aT_0&\dfrac{1}{\beta}\end{bmatrix},$$
$$\boldsymbol{d}(\boldsymbol{k},t)=\begin{bmatrix}-\dfrac{4}{3}a\rho_0T_0k^2\psi(\boldsymbol{k},t)\\\dfrac{5}{2}a^2\rho_0T_0k^2T(\boldsymbol{k},t)\end{bmatrix},\quad\mathfrak{a}(\boldsymbol{k},t)=\begin{bmatrix}\mathfrak{v}(\boldsymbol{k},t)\\\mathfrak{w}(\boldsymbol{k},t)\end{bmatrix}, \tag{30}$$

and then we obtain the eigenvalues and the components of the corresponding eigenvectors of the matrix $\boldsymbol{M}$ as follows:

$$\begin{aligned} \lambda_{1,2} &= \frac{\alpha+\beta \mp \sqrt{(\alpha-\beta)^2 - \frac{32}{15}aT_0k^2\alpha^2\beta^2}}{2\alpha\beta}, \\ u_{1,2} &= \frac{\alpha-\beta \pm \sqrt{(\alpha-\beta)^2 - \frac{32}{15}aT_0k^2\alpha^2\beta^2}}{2aT_0\alpha\beta}. \end{aligned} \tag{31}$$

It is noticeable that the order of magnitude of λ_1^{-1} and λ_2^{-1} is the same as that of the relaxation times α and β.

Denoting $\mathfrak{b} = [\mathfrak{g},\ \mathfrak{h}]^T$, we have the following relation up to the leading term with respect to α and β:

$$\begin{bmatrix} \tau(\boldsymbol{k},t) \\ \varphi(\boldsymbol{k},t) \end{bmatrix} = \begin{bmatrix} -\dfrac{4}{3}a\rho_0T_0k^2\alpha\psi(\boldsymbol{k},t) + \mathfrak{g}(\boldsymbol{k},t) \\ \dfrac{5}{2}a^2\rho_0T_0k^2\beta T(\boldsymbol{k},t) + \mathfrak{h}(\boldsymbol{k},t) \end{bmatrix}. \tag{32}$$

The random forces $\mathfrak{g}$ and $\mathfrak{h}$ have null means and correlations:

$$\begin{aligned} &\left\langle \mathfrak{g}(\boldsymbol{k},t)\mathfrak{g}(\boldsymbol{k}',t') \right\rangle = \frac{1}{3\pi^3}k_Ba\rho_0T_0^2k^4\alpha\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'), \\ &\left\langle \mathfrak{h}(\boldsymbol{k},t)\mathfrak{h}(\boldsymbol{k}',t') \right\rangle = \frac{5}{8\pi^3}k_Ba^2\rho_0T_0^3k^2\beta\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'), \\ &\left\langle \mathfrak{g}(\boldsymbol{k},t)\mathfrak{h}(\boldsymbol{k}',t') \right\rangle = 0. \end{aligned} \tag{33}$$

System-T: The quantities in Eq. (20) are given by

$$\begin{aligned} &\boldsymbol{y}(\boldsymbol{k},t) = \begin{bmatrix} \sigma_i(\boldsymbol{k},t) \\ \pi_i(\boldsymbol{k},t) \end{bmatrix}, \quad \boldsymbol{M}(\boldsymbol{k}) = \begin{bmatrix} \dfrac{1}{\alpha} & \dfrac{2}{5}k^2 \\ -aT_0 & \dfrac{1}{\beta} \end{bmatrix}, \\ &\boldsymbol{d}(\boldsymbol{k},t) = \begin{bmatrix} -a\rho_0T_0k^2\omega_i(\boldsymbol{k},t) \\ 0 \end{bmatrix}, \quad \mathfrak{a}(\boldsymbol{k},t) = \begin{bmatrix} \mathfrak{x}_i(\boldsymbol{k},t) \\ \mathfrak{y}_i(\boldsymbol{k},t) \end{bmatrix}. \end{aligned} \tag{34}$$

Then we obtain

$$\begin{aligned} \lambda_{1,2} &= \frac{\alpha+\beta \mp \sqrt{(\alpha-\beta)^2 - \frac{8}{5}aT_0k^2\alpha^2\beta^2}}{2\alpha\beta}, \\ u_{1,2} &= \frac{\alpha-\beta \pm \sqrt{(\alpha-\beta)^2 - \frac{8}{5}aT_0k^2\alpha^2\beta^2}}{2aT_0\alpha\beta}. \end{aligned} \tag{35}$$

