Eigenvalues of Laplacians for Kähler Graphs

2016

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Introduction

A graph is a pair of a set of vertices and a set of edges, and forms a 1-dimensional CW-complex. From geometrical point of view, we can consider graphs as discrete models of surfaces and more generally as discrete models of Riemannian manifolds. Chains of edges, which are called paths, on a graph are considered to be correspond to geodesics on a Riemannian manifold. For a graph, we have adjacency and transition operators acting on the set of all square-summable functions on the set of vertices. The adjacency operator of a graph shows how edges in this graph are settled between vertices, hence is the generating operator of paths. The transition operator shows how cargoes placed at vertices are transfered through edges, hence is the generating operator of paths attached with probabilities. Thus we can say that properties of these operators show properties of the underlying graph. Many mathematicians therefore have studied spectrum of these operators and those of Laplacians corresponding to them.

In this paper we study Kähler graphs which were introduced by T. Adachi [**2**]. A Kähler graph is a graph whose set of edges are divided into two subsets. One is the set of principal edges and the other is the set of auxiliary edges. We may say that a Kähler graph is a compound of two kinds of graphs having a common set of vertices. From geometrical point of view, we can explain Kähler graph as discrete models of Riemannian manifold admitting magnetic fields. We consider paths on the principal graph of a Kähler graph as geodesics which are motions of electric charged particles without influence of magnetic fields. Under the influence of a magnetic field, we consider that each path on the principal graph is bended to directions of edges in the auxiliary graph. More precisely, we consider a p-step path in the principal graph followed by a q-step path in the auxiliary graph as a trajectory of a charged particle under the influence of magnetic field of strength q/p.

In this thesis, starting with summarizing some basic notions and properties of ordinary graphs, we introduce the notion of Kähler graphs following to [**2**], and study some basic properties. In §2 we give some examples of Kähler graphs; Kähler graphs of n-dimensional complex lattice, Cayley Kähler graphs, complement filled Kähler graphs, Kähler graphs of product types, and some other typical Kähler graphs obtained from Petersen graphs, Heawood graphs and so on. In §3 we define adjacency and transition operators on a Kähler graph which are associated with bicolored paths, paths formed by paths on principal graphs and paths on auxiliary graphs. Roughly speaking, for paths on principal graphs we attach either adjacency operators or transition operators, and for paths on auxiliary graphs we attach probabilistic transition operators. Here, probabilistic transition operators coincide with transition operators when we consider 1-step paths on auxiliary graphs. But they are different from iteration of transition operators when we consider paths of two and more steps on auxiliary graphs. By our definition these operators for Kähler graphs are not selfadjoint, in general. In §4 we study eigenvalues of Laplacians corresponding to these operators. When a graph is finite, the set of square-summable functions on the set of vertices coincides with the set of all functions on this set, and spectrum of these operators are the sets of eigenvalues of corresponding matrices. We mainly study the case that the adjacency operators of principal and auxiliary graphs are commutative, and show the relationship between the eigenvalues of Kähler graphs and those of their principal and auxiliary graphs. As an application we study isospectral problem on Kähler graphs and give some example of pairs of Kähler graphs which have the same eigenvalues.

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CHAPTER 1

Graphs

1. Some fundamental notions and results on graphs

1.1. Vertices and edges. A *graph* G consists of a set V of vertices and a set E of edges. A graph is represented as a 1-dimensional CW-complex. For the set of vertices of a graph G we denote it by $V(G)$ or simply by V. According that the cardinality of the set V of vertices is finite (see Fig. 1) or infinite (see Fig. 2), we classify graphs into two "classes". For a finite graph, we denote the set of vertices as $V = \{v_1, v_2, \ldots, v_{n_G}\}$, where n_G denotes the cardinality of the set of V, and for an infinite graph, we denote as $V = \{v_\lambda\}_{\lambda\in\Lambda}$.

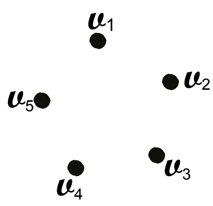


FIG. 1. finite vertices

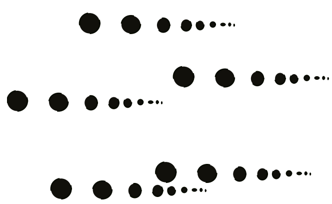


FIG. 2. infinite vertices

For the set of edges of a graph G each of which joins two vertices, we denote it by $E(G)$ or E. According that the cardinality of the set E of edges is finite (see Fig. 1) or infinite (see Fig. 2), we classify graphs into two "classes". When both of the set of vertices and the set of edges are finite, we call this a *finite graph*.

EXAMPLE 1.1. Fig. 3 shows a finite graph. Its set of vertices is $V = \{v_1, \ldots, v_5\}$ and its set of edges is $E = \{e_1, e_2, e_3, e_4, e_5\}$. Fig. 4 shows an infinite graph. It has an infinite set of vertices $V = \{v_\lambda\}_{\lambda\in\Lambda}$ and an infinite set of edges $E = \{e_\mu\}_{\mu\in A}$.

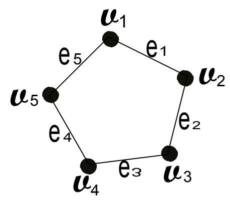


Fig. 3. (finite edges)

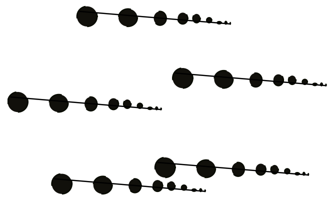


Fig. 4. (infinite edges)

When we consider graphs, we sometimes give an orientation on each edges. When we consider orientations on all edges of a graph, we say it is an *oriented* graph or a *directed* graph. In order to make clear that we do not consider orientations of edges, we call this graph *non-oriented* or undirected. Given an edge $e \in E$ of an oriented graph, we denote by $o(e)$ its origin and by $t(e)$ its terminus. For an edge $e \in E$ of a non-directed graph, we denote its vertices at its ends by $o(e), t(e)$. In this case we do not distinguish the origin and the terminus. We say two vertices v, w to be *adjacent* to each other if there is an edge joining them. In this case we denote as $v \sim w$. An edge $e \in E$ which joins a vertex and itself (i.e. $o(e) = t(e)$) (see Fig. 5) is called a *loop*. When two or more edges are attached to given two vertices (which may coincide with each other) we call them *multiple edges*. If a graph has multiple edges but not loops then it is called a *multiple graph* (see Fig. 6). If a graph does not have loops and multiple edges, we call it *simple*.

From now on, through out this paper we just say a graph for a non-oriented graph. An edge e of a graph without multiple edges can be expressed by its both ends as $e = \{o(e), t(e)\}$. We express an edge e of a directed graph as $e = \big(o(e), t(e)\big)$.

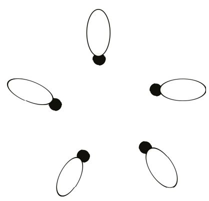


Fig. 5. loops

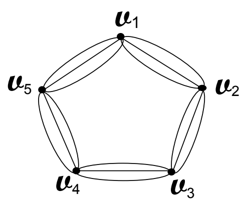


Fig. 6. multiple edges

Let $G = (V, E)$ be a graph which may have loops and multiple edges. Given a vertex $v \in V$ we denote by $d_G(v)$ the cardinality of the set of edges emanating from v, and call it the *degree* at v. We note that when there is a loop $e = \{v, v\}$ we compute this edge twice. If the degree at v is $d(v) = 0$ we call this vertex an *isolated point* (see Fig. 1), and if $d(v) = 1$ we call it a *terminal point*. If one of the end point of an edge is a terminal point, we call this edge a *hair*.

For a finite graph G, we can consider a sequence of degrees $(d_G(v_1), d_G(v_2), \cdots, d_G(v_n))$ at its vertices. At a vertex v of a directed graph G, we set $d_G^-(v)$ the cardinality of the set of edges having v as their terminus, and set $d_G^+(v)$ the cardinality of the set of edges having v as their origin.

PROPOSITION 1.1. *For a simple finite graph G, the degree $d(v)$ at each vertex v satisfies $d(v) \leq n_G - 1$.*

PROOF. We consider at a vertex $v \in V(G)$. Since G does not have loops, this vertex v can be joined at most $n_G - 1$ vertices. As G does not have multiple edges, if two distinct vertices are adjacent to each other, then there is only one edge joining then. Therefore we have $d_G(v) \leq n_G - 1$. □

PROPOSITION 1.2 (Hand shaking lemma). *Let $G = (V, E)$ be an undirected finite graph which may have loops and multiple edges. Then the cardinality $\sharp E$ of the set of edges and degrees satisfy the following relation*:

$$\sum_{v \in V} d_G(v) = 2\sharp E$$

PROOF. For each edge $e = \{v, w\}$, we can attach two vertices $v, w \in V$. So when we compute degrees at these vertices, this edge is counted twice. We hence get the conclusion. □

As a consequence of the above propositions we have the following.

LEMMA 1.1. . *For a finite simple graph G, the cardinality $\sharp E(G)$ of the set of edges is not greater than $\dfrac{n_G(n_G - 1)}{2}$.*

A graph $G = (V, E)$ is said to be *regular* if all the vertices of G have the same degree (see Fig. 9). When each vertex has the same degree r, we call it a regular graph of degree r. A regular graph of degree 0 is called an *empty graph*. By Proposition 1.2 we have the following.

COROLLARY 1.1. *When $G = (V, E)$ is a regular graph of degree r, the cardinality of its set of edges is given as* $\sharp E = \dfrac{1}{2} r n_G$

A *complete graph* is a simple graph all of whose pairs of vertices are joined by edges. A complete graph having n vertices is denoted by K_n. Clearly it is a regular graph of degree $(n-1)$.

EXAMPLE 1.2. We take the following three graphs having five vertices $V = \{v_1, v_2, v_3, v_4, v_5\}$.

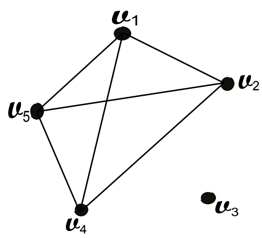


FIG. 7

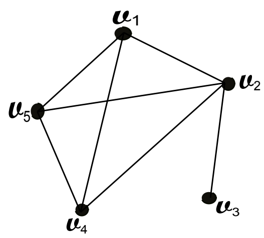


FIG. 8

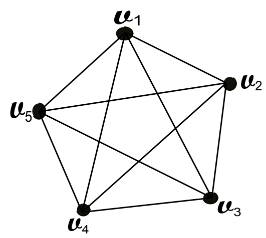


FIG. 9

(1) In Fig. 7, the vertex v_3 is an isolated point, i.e. $d_G(v_3) = 0$. Other vertices have the same degrees.

(2) In Fig. 8, the vertex v_3 is a terminal point, i.e. $d(v_3) = 1$. The sum of degrees is

$$\sum_{v \in V} d(v) = d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) = 3 + 4 + 1 + 3 + 3 = 14,$$

which is the twice of the cardinality 7 of edges.

(3) Fig. 9 shows a complete graph K_5. As it is regular of degree 4, we have

$$\sharp E = \frac{1}{2} r \sharp V = \frac{1}{2} 4 \times 5 = 10.$$

1.2. Paths. Two edges e_1, e_2 are said to be adjacent to each other if they have a common vertex ($e_1 \cap e_2 \neq \emptyset$) and $e_1 \neq e_2$. A sequence $\gamma = (e_1, e_2, v_3, \ldots e_m)$ of the adjacent edges, that is, e_i and e_{i+1} are adjacent to each other for $i = 1, \cdots, m-1$, is said to be a road or a *path* in this graph G. A path is sometimes represented as $\gamma = (v_0, v_1, v_2, \cdots v_m)$ by use of vertices. In this case, we have $v_i \sim v_{i+1}$ for all $i (0 \leqq i \leqq m-1)$. We denote the origin v_0 of γ by $o(\gamma)$, and the terminus v_m of γ by $t(\gamma)$. We say that the length of this path γ is m and denote as $\text{length}(\gamma) = m$. We say a path of length m also a path of m-step. When the origin v_0 and the terminus v_m of a path coincide with each other, we call it a *closed path*.

Example 1.3. We study the following graph having six vertices. In Fig. 10 we mark vertices and in Fig. 11 we mark edges. We show all paths from v_1 to v_6 which does not pass through the same vertex twice by two ways of expression.

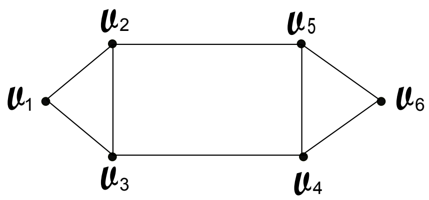


Fig. 10

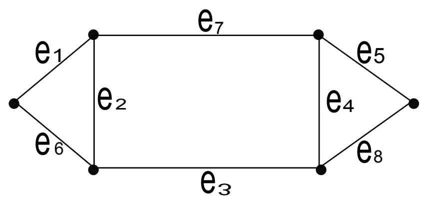


Fig. 11

(v_1, v_2, v_5, v_6)	(e_1, e_3, e_5)
(v_1, v_3, v_4, v_6)	(e_6, e_3, e_8)
$(v_1, v_2, v_3, v_4, v_6)$	(e_1, e_2, e_3, e_8)
$(v_1, v_3, v_2, v_5, v_6)$	(e_6, e_2, e_7, e_5)
$(v_1, v_2, v_3, v_4, v_5, v_6)$	$(e_1, e_2, e_3, e_4, e_5)$
$(v_1, v_3, v_2, v_5, v_4, v_6)$	$(e_6, e_2, e_7, e_4, e_8)$

There are six such paths. They are two paths of 3-step, two paths of 4-step and two paths of 5-step.

We here give operations of paths. Given two paths γ_1, γ_2 with $t(\gamma_1) = o(\gamma_2)$, we define their *join* $\gamma_1 \cdot \gamma_2$ as a joined path. That is, if $\gamma_1 = (v_0, v_1, \ldots, v_m)$ and $\gamma_2 = (w_0, w_1, \ldots, w_n)$ with $v_m = w_0$, we set $\gamma_1 \cdot \gamma_2 = (v_0, \ldots, v_m w_1, \ldots, w_n)$. Hence when γ_1 is of m-setp and γ_2 is of n-step we have $\gamma_1 \cdot \gamma_2$ is of $(m+n)$-step. For a path $\gamma = (v_0, v_1, \ldots, v_n)$ we define its *reversed path* γ^{-1} by $\gamma^{-1} = (v_n, v_{n-1}, \ldots, v_0)$. For example, in Example 1.3 for a path $\gamma_1 = (v_1, v_2, v_5, v_6)$ its reverse is $\gamma^{-1} = (v_6, v_5, v_2, v_1)$.

When a path γ^* is included in a longer path γ, that is, if $\gamma = (v_0, v_1, \ldots, v_n)$ and $\gamma^* = (v_i, v_{i+1}, \ldots, v_k)$ for some i and k satisfying $0 \le i < k \le n$, we say this path γ^* to be a *subpath* of γ. For a path $\gamma = (v_0, v_1 \ldots v_{i-2}, v_{i-1}, v_i, v_{i+1})$, we say it has a *backtraking* if there is i_0 satisfying $v_{i_0-1} = v_{i_0+1}$, and we say it do not have backtraking if vertices v_{i-1} and v_{i+1} does not coincide for all i.

1.3. Connected components. Given two vertices $v, w \in V$ of a graph $G = (V, E)$, we say they are connected by paths if there is a path joining them, that is we have a path γ with $o(\gamma) = v$ and $t(\gamma) = w$. We call this graph G *connected* if every pair of distinct vertices are connected by paths. We denote as v—w either if $v = w$ or v, w are connected by paths.

We here show that this relation v—w gives an equivalence relation on the set V.

(1) When $v = w$ we have v—w by definition.

(2) Suppose v—w. When $v = w$, we clearly have w—v. When $v \neq w$, there is a path $\gamma = (v_0, v_1, \ldots, v_{m-1}, v_m)$ from v to w. If we take its reversed path $\gamma^{-1} = (v_m, v_{m-1}, \cdots, v_1, v_0)$, then we have $o(\gamma^{-1}) = t(\gamma) = w,\ t(\gamma^{-1}) = o(\gamma) = v$, hence find w—v.

(3) Suppose u—v and v—w. When either $u = v$ or $v = w$, we have v—w When $u \neq v$ and $v \neq w$, there are paths $\sigma = (v_0, v_1, \ldots, v_n)$ from u to v and $\gamma = (v'_0, v'_1, \ldots, v'_m)$ from v to w. Since $t(\sigma) = v = o(\gamma)$, we can take the joined path $\sigma \cdot \gamma = (v_0, v_1, \ldots, v_n, v'_1, v'_2, \ldots, v'_m)$ which is a path from u to w. We hence get u—w.

By these $(1), (2), (3)$ we find that the relation — is an equivalence relation.

We decompose V into equivalence classes $V = \sum_i V_i$. We put E_i the set of edges one of whose ends belongs to V_i. If we suppose $E_i \cap E_j \neq \emptyset$, we have an edge e with $o(e) \in V_i$ and $t(e) \in V_j$. Then these two vertices are connected by paths, hence they belong to the same equivalence class. We therefore have $i = j$. Thus we have a disjoint decomposition $E = \sum E_i$ of E and get connected graphs $G_i = (V_i, E_i)$. We call each G_i a connected component of G, and call $G = \sum G_i$ the decomposition of G into connected components.

1.4. Graph isomorphisms. Let $G = (V, E)$, $G' = (V', E')$ be two graphs. A map $f : V \to V'$ is said to be an *homomorphism* of G to G' if it satisfies $f(v) \sim f(v')$ for arbitrary $v, v' \in V$ with $v \sim v'$. A bijection $f : V \to V'$ is called an *isomorphism* of G to G' if it and its inverse $f^{-1} : V' \to V$ are homomorphisms. When we have an isomorphism between G and G', we say these graphs are isomorphic.

