A note on Chebyshev polynomials, cyclotomic polynomials and twin primes

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Abstract

We revisit Stephen P. Humphries' results indicating some connections between Chebyshev polynomials and twin primes, by using Chebyshev polynomials of the third and fourth kinds and cyclotomic polynomials. We then give counterexamples to a conjecture of Humphries'. We also remark another characterization of twin primes in terms of Chebyshev polynomials of the second kind.

Keywords: Chebyshev polynomial; cyclotomic polynomial; twin prime

1 Introduction

In a series of papers [3, 4, 5, 6], Stephen P. Humphries defined and investigated certain operators to determine the geometric and algebraic intersection number functions associated to a simple closed curve on a surface. A prominent role was played by Chebyshev polynomials.

In this note, we are particularly interested in number theoretical aspects of his results. Specifically, he indicated some connections between Chebyshev polynomials and twin primes in [6]. In section 3 we give short proofs of some key results in [6] by using Chebyshev polynomials of the third and fourth kinds and cyclotomic polynomials. Our proofs are essentially the same as the original ones, but an effective use of such polynomials makes the discussion more transparent and the computation easier. We point out two facts (Propositions 3.1 and 3.2) which seem to have been overlooked, though the result of combining them has been noticed, in [6].

In section 4 we give counterexamples to [6, Conjecture 1.21].

In the final section we remark another characterization of twin primes in terms of Chebyshev polynomials of the second kind.

2 Chebyshev polynomials and cyclotomic polynomials

We refer the reader to [7, 8] for Chebyshev polynomials.

For each integer n, the Chebyshev polynomials $T_n(x), U_n(x), V_n(x)$ and $W_n(x)$ of the first, second, third and fourth kinds, respectively, are characterized by

$$T_n(\cos\theta) = \cos n\theta, \qquad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$
$$V_n(\cos\theta) = \frac{\cos(n+1/2)\theta}{\cos\theta/2}, \qquad W_n(\cos\theta) = \frac{\sin(n+1/2)\theta}{\sin\theta/2}.$$

They all satisfy the same recurrence relation

$$f_{n+2}(x) = 2xf_{n+1}(x) - f_n(x),$$

with different initial terms

$$T_0(x) = U_0(x) = V_0(x) = W_0(x) = 1,$$

$$T_1(x) = x,\ U_1(x) = 2x,\ V_1(x) = 2x-1,\ W_1(x) = 2x+1.$$

It follows that they have integral coefficients and the indices, when non-negative, represent the degrees of the polynomials. Schur's notation $\mathscr{U}_n(x) = U_{n-1}(x)$ is sometimes useful.

One can easily prove the following identities. Some of them will not be used in this note, but are here for aesthetic reasons.

Lemma 2.1. *(i)* $T_{-n}(x) = T_n(x), \mathscr{U}_{-n}(x) = -\mathscr{U}_n(x), V_{-n}(x) = V_{n-1}(x), W_{-n}(x) = -W_{n-1}(x)$.

(ii) $T_n(-x) = (-1)^n T_n(x), U_n(-x) = (-1)^n U_n(x), V_n(-x) = (-1)^n W_n(x)$.

(iii) $V_n(x) = U_n(x) - U_{n-1}(x), W_n(x) = U_n(x) + U_{n-1}(x)$.

(iv) $T_n(1) = 1, \mathscr{U}_n(1) = n$.

(v) $\mathscr{U}_n(0) = (-1)^{(n-1)/2}$ *if* n *is odd,* $\mathscr{U}_n(0) = 0$ *if* n *is even.*

(vi) $\mathscr{U}_n'(0) = 0$ *if* n *is odd,* $\mathscr{U}_n'(0) = -(-1)^{n/2} n$ *if* n *is even.*

(vii) $\int_{-1}^{1} \mathscr{U}_n(x)dx = 2/n$ *if* n *is odd,* $\int_{-1}^{1} \mathscr{U}_n(x)dx = 0$ *if* n *is even.*

Lemma 2.2.