Denoting $\mathfrak{b} = [\mathfrak{k}_i, \mathfrak{l}_i]^T$, we obtain the following relations in a similar way as above:

$$\begin{bmatrix} \sigma_i(\boldsymbol{k},t) \\ \pi_i(\boldsymbol{k},t) \end{bmatrix} = \begin{bmatrix} -a\rho_0T_0k^2\alpha\omega_i(\boldsymbol{k},t) + \mathfrak{k}_i(\boldsymbol{k},t) \\ \mathfrak{l}_i(\boldsymbol{k},t) \end{bmatrix}. \tag{36}$$

Note that there is no deterministic part in $\pi_i(\boldsymbol{k},t)$, therefore, only the random force plays a role. The correlations between the zero-mean random forces are given by

$$\begin{aligned} \left\langle \mathfrak{k}_i(\boldsymbol{k},t)\mathfrak{k}_m(\boldsymbol{k}',t') \right\rangle &= \frac{1}{4\pi^3}k_Ba\rho_0T_0^2k^4\alpha\left(\delta_{im} - \frac{k_ik_m}{k^2}\right) \\ &\qquad \times\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'), \\ \left\langle \mathfrak{l}_i(\boldsymbol{k},t)\mathfrak{l}_m(\boldsymbol{k}',t') \right\rangle &= \frac{5}{8\pi^3}k_Ba^2\rho_0T_0^3k^2\beta\left(\delta_{im} - \frac{k_ik_m}{k^2}\right) \\ &\qquad \times\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'), \\ \left\langle \mathfrak{k}_i(\boldsymbol{k},t)\mathfrak{l}_m(\boldsymbol{k}',t') \right\rangle &= 0. \end{aligned} \tag{37}$$

The relationship between the present theory and the LL theory: We can now confirm that the expressions in Eqs. (32), (33), (36) and (37) are exactly the same as those derived from the LL theory where the shear viscosity μ and the heat conductivity κ are identified by the relations [26]:

$$\mu = a\rho_0T_0\alpha, \quad \kappa = \frac{5}{2}a^2\rho_0T_0\beta. \tag{38}$$

Thus we have proved that the LL theory can be derived from the present theory by using the coarse graining approximation.

Two theories belong to the two different levels of description of fluctuating hydrodynamics. The rapidly changing deterministic modes in ET have been consistently renormalized into the random forces in the LL theory. Therefore, from a physical point of view, the delta-functions appeared in the correlations have their own validity range depending on the spatio-temporal resolution of their description level.

5. Discussion and concluding remarks

In the present paper, we have made clear the link between the two levels of description of fluctuating hydrodynamics, that is, the theory based on ET-13 and the LL theory. Generally speaking, there are many such levels. Boillat and Ruggeri [26, 32] found the hierarchy structure of ET and the important concept called the "main subsystem" of field equations. Each main subsystem gives us one level of description with different resolution from each other. And, in a similar way as above, we can develop the corresponding fluctuating hydrodynamics basing on such a main subsystem. Details of the hierarchy structure in the hydrodynamic fluctuations will be presented in the next paper.

Lastly we summarize the concluding remarks:

(i) In ET, Navier-Stokes and Fourier constitutive equations are obtained as its limit case by using an iterative scheme called the Maxwellian iteration [26]. If we apply this scheme formally to the present basic system with random forces, we can also obtain the results of the LL theory. The successive approximation scheme adopted in this letter is, in our opinion, easier to understand the physical meaning of the approximation process. In this respect, it is interesting, as a next study, to study the second-order approximation in the successive scheme and to compare it with that of the second-order Maxwellian iteration.

(ii) In the present paper, we have studied a monatomic rarefied gas only. Fluctuating hydrodynamics can also be established in a similar way by using recently-developed ET for a polyatomic rarefied gas and for a real gas [33] where the dynamic pressure exists.

(iii) As the basic system of equations in ET is of hyperbolic type, the propagation speed of information is finite. In this respect, ET is in sharp contrast to the traditional theory of Navier-Stokes and Fourier type that predicts infinite speeds for the propagation of heat and shear stress. It is, therefore, quite reasonable to adopt ET in order to develop the relativistic fluctuating hydrodynamics. See the pioneering work by Calzetta [34].

(iv) Numerical analyses based on the present theory in various situations are highly expected. We can expect qualitatively different effects between the LL theory and the present theory, especially, in a small spatio-temporal scale.

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