Example 1.4. We give two graphs (V, E) and (V', E') in the following manner:

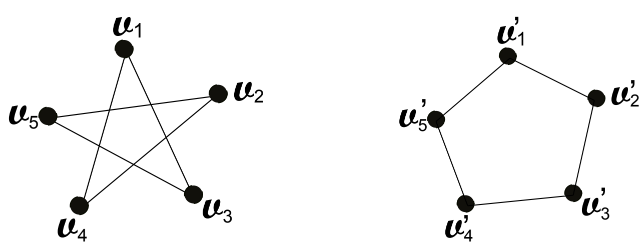
$$V = \{v_1, v_2, v_3, v_4, v_5\}, \quad E = \big\{\{v_1, v_3\}, \{v_3, v_5\}, \{v_5, v_2\}, \{v_2, v_4\}, \{v_4, v_1\}\big\},$$

$$V' = \{v_1', v_2', v_3', v_4', v_5'\}, \quad E' = \big\{\{v_1', v_2'\}, \{v_2', v_3'\}, \{v_3', v_4'\}, \{v_4', v_5'\}, \{v_5', v_1'\}\big\}.$$

We find that a bijection $f : V \to V'$

$$v_1 \mapsto v_1',\ v_3 \mapsto v_2',\ v_5 \mapsto v_3',\ v_2 \mapsto v_4',\ v_4 \mapsto v_5'$$

is an isomorphism between these two graphs.



Proposition 1.3. *If two finite complete graphs have the same cardinalities of their sets of vertices, then they are isomorphic.*

PROOF. Let $G = (V, E)$ and $G' = (V', E')$ be two complete graphs with $\sharp V = \sharp V'$. We denote as $V = \{v_1, \ldots, v_n\}, V' = \{v'_1, \ldots, v'_n\}$ and define a bijection $f : V \longrightarrow V'$ by $f(v_i) = v'_i$. If $j \neq i$ we see $v_i \sim v_j$ and $v'_i \sim v'_j$ because G and G' are complete. Hence $v_i \sim v_j$ shows $f(v_i) \sim f(v_j)$ and $v'_i \sim v'_j$ shows $f^{-1}(v'_i) \sim f(v'_j)$. Thus f is an isomorphism, hence we get the conclusion. □

We call a graph G *vertex-transitive* if for arbitrary distinct two vertices $v, v' \in V$ there is an isomorphism (automorphism) $f : V \to V$ of G satisfying $f(v) = v^{'}$. It is trivial that a vertex-transitive graph is regular. A typical example of a vertex of transitive graph is a Cayle graph. Let $\mathcal{G}$ is a group and $\mathcal{S}$ is subset of $\mathcal{G}$ which does not contain the identity $1_{\mathcal{G}}$ and that is invariant under the action of the inverse operation. That is, $\mathcal{S} = \mathcal{S}^{-1} = \{s^{-1} \mid s \in \mathcal{S}\}$. If we put $V = \mathcal{G}$ and define $E = E(\mathcal{G}; \mathcal{S})$ as the of set pairs $g, h \in \mathcal{G}$ satisfing $gh^{-1} \in \mathcal{S}$, then we obtain a graph $G(\mathcal{G}; \mathcal{S})$.

PROPOSITION 1.4. *A Cayley graph* $G(\mathcal{G}; \mathcal{S}) = (V, E)$ *is vertex-transitive.*

PROOF. We take arbitrary two elements $g_1, g_2 \in \mathcal{G}$. We have an element $x \in \mathcal{G}$ satisfying $g_2 = g_1 x$. That is $x = {g_1}^{-1} g_2$. We define a map $f_{g_1 . g_2} : \mathcal{G} \longrightarrow \mathcal{G}$ by $f_{g_1, g_2}(g) = gx$. We shall show that this map f is an isomorphism.

We suppose two distinct elements $g, h \in \mathcal{G}$ satisfy $g \sim h$. Then we have an element $s \in \mathcal{S}$ with $gh^{-1} = s$. That is $s^{-1} g = h$. As we have

$$s^{-1} f(g) = s^{-1}(gx) = (s^{-1} g)x = f(s^{-1} g) = f(h)$$

we see $f(g)(f(h))^{-1} = s$. Hence $f(g) \sim f(h)$ and we find that f is an homomorphism. The inverse map $f^{-1} : \mathcal{G} \longrightarrow \mathcal{G}$ is given by $f^{-1}(g) = gx^{-1}$. If $g, h \in \mathcal{G}$ satisfy $g \sim h$, we have $gh^{-1} = s \in \mathcal{S}$, hence we see

$$f^{-1}(g)\{f^{-1}(h)\}^{-1} = gx^{-1}(hx^{-1})^{-1} = gx^{-1}xh^{-1} = gh^{-1} = s.$$

Hence $f^{-1}(g) \sim f^{-1}(h)$ and we find that f^{-1} is an homomorphism. □

1.5. Cycle graphs. A *cycle graph* is a graph consists of a closed path, that is a connected regular graph of degree 2. When a cycle have N vertices we call it an N-cycle. It is also called a *circuit*. Since we suppose graphs are simple, the cardinality of the set of vertices N of a cycle graph is more than 2.

PROPOSITION 1.5. *Cycle graphs of N vertices are is isomorphic to each other.*

PROOF. Let (V, E) be an N-cycle. We choose an arbitrary vertex $v_0 \in V$. Take $v_1 \in V \setminus \{v_0\}$ so that it is adjacent to v_0 (i.e. $\{v_0, v_1\} \in E$). Because (V, E) is a regular graph of degree two, we can choose $v_2 \in V \setminus \{v_0, v_1\}$ so that it is adjacent to v_1. Inductively, for $3 \le i \le N-1$ we can choose $v_i \in V \setminus \{v_{i-2}, v_{i-1}\}$ so that it is adjacent to v_{i-1} for $i \le N-1$.

Here, we show that $v_i \ne v_0, \dots, v_{i-1}$ by mathematical induction. We suppose this condition holds for all i with $1 \le i \le i_*$ $(\le N-2)$. If we suppose $v_{i_*+1} = v_r$ with some r with $1 \le r \le i_* - 2$, then v_{i_*} is adjacent to v_r, hence it is either v_{r-1} or v_{r+1}, which is a contradiction to the assumption (see Fig. 12). If we suppose $v_{i_*+1} = v_0$, then $(v_0, \dots, v_{i_*}, v_0)$ is a closed path (without backtrackings). Since the degree at each vertex is 2 it is a connected component. As $i_* \le N-2$ it is also a contradiction. Thus the condition holds for $i_* + 1$.

By the above operation we get a path $(v_0, \dots, v_{N-1})$ without backtracking all of whose vertices are distinct. As $n_G = N$ we find that v_0 and v_{N-1} are adjacent to each other. Hence we obtain that an N-cycle is a graph of N-step closed path without backtracking all of whose vertices are different (see Fig. 13).

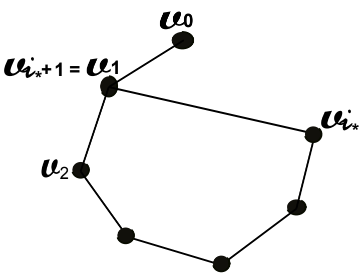


FIG. 12. unclosed path

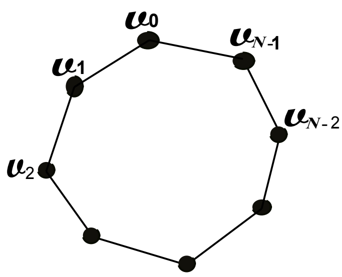


FIG. 13. closed path

If we have two N-cycles $(v_0, \dots, v_{N-1})$, $(w_0, \dots, w_{N-1})$ which is formed by N-step closed path without backtracking all of whose vertices are different, then the map f defined by $v_i \mapsto w_i$ is an isomorphism. □

By the above proposition, we denote by C_N an N-cycle graph.

2. Laplacians of graphs

2.1. Adjacency and transition operators of a graph. Given a locally finite graph $G = (V, E)$ we denote by $C(V; \mathbb{R})$ the set of all real valued functions of V, that is, $C(V; \mathbb{R}) = \{f : V \to \mathbb{R}\}$. We define its *adjacency operator* $\mathcal{A}_G$ and its *transition operator* $\mathcal{P}_G$ acting on $C(V; \mathbb{R})$ by

$$\mathcal{A}_G f(v) = \sum_{e \in E : o(e) = v} f\big(t(e)\big), \quad \mathcal{P}_G f(v) = \frac{1}{d_G(v)} \sum_{e \in E : o(e) = v} f\big(t(e)\big),$$

respectively. When the degree $d_G(v)$ at vertex v does not depend on the choice of vertices, that is, the degree function d_G is a constant function, those operators satisfies the following relation

$$\mathcal{P}_G = \frac{1}{d_G} \mathcal{A}_G. \tag{2.1}$$

When G is simple, these operators are expressed as

$$\mathcal{A}_G f(v) = \sum_{w \in V : w \sim v} f(w), \quad \mathcal{P}_G f(v) = \frac{1}{d_G(v)} \sum_{w \in V : w \sim v} f(w),$$

respectively.

We here express the adjacency operator $\mathcal{A}_G$ by a matrix in the case that G is a finite graph. When G is a finite graph, for a pair (v, w) of vertices in G, we set

$$a_{vw} = (\text{number of edges which join } v \text{ and } w),$$

and define a symmetric matrix A_G by $A_G = (a_{vw})$. We call this the *adjacent matrix* of G. When the cardinality of the set of vertices is n, then the adjacency matrix is an $n \times n$ symmetric matrix. When a graph G is simple graph, then we have $a_{vw} = 1$ for two vertices which are adjacent to each other and $a_{vw} = 0$ for two vertices which are not adjacent to each other, and moreover we have $a_{vv} = 0$. Therefore, for a simple graph its adjacency matrix is a symmetric matrix each of whose entries is either 0 or 1 and whose diagonal complements are 0. This adjacent matrix is a matrix representation of the adjacency operator. For each vertex $v \in V$ we define a function $\delta_v : V \to \mathbb{R}$ by

$$\delta_v(w) = \begin{cases} 1, & \text{when } w = v, \\ 0, & \text{when } w \neq v. \end{cases}$$

Then $\{\delta_v \mid v \in V\}$ forms a basis of $C(V;\mathbb{R})$. As we have

$$\mathcal{A}_G\delta_v(u) = \sum_{e\in E:o(e)=u} \delta_v\big(t(e)\big) = \sharp\{e \in E \mid e \text{ joins } u \text{ and } v\} = a_{uv},$$

where for a set S we denote by $\sharp S$ its cardinality, we find that

$$\mathcal{A}_G\delta_v = \sum_{w\in V} a_{vw}\delta_w.$$

Thus A_G is the matrix representation of $\mathcal{A}_G$ with respect to the basis $\{\delta_v \mid v \in V\}$.

Example 1.5. We take a graph $G = (V, E)$ which is given by

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\},$$

$$E = \big\{e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, e_3 = \{v_3, v_4\}, e_4 = \{v_4, v_5\},$$

$$e_5 = \{v_5, v_6\}, e_6 = \{v_1, v_3\}, e_7 = \{v_2, v_5\}, e_8 = \{v_4, v_6\}\big\}.$$

Then its adjacency matrix is as follows:

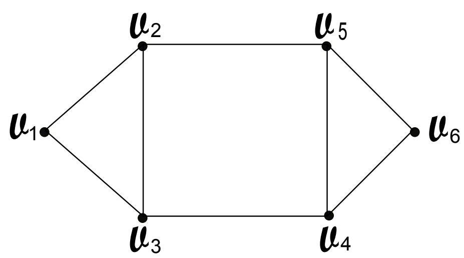


Fig. 14

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	1	1	0	0	0
v_2	1	0	1	0	1	0
v_3	1	1	0	1	0	0
v_4	0	0	1	0	1	1
v_5	0	1	0	1	0	1
v_6	0	0	0	1	1	0

Fig. 15

Next we consider vertices and edges adjacency of a finite graph. We denote by n the cardinality of the set of vertices, and by m that of the set of edges. We define an $n \times m$-matrix $B = (b_{ve})$ by setting $b_{ve} = 1$ when a vertex v and an edge e are adjacent to each other and $b_{ve} = 0$ when they are not adjacent to each other. We call it the incident matrix of this graph (see Fig. 17).

Example 1.6. For the graph in Example 1.5, its incident matrix is given as follows:

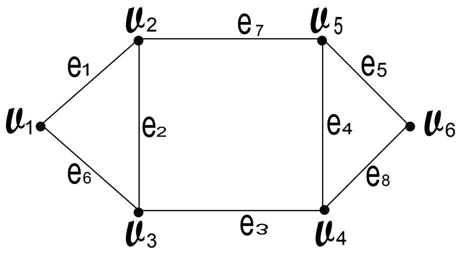


FIG. 16

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	1	0	0	0	0	1	0	0
v_2	1	1	1	0	0	0	0	0
v_3	0	1	0	0	0	1	1	0
v_4	0	0	0	1	0	0	1	1
v_5	0	0	0	1	1	0	1	0
v_6	0	0	0	0	1	0	0	1

FIG. 17

For two vertices v and w of a finite graph $G = (V, E)$, we define its *transition matrix* $P_G = (p_{vw})$ by using adjacency matrix $A_G = (a_{vw})$ as

$$p_{vw} = \frac{a_{vw}}{d_G(v)} = \frac{\text{numbers of the adjacent edges between } v \text{ and } w}{\text{degree at vertex } v}.$$

As we have

$$\mathcal{P}_G \delta_v(u) = \frac{1}{d_G(v)} \sum_{e \in E : o(e) = u} \delta_v\big(t(e)\big) = \frac{a_{uv}}{d_G(v)} = p_{uv},$$

we see

$$\mathcal{P}_G \delta_v = \sum_{w \in V} p_{vw} \delta_w.$$

Hence, P_G is the matrix representation of $\mathcal{P}_G$ with respect to the basis $\{\delta_v \mid v \in V\}$. Transition matrix is used to describe the probabilities of moving from each vertex to other vertices. That is, when we have baggage of amount k at a vertex v at first, then next time they are transferred to vertices adjacent to v and the amount at w received from v is $p_{uv} \times k$.

EXAMPLE 1.7. For the graph in Example 1.5, its transition matrix is given as follows:

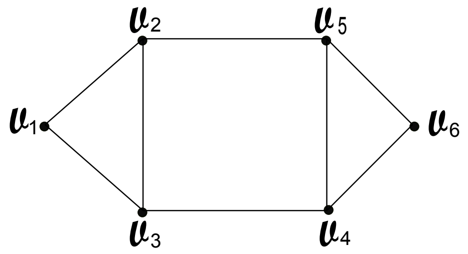


Fig. 18

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
v_2	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0
v_3	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0
v_4	0	0	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$
v_5	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$
v_6	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0

Fig. 19

Proposition 1.6. *The sum of components in the each row of the transition matrix $P_G = (p_{vw})$ of a finite graph $G = (V, E)$ is equal to 1, that is $\sum_w p_{vw} = 1$ for each $v \in V$.*

Proof. According to the definition of $\deg(v)$, we have

$$\sum_{w \in V} p_{vw} = \sum_{w \in V} \frac{a_{vw}}{\deg(v)} = \frac{1}{\deg(v)} \sum_{w \in V} a_{vw} = 1.$$

□

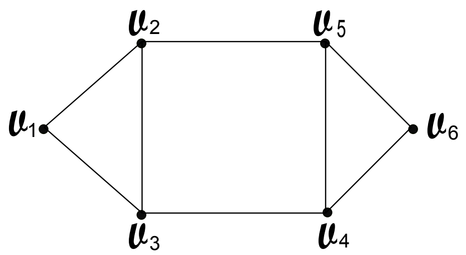


Fig. 20

v_1	$0 + \frac{1}{2} + \frac{1}{2} + 0 + 0 + 0 = 1$
v_2	$\frac{1}{3} + 0 + \frac{1}{3} + 0 + \frac{1}{3} + 0 = 1$
v_3	$\frac{1}{3} + \frac{1}{3} + 0 + \frac{1}{3} + 0 + 0 = 1$
v_4	$0 + 0 + \frac{1}{3} + 0 + \frac{1}{3} + \frac{1}{3} = 1$
v_5	$0 + \frac{1}{3} + 0 + \frac{1}{3} + 0 + \frac{1}{3} = 1$
v_6	$0 + 0 + 0 + \frac{1}{2} + \frac{1}{2} + 0 = 1$

Fig. 21

2.2. Laplacian of graph. For a locally finite graph $G = (V, E)$, we define its degree operator $\mathcal{D}_G$ acting on $C(V, \mathbb{R})$ by

$$\mathcal{D}_G f(v) = d_G(v) f(v).$$

When G is a finite graph, it is represented by a diagonal matrix D_G whose diagonal componetns are $d_G(v)$ $(v \in V)$. That is, if we denote as $D_G = (d_{vw})$ we have

$$d_{vw} = \begin{cases} \deg_G(v), & \text{if } v = w, \\ 0, & \text{if } v \neq w. \end{cases}$$

We define the *combinatorial Laplacian* $\Delta_{\mathcal{A}_G}$ and the *transitional Laplacian* $\Delta_{\mathcal{P}_G}$ acting on $C(V, \mathbb{R})$ by $\Delta_{\mathcal{A}_G} = \mathcal{D}_G - \mathcal{A}_G$ and $\Delta_{\mathcal{P}_G} = \mathcal{I} - \mathcal{P}_G$, respectively. Here, $\mathcal{I}$ denotes the identity operator. Thus we have

$$\Delta_{\mathcal{A}_G} f(v) = d_G(v) f(v) - \mathcal{A}_G f(v) \qquad \textit{and} \qquad \Delta_{\mathcal{P}_G} f(v) = f(v) - \mathcal{P}_G f(v)$$

for $f \in C(V, \mathbb{R})$. When the graph $G = (V, E)$ is regular, that is its degree-function d_G does not depend on the choice of vertices, by (2.1) these Laplacians are related with each other as

$$\Delta_{\mathcal{A}_G} = d_G \Delta_{\mathcal{P}_G}.$$

When G is finite, by using the canonical basis $\{\delta_v \mid v \in V\}$ of $C(V, \mathbb{R})$, we can represent these Laplacians by matrices. Let D_G denote the matrix representation of $\mathcal{D}_G$. By using the matrix representations A_G, P_G, D_G of $\mathcal{A}_G$, $\mathcal{P}_G$, $\mathcal{D}_G$, we find that the matrix representations Δ_{A_G}, Δ_{P_G} of $\Delta_{\mathcal{A}_G}, \Delta_{\mathcal{P}_G}$ are given as $\Delta_{A_G} = D_G - A_G$ and $\Delta_{P_G} = I - P_G$, respectively, where I denotes the identity matrix.