$$
\begin{aligned}
2T_m(x)T_n(x) &= T_{m+n}(x) + T_{m-n}(x), \\
2(x^2-1)\mathscr{U}_m(x)\mathscr{U}_n(x) &= T_{m+n}(x) - T_{m-n}(x), \\
(x+1)V_m(x)V_n(x) &= T_{m+n+1}(x) + T_{m-n}(x), \\
(x-1)W_m(x)W_n(x) &= T_{m+n+1}(x) - T_{m-n}(x), \\
2T_m(x)\mathscr{U}_n(x) &= \mathscr{U}_{m+n}(x) - \mathscr{U}_{m-n}(x), \\
V_m(x)W_n(x) &= \mathscr{U}_{m+n+1}(x) - \mathscr{U}_{m-n}(x), \\
2T_m(x)V_n(x) &= V_{m+n}(x) + V_{m-n-1}(x) \quad = V_{m+n}(x) + V_{n-m}(x), \\
2T_m(x)W_n(x) &= W_{m+n}(x) - W_{m-n-1}(x) = W_{m+n}(x) + W_{n-m}(x), \\
2(x+1)\mathscr{U}_m(x)V_n(x) &= W_{m+n}(x) + W_{m-n-1}(x) = W_{m+n}(x) - W_{n-m}(x), \\
2(x-1)\mathscr{U}_m(x)W_n(x) &= V_{m+n}(x) - V_{m-n-1}(x) \quad = V_{m+n}(x) - V_{n-m}(x).
\end{aligned}
$$

Lemma 2.3.

$$\begin{aligned}
2\mathscr{U}_d(x)\textstyle\sum_{j=0}^{n} T_{a+2dj}(x) &= \mathscr{U}_{a+d+2dn}(x) - \mathscr{U}_{a-d}(x),\\
2(x^2-1)\mathscr{U}_d(x)\textstyle\sum_{j=0}^{n} \mathscr{U}_{a+2dj}(x) &= T_{a+d+2dn}(x) - T_{a-d}(x),\\
2(x+1)\mathscr{U}_d(x)\textstyle\sum_{j=0}^{n} V_{a+2dj}(x) &= W_{a+d+2dn}(x) - W_{a-d}(x),\\
2(x-1)\mathscr{U}_d(x)\textstyle\sum_{j=0}^{n} W_{a+2dj}(x) &= V_{a+d+2dn}(x) - V_{a-d}(x),\\
2W_d(x)\textstyle\sum_{j=0}^{n} T_{a+(2d+1)j}(x) &= W_{a+d+(2d+1)n}(x) - W_{a-d-1}(x),\\
2(x-1)W_d(x)\textstyle\sum_{j=0}^{n} \mathscr{U}_{a+(2d+1)j}(x) &= V_{a+d+(2d+1)n}(x) - V_{a-d-1}(x),\\
W_d(x)\textstyle\sum_{j=0}^{n} V_{a+(2d+1)j}(x) &= \mathscr{U}_{a+d+1+(2d+1)n}(x) - \mathscr{U}_{a-d}(x),\\
(x-1)W_d(x)\textstyle\sum_{j=0}^{n} W_{a+(2d+1)j}(x) &= T_{a+d+1+(2d+1)n}(x) - T_{a-d}(x).
\end{aligned}$$

Let

$$\Phi_n(x) = \prod_{d|n}(x^d-1)^{\mu(n/d)} \tag{1}$$

be the nth cyclotomic polynomial, where d ranges over all positive divisors of n and μ is the Möbius function. For $n \geq 3$, there exists a unique polynomial $\Psi_n(x) \in \mathbb{Z}[x]$ such that

$$\Psi_n(x+x^{-1}) = x^{-\phi(n)/2}\Phi_n(x),$$

where ϕ is the Euler function. As $\Phi_n(x)$ is irreducible (in this note, we consider irreducibility over $\mathbb{Z}$), so is $\Psi_n(x)$. These polynomials appear in the factorization of Chebyshev polynomials as follows.

Proposition 2.4. *(i)* $2T_n(x/2) = \prod_{\substack{d|n,\\ n/d:odd}} \Psi_{4d}(x)$.

(ii) $\mathscr{U}_n(x/2) = \prod_{2<d|2n} \Psi_d(x)$.

(iii) $V_n(x/2) = \prod_{1<d|2n+1} \Psi_{2d}(x)$.

(iv) $W_n(x/2) = \prod_{1<d|2n+1} \Psi_d(x)$.

Proof. See [8, section 5.2] for (i) and (ii). One can prove the remaining similarly, i.e., by comparing the zeros of both sides. □

By the inversion formula, one obtains the following expression.