Example 1.8. Let $G = (V, E)$ be a graph in Fig. 22. We take a function $f \in C(V; \mathbb{R})$ given by

$$f(v_1) = 1,\ f(v_2) = 3,\ f(v_3) = -7,\ f(v_4) = 4,\ f(v_5) = -13.$$

Then we have

$$\Delta_{\mathcal{A}_G} f(v_1) = d_G(v_1) f(v_1) - \{f(v_2) + f(v_3) + f(v_4) + f(v_5)\} = 17,$$

$$\Delta_{\mathcal{A}_G} f(v_2) = 18,\ \Delta_{\mathcal{A}_G} f(v_3) = -29,\ \Delta_{\mathcal{A}_G} f(v_4) = 20,\ \Delta_{\mathcal{A}_G} f(v_5) = -31.$$

and

$$\Delta_{\mathcal{P}_G} f(v_1) = f(v_1) - \frac{1}{d_G(v_1)} \{f(v_2) + f(v_3) + f(v_4) + f(v_5)\} = \frac{17}{4},$$

$$\Delta_{\mathcal{P}_G} f(v_2) = \frac{18}{2},\ \Delta_{\mathcal{P}_G} f(v_3) = \frac{-29}{3},\ \Delta_{\mathcal{P}_G} f(v_4) = \frac{20}{3},\ \Delta_{\mathcal{P}_G} f(v_5) = \frac{-31}{2}.$$

If we represent them by matrices with respect to the canonical basis $\{\delta_{v_1}, \delta_{v_2}, \delta_{v_3}, \delta_{v_4}, \delta_{v_5}\}$, we have

$$f = \delta_{v_1} + 3\delta_{v_2} + (-7)\delta_{v_3} + 4\delta_{v_4} + (-13)\delta_{v_5} \longleftrightarrow \begin{pmatrix} 1 \\ 3 \\ -7 \\ 4 \\ -13 \end{pmatrix}$$

and

$$\Delta_{\mathcal{A}_G} f \Leftrightarrow \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -7 \\ 4 \\ -13 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -7 \\ 4 \\ -13 \end{pmatrix} = \begin{pmatrix} 17 \\ 12 \\ -29 \\ 31 \\ -31 \end{pmatrix}$$

$$\Delta_{\mathcal{P}_G} f \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -7 \\ 4 \\ -13 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -7 \\ 4 \\ -13 \end{pmatrix} = \begin{pmatrix} \frac{17}{4} \\ \frac{12}{2} \\ \frac{-29}{3} \\ \frac{31}{3} \\ \frac{-31}{2} \end{pmatrix}.$$

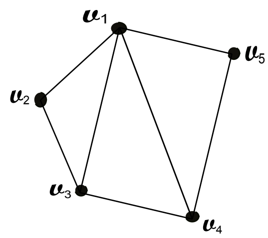


FIG. 22

In order to show properties of graphs it is a way to study their eigenvalues of Laplacians. We here briefly recall definitions of eigenvalues and eigenvectors, and some of their basic properties.

If a square matrix B satisfies $B\boldsymbol{v} = \lambda\boldsymbol{v}$ with a non-null vector $\boldsymbol{v}$ and a constant λ, we call λ an eigenvalue of B and call $\boldsymbol{v}$ an eigenvector of B corresponding to λ.

NOTE 1.1. *Let A be a real symmetric matrix.*

(1) *All eigenvalues of A are real, hence we can choose a real eigenvector for each eigenvalue.*

(2) *For its two distinct eigenvalues λ, μ, we take eigenvectors v, w corresponding to each of them. Then they are orthogonal to each other.*

PROOF. Let n denote the size of A, which means that A is an $n \times n$ matrix. We consider a Hermitian inner product on $\mathbb{C}^n$ which is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = {}^t\boldsymbol{x}\overline{\boldsymbol{y}} = x_1\overline{y}_1 + x_2\overline{y}_2 + \cdots + x_n\overline{y}_n$$

for $\boldsymbol{x} = {}^t(x_1, \ldots, x_n)$, $\boldsymbol{y} = {}^t(y_1, \ldots, y_n) \in \mathbb{C}^n$, where for a complex number $z = a + \sqrt{-1}b$ we set its complex conjugate by $\overline{z} = a - \sqrt{-1}b$, and for a matrix B we denote by ${}^tB$ its transposed matrix.

(1) We take an eigenvalue λ and an eigenvector $\boldsymbol{v}$ corresponding to λ. Since A is a real symmetric matrix, we have

$$\begin{aligned}\lambda\|\boldsymbol{v}\|^2 &= \lambda\langle \boldsymbol{v}, \boldsymbol{v} \rangle = \langle \lambda\boldsymbol{v}, \boldsymbol{v} \rangle = \langle A\boldsymbol{v}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, {}^t\overline{A}\boldsymbol{v} \rangle \\ &= \langle \boldsymbol{v}, A\boldsymbol{v} \rangle = \langle \boldsymbol{v}, \lambda\boldsymbol{v} \rangle = \overline{\lambda}\langle \boldsymbol{v}, \boldsymbol{v} \rangle = \overline{\lambda}\|\boldsymbol{v}\|^2,\end{aligned}$$

where $\overline{A} = (\overline{a_{ij}})$ for the matrix $A = (a_{ij})$. As $\boldsymbol{v}$ is not a null vector, we find $\lambda = \overline{\lambda}$, which shows that λ is real.

We take an eigenvector $\boldsymbol{v} \in \mathbb{C}^n$ corresponding to λ and denote as $\boldsymbol{v} = \boldsymbol{x} + \sqrt{-1}\boldsymbol{y}$, where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. As we have

$$\lambda\boldsymbol{x} + \sqrt{-1}\lambda\boldsymbol{y} = \lambda\boldsymbol{v} = A\boldsymbol{v} = A\boldsymbol{x} + \sqrt{-1}A\boldsymbol{y}$$

and $A\boldsymbol{x}, A\boldsymbol{y} \in \mathbb{R}^n$, we see both $\boldsymbol{x}$ and $\boldsymbol{y}$ are eigenvectors corresponding to λ if they are not null vectors. As $\boldsymbol{v}$ is not a null vector, either $\boldsymbol{x}$ or $\boldsymbol{y}$ is not null.

(2) We have

$$\lambda\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \lambda\boldsymbol{v}, \boldsymbol{w} \rangle = \langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \mu\boldsymbol{w} \rangle = \mu\langle \boldsymbol{v}, \boldsymbol{w} \rangle.$$

As $\lambda \neq \mu$ we find $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$, so that two eigenvectors $\boldsymbol{v}, \boldsymbol{w}$ are orthogonal to each other. □

For an eigenvalue λ of a square matrix B, we denote by $m_B(\lambda)$ its multiplicity, which is the dimension of the eigenspace $\{\boldsymbol{v} \in \mathbb{C}^n \mid B\boldsymbol{v} = \lambda\boldsymbol{v}\}$. The following is well known.

NOTE 1.2. *A symmetric matrix A is diagonalizable by some orthogonal matrix R, that is ${}^tRAR$ turns to be a diagonal matrix. In particular, the sum $\sum m_A(\lambda)$ of multiplicities of all distinct eigenvalues coincides with the size n of A.*

This means that there is an orthonormal basis $(\boldsymbol{v}_1, \boldsymbol{v}_2 \cdots, \boldsymbol{v}_n)$ which is formed by eigenvectors.

NOTE 1.3. *Let A, B are symmetric matrices of the same size. If they are commutative (i.e. $AB = BA$), then they are simultaneously diagonalizable.*

PROOF. When $\boldsymbol{v}$ is an eigenvector of A associated with an eigenvalue λ, we have

$$AB\boldsymbol{v} = BA\boldsymbol{v} = \lambda B\boldsymbol{v}.$$

Thus $B\boldsymbol{v}$ is also an eigenvector associated with λ.

If $m_A(\lambda) = k$, we take linearly independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ associated with λ. Then we can represent $B\boldsymbol{v}_j$ as

$$B\boldsymbol{v}_j = c_{1j}\boldsymbol{v}_1 + \cdots + c_{kj}\boldsymbol{v}_k.$$

If we define a matrix of size k by $C = (c_{ij})$, we have $B\begin{pmatrix}\boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k\end{pmatrix} = \begin{pmatrix}\boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k\end{pmatrix} C$. Thus if we take an orthogonal matrix P satisfying that

$${}^tPAP = \begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_r & & \\ & & & & & \ddots & \\ & & & & & & \lambda_r \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_r$ are mutually distinct eigenvalues of A, as the low vectors of P are eigenvectors of A, we have

$$BP = P \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_r \end{pmatrix},$$

where C_ℓ is a square matrix of size $m_A(\lambda_\ell)$.

Since ${}^tPBP$ is symmetric, we find that each C_ℓ is also symmetric. Therefore we have orthogonal matrices Q_ℓ satisfying that ${}^tQ_\ell C_\ell Q_\ell$ are diagonal matrices by Note 1.2. We set

$$Q = \begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_r \end{pmatrix}.$$

Then we have

$$\begin{aligned} {}^t(PQ)B(PQ) &= {}^tQ({}^tPBP)Q \\ &= \begin{pmatrix} {}^tQ_1 & & \\ & \ddots & \\ & & {}^tQ_r \end{pmatrix} \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_r \end{pmatrix} \begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_r \end{pmatrix} \\ &= \begin{pmatrix} {}^tQ_1C_1Q_1 & & \\ & \ddots & \\ & & {}^tQ_rC_rQ_r \end{pmatrix} \end{aligned}$$

is a diagonal matrix. On the other hand, if we denote $P = \begin{pmatrix} \boldsymbol{p}_1 & \cdots & \boldsymbol{p}_n \end{pmatrix}$, we find that the low vectors obtained by $\begin{pmatrix} \boldsymbol{p}_1 & \cdots & \boldsymbol{p}_{M_A(\lambda_1)} \end{pmatrix} Q_1$ are eigenvectors associated with λ_1, the low vectors obtained by $\begin{pmatrix} \boldsymbol{p}_{m_A(\lambda_1)+1} & \cdots & \boldsymbol{p}_{M_A(\lambda_1)+m_A(\lambda_2)} \end{pmatrix} Q_2$ are eigenvectors associated with λ_2 and so on. Hence we obtain that ${}^t(PQ)A(PQ)$ is a diagonal matirx. Thus find both ${}^t(PQ)A(PQ)$ and ${}^t(PQ)B(PQ)$ are diagonal matrices, and we get the conclusion. □

Remark 1.1. If A and B are simultaneously diagonalizable, then there exists a basis $v_1, \dots, v_n$ consists of eigenvectors of both of them (i.e. $Av_i = \lambda_i v_i$ and $Av_i = \eta_i v_i$ for all i).

We now come back to study Laplacians of graphs. Let $G = (V, E)$ be a finite non-oriented graph. For each edge $e \in E$ we give a direction and consider an oriented graph (V, E^+). For an non-oriented edge $e \in E$ we denote by $\vec{e} \in E^+$ the edge with considered orientation. Let $C(E^+)$ be a set of all (real valued) functions of the set E^+ of oriented edges. We define a map $\nabla : C(V) \to C(E^+)$ by $\nabla f\big((v,w)\big) = f(w) - f(v)$ for each $f \in C(V)$, and call it *coboundary operator.* In order to study the relationship between Laplacians and the coboundary operator, we define an inner product $\langle\ ,\ \rangle$ and a weighted inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $C(V)$ by

$$\langle f, g\rangle = \sum_{v\in V} f(v)g(v),$$
$$\langle\!\langle f, g\rangle\!\rangle = \sum_{v\in V} d_G(v)f(v)g(v)$$

for $f, g \in C(V)$. Also we define an inner product $\langle\ ,\ \rangle$ on $C(E^+)$ by

$$\langle \varphi, \psi\rangle = \sum_{\vec{e}\in E^+} \varphi(\vec{e})\psi(\vec{e})$$

for $\varphi, \psi \in C(E^+)$.

For each edge $e \in E$, we give the reversed direction and consider another oriented graph (V, E^-). This means that an oriented edge $\vec{e} \in E^+$ if and only if its reversed edge $\vec{e}^{\,-1} \in E^-$. In particular, we have a bijection $E^+ \ni \vec{e} \mapsto \vec{e}^{\,-1} \in E^-$. We define an inner product $\langle\ ,\ \rangle$ on $C(E^-)$ by

$$\langle \hat{\varphi}, \hat{\psi}\rangle = \sum_{\hat{e}\in E^-} \hat{\varphi}(\hat{e})\hat{\psi}(\hat{e})$$

for $\hat{\varphi}, \hat{\psi} \in C(E^-)$. For a function $\varphi \in C(E^+)$ we define a function $\hat{\varphi} \in C(E^-)$ by $\hat{\varphi}(\vec{e}^{\,-1}) = -\varphi(\vec{e})$. We then have

$$\begin{aligned}\langle \varphi, \psi\rangle &= \sum_{\vec{e}\in E^+} \varphi(\vec{e})\psi(\vec{e}) = \sum_{\vec{e}\in E^+} \big(-\varphi(\vec{e})\big)\big(-\psi(\vec{e})\big) = \sum_{\vec{e}\in E^+} \hat{\varphi}(\vec{e}^{\,-1})\hat{\psi}(\vec{e}^{\,-1}) \\ &= \sum_{\hat{e}\in E^-} \hat{\varphi}(\hat{e})\hat{\psi}(\hat{e}) = \langle \hat{\varphi}, \hat{\psi}\rangle.\end{aligned}$$

By using this duality, we show the following.

PROPOSITION 1.7. *For functions $f, g \in C(V)$ we have*

$$\langle \Delta_{\mathcal{A}_G} f, g\rangle = \langle \nabla f, \nabla g\rangle = \langle f, \Delta_{\mathcal{A}_G} g\rangle,$$
$$\langle\!\langle \Delta_{\mathcal{P}_G} f, g\rangle\!\rangle = \langle \nabla f, \nabla g\rangle = \langle\!\langle f, \Delta_{\mathcal{P}_G} g\rangle\!\rangle.$$

Proof. By using the duality we have

$$\begin{aligned} 2\langle \nabla f, \nabla g\rangle &= \sum_{\vec{e}\in E^+} \nabla f(\vec{e})\nabla g(\vec{e}) + \sum_{\hat{e}\in E^-} \nabla f(\hat{e})\nabla g(\hat{e}) \\ &= \sum_{\vec{e}\in E^+} \{f\big(t(\vec{e})\big) - f\big(o(\vec{e})\big)\}\{g\big(t(\vec{e})\big) - g\big(o(\vec{e})\big)\} \\ &\quad + \sum_{\vec{e}\in E^+} \{f\big(o(\vec{e})\big) - f\big(t(\vec{e})\big)\}\{g\big(o(\vec{e})\big) - g\big(t(\vec{e})\big)\}. \end{aligned}$$

On the other hand, by direct computation we see

$$\begin{aligned} \langle \Delta_{\mathcal{A}_G} f, g\rangle &= \sum_{u\in V}\big\{d_G(u)f(u) - \sum_{u\sim v} f(v)\big\}g(u) \\ &= \sum_{u\in V}\big\{d_G(u)f(u)g(u) - \sum_{u\sim v} f(v)g(u)\big\} \\ &= \sum_{u\in V}\sum_{v\sim u}\big\{f(u) - f(v)\big\}g(u). \end{aligned}$$

If we consider $u \in V$ as an origin of a non-oriented edge e, then the vertex v with $v \sim u$ is the terminus of this edge, and if we consider u as a terminus of e, then v is the origin of e. We therefore have

$$\begin{aligned} &\sum_{u\in V}\sum_{v\sim u}\big\{f(u) - f(v)\big\}g(u) \\ &\quad = \sum_{u\in V}\sum_{e\in E, o(e)=u} \big\{f\big(o(e)\big) - f\big(t(e)\big)\big\}g\big(o(e)\big) \\ &\qquad + \sum_{u\in V}\sum_{e\in E, t(e)=u} \big\{f\big(t(e)\big) - f\big(o(e)\big)\big\}g\big(t(e)\big) \\ &\quad = \sum_{e\in E}\big\{f\big(o(e)\big) - f\big(t(e)\big)\big\}g\big(o(e)\big) + \sum_{e\in E}\big\{f\big(t(e)\big) - f\big(o(e)\big)\big\}g\big(t(e)\big) \\ &\quad = \sum_{e\in E}\big\{f\big(t(e)\big) - f\big(o(e)\big)\big\}\big\{g\big(t(e)\big) - g\big(o(e)\big)\big\}. \end{aligned}$$

We should note that both E^+, E^- are bijective to E. As we consider each edge $e \in E$ its (temporary) orientation, we find that $\langle \Delta_{\mathcal{A}_G} f, g\rangle = \langle \nabla f, \nabla g\rangle$. Next we study $\Delta_{\mathcal{P}_G}$.

$$\begin{aligned} \langle\!\langle \Delta_{\mathcal{P}_G} f, g\rangle\!\rangle &= \sum_{u\in V} d_G(u)\big\{f(u) - \frac{1}{d_G(u)}\sum_{u\sim v} f(v)\big\}g(u) \\ &= \sum_{u\in V}\Big\{d_G(u)f(u)g(u) - \sum_{u\sim v} f(v)g(u)\Big\} \\ &= \sum_{u\in V}\sum_{u\sim v}\big\{f(u) - f(v)\big\}g(u) = \langle \Delta_{\mathcal{A}_G} f, g\rangle. \end{aligned}$$

Hence we have $\langle\!\langle \Delta_{\mathcal{P}_G} f, g\rangle\!\rangle = \langle \nabla f, \nabla g\rangle$, and get the conclusion. $\square$

By using this we find the following result.

Proposition 1.8. *Let $G = (V, E)$ be a finite graph.*

(1) *Every eigenvalue of $\Delta_{\mathcal{A}_G}$ and $\Delta_{\mathcal{P}_G}$ are nonnegative.*
(2) 0 *is an eigenvalue of both $\Delta_{\mathcal{A}_G}$ and $\Delta_{\mathcal{P}_G}$.*
(3) *The multiplicity of* 0 *coincides with the number k_G of connected component of G. Eigenfunctions associated with* 0 *are functions which are constant on each component of G.*

Proof. (1) Let f be an eigenfunction of $\Delta_{\mathcal{A}_G}$ associated with λ. As we have $\Delta_{\mathcal{A}_G} f = \lambda f$, we see

$$\lambda\langle f, f\rangle = \langle \lambda f, f\rangle = \langle \Delta_{\mathcal{A}_G} f, f\rangle = \langle \nabla f, \nabla f\rangle.$$

Since f is not the null function, we have $\langle f, f\rangle \neq 0$. Therefore we have $\lambda = \frac{\langle \nabla f, \nabla f\rangle}{\langle f, f\rangle} \geq 0$, and $\lambda = 0$ if and only if $\langle \nabla f, \nabla f\rangle = 0$, which means $\nabla f(e) = 0$ for all $e \in E$.

Similarly if we take an eigenfunction of $\Delta_{\mathcal{P}_G}$ associated with λ, we have

$$\lambda\langle\!\langle f, f\rangle\!\rangle = \langle\!\langle \lambda f, f\rangle\!\rangle = \langle\!\langle \Delta_{\mathcal{P}_G} f, f\rangle\!\rangle = \langle \nabla f, \nabla f\rangle.$$

Hence we obtain $\lambda = \frac{\langle \nabla f, \nabla f\rangle}{\langle\!\langle f, f\rangle\!\rangle} \geq 0$, and $\lambda = 0$ if and only if $\langle \nabla f, \nabla f\rangle = 0$.

(2) We take a function f on V which is constant on each connected component of G. We decompose V into $V_1 + \cdots + V_{k(G)}$ components. Then we see $f(v) = a_i$ for all $v \in V_i$ $(i = 1, \ldots, k(G))$. If $v \in V_i$, we have

$$\Delta_{\mathcal{A}_G} f(v) = d_G(v) f(v) - \sum_{w \sim v} f(w) = a_i\big(d_G(v) - \sum_{w \sim v} 1\big) = 0,$$
$$\Delta_{\mathcal{P}_G} f(v) = f(v) - \frac{1}{d_G(v)} \sum_{w \sim v} f(w) = a_i\Big(1 - \frac{1}{d_G(v)} \sum_{w \sim v} 1\Big) = 0.$$

Hence 0 is an eigenvalue of both $\Delta_{\mathcal{A}_G}$ and $\Delta_{\mathcal{P}_G}$.

(3) Given two vertices v, w in the same connected component of G, there is a path $\gamma = (v_0, v_1, \ldots, v_n)$ joining them. That is, $v_0 = v$ and $v_n = w$. When f is

an eigenfunction associated with 0, as $\nabla f(e) = 0$ for all $e \in E$, which means that $f\big(t(e)\big) = f\big(o(e)\big)$, we find that

$$f(v) = f(v_1) = \cdots = f(v_{n-1}) = f(w).$$

Therefore every eigenfunction associated with 0 is constant on each connected component.

On the other hand, we take functions f_i $(i = 1, \ldots, k(G))$ defined by

$$f_i(v) = \begin{cases} 1, & \text{if } v \in V_i, \\ 0, & \text{if } v \notin V_i. \end{cases}$$

Then they are eigenfunctions associated with 0. These functions are linearly independent. As a matter of fact, if $a_1 f_1 L + \cdots + a_{k(G)} f_{k(G)}$ is the null function with some real numbers $a_1, \ldots, a_{k(G)}$, then by taking a vertex $v_i \in V_i$ for each i we find

$$0 = a_1 f_1(v_i) + \cdots + a_{k(G)} f_{k(G)}(v_i) = a_i.$$

Since every function g which is constant on each component, say $g \equiv b_i$ on V_i for every i, we have $g = b_1 f_1 L + \cdots + b_{k(G)} f_{k(G)}$. Hence the dimension of eigenfunctions associated with 0 is $k(G)$. Thus the multiplicity of 0 is $k(G)$. □

CHAPTER 2

Kähler graphs

1. Definition and Examples of Kähler graphs

A Kähler graph is a graph which possesses two different kind of adjacencies. We say a graph $G = (V, E)$ to be *Kähler* if its set of edge E is divided into two disjoint subsets $E^{(p)}$ and $E^{(a)}$ and it satisfies the following condition:

At each vertex $v \in V$ there are at least four edges emanating from v,

two of them are contained in $E^{(p)}$ and two of them are contained in $E^{(a)}$.

We then get two graphs $G^{(p)} = (V, E^{(p)})$ and $G^{(a)} = (V, E^{(a)})$ which share the same set of vertices V. We call them the *principal graph* and the *auxiliary graph* of a Kähler graph G, respectively. Correspondingly, we call an edge belonging to $E^{(p)}$ to be principal and that belonging to $E^{(a)}$ to be auxiliary. In order to clarify the structure of Kähler graph, we usually denote a Kähler graph as $G = (V, E^{(p)} \cup E^{(a)})$. For a vertex $v \in V$ of a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$, we denote by $d_G^{(p)}(v)$ the degree of the principal graph $G^{(p)}$ at v, and by $d_G^{(a)}(v)$ the degree of the auxiliary graph $G^{(a)}$ at v. We call these $d_G^{(p)}(v)$ and $d_G^{(a)}(v)$ the principal and auxiliary degrees at v, respectively. Clearly we have $d_G(v) = d_G^{(p)}(v) + d_G^{(a)}(v)$. For distinct two vertices $v, w \in V$, we denote by $v \sim_p w$ their adjacency in the principal graph, and denote by $v \sim_a w$ their adjacency in the auxiliary graph.

In this paper, when we draw figures of Kähler graphs, we draw principal edges by lines and draw auxiliary edges by dotted lines (see Figs. 1, 3). One may use two kinds of colors to show these edges. To distinguish Kähler graphs from other graphs we sometimes call graphs as ordinary graphs.

Example 2.1. We define a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ as

$$V = \{v_1, v_2, v_3, v_4, v_5\},$$
$$E^{(p)} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_1\}\}$$
$$E^{(a)} = \{\{v_1, v_3\}, \{v_3, v_5\}, \{v_5, v_2\}, \{v_2, v_4\}, \{v_4, v_5\}\}.$$

If we draw figures of this Kähler graph and its principal and auxiliary graphs, we have as follows.

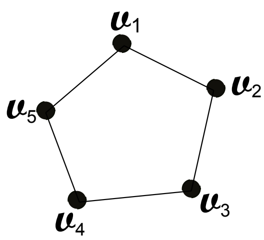


Fig. 1

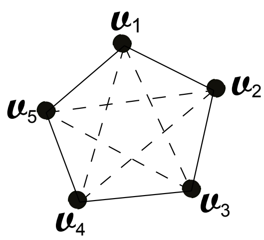


Fig. 2

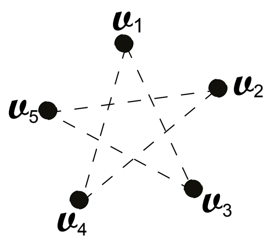


Fig. 3

This example suggests us a way of constructing Kähler graphs. For an ordinary finite graph $G = (V, E)$ we take its complement graph $G^c = (V, E^c)$. Here, we define E^c in the following manner: For distinct two vertices $v, w \in V$ we define $v \sim w$ in G^c if and only if $v \not\sim w$ in G. Here, for two vertices v, w we show as $v \not\sim w$ if they are not adjacent to each other. By the definition of complement graphs, we see $E \cap E^c = \emptyset$. Under the condition that $2 \le d_G(v) \le n_G - 3$, we have $2 \le d_{G^c}(v) \le n_{G^c} - 3$ because $d_G(v) + d_{G^c}(v) = n_G - 1$, where $n_G = n_{G^c}$ denote the cardinality of the set of V. Thus we obtain a Kähler graph $G^K = (V, E \cup E^c)$ which is complete as an ordinary graph. We call this the *complement-filled* Kähler graph of G.

We here give some other examples of Kähler graph.

Example 2.2. We denote by $\mathbb{Z}$ the set of integers and by $\mathbb{R}$ the set of real numbers. We take the set of lattice points $V = \mathbb{Z}^2 = \{(a, b) \mid a, b \in \mathbb{Z}\}$ in a Euclidean plane $\mathbb{R}^2$. We set principal edges so that lines which are parallel to the x-axis are formed by them, and set auxiliary edges so that lines parallel to the y-axis are formed by them.

That is, we set

$$E_1^{(p)} = \{\{(a,b),\,(a+1,b)\},\ \{(a,b),\,(a-1,b)\} \mid a,b\in\mathbb{Z}\},$$
$$E_1^{(a)} = \{\{(a,b),\,(a,b+1)\},\ \{(a,b),\,(a,b-1)\} \mid a,b\in\mathbb{Z}\}.$$

We then obtain a Kähler graph $(V, E_1^{(p)}\cup E_1^{(a)})$ (see Fig 4,Fig 5).

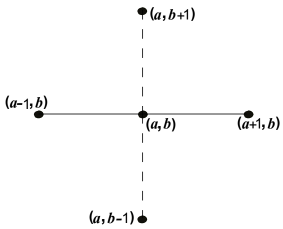


FIG. 4

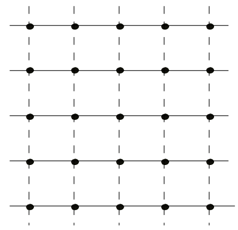


FIG. 5. $(V, E_1^{(p)}\cup E_1^{(a)})$

If we set

$$E_2^{(p)} = \left\{ \begin{array}{c} \{(a,b),\,(a+1,b)\},\ \{(a,b),\,(a-1,b)\}, \\ \{(a,b),\,(a,b+1)\},\ \{(a,b),\,(a,b-1)\} \end{array} \;\middle|\; a,b\in\mathbb{Z} \right\},$$
$$E_2^{(a)} = \left\{ \begin{array}{c} \{(a,b),\,(a+1,b+1)\},\ \{(a,b),\,(a-1,b-1)\}, \\ \{(a,b),\,(a-1,b+1)\},\ \{(a,b),\,(a+1,b-1)\} \end{array} \;\middle|\; a,b\in\mathbb{Z} \right\},$$

we obtain another Kähler graph $(V, E_2^{(p)}\cup E_2^{(a)})$ (see Figs. 6, 7). Its principal graph is connected.

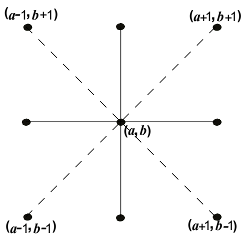


FIG. 6

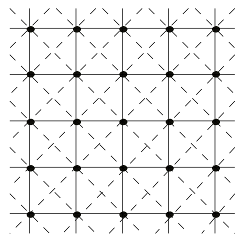


FIG. 7. $(V, E_2^{(p)}\cup E_2^{(a)})$

Similarly if we set

$$E_3^{(p)} = \left\{ \begin{array}{c} \{(a,b),\,(a+1,b)\},\ \{(a,b),\,(a-1,b)\}, \\ \{(a,b),\,(a+1,b+1)\},\ \{(a,b),\,(a-1,b-1)\} \end{array} \;\middle|\; a,b\in\mathbb{Z} \right\},$$
$$E_3^{(a)} = \left\{ \begin{array}{c} \{(a,b),\,(a,b+1)\},\ \{(a,b),\,(a,b-1)\}, \\ \{(a,b),\,(a-1,b+1)\},\ \{(a,b),\,(a+1,b-1)\} \end{array} \;\middle|\; a,b\in\mathbb{Z} \right\},$$

we obtain a Kähler graph $(V, E_3^{(p)}\cup E_3^{(a)})$ (see Figs 8, 9). Its principal and auxiliary graphs are connected.

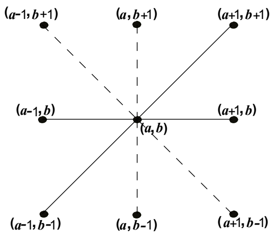


FIG. 8

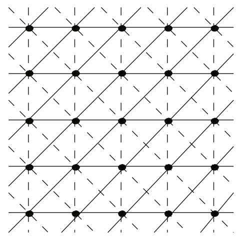


FIG. 9. $(V, E_3^{(p)} \cup E_3^{(a)})$

We note that V can be identified with the set of lattice points $\{a + \sqrt{-1}b \mid a, b \in \mathbb{Z}\}$ in the field $\mathbb{C}$ of complex numbers. We call these Kähler graphs $(V, E_1^{(p)} \cup E_1^{(a)})$, $(V, E_2^{(p)} \cup E_2^{(a)})$, $(V, E_3^{(p)} \cup E_3^{(a)})$ a Kähler graph of complex lattice, a complex line of Cartesian-tensor product type, and a Cayley complex line, respectively.

We can extend the above examples of Kähler graphs to Kähler graphs of lattice points in a complex m dimensional Euclidean space $\mathbb{C}^m$.

EXAMPLE 2.3. We take the set of lattice points

$$V = \big\{(a_1 + \sqrt{-1}b_1, \dots, a_m + \sqrt{-1}b_m) \bigm| a_i, b_i \in \mathbb{Z} \text{ for all } i = 1, \dots, m\big\}.$$

We define $(V, E_1^{(p)} \cup E_1^{(a)})$ as follows:

1) Two vertices

$$\boldsymbol{z} = (a_1+\sqrt{-1}b_1, \dots, a_m+\sqrt{-1}b_m),\ \boldsymbol{z}' = (a_1'+\sqrt{-1}b_1', \dots, a_m'+\sqrt{-1}b_m') \in V$$

are adjacent to each other in the principal graph if and only if there is i_0 $(1 \leq i_0 \leq m)$ satisfying that

   i) $a_{i_0}' = a_{i_0} + 1$ or $a_{i_0}' = a_{i_0} - 1$,
   ii) $a_i' = a_i$ for $i \neq i_0$,
   iii) $b_i' = b_i$ for all i;

2) Two $\boldsymbol{z}, \boldsymbol{z}' \in V$ are adjacent to each other in the auxiliary graph if and only if there is i_0 $(1 \leq i_0 \leq m)$ satisfying that

   i) $b_{i_0}' = b_{i_0} + 1$ or $b_{i_0}' = b_{i_0} - 1$,
   ii) $a_i' = a_i$ for all i,

iii) $b'_i = b_i$ for $i \neq i_0$.

We call this graph a Kähler graph of m-dimensional complex lattice.

We define $(V, E_2^{(p)} \cup E_2^{(a)})$ as follows:

1) Two vertices

$$\boldsymbol{z} = (a_1+\sqrt{-1}b_1, \dots, a_m+\sqrt{-1}b_m),\ \boldsymbol{z}' = (a'_1+\sqrt{-1}b'_1, \dots, a'_m+\sqrt{-1}b'_m) \in V$$

are adjacent to each other in the principal graph if and only if there is i_0 $(1 \leq i_0 \leq m)$ satisfying either the following i), ii), iii) or i'), ii'), iii'):

i) $a'_{i_0} = a_{i_0} + 1$ or $a'_{i_0} = a_{i_0} - 1$,

ii) $a'_i = a_i$ for $i \neq i_0$,

iii) $b'_i = b_i$ for all i;

i') $b'_{i_0} = b_{i_0} + 1$ or $b'_{i_0} = b_{i_0} - 1$,

ii') $a'_i = a_i$ for all i,

iii') $b'_i = b_i$ for $i \neq i_0$;

2) Two $\boldsymbol{z}, \boldsymbol{z}' \in V$ are adjacent to each other in the auxiliary graph if and only if there is i_0 $(1 \leq i_0 \leq m)$ satisfying that

i) one of the following holds:

a) $a'_{i_0} = a_{i_0} + 1$ and $b'_{i_0} = b_{i_0} + 1$,

b) $a'_{i_0} = a_{i_0} + 1$ and $b'_{i_0} = b_{i_0} - 1$,

c) $a'_{i_0} = a_{i_0} - 1$ and $b'_{i_0} = b_{i_0} + 1$,

d) $a'_{i_0} = a_{i_0} - 1$ and $b'_{i_0} = b_{i_0} - 1$,

ii) $a'_i = a_i$ for $i \neq i_0$,

iii) $b'_i = b_i$ for $i \neq i_0$.

We call this a Kähler graph of m-dimensional complex lattice of Cartesian-tensor product type.