Proposition 2.5 (Cf. [1]).

$$\Psi_n(x) = \begin{cases} \prod_{d|n} W_{(d-1)/2}(x/2)^{\mu(n/d)} & (n \equiv 1 \pmod 2) \\ \prod_{d|n/2} V_{(d-1)/2}(x/2)^{\mu(n/2d)} & (n \equiv 2 \pmod 4) \\ \prod_{d|n/2} \mathscr{U}_d(x/2)^{\mu(n/2d)} & (n \equiv 0 \pmod 4). \end{cases}$$

3 Humphries' results, revisited

For $n \geq 1$ we define $\Theta_n \in \mathbb{Z}[x, y]$ by

$$\Theta_n = \begin{cases} W_{(n-1)/2}(x/2) & (n : \text{odd}) \\ y\,\mathscr{U}_{n/2}(x/2) & (n : \text{even}). \end{cases}$$

By Lemma 2.3 (the first identity with $(a, d) = (0, 1)$ and the fifth identity with $(a, d) = (0, 0)$), this definition is equivalent to Humphries' in [6, Theorem 1.7], where x and y are written as $-r$ and $-s$, respectively. We also define

$$\Gamma_n = \prod_{d|n} \Theta_d^{\mu(n/d)}.$$

For example, $\Gamma_1 = 1, \Gamma_2 = y$.

Proposition 3.1. $\Gamma_n = \Psi_n(x)$ $(n \geq 3)$.

Proof. The claim follows from Proposition 2.5 and the definition of Γ_n. This is clear if n is odd. Suppose $n \equiv 2 \pmod 4$. Treating odd and even divisors of n separately and using the sixth identity of Lemma 2.2, we compute

$$\begin{aligned} \Gamma_n &= \prod_{d|n/2} W_{(d-1)/2}(x/2)^{\mu(n/d)} \prod_{d|n/2} (y\,\mathscr{U}_d(x/2))^{\mu(n/2d)} \\ &= \prod_{d|n/2} \left(W_{(d-1)/2}(x/2)^{-1}\mathscr{U}_d(x/2)\right)^{\mu(n/2d)} \\ &= \prod_{d|n/2} V_{(d-1)/2}(x/2)^{\mu(n/2d)} \\ &= \Psi_n(x). \end{aligned}$$

In the case $n \equiv 0 \pmod 4$, we have only to note that $\mu(n/d) = 0$ for odd divisors d of n. □

As a corollary, we see that Γ_n is irreducible for $n \geq 1$, and $\Gamma_n \in \mathbb{Z}[x]$, $\deg(\Gamma_n) = \phi(n)/2$ for $n \geq 3$. By the inversion formula, we have

$$\Theta_n = \prod_{d|n} \Gamma_d. \tag{2}$$

These facts together with Proposition 3.2 below imply Theorems 1.10 and 1.11 of [6].

We define $q_{n,i} \in \mathbb{Z}[x]$ by the identity

$$\sum_{i=0}^{2n+1} q_{n,i} t^i = (t+1) \prod_{j=1}^{n} (t^2 + 2T_j(x/2)t + 1). \tag{3}$$

Our definition of $q_{n,i}$ differs from Humphries' by a factor of $(-1)^{i+1}$ (recall also that $x = -r$). We further define

$$\gamma_n = \gcd(\{q_{n,i}; 1 \leq i \leq 2n\}).$$

Proposition 3.2. $\gamma_n = \Gamma_{2n+1}$.

Proof. It is proved in [6, Theorem 1.7] that

$$q_{n,i} = \frac{\Theta_{2n+1}!}{\Theta_i! \Theta_{2n+1-i}!}, \tag{4}$$

where $\Theta_k!$ is defined to be $\Theta_1 \Theta_2 \ldots \Theta_k$. Substituting (2) into (4), we obtain

$$q_{n,i} = \prod_{d=1}^{2n+1} \Gamma_d^{m(n,i,d)}$$

so that

$$\gamma_n = \prod_{d=1}^{2n+1} \Gamma_d^{\min\{m(n,i,d); 1 \leq i \leq 2n\}},$$

where

$$m(n,i,d) = \left\lfloor \frac{2n+1}{d} \right\rfloor - \left\lfloor \frac{i}{d} \right\rfloor - \left\lfloor \frac{2n+1-i}{d} \right\rfloor.$$

Since $m(n,i,2n+1) = 1$ for $1 \leq i \leq 2n$ and $m(n,d,d) = 0$ for $1 \leq d \leq 2n$, we complete the proof. □

Remark 3.3. The result of combining Propositions 3.1 and 3.2 is nothing but [6, Theorem 1.18(iii)]. However, it seems that these two facts have not been noticed separately.

The following is the key to the proof of the main result in [6]. It should be noted that Γ_{km} is mistyped as γ_{km} in [6].

Proposition 3.4 ([6, Proposition 6.2]). *We define*

$$\Sigma_{n,m,s} = \sum_{i \equiv s \pmod{m}} (-1)^i q_{n,i},$$

$$\gamma_{n,m} = \gcd(\{\Sigma_{n,m,s}\,;\, 1 \leq s \leq m\}).$$

Then we have:

(i) $(x+2) \mid \gamma_{n,2}$.

(ii) If n, m, k *are integers such that* $3 \leq km \leq n$*, then* $\Gamma_{km} \mid \gamma_{n,m}$.

Proof. (i) Upon substituting $x = -2$ in (3), we have

$$\sum_{i=0}^{2n+1} q_{n,i} t^i = (t^2-1)^n (t + (-1)^n).$$

Substituting $t = \pm 1$ into this equality and taking the difference (resp. sum), we obtain $2\Sigma_{n,2,1} = 0$ (resp. $2\Sigma_{n,2,2} = 0$).

(ii) We may suppose $m \geq 2$ since $\Sigma_{n,1,s} = 0$. Multiplying (3) by t^{-s}, substituting $t = -\zeta^l$ where $\zeta = \exp(2\pi\sqrt{-1}/m)$ and summing over $0 \leq l \leq m-1$, we obtain

$$m\Sigma_{n,m,s} = \sum_{l=0}^{m-1} \zeta^{-ls}(-\zeta^l + 1) \prod_{j=1}^{n} (\zeta^{2l} - 2T_j(x/2)\zeta^l + 1).$$

On the other hand, we have

$$\Gamma_{km} = \prod_{\substack{1 \leq a \leq km/2 \\ \gcd(a,km)=1}} \left(x - 2\cos\frac{2a\pi}{km} \right)$$

by Proposition 3.1. Let a be an arbitrary integer such that $1 \leq a \leq km/2, \gcd(a, km) = 1$. It suffices to show that, for each l, there exists an integer j in the range $1 \leq j \leq n$ such that $\zeta^{2l} - 2T_j(x/2)\zeta^l + 1$ vanishes for $x = 2\cos(2a\pi/km)$. This last condition is equivalent to

$\cos(2l\pi/m) = \cos(2aj\pi/km)$, that is, $kl \equiv \pm aj \pmod{km}$. By the assumptions on n, m, k and a, the set $\{\pm aj\,;\, 1 \leq j \leq n\}$ modulo km exhausts the residue classes modulo km. This completes the proof.

□

Now we turn to some connections with twin primes. Let $\nu_2 : \mathbb{Z}[x, y] \to \mathbb{Z}$ denote the ring homomorphism obtained by evaluation at $(x, y) = (2, 2)$.

Proposition 3.5 ([6, Theorem 1.12]). *(i)* $\nu_2(\Theta_n) = n$.

(ii) If $n = p^k$ *where* p *is a prime and* $k \geq 1$, *then* $\nu_2(\Gamma_n) = p$. *Otherwise,* $\nu_2(\Gamma_n) = 1$.

Proof. (i) $\nu_2(\Theta_{2k+1}) = W_k(1) = U_k(1) + U_{k-1}(1) = 2k + 1$, and $\nu_2(\Theta_{2k}) = 2\mathscr{U}_k(1) = 2k$.

(ii) $\nu_2(\Gamma_n) = \Psi_n(2) = \Phi_n(1)$. The claim follows from this and the definition (1) of $\Phi_n(x)$.

□

From the last property, Humphries noticed the following connection with twin primes. For positive integers $n_1, n_2, \ldots, n_r$, we define

$$\Lambda_{n_1,n_2,\ldots,n_r} = \sum_{j=1}^{r} (\Gamma_{n_j} - n_j).$$

Corollary 3.6 ([6, Theorems 1.13 and 1.14]). *(i) Suppose* $n > m > 1$. *Then* $\nu_2(\Gamma_n) - \nu_2(\Gamma_m) = 2$ *if and only if there exist twin primes* $p, p+2$ *such that* n *and* m *are powers of* $p + 2$ *and* p, *respectively.*

(ii) If $p, p + k$ *are primes, then* $\gcd(\Lambda_p, \Lambda_{p+k}) \neq 1$.