We define $(V, E_3^{(p)} \cup E_3^{(a)})$ as follows:

1) Two vertices

$$\boldsymbol{z} = (a_1+\sqrt{-1}b_1, \dots, a_m+\sqrt{-1}b_m),\ \boldsymbol{z}' = (a'_1+\sqrt{-1}b'_1, \dots, a'_m+\sqrt{-1}b'_m) \in V$$

are adjacent to each other in the principal graph if and only if there is i_0 $(1 \leq i_0 \leq m)$ satisfying either

i) $a'_{i_0} = a_{i_0} + 1$ or $a'_{i_0} = a_{i_0} - 1$,

ii) $a'_i = a_i$ for $i \neq i_0$,

iii) $b'_i = b_i$ for all i,

or

i) either $a'_{i_0} = a_{i_0} + 1$ and $b'_{i_0} = b_{i_0} + 1$, or $a'_{i_0} = a_{i_0} - 1$ and $b'_{i_0} = b_{i_0} - 1$,

ii) $a'_i = a_i$ for $i \neq i_0$,

iii) $b'_i = b_i$ for $i \neq i_0$;

2) Two $\boldsymbol{z}, \boldsymbol{z}' \in V$ are adjacent to each other in the auxiliary graph if and only if there is i_0 $(1 \leq i_0 \leq m)$ satisfying either

i) $b'_{i_0} = b_{i_0} + 1$ or $b'_{i_0} = b_{i_0} - 1$,

ii) $a'_i = a_i$ for all i,

iii) $b'_i = b_i$ for $i \neq i_0$.

or

i) either $a'_{i_0} = a_{i_0} + 1$ and $b'_{i_0} = b_{i_0} - 1$, or $a'_{i_0} = a_{i_0} - 1$ and $b'_{i_0} = b_{i_0} + 1$,

ii) $a'_i = a_i$ for $i \neq i_0$,

iii) $b'_i = b_i$ for $i \neq i_0$.

We call this graph a Kähler graph of m-dimensional Cayley complex lattice.

We here give concrete examples of Kähler graphs of higher dimensional complex lattice, of higher dimensional complex lattice of Cartesian-tensor product type and of higher dimensional Cayley complex lattice in order to help readers to understand.

Example 2.4. We take the set

$$V = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \;\middle|\; z_i = a_i + \sqrt{-1}b_i,\ a_i, b_i \in \mathbb{Z}\right\}$$

of lattice points in $\mathbb{C}^3$.

(1) In a Kähler graph of 3-dimensional complex lattice, each point $(z_1, z_2, z_3) \in V$ is principally adjacent to the following six points

$$(z_1 \pm 1, z_2, z_3),\ (z_1, z_2 \pm 1, z_3),\ (z_1, z_2, z_3 \pm 1),$$

and is auxiliary adjacent to the following six points

$$\big(z_1 \pm \sqrt{-1}, z_2, z_3\big),\ \big(z_1, z_2 \pm \sqrt{-1}, z_3\big),\ \big(z_1, z_2, z_3 \pm \sqrt{-1}\big).$$

(2) In a Kähler graph of 3-dimensional complex lattice of Cartesian-tensor product type, each point $(z_1, z_2, z_3) \in V$ is principally adjacent to the following 12 points

$$\big(z_1 \pm 1, z_2, z_3\big),\ \big(z_1, z_2 \pm 1, z_3\big),\ \big(z_1, z_2, z_3 \pm 1\big),$$
$$\big(z_1 \pm \sqrt{-1}, z_2, z_3\big),\ \big(z_1, z_2 \pm \sqrt{-1}, z_3\big),\ \big(z_1, z_2, z_3 \pm \sqrt{-1}\big).$$

and is auxillary adjacent to the following 12 points

$$\big(z_1 \pm (1+\sqrt{-1}), z_2, z_3\big),\ \big(z_1, z_2 \pm (1+\sqrt{-1}), z_3\big),\ \big(z_1, z_2, z_3 \pm (1+\sqrt{-1})\big),$$
$$\big(z_1 \pm (1-\sqrt{-1}), z_2, z_3\big),\ \big(z_1, z_2 \pm (1-\sqrt{-1}), z_3\big),\ \big(z_1, z_2, z_3 \pm (1-\sqrt{-1})\big).$$

(3) In a Kähler graph of 3-dimensional Cayley complex lattice, each point $z_1, z_2, z_3 \in V$ is principally adjacent to the following 12 points

$$\big(z_1 \pm 1, z_2, z_3\big),\ \big(z_1, z_2 \pm 1, z_3\big),\ \big(z_1, z_2, z_3 \pm 1\big),$$
$$\big(z_1 \pm (1+\sqrt{-1}), z_2, z_3\big),\ \big(z_1, z_2 \pm (1+\sqrt{-1}), z_3\big),\ \big(z_1, z_2, z_3 \pm (1+\sqrt{-1})\big).$$

and is auxillary adjacent to the following 12 points

$$\big(z_1 \pm \sqrt{-1}, z_2, z_3\big),\ \big(z_1, z_2 \pm \sqrt{-1}, z_3\big),\ \big(z_1, z_2, z_3 \pm \sqrt{-1}\big)$$
$$\big(z_1 \pm (1-\sqrt{-1}), z_2, z_3\big),\ \big(z_1, z_2 \pm (1-\sqrt{-1}), z_3\big),\ \big(z_1, z_2, z_3 \pm (1-\sqrt{-1})\big).$$

We can associate graphs to groups. For a group $\mathcal{G}$ we take two disjoint nonempty finite subsets $\mathcal{S}^{(p)}$ and $\mathcal{S}^{(a)}$ of $\mathcal{G}$ which do not contain the identity and that are invarint under the action of the inverse operation. Since we get two Cayley graphs $\big(\mathcal{G}, E(\mathcal{G};\mathcal{S}^{(p)})\big)$ and $\big(\mathcal{G}, E(\mathcal{G};\mathcal{S}^{(a)})\big)$, where

$$E(\mathcal{G};\mathcal{S}^{(p)}) = \big\{\{g,h\} \bigm| g^{-1}h \in \mathcal{S}^{(p)}\big\} \quad \text{and} \quad E(\mathcal{G};\mathcal{S}^{(a)}) = \big\{\{g,h\} \bigm| g^{-1}h \in \mathcal{S}^{(a)}\big\},$$

we obtain a locally finite Kähler graph $\big(\mathcal{G}, E(\mathcal{G};\mathcal{S}^{(p)}) \cup E(\mathcal{G};\mathcal{S}^{(a)})\big)$. We call this graph a *Cayley Kähler graph.* The Kähler graphs in Example 2.2 are Cayley Kähler graphs.

Example 2.5. We take a dihedral group

$$\begin{aligned} D_4 &= \big\langle a, b \bigm| a^4 = b^2 = 1,\ ab = ba^3 \big\rangle \\ &= \big\langle b, c \bigm| b^2 = c^2 = 1,\ bcbc = cbcb \big\rangle. \end{aligned}$$

where $c = ab$. If we set $\mathcal{S}^{(p)} = \{b, c\}$ and $\mathcal{S}^{(a)} = \{a, a^3\}$, we get a regular Kähler graph as like Fig. 10. By the construction of this Kähler graph we find that the principal degree and the auxiliary degree are 2.

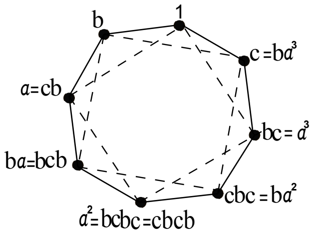


Fig. 10

A Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ is said to be regular if both the principal and auxiliary graphs are regular. That is, both the principal and the auxiliary degrees do not depend on the choice of vertex $v \in V$. When we consider Kähler graphs of $n_G = 5$, we see they are complete by the condition of Kähler graphs. In order to show more examples on forms of Kähler graph, we here consider Kähler graphs of $n_G \geq 6$.

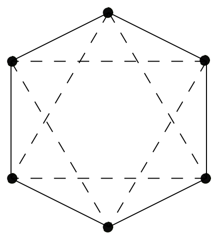


Fig. 11

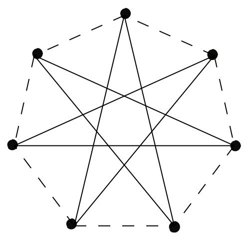


Fig. 12

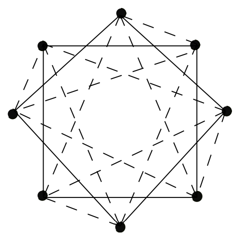


Fig. 13

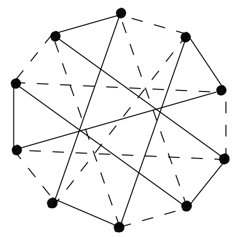


Fig. 14

In the Figs. 11, 12, 13 and 14), we give regular Kähler graphs whose sets of vertices have cardinality $n_G = 6, 7, 8, 10$, respectively. Their principal and auxiliary degrees are the same $d^{(p)}(v) = d^{(a)}(v) = 2$ in Figs. 11, 12, 14, and are different $d^{(p)}(v) = 2, d^{(a)}(v) = 3$ in Fig. 13. We discuss in §2.3 more detail on the relationship between the cardinality of the set of vertices and principal and auxiliary degrees.

We here note the following:

1) When G is a finite graph then $d_{G^c}(v) = n_G - d_G(v) - 1$;

2) In particular, when G is a finite graph, G is regular if and only if G^c is regular.

Therefore if a finite ordinary graph G is regular and satisfies $2 \leq d_G \leq n_G - 3$, then its complement-filled Kähler graph G^K is a regular Kähler graph.

2. Kähler graphs of product type

A Kähler graph of complex lattice consists of horizontal lines for the principal graph and vertical lines for the auxiliary graph. In other words, it is a product of a principal graph of real lattice and an auxiliary graph of real lattice. In this section we show some product operations to get Kähler graphs by using ordinary graphs.

It is known that we have four typical ways of product operation of graphs; Cartesian product, strong product, semi-tensor product and lexicographical product. Given two ordinary graphs $G = (V, E)$ and $H = (W, F)$, we define their Cartesian product $G \square H$, strong product $G \boxtimes H$, semi-tensor product $G \otimes H$ and lexicographical product $G \vdash H$ in the following manner:

1) Their sets of vertices are the product $V \times W$;
2) Two vertices $(v, w), (v', w') \in V \times W$ are adjacent to each other if they satisfy the following conditions:
   (a) either $v \sim v'$ in G and $w = w'$ or $v = v'$ and $w \sim w'$ in H for $G \square H$
   (b) they satisfy one of the conditions in $G \boxtimes H$;
      b-i) $v \sim v'$ in G and $w = w'$,
      b-ii) $v = v'$ and $w \sim w'$ in H,
      b-iii) $v \sim v'$ in G and $w \sim w'$ for H;
   (c) $v \sim v'$ in G and $w \sim w'$ in H for $G \otimes H$;
   (d) either $v \sim v'$ in G and $w = w'$ or $w \sim w'$ in H for $G \vdash H$.

Corresponding to these operations and the operations of complement we give some product operations of ordinary graphs to get Kähler graphs. Through out this section $G = (V, E)$ and $H = (W, F)$ are ordinary graphs.

2.1. Kähler graphs of product type whose principal graphs are unions of copies of original graphs. First we consider product operations satisfying that the constructed Kählar graphs have principal graphs each of whose connected components is isomorphic to the original graph.

[1] Kähler graphs of Cartesian product type

Given two ordinary graphs $G = (V, E)$ and $H = (W, F)$, we define their Kähler graph of *Cartesian product type* $G \widehat{\square} H$ as follows:

i) Its set of vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if $v \sim v'$ in G and $w = w'$;

iii) Two vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if $w \sim w'$ in H and $v = v'$.

Example 2.6. If we take G and H as graphs of real lines, then their Kähler graph of *Cartesian product type* is a graph of complex line. If we represent G by a horizontal line and H by a vertical line, then $G \widehat{\square} H$ is represented as Fig. 16.

Fig. 15. $G = H$

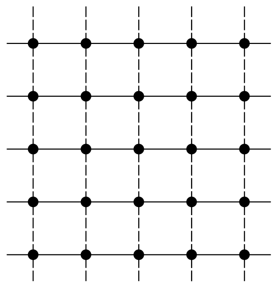


Fig. 16. $G \widehat{\square} H$

When G and H are locally finite graphs, their Kählar graph of Cartesian product type is also locally finite. Its principal and auxiliary degrees are given as

$$d^{(p)}_{G \widehat{\square} H}(v, w) = d_G(v) \quad \text{and} \quad d^{(a)}_{G \widehat{\square} H}(v, w) = d_H(w).$$

In particular, when G and H are regular, their Kählar graph of Cartesian product type is also regular.

[2] Kähler graphs of strong product type

Given two ordinary graphs $G = (V, E)$ and $H = (W, F)$, we define their Kähler graph of *strong product type* $G \widehat{\boxtimes} H$ as follows:

i) Its set of vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v,w), $(v',w') \in V \times W$ are adjacent to each other by a principal edge if and only if $v \sim v'$ in G and $w = w'$;

iii) Two distinct vertices (v,w), $(v',w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if they satisfy one of the following conditions;

(a) $v \sim v'$ in G and $w = w'$,

(b) $v = v'$ and $w \sim w'$ in H,

(c) $v \sim v'$ in G and $w \sim w'$ in H.

Example 2.7. If we take G and H as graphs of real lines, then their Kähler graph of strong product type is like the following figures.

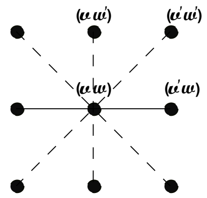


Fig. 17. adjacency at a vertex in $G\widehat{\boxtimes}H$

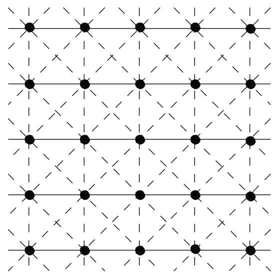


Fig. 18. $G\widehat{\boxtimes}H$

When G and H are locally finite graphs, their Kählar graph of strong product type is also locally finite. Its principal and auxiliary degrees are given as

$$d^{(p)}_{G\widehat{\boxtimes}H}(v,w) = d_G(v) \quad \text{and} \quad d^{(a)}_{G\widehat{\boxtimes}H}(v,w) = d_H(w)\{d_G(v)+1\}.$$

In particular, when G and H are regular, their Kählar graph of strong product type is also regular.

[3] Kähler graphs of semi-tensor product type

For two ordinary graphs $G = (V,E)$ and $H = (W,F)$, we define their Kähler graph of *semi-tensor product type* $G\widehat{\otimes}H$ as follows;

i) Its set of vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if $v \sim v'$ in G and $w = w'$;

iii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if $v \sim v'$ in G and $w \sim w'$ in H.

Example 2.8. If we take G and H as graphs of real lines, then their Kähler graph of semi-tensor product type is like the following figures.

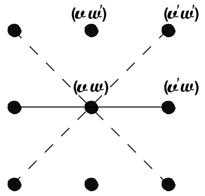


Fig. 19. adjacency at a vertex in $G \widehat{\otimes} H$

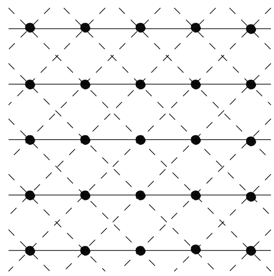


Fig. 20. $G \widehat{\otimes} H$

By definitions if we take both the auxiliary edges of the Kähler graph of semi-tensor product type and those of the Kähler graph of Cartesian product type, we get the auxiliary edges of the Kähler graph of strong product type.

When G and H are locally finite graphs, their Kählar graph of semi-tensor product type is also locally finite. Its principal and auxiliary degrees are given as

$$d^{(p)}_{G \widehat{\otimes} H}(v, w) = d_G(v) \quad \text{and} \quad d^{(a)}_{G \widehat{\otimes} H}(v, w) = d_G(v) d_H(w).$$

In particular, when G and H are regular, their Kählar graph of semi-tensor product type is also regular.

[4] Kähler graphs of lexicographical product type

Given two ordinary graphs $G = (V, E)$ and $H = (W, F)$, we define their Kähler graph $G \triangleright H$ of *lexicographical product type* as follows:

i) Its set of vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if $v \sim v'$ in G and $w = w'$;

iii) Two distinct vertices (v,w), $(v',w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if $w \sim w'$ in H.

EXAMPLE 2.9. If we take G and H as graphs of real lines, then their Kähler graph of lexicographical product type is like the following figures.

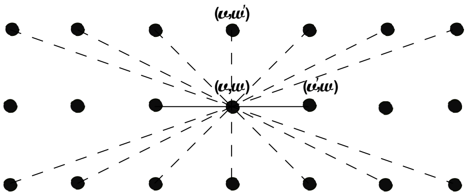


FIG. 21. adjacency at a vertex in $G \triangleright H$

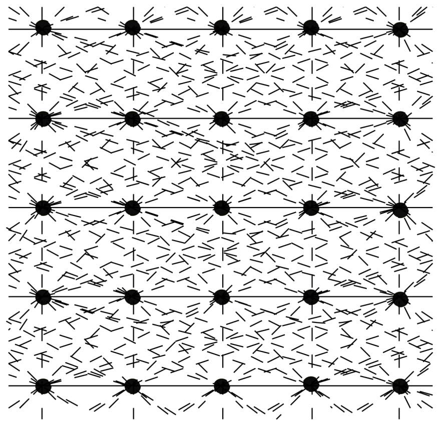


FIG. 22. $G \triangleright H$

When G is a finite graph and H is locally finite, then their Kähler graph of lexicographical product type is locally finite. Its principal and auxiliary degrees are given as

$$d^{(p)}_{G\triangleright H}(v,w) = d_G(v) \quad \text{and} \quad d^{(a)}_{G\triangleright H}(v,w) = n_G d_H(w).$$

In particular, when G and H are regular, their Kählar graph of lexicographical product type is also regular. We note that when G is a complete graph then a Kähler graph $G\widehat{\boxtimes}H$ of strong product type coincides with a Kähler graph $G \triangleright H$ of lexicographical product type.