(iii) Suppose $n_1, n_2, \ldots, n_r$ *are odd positive integers. Then* $\nu_2(\Lambda_{n_1,n_2,\ldots,n_r}) = 0$ *if and only if* $n_1, n_2, \ldots, n_r$ *are primes.*

Proof. These are immediate from Proposition 3.5. □

Another connection that Humphries noticed is the following.

Proposition 3.7 ([6, Proposition 7.2]). *If* $p, p + 2k$ *are primes, then* $\Gamma_{p+k} \mid \Gamma_p + \Gamma_{p+2k}$.

Proof. Define $\Theta'_n \in \mathbb{Z}[x]$ by $\Theta'_n = \Theta_n$ if n is odd, $\Theta'_n = y^{-1}\Theta_n$ if n is even. It suffices to show that $\Theta'_n \mid \Theta'_{n-k} + \Theta'_{n+k}$ for $n > k \geq 1$. By Lemma 2.2, the quotient $(\Theta'_{n-k} + \Theta'_{n+k})/\Theta'_n$ is as follows:

- If $n = 2a+1, k = 2b+1$, it is $(\mathscr{U}_{a-b} + \mathscr{U}_{a+b+1})/W_a = V_b$.
- If $n = 2a+1, k = 2b$, it is $(W_{a-b} + W_{a+b})/W_a = 2T_b$.
- If $n = 2a, k = 2b+1$, it is $(W_{a-b-1} + W_{a+b})/\mathscr{U}_a = 2(x+1)V_b$.
- If $n = 2a, k = 2b$, it is $(\mathscr{U}_{a-b} + \mathscr{U}_{a+b})/\mathscr{U}_a = 2T_b$.

Here we abbreviate $W_a = W_a(x/2)$ and so on. This completes the proof. □

4 Humphries' conjectures

Humphries made two sets of conjectures in [6].

Conjecture 4.1 ([6, Conjecture 1.20])**.** In the following, all numbers are supposed to be positive integers.

(i) $\gcd(\Lambda_n, \Lambda_{n+k}) \neq 1$ if and only if $n, n+k$ are primes.

(ii) Suppose $n > 8$. Then $\Lambda_{n,n+2}$ is reducible if and only if $n, n+2$ are primes.

(iii) Suppose $n_1, n_2, \ldots, n_r$ are distinct odd integers greater than 8. Then $\Lambda_{n_1,n_2,\ldots,n_r}$ is reducible if and only if $n_1, n_2, \ldots, n_r$ are primes.

(iv) Suppose n is odd. Then $\Gamma_{n+k} \mid \Gamma_n + \Gamma_{n+2k}$ if and only if $n, n+2k$ are primes.

(v) Γ_n is irreducible.

The "if" part is true in each case as we have seen above. One can not drop the condition $n \neq 8$ in (ii) since $\Lambda_{8,10} = (x-3)(2x+7)$. We note that (v) is true by Proposition 3.1.

Now we define, changing the sign of those defined in [6],

$$\Delta_n(x) = \sum_{d|n} \mathscr{U}_d(x/2), \quad d_n(x) = \Delta_n(x) + n.$$

Conjecture 4.2 ([6, Conjecture 1.21])**.** The following conditions are equivalent for odd $n > 1$:

(i) $n, n+2$ are primes.

(ii) $d_{n+2}(x) - d_n(x)$ has a multiple root.

(iii) $d_{n+2}(x) \geq d_n(x)$ for all real numbers x.

(iv) $d_{n+2}(2) - d_n(2) = 4$.

(v) $\int_{-2}^{2}(d_{n+2}(x) - d_n(x))\,dx = 8 - 8/n(n+2)$.

(vi) $d_{n+2}(x) - d_n(x)$ is a perfect square.

By the identity $\mathscr{U}_{n+2}(x) - \mathscr{U}_n(x) + 2 = 4T_{(n+1)/2}(x)^2$, the implication (i) $\Rightarrow$ (vi) holds true. The implications (vi) $\Rightarrow$ (ii) and (vi) $\Rightarrow$ (iii) are clear. Let $\sigma_k(n) = \sum_{d \nmid n} d^k$ denote the divisor function. If $n, n+2$ are primes, we have clearly

$$\sigma_k(n+2) - (n+2)^k = \sigma_k(n) - n^k. \tag{5}$$

The conditions (iv) and (v) are special cases of (5) since $\Delta_n(2) = \sigma_1(n)$ and $\int_{-2}^{2} \Delta_n(x)dx = 4\sigma_{-1}(n)$ by Lemma 2.1. More generally, one might ask, for each $k \neq 0$, whether (5) implies the primality of n and $n+2$.