By the definition of Kähler graphs of lexicographical product type, we see that each of its vertex (v,w) is completely adjacent to vertices whose second components are adjacent to w in the graph of second components,

[5] Kähler graphs of co-Cartesian product type

Let $G=(V,E)$ and $H=(W,F)$ be ordinary graphs. We define their Kähler graph of *co-Cartesian product type* $G \overset{c}{\square} H$ as follows:

i) Its set of vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if $v \sim v'$ in G and $w = w'$;

iii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if $v \neq v'$ and $w \sim w'$ in H.

EXAMPLE 2.10. If we take G and H as graphs of real lines, then their Kähler graph of co-Cartesian product type is like the following figures.

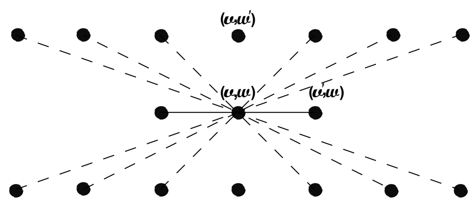


FIG. 23. adjacency at a vertex in $G \overset{c}{\Box} H$

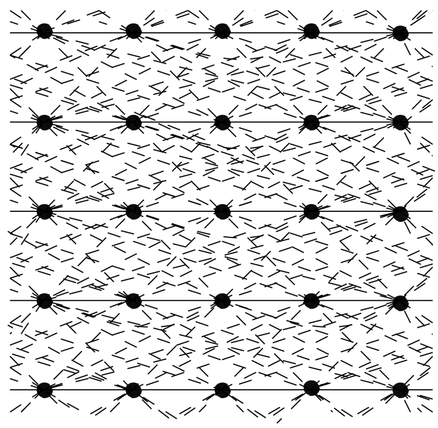


FIG. 24. $G \overset{c}{\Box} H$

When G is finite and H is locally finite, then their Kähler graph of co-Cartesian product type is locally finite. Its principal and auxiliary degrees are given as

$$d^{(p)}_{G \overset{c}{\Box} H}(v, w) = d_G(v) \quad \text{and} \quad d^{(a)}_{G \overset{c}{\Box} H}(v, w) = (n_G - 1) d_H(w).$$

In particular, when G is finite and regular and H is regular, their Kählar graph of co-Cartesian product type is also regular.

[6] Kähler graphs of co-tensor product type

Let $G = (V, E)$ and $H = (W, F)$ be ordinary graphs. We define their Kähler graph of *co-tensor product type* $G \overset{c}{\otimes} H$ as follows:

i) Its set of vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if $v \sim v'$ in G and $w = w'$;

iii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if $v \not\sim v'$ in G and $w \sim w'$ in H.

Example 2.11. If we take G and H as graphs of real lines, then their Kähler graph of co-tensor product type is like the following figures.

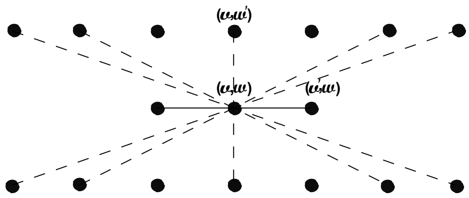


Fig. 25. adjacency at a vertex in $G \overset{c}{\otimes} H$

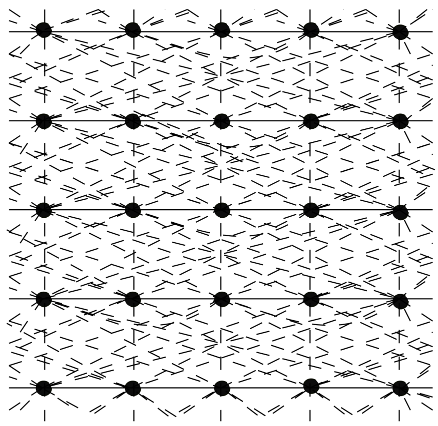


Fig. 26. $G \overset{c}{\otimes} H$

When G is finite and H is locally finite, then their Kähler graph of co-tensor product type is locally finite. Its principal and auxiliary degrees are given as

$$d^{(p)}_{G \overset{c}{\otimes} H}(v, w) = d_G(v) \quad \text{and} \quad d^{(a)}_{G \overset{c}{\otimes} H}(v, w) = \big(n_G - d_G(v)\big) d_H(w).$$

In particular, when G is finite and regular and H is regular, their Kähler graph of co-tensor product type is also regular.

[7] Kähler graphs of co-strong product type

Let $G = (V, E)$ and $H = (W, F)$ be ordinary graphs. Suppose that for each vertex $v \in V$ there exists at least one vertex which is different from v and is not adjacent to v in G. We define their Kähler graph of *co-strong product type* $G \overset{c}{\boxtimes} H$ as follows:

i) Its set of vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v,w), $(v',w') \in V \times W$ are adjacent to each other by a principal edge if and only if $v \sim v'$ in G and $w = w'$;

iii) Two distinct vertices (v,w), $(v',w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if $v \neq v'$, $v \not\sim v'$ in G and $w \sim w'$ in H.

Example 2.12. If we take G and H as graphs of real lines, then their Kähler graph of co-strong product type is like the following figures.

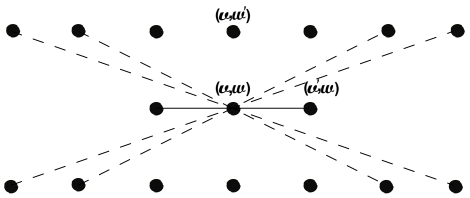


Fig. 27. adjacency at a vertex in $G \overset{c}{\boxtimes} H$

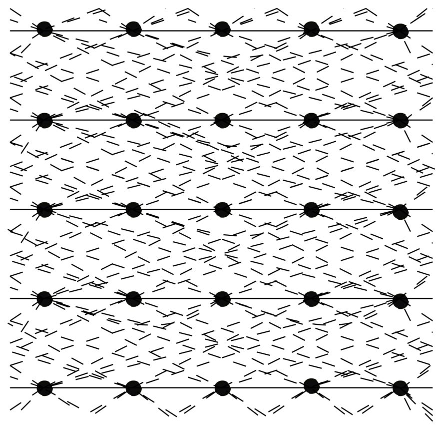


Fig. 28. $G \overset{c}{\boxtimes} H$

When G is finite and H is locally finite, then their Kähler graph of co-strong product type is locally finite. Its principal and auxiliary degrees are given as

$$d^{(p)}_{G \overset{c}{\boxtimes} H}(v,w) = d_G(v) \quad \text{and} \quad d^{(a)}_{G \overset{c}{\boxtimes} H}(v,w) = \big(n_G - d_G(v) - 1\big) d_H(w).$$

In particular, when G is finite and regular and H is regular, their Kählar graph of co-strong product type is also regular.

We note that if we define a Kählar graph of "co-lexicographical product" type it is nothing but a union of copies of G because we can not add auxiliary edges.

We here point out that we can do both the product operation and the complement-filling operation. Given two ordinary graphs $G = (V,E)$ and $H = (W,F)$, we define a Kähler graph $G\,\widehat{\Box}^K H$ as follows:

i) Its set of the vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if $v \sim v'$ in G and $w = w'$;

iii) Two vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if either $w \sim w'$ in H and $v = v'$, or $v \neq v'$, $v \not\sim v'$ in G and $w = w'$.

We call $G \,\widehat{\square}^K H$ a Kähler graph of complement-filled Cartesian product type. We can obtain $G \,\widehat{\square}^K H$ from $G \,\widehat{\square}\, H$ by adding auxiliary edges according to the rule that

[**rule** K]: $(v, w) \sim_a (v', w')$ if $v \neq v'$, $v \not\sim v'$ in G and $w = w'$.

We note that when G is a complete graph then we have $G \,\widehat{\square}^K H = G \,\widehat{\square}\, H$.

Similarly, by using other Kähler graphs of product type and by adding auxiliary edges according to [rule K], we get six Kähler graphs $G \widehat{\boxtimes}^K H$, $G \widehat{\otimes}^K H$, $G \rhd^K H$, $G \overset{c}{\square}^K H$, $G \overset{c}{\boxtimes}{}^K H$ and $G \overset{c}{\otimes}{}^K H$. When G is finite and H is locally finite, these Kähler graphs are also locally finite. Their principal and auxiliary degrees are given as

$$\begin{aligned} d^{(p)}_{G \widehat{\square}^K H}(v, w) &= d^{(p)}_{G \widehat{\boxtimes}^K H}(v, w) = d^{(p)}_{G \widehat{\otimes}^K H}(v, w) = d^{(p)}_{G \rhd^K H}(v, w) \\ &= d^{(p)}_{G \overset{c}{\square}^K H}(v, w) = d^{(p)}_{G \overset{c}{\otimes}^K H}(v, w) = d^{(p)}_{G \overset{c}{\boxtimes}^K H}(v, w) = d_G(v), \end{aligned}$$

and

$$\begin{aligned} d^{(a)}_{G \widehat{\square}^K H}(v, w) &= n_G + d_H(w) - d_G(v) - 1, \\ d^{(a)}_{G \widehat{\boxtimes}^K H}(v, w) &= n_G + \{d_H(w) - 1\}\{d_G(v) + 1\}, \\ d^{(a)}_{G \widehat{\otimes}^K H}(v, w) &= n_G + d_G(v)\{d_H(w) - 1\} - 1, \\ d^{(a)}_{G \rhd^K H}(v, w) &= n_G\{d_H(w) + 1\} - d_G(v) - 1, \\ d^{(a)}_{G \overset{c}{\square}^K H}(v, w) &= n_G\{d_H(w) + 1\} - d_H(w) - d_G(v) - 1, \\ d^{(a)}_{G \overset{c}{\otimes}^K H}(v, w) &= \{n_G - d_G(v)\}\{d_H(w) + 1\} - 1, \\ d^{(a)}_{G \overset{c}{\boxtimes}^K H}(v, w) &= \{d_H(w) + 1\}\{n_G - d_G(v) - 1\}. \end{aligned}$$

When G and H are finite graphs, if we consider the operation $G \overset{c}{\rhd}{}^K H$, then it is an n_H-copies of the complement-filled Kähler graph G^K.

Example 2.13. If we take G and H as graphs of real lines, then their Kähler graphs of complement-filled product type is like the following figures.

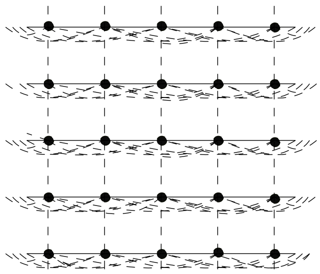


Fig. 29. $G\,\widehat{\square}^K H$

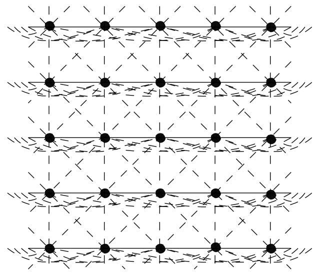


Fig. 30. $G\widehat{\boxtimes}^K H$

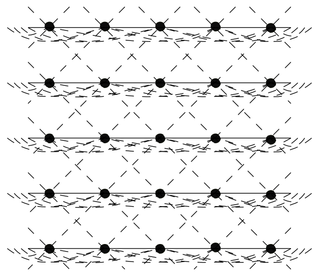


Fig. 31. $G\widehat{\otimes}^K H$

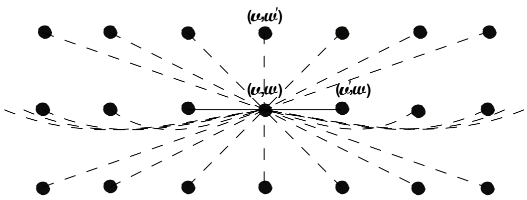


Fig. 32. adjacency at a vertex in $G \rhd^K H$

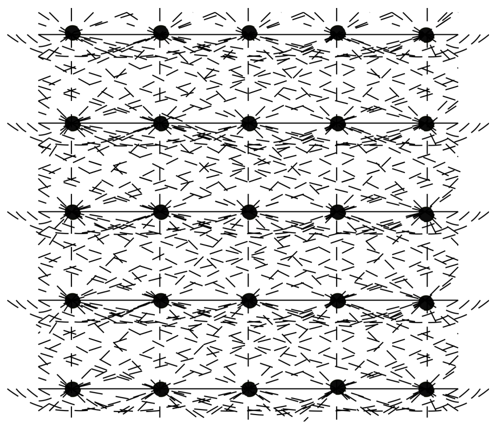


Fig. 33. $G \rhd^K H$

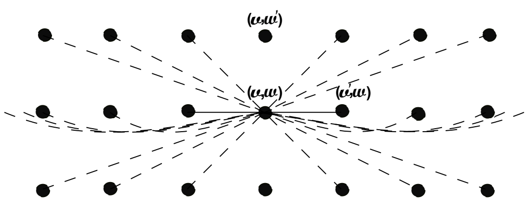


Fig. 34. adjacency at a vertex in $G \overset{c}{\square}{}^K H$

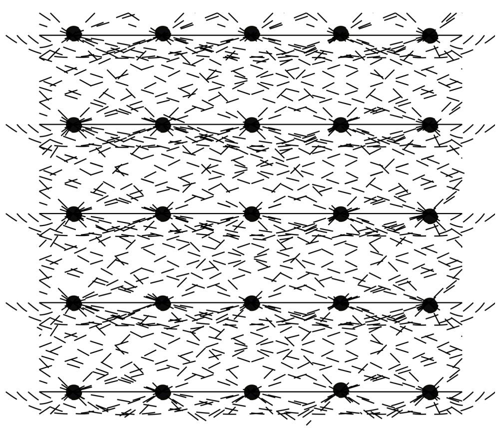


Fig. 35. $G \overset{c}{\square}{}^K H$

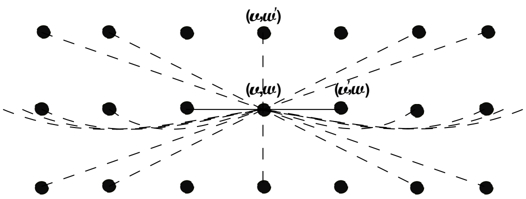


FIG. 36. adjacency at a vertex in $G \overset{c}{\otimes}{}^K H$

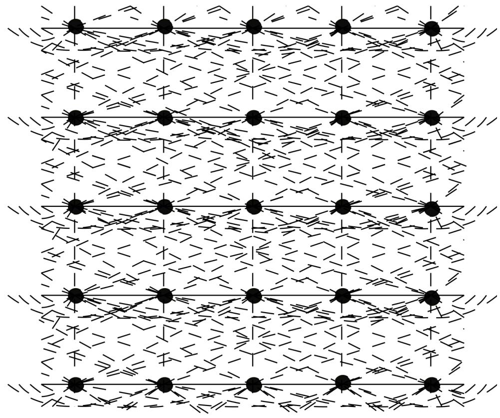


FIG. 37. $G \overset{c}{\otimes}{}^K H$

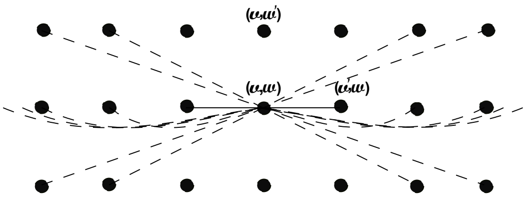


FIG. 38. adjacency at a vertex in $G \overset{c}{\boxtimes}{}^K H$

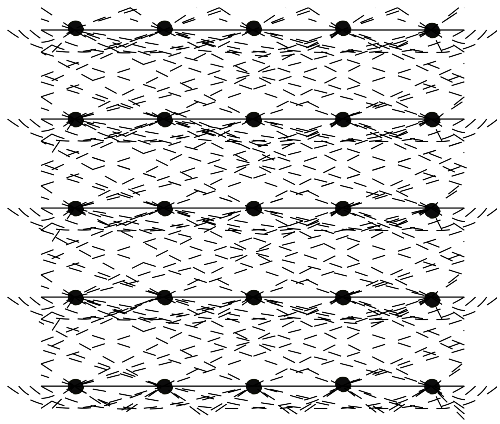


FIG. 39. $G \overset{c}{\boxtimes}{}^K H$

We here give an operation of getting Kähler graphs which is related with the product operation of lexicographic type. Let $H = (W, F)$ be an ordinary graph which may have hairs. We express the set W by $\{w_\alpha \mid \alpha \in A\}$. Let G_α $(\alpha \in A)$ be ordinary graphs. We define their *Kähler extension* $H^K(G_\alpha; \alpha \in A)$ in the following manner:

i) Its set of the vertices is the sum $\bigcup_{\alpha\in A} V_\alpha \times \{w_\alpha\}$;
ii) Two distinct vertices $(v, w_\alpha), (v', w_\beta) \in \bigcup_{\alpha\in A} V_\alpha \times \{w_\alpha\}$ are adjacent to each other by a principal edge if and only if $\alpha = \beta$ and $v \sim v'$ in G_α;
iii) Two distinct vertices $(v, w_\alpha), (v', w_\beta) \in \bigcup_{\alpha\in A} V_\alpha \times \{v_\alpha\}$ are adjacent to each other by an auxiliary edge if and only if $w_\alpha \sim w_\beta$ in H.

When all G_α are finite and H is locally finite, then $H^K(G_\alpha;\alpha\in A)$ is locally finite and its principal and auxiliary degrees are

$$d^{(p)}_{H^K(G_\alpha;\alpha\in A)}(v,w_\alpha)=d_{G_\alpha}(v),\quad d^{(a)}_{H^K(G_\alpha;\alpha\in A)}(v,w_\alpha)=\sum_{\beta:w_\beta\sim w_\alpha} n_{G_\beta}.$$

When all G_α are the same (i.e. $G_\alpha=G$), we have $H^K(G_\alpha;\alpha\in A)=G\rhd H$. When H is a complete graph of $n_H=2$ (hence $d_H=1$), we denote $H^K(G_1,G_2)$ also by $G_1\widehat{+}G_2$ and call it the *join* of G_1 and G_2.

When H is a finite complete graph, we sometimes write $H^K(G_1,\dots,G_{n_H})$ by $G_1\widehat{+}G_2\widehat{+}\cdots\widehat{+}G_{n_H}$. When all $G_1,\dots,G_{n_H}$ are complete ordinary graphs, then the graph $H^K(G_1,\dots,G_{n_H})$ is also complete as an ordinary graph.

EXAMPLE 2.14. If we take a 3-circuit G_1 and a 4-circuit G_2, then the graph $G_1\widehat{+}G_2$ is not a complete graph as an ordinary graph.

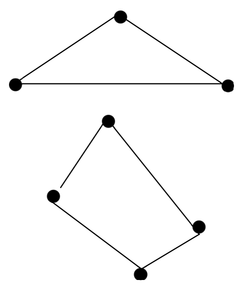


FIG. 40. $G_1\cup G_2$

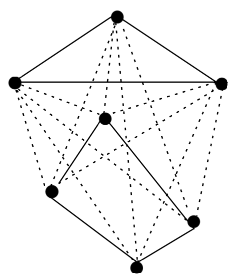


FIG. 41. $G_1\widehat{+}G_2$

EXAMPLE 2.15. If we take three complete graphs K_3, K_4 and K_5, the graph $K_3\widehat{+}K_4\widehat{+}K_5$ is like Fig. 43. We note

$$\begin{cases} d^{(p)}_{K_3\widehat{+}K_4\widehat{+}K_5}(v)=2, & d^{(a)}_{K_3\widehat{+}K_4\widehat{+}K_5}(v)=9, \quad \text{when } v\in K_3,\\ d^{(p)}_{K_3\widehat{+}K_4\widehat{+}K_5}(v)=3, & d^{(a)}_{K_3\widehat{+}K_4\widehat{+}K_5}(v)=8, \quad \text{when } v\in K_4,\\ d^{(p)}_{K_3\widehat{+}K_4\widehat{+}K_5}(v)=4, & d^{(a)}_{K_3\widehat{+}K_4\widehat{+}K_5}(v)=7, \quad \text{when } v\in K_5. \end{cases}$$

Obviously, we can do both the extending operation and complement-filling operation. When at least one of G_α $(\alpha\in A)$ is not complete, we define $H^K_c(G_\alpha;\alpha\in A)$ in the following manner:

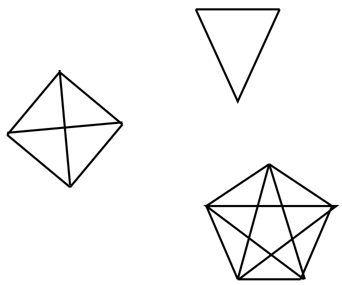


FIG. 42. $K_3 \cup K_4 \cup K_5$

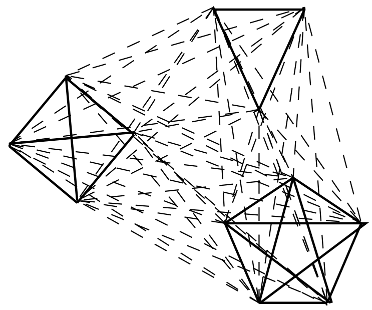


FIG. 43. $K_3 \widehat{+} K_4 \widehat{+} K_5$

i) Its set of the vertices is the sum $\bigcup_{\alpha \in A} V_\alpha \times \{w_\alpha\}$;

ii) Two distinct vertices $(v, w_\alpha), (v', w_\beta) \in \bigcup_{\alpha \in A} V_\alpha \times \{w_\alpha\}$ are adjacent to each other by a principal edge if and only if $\alpha = \beta$ and $v \sim v'$ in G_α;

iii) Two distinct vertices $(v, w_\alpha), (v', w_\beta) \in \bigcup_{\alpha \in A} V_\alpha \times \{v_\alpha\}$ are adjacent to each other by an auxiliary edge if and only if either $w_\alpha \sim w_\beta$ in H or $w_\alpha = w_\beta$ and $v \neq v'$, $v \not\sim v'$.

When all G_α are finite and H is locally finite, then $H^K_c(G_\alpha; \alpha \in A)$ is locally finite and its principal and auxiliary degrees are

$$d^{(p)}_{H^K_c(G_\alpha;\alpha\in A)}(v, w_\alpha) = d_{G_\alpha}(v), \quad d^{(p)}_{H^K_c(G_\alpha;\alpha\in A)}(v, w_\alpha) = n_{G_\alpha} - d_{G_\alpha} - 1 + \sum_{\beta : w_\beta \sim w_\alpha} n_{G_\beta}.$$

2.2. Product operations which are commutative. In the previous subsection, we constructed Kähler graphs whose principal graphs are unions of copies of given ordinary graphs. That is, for given graphs G and H, the principal graphs of their Kähler graphs of product type given in the previous subsection are unions of n_H-copies of G. We will explain the geometric meaning of Kähler graphs in §3.1, but if we say a bit on these Kähler graphs of product type, they show motions of charged particles which are just moving in the horizontal component G.

We should note that those seven Kähler graphs of product types are not connected. Moreover, those product operations are not commutative in general, that is

$$G \,\widehat{\square}\, H \neq H \,\widehat{\square}\, G, \quad G \widehat{\boxtimes} H \neq H \widehat{\boxtimes} G, \quad G \widehat{\otimes} H \neq H \widehat{\otimes} G, \quad G \triangleright H \neq H \triangleright G,$$

$$G \overset{c}{\square} H \neq H \overset{c}{\square} G, \quad G \overset{c}{\boxtimes} H \neq H \overset{c}{\boxtimes} G, \quad G \overset{c}{\otimes} H \neq H \overset{c}{\otimes} G$$

In this subsection we give some product operations which are commutative. These Kähler graphs show motions of charged particles which are moving both in the horizontal component G and in the vertical component H.

[1] Kähler graphs of Cartesian-tensor product type

Given two ordinary graphs $G = (V, E)$ and $H = (W, F)$ we define their Kähler graph of *Cartesian-tensor product type* $G \boxplus H$ as follows;

i) Its set of vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if either $v = v'$ and $w \sim w'$ in H or $w = w'$ and $v \sim v'$ in G;

iii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if $v \sim v'$ in G and $w \sim w'$ in H.

Example 2.16. If we take G and H as graphs of real lines, then their Kähler graph of Cartesian-tensor product type is like the following figures.

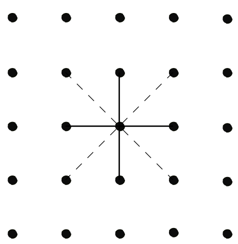


Fig. 44. adjacency at a vertex in $G \boxplus H$

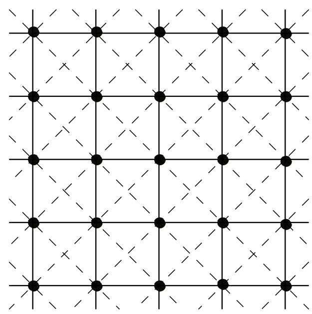


Fig. 45. $G \boxplus H$

When G and H are locally finite graphs, their Kähler graph of Cartesian-tensor product type is also locally finite. Its principal and auxiliary degrees are given as

$$d^{(p)}_{G \boxplus H}(v) = d_G(v) + d_H(w) \quad \text{and} \quad d^{(a)}_{G \boxplus H}(v) = d_G(v) d_H(w).$$

By definition, the operation of Cartesian-tensor product is commutative (i.e. $G \boxplus H = H \boxplus G$).

[2] Kähler graphs of Cartesain-complement product type

Let $G = (V, E)$ and $H = (W, F)$ be two ordinary graphs. We suppose the following:

(a) For each vertex $v \in V$, there exists at least one vertex which is different from v and is not adjacent to v in G;

(b) For each vertex $w \in W$, there exists at least one vertex which is different from w and is not adjacent to w in H.

We define their Kähler graph of *Cartesian-complement product type* $G \boxdot H$ as follows:

i) Its set of vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if either $v = v'$ and $w \sim w'$ in H or $w = w'$ and $v \sim v'$ in G;

iii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by an auxiliary edges if and only if either $v \neq v'$, $v \not\sim v'$ in G and $w \sim w'$ in H, or $w \neq w'$, $w \not\sim w'$ in H and $v \sim v'$ in G.

We note that if either the condition (a) or the condition (b) holds we can get a new Kähler graph of product type.

Example 2.17. If we take G and H as graphs of real lines, then their Kähler graph of Cartesian-complement product type is like the following figures.

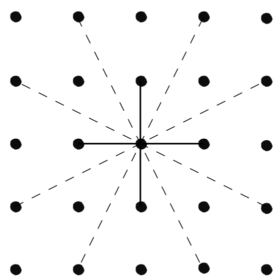


Fig. 46. adjacency at a vertex in $G \boxdot H$

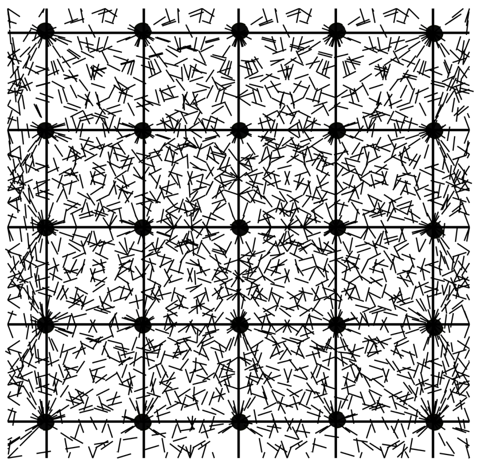


Fig. 47. $G \boxdot H$

When both G and H are finite, their Kähler graph of Cartesian-complement product type is finite. Its principal and auxiliary degrees are given as

$$d^{(p)}_{G \boxdot H}(v) = d_G(v) + d_H(w),$$

$$d^{(a)}_{G \boxdot H}(v) = d_G(v)\{n_H - d_H(w) - 1\} + d_H(w)\{n_G - d_G(v) - 1\}.$$

By definition, the operation of Cartesian-complement product is commutative (i.e. $G \boxdot H = H \boxdot G$).

[3] Kähler graphs of Cartesian-lexicographical product type

Given two ordinary graphs $G = (V, E)$ and $H = (W, F)$, we define their Kähler graph of *Cartesian-lexicographical product type* $G \Diamond H$ as follows:

i) Its set of vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if either $v = v'$ and $w \sim w'$ in H or $w = w'$ and $v \sim v'$ in G;

iii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if either $v \neq v'$ and $w \sim w'$ in H or $w \neq w'$ and $v \sim v'$ in G.

Example 2.18. If we take G and H as graphs of real lines, then their Kähler graph $G \Diamond H$ of Cartesian-lexicographical product type is like the following figure.

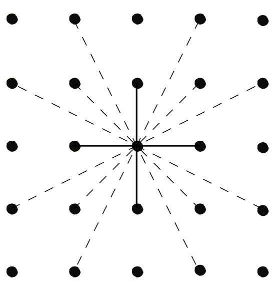


Fig. 48. adjacency at a vertex in $G \Diamond H$

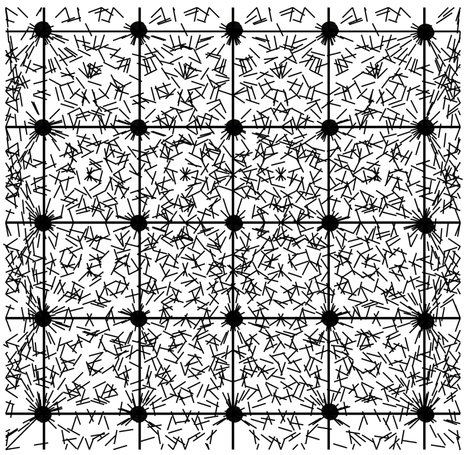


Fig. 49. $G \Diamond H$

When both G and H are finite, their Kähler graph of Cartesian-lexicographical product type is finite. Its principal and auxiliary degrees are given as

$$d^{(p)}_{G\Diamond H}(v) = d_G(v) + d_H(w) \quad \text{and} \quad d^{(a)}_{G\Diamond H}(v) = d_H(w)\{n_G - 1\} + d_G(v)\{n_H - 1\}.$$

By definition we see the operation of Cartesian-lexicographical product is commutative (i.e. $G\Diamond H = H\Diamond G$).

[4] Kähler graphs of strong-complement product type

Let $G = (V, E)$ and $H = (W, F)$ be two ordinary graphs. We suppose the following conditions which are the same as the conditions in the operation of Cartesian-complement product.

(a) For each vertex $v \in V$, there exists at least one vertex which is different from v and is not adjacent to v in G;

(b) For each vertex $w \in W$, there exists at least one vertex which is different from w and is not adjacent to w in H.

We define their Kähler graph of *strong-complement product type* $G \circledast H$ as follows:

i) Its set of vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if they satisfy one of the following conditions;

ii-a) $w = w'$ and $v \sim v'$ in G,

ii-b) $v = v'$ and $w \sim w'$ in H,

ii-c) $v \sim v'$ in G and $w \sim w'$ in H;

iii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if

ii-a) $v \neq v'$, $v \not\sim v'$ in G and $w \sim w'$ in H,

ii-b) $w \neq w'$, $w \not\sim w'$ in H and $v \sim v'$ in G.

Example 2.19. If we take G and H as graphs of real lines, then their Kähler graph of strong-complement product type is like the following figure.

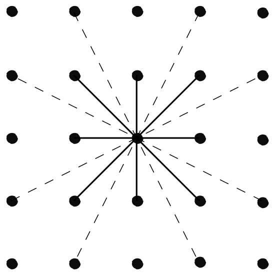


FIG. 50. adjacency at a vertex in $G \circledast H$

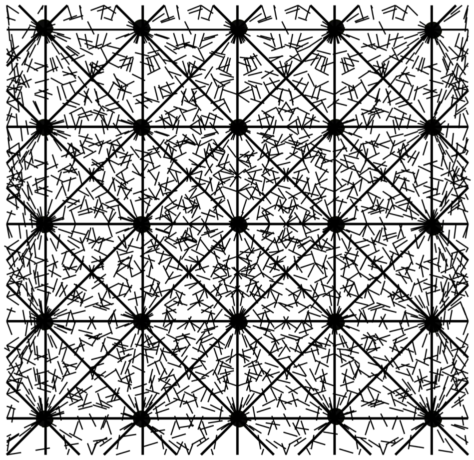


FIG. 51. $G \circledast H$

When G and H are finite, then $G \circledast H$ is also finite. Its principal and auxiliary degrees are given as

$$d^{(p)}_{H \circledast G} = d_G(v) + d_H(w) + d_G(v)d_H(w),$$
$$d^{(a)}_{H \circledast G} = d_G(v)\{n_H - d_H(w) - 1\} + d_H(w)\{n_G - d_G(v) - 1\}.$$

By definition we see that this strong-complement product operation is commutative (i.e. $G \circledast H = H \circledast G$).

[5] Kähler graphs of complement-tensor product type

Let $G = (V, E)$ and $H = (W, F)$ be two ordinary graphs. We suppose the following conditions which are the same as the conditions in the operations of Cartesian-complement product and of strong-complement product.

(a) For each vertex $v \in V$, there exists at least one vertex which is different from v and is not adjacent to v in G;

(b) For each vertex $w \in W$, there exists at least one vertex which is different from w and is not adjacent to w in H.

We define their Kähler graph of *complement-tensor product type* $G \spadesuit H$ as follows;

i) Its set of vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if either $v \sim v'$ in G and $w \not\sim w'$ in H, or $v \not\sim v'$ in G and $w \sim w'$ in H;

iii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by an auxiliary edges if and only if $v \sim v'$ in G and $w \sim w'$ in H.

Example 2.20. If we take G and H as the graphs of real lines, then their Kähler graph of complement-tensor product type is like the following figure.

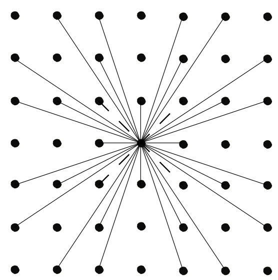


Fig. 52. adjacency at a vertex in $G\spadesuit H$

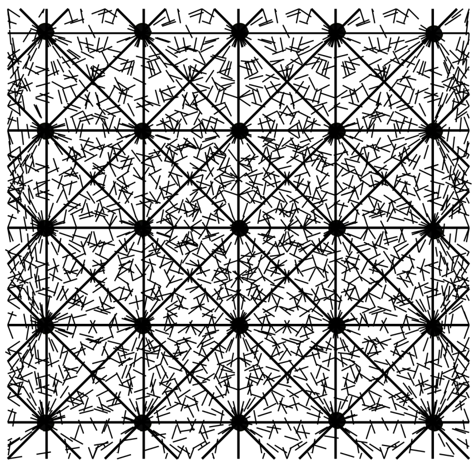


Fig. 53. $G\spadesuit H$

When G and H are finite graphs, then their Kähler graph $G\spadesuit H$ of complement-tensor product type is also finite. Its principal and auxiliary degrees are given as

$$d^{(p)}_{H\spadesuit G} = d_G(v)\{n_H - d_H(w)\} + d_H(w)\{n_G - d_G(v)\} \quad \text{and} \quad d^{(a)}_{H\spadesuit G} = d_G(v)d_H(w).$$

By definition we see that the complement-tensor product operation is commutative (i.e. $G\spadesuit H = H\spadesuit G$).