We give counterexamples to Conjecture 4.2. First, there exist infinitely many counterexamples to the implication (ii) $\Rightarrow$ (i) as the next theorem shows.

Theorem 4.3. *Let $n > 1$ be odd. Then $x = 0$ is a multiple root of $d_{n+2}(x) - d_n(x)$ if and only if n is of the form $p_0(p_1p_2\dots p_r)^2$, where $p_0, p_1, \dots, p_r$ are (not necessarily distinct) primes such that $p_0 \equiv 1 \pmod 4$ and $p_1 \equiv \cdots \equiv p_r \equiv 3 \pmod 4$.*

Proof. Let χ denote the non-trivial Dirichlet character modulo 4. It follows from Lemma 2.1 that $\Delta_n(0) = \sum_{d|n} \chi(d)$, which we will write as $\sigma_\chi(n)$. By the multiplicativity of χ, we have $\sigma_\chi(n_1n_2) = \sigma_\chi(n_1)\sigma_\chi(n_2)$ if $\gcd(n_1, n_2) = 1$. If p is a prime and $e \geq 1$, we have

$$\sigma_\chi(p^e) = \begin{cases} 1 & (p = 2) \\ e+1 & (p \equiv 1 \pmod 4) \\ 0 & (p \equiv 3 \pmod 4), e : \text{odd}) \\ 1 & (p \equiv 3 \pmod 4)), e : \text{even}). \end{cases}$$

A general formula of $\sigma_\chi(n)$ follows from these facts. In particular, the condition on n in the statement of the theorem is equivalent to $\sigma_\chi(n) = 2$. Now let $n > 1$ be odd and put $g(x) = d_{n+2}(x) - d_n(x)$. Since $\sigma_\chi(k) = 0$ if $k \equiv 3 \pmod 4$, we have $g(0) = 2 + \sigma_\chi(n+2) - \sigma_\chi(n) = 0$ if and only if $\sigma_\chi(n) = 2$. This completes the proof since $g'(0) = 0$ holds by Lemma 2.1. $\square$

Except for these cases, we have found no counterexamples to (ii) $\Rightarrow$ (i) in the range $n < 10^4$. Maybe (ii) should be replaced by the following:

(ii)' $d_{n+2}(x) - d_n(x)$ has a multiple root except possibly $x = 0$.

Second, as a result of computer search, we have found two counterexamples to the implication (iv) $\Rightarrow$ (i) in the range $n < 10^{10}$; they are $n = 8575$ and $n = 8825$. If we allow n to be even, we have one more counterexample $n = 434$ in the same range.

We finally remark that if n is a counterexample to (v) $\Rightarrow$ (i), we must have $\sigma_1(n+2) \equiv 1 \pmod{n+2}$ and $\sigma_1(n) \equiv 1 \pmod n$. First candidates would be positive integers n satisfying $\sigma_1(n) = 2n+1$, the so called quasiperfect numbers. However, no quasiperfect numbers have been found so far, and it is known that if there are any they must be odd squares with at least seven distinct prime factors and must exceed 10^{35} (cf. [2]).

5 Another characterization of twin primes

Here is another characterization of twin primes.

Proposition 5.1. *Let $n > 1$ be odd. The following conditions are equivalent:*

(i) $n, n+2$ *are primes.*

(ii) $\mathscr{U}_{n+1}(x/2) + 1$ *has exactly two irreducible factors.*

(iii) $\mathscr{U}_{n+1}(x/2) - 1$ *has exactly two irreducible factors.*

Proof. By the sixth identity of Lemma 2.2, we have

$$\mathscr{U}_{n+1}(x/2) - 1 = V_{(n+1)/2}(x/2)W_{(n-1)/2}(x/2).$$

By Proposition 2.4, $W_{(n-1)/2}(x/2)$ is irreducible if and only if n is a prime, and $V_{(n+1)/2}(x/2) = (-1)^{(n+1)/2}W_{(n+1)/2}(-x/2)$ is irreducible if and only if $n+2$ is a prime. This shows (i) $\iff$ (iii). Replacing x by $-x$, we have (i) $\iff$ (ii). $\square$

Acknowledgments

This work was supported by JSPS KAKENHI (No. 23540014).

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