[6] Kähler graphs of tensor-complement product type

Let $G = (V, E)$ and $H = (W, F)$ be two ordinary graphs. We suppose the following conditions as usual.

(a) For each vertex $v \in V$, there exists at least one vertex which is different from v and is not adjacent to v in G;

(b) For each vertex $w \in W$, there exists at least one vertex which is different from w and is not adjacent to w in H.

We define their Kähler graph of tensor-complement product type $G\clubsuit H$ as follows;

i) Its set of vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by an principal edge if and only if $v \sim v'$ in G and $w \sim w'$ in H;

iii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if either $w \neq w'$, $w \not\sim w'$ in H and $v \sim v'$ in G, or $v \neq v'$, $v \not\sim v'$ in G and $w \sim w'$ in H.

EXAMPLE 2.21. If we take G and H as graphs of real lines, then their Kähler graph of tensor-complement product type is like the following figure.

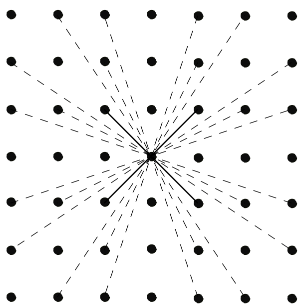


FIG. 54. adjacency at a vertex in $G\clubsuit H$

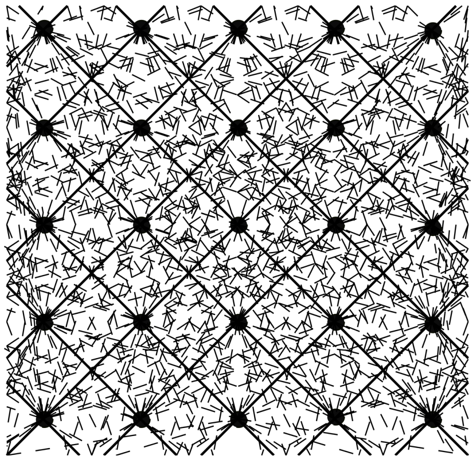


FIG. 55. $G\clubsuit H$

When G and H are finite graphs, then their Kähler graph $G\clubsuit H$ of tensor-complement product type is also finite. Its principal and auxiliary degrees are given as

$$d^{(p)}_{H\clubsuit G} = d_G(v)d_H(w) \quad \text{and} \quad d^{(a)}_{H\clubsuit G} = d_G(v)\{n_H - d_H(w) - 1\} + d_H(w)\{n_G - d_G(v) - 1\}.$$

By definition we see that the complement-tensor product operation is commutative (i.e. $G\clubsuit H = H\clubsuit G$).

For a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ we set $F^{(p)} = E^{(a)}, F^{(a)} = E^{(p)}$ and $G^* = (V, F^{(p)} \cup F^{(a)})$. We call G^* the *dual* Kähler graph of G. By taking the duals of $G \boxplus H$, $G \boxdot H$, $G \Diamond H$, $G \divideontimes H$ and $G \clubsuit H$ we get other Kähler graphs of product type by commutative operations.

Proposition 2.1. *If G and H are connected, then the principal graphs of their Kähler graphs of product type $G \boxplus H$, $G \boxdot H$, $G \Diamond H$, $G \divideontimes H$ are also connected.*

Proof. We take two distinct vertices (v, w) and (v', w') in the Kähler graph of product type in the assertion. Since G is connected, if $v \neq v'$ we have a path γ joining v and v' $(o(\gamma) = v$ and $t(\gamma) = v')$. Similarly as H is connected, if $w \neq w'$ we have a path σ joining w and w' $(o(\sigma) = w$ and $t(\sigma) = w')$. If we denote $\gamma = (v_0, \dots, v_n$ and $\sigma = (w_0, \dots, w_m)$, then the curve $\hat{\gamma} \cdot \hat{\sigma}$ with $\hat{\gamma} = \big((v_0, w), \dots, (v_n, w)\big)$ and $\hat{\sigma} = \big((v', w_0), \dots, (v', w_m)\big)$ joins (v, w) and (v', w'). When either $v = v'$ or $w = w'$, the curve $\hat{\sigma}$ or the curve $\hat{\gamma}$ joins (v, w) and (v', w'). □

Here, we note that we can do the product operations and the complement-filling operation in the same time. Given two ordinary graphs $G = (V, E)$ and $H = (W, F)$ we define a Kähler graph $G \boxplus^{\circledR} H$ as follows;

i) Its set of the vertices is the product $V \times W$ of their sets of vertices;

ii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if either $v = v'$ and $w \sim w'$ in H or $w = w'$ and $v \sim v'$ in G;

iii) Two distinct vertices (v, w), $(v', w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if they satisfy one of the following conditions;

iii-a) $v \sim v'$ in G and $w \sim w'$ in H,

iii-b) $v = v'$, $w \neq w'$ and $w \not\sim w'$ in H,

iii-c) $w = w'$, $v \neq v'$ and $v \not\sim v'$ in G.

We call this graph a Kähler graph of commutatively complement-filled Cartesian-tensor product type. We note that both G and H are complete graphs we have

$G \boxplus^{\textcircled{K}} H = G \boxplus H$. We can obtain $G \boxplus^{\textcircled{K}} H$ from $G \boxplus H$ by adding auxiliary edges according to the rule that

[**rule** $\textcircled{K}$]: $(v, w) \sim_a (v', w')$ if either $v \neq v'$, $v \not\sim v'$ in G and $w = w'$,
or $v = v'$, $w \neq w'$ and $w \not\sim w'$ in H.

Similarly, by using other Kähler graphs of product type and by adding auxiliary edges according to [rule $\textcircled{K}$], we get five Kähler graphs $G \boxdot^{\textcircled{K}} H$, $G\Diamond^{\textcircled{K}}H$, $G \divideontimes^{\textcircled{K}} H$, $G\spadesuit^{\textcircled{K}}H$ and $G\clubsuit^{\textcircled{K}}H$. When G and H are finite graphs, then these six Kähler graphs are finite. Their principal and the auxiliary degrees are given as

$$d^{(p)}_{G\boxplus^{\textcircled{K}}H}(v,w) = d^{(p)}_{G\boxdot^{\textcircled{K}}H}(v,w) = d^{(p)}_{G\Diamond^{\textcircled{K}}H}(v,w) = d_G(v) + d_H(w),$$

$$d^{(p)}_{G\divideontimes^{\textcircled{K}}H}(v,w) = d_G(v) + d_H(w) + d_G(v)d_H(w),$$

$$d^{(p)}_{G\spadesuit^{\textcircled{K}}H}(v,w) = d_G(v)\{n_H - d_H(w)\} + d_H(w)\{n_G - d_G(v)\},$$

$$d^{(p)}_{G\clubsuit^{\textcircled{K}}H}(v,w) = d_G(v)d_H(w)$$

and

$$d^{(a)}_{G\boxplus^{\textcircled{K}}H}(v,w) = d_G(v)d_H(w) + n_G + n_H - d_G(v) - d_H(w) - 2,$$

$$d^{(a)}_{G\boxdot^{\textcircled{K}}H}(v,w) = (d_G(v)+1)\{n_H - d_H(w) - 1\} + (d_H(w)+1)\{n_G - d_G(v) - 1\},$$

$$d^{(a)}_{G\Diamond^{\textcircled{K}}H}(v,w) = n_G(d_H(w)+1) + n_H(d_G(v)+1) - 2\{d_G(v) + d_H(w) + 1\},$$

$$d^{(a)}_{G\divideontimes^{\textcircled{K}}H}(v,w) = (d_G(v)+1)\{n_H - d_H(w) - 1\} + (d_H(w)+1)\{n_G - d_G(v) - 1\},$$

$$d^{(a)}_{G\spadesuit^{\textcircled{K}}H}(v,w) = n_G + n_H + d_G(v)d_H(w) - d_G(v) - d_H(w) - 2,$$

$$d^{(a)}_{G\clubsuit^{\textcircled{K}}H}(v,w) = n_G(d_H+1) + n_H(d_G+1) - 2(d_G+1)(d_H+1).$$

By definition, it is clear that these operations are commutative:

$$G \boxdot^{\textcircled{K}} H = H \boxdot^{\textcircled{K}} G, \quad G\Diamond^{\textcircled{K}}H = H\Diamond^{\textcircled{K}}G, \quad G \divideontimes^{\textcircled{K}} H = H \divideontimes^{\textcircled{K}} G,$$

$$G\spadesuit^{\textcircled{K}}H = H\spadesuit^{\textcircled{K}}G, \qquad G\clubsuit^{\textcircled{K}}H = H\clubsuit^{\textcircled{K}}G$$

3. Vertex-transitive Kähler graphs

In this section we give a condition that we can construct a "symmetric" Kähler graph of given cardinality of the set of vertices. Here, the word "symmetric" is vague. We shall explain this later, and at first we study regular Kähler graphs.

3.1. A condition on regular Kähler graphs. We shall start by considering experimentally the situation of small cardinality of the set of vertices. Let $G = (V, E^{(p)} \cup E^{(a)})$ be a Kähler graph.

(1) If $n_G = 1$, as we suppose it does not have loops, it is a graph of an isolated point and does not have edges, hence it is not a Kähler graph.

(2) If $n_G = 2$, as we suppose it does not have loops and multiple edges (i.e. simple), it is either a graph of two isolated points or a graph of an edge and its end points, hence it is not a Kähler graph.

(3) If $n_G = 3$, as it is a simple graph, the degree at each vertex is less than three. Thus we can not construct a Kähler graph of $n_G = 3$ by the condition $d^{(p)}(v) \geq 2, d^{(a)}(v) \geq 2$. Even if we weaken the condition on degrees to $d^{(p)}(v) \geq 1, d^{(a)}(v) \geq 1$, we need at least one pair of multiple edges. (see Fig. 56)

(4) When $n_G = 4$, we can not construct Kähler graphs by the condition on degrees. If we weaken the condition on degrees to $d^{(p)}(v) \geq 1, d^{(a)}(v) \geq 1$, we get a graph of constant degrees $d_G^{(p)} = d_G^{(a)} = 1$ (see Fig. 57).

If we allow us to use loops an multiple edges, we have "extended" Kähler graphs like Figs. 58, 59 .

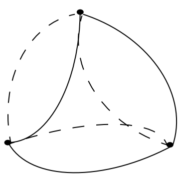


FIG. 56

FIG. 57

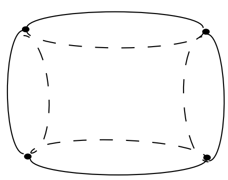


FIG. 58

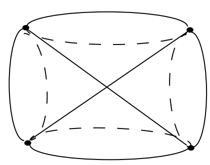


FIG. 59

Under the above study we give a condition on the cardinality of the set of vertices and the principal and the auxiliary degrees of a regular Kähler graph.

Proposition 2.2. *If* $G = (V, E^{(p)} \cup E^{(a)})$ *is a finite Kähler graph, then its cardinality* n_G *of the set of vertices, its principal degree* $d_G^{(p)}$ *and its auxiliary degree* $d_G^{(a)}$ *satisfy the following conditions*:

1) $n_G \geq 5$;

2) $d_G^{(p)}(v) \geq 2,\ d_G^{(a)}(v) \geq 2,\ d_G^{(p)}(v) + d_G^{(a)}(v) \leq n_G - 1$;

Moreover, if G *is regular, they additionally satisfy the following condition*:

(3) *When* n_G *is odd, both* $d_G^{(p)}$ *and* $d_G^{(a)}$ *are even.*

Proof. Since G is simple, the total degree $d_G(v) = d_G^{(p)}(v) + d_G^{(a)}(v)$ is less than n_G. Hence the second condition comes from the definition of Kähler graphs. In particular, we have $n_G \geq d_G^{(p)}(v) + d_G^{(a)}(v) + 1 \geq 5$.

When G is regular, by hand shaking lemma (Proposition 1.2), the cardinalities of the sets of principal and auxiliary edges satisfy $2\sharp E^{(p)} = n_G d_G^{(p)}$ and $2\sharp E^{(a)} = n_G d_G^{(a)}$. We hence get the third condition. □

In this section we show the converse of this proposition.

3.2. Kähler graph isomorphisms. Though the regularity condition shows some "symmetric" property of a Kähler graph, it is a very weak condition. We hence introduce another notion which shows more on "symmetry" of Kähler graphs. Let $G_1 = (V_1, E_1^{(p)} \cup E_1^{(a)})$, $G_2 = (V_2, E_2^{(p)} + E_2^{(a)})$ be two Kähler graphs. A map $f : V_1 \to V_2$ is said to be a *homomorphism* of G_1 to G_2 if it induces homomorphisms between principal graphs and between auxiliary graphs. That is, if two vertices $v, w \in V$ satisfy $v \sim_p w$ in G_1 then $f(v) \sim_p f(w)$ in G_2, and if they satisfy $v \sim_a w$ in G_1 then $f(v) \sim_a f(w)$ in G_2. We shall denote a homomorphism between two Kähler graphs G_1 and G_2 as $f : G_1 \to G_2$. When f is a bijective homomorphism and its inverse $f^{-1} : V_2 \to V_1$ is also a homomorphism between Kähler graphs, we call it an *isomorphism* of a Kähler graph.

Lemma 2.1. *Let $f : G_1 \to G_2$ be an isomorphism between locally finite Kähler graphs. For each vertex $v \in V_1$ we have $d^{(p)}_{G_2}\big(f(v)\big) = d^{(p)}_{G_1}(v)$ and $d^{(a)}_{G_2}\big(f(v)\big) = d^{(a)}_{G_1}(v)$.*

Proof. We denote as $G_1 = (V_1, E_1^{(p)} \cup E_1^{(a)})$ and $G_2 = (V_2, E_2^{(p)} \cup E_2^{(a)})$. For $v \in V_1$ we take all vertices $v_1, \dots, v_{d^{(p)}_{G_1}(v)} \in V_1$ which are principally adjacent to v (i.e. $v_j \sim_p v$), and all vertices $v'_1, \dots, v'_{d^{(a)}_{G_1}(v)} \in V_1$ which are auxiliary adjacent to v (i.e. $v'_\ell \sim_a v$). Since f is a homomorphism, we have $f(v_j) \sim_p f(v)$ and $f(v'_\ell) \sim_a f(v)$ in G_2. As f is a bijection these $f(v_1), \dots, f\big(v_{d^{(p)}_{G_1}(v)}\big), f(v'_1), \dots, f\big(v'_{d^{(a)}_{G_1}(v)}\big)$ are mutually different. Hence we have $d^{(p)}_{G_1}(v) \le d^{(p)}_{G_2}\big(f(v)\big)$ and $d^{(a)}_{G_1}(v) \le d^{(a)}_{G_2}\big(f(v)\big)$. Since f^{-1} is also a bijective homomorphism, by the same argument we have $d^{(p)}_{G_1}(v) \ge d^{(p)}_{G_2}\big(f(v)\big)$ and $d^{(a)}_{G_1}(v) \ge d^{(a)}_{G_2}\big(f(v)\big)$ because $f^{-1}\big(f(v)\big) = v$. Thus we get the conclusion. □

We call a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ *vertex-transitive* if for arbitrary distinct vertices $v, w \in V$ there is an isomorphism (automorphism) $f : V \to V$ of G satisfying $f(v) = w$. By Lemma 2.1, we find that a vertex-transitive Kähler graph is regular.

We have many vertex-transitive Kähler graphs. But regular graphs are not necessarily vertex-transitive.

Example 2.22. A Kähler graph of m-dimensional complex Euclidean lattice is vertex-transitive.

As a matter of fact, we take arbitrary distinct vertices $\boldsymbol{z}, \boldsymbol{z}' \in \mathbb{Z}^{2m} \subset \mathbb{R}^{2m} \cong \mathbb{C}^m$. We define a bijection $\varphi = \varphi_{\boldsymbol{z},\boldsymbol{z}'}$ as a translation $\varphi(\boldsymbol{w}) = \boldsymbol{w} + (\boldsymbol{z}' - \boldsymbol{z})$. Clearly, we have $\varphi(\boldsymbol{z}) = \boldsymbol{z}'$. Suppose $\boldsymbol{w} \sim_p \boldsymbol{w}'$. We denote as

$$\boldsymbol{z} = (a_1 + \sqrt{-1}b_1, \dots, a_m + \sqrt{-1}b_m), \quad \boldsymbol{z}' = (a'_1 + \sqrt{-1}b'_1, \dots, a'_m + \sqrt{-1}b'_m),$$
$$\boldsymbol{w} = (c_1 + \sqrt{-1}d_1, \dots, c_m + \sqrt{-1}d_m), \quad \boldsymbol{w}' = (c'_1 + \sqrt{-1}d'_1, \dots, c'_m + \sqrt{-1}d'_m).$$

Then, there is i_0 satisfying that $c'_{i_0} = c_{i_0} \pm 1$, $c'_i = c_i$ for $i \ne i_0$ and $d'_i = d_i$ for all i. As we have

$$\varphi(\boldsymbol{w}) = \big((c_1 + a'_1 - a_1) + \sqrt{-1}(d_1 + b'_1 - b_1), \dots, (c_m + a'_m - a_m) + \sqrt{-1}(d_m + b'_m - b_m)\big),$$
$$\varphi(\boldsymbol{w}') = \big((c'_1 + a'_1 - a_1) + \sqrt{-1}(d'_1 + b'_1 - b_1), \dots, (c'_m + a'_m - a_m) + \sqrt{-1}(d'_m + b'_m - b_m)\big),$$

we see $\varphi(\boldsymbol{w}) \sim_p \varphi(\boldsymbol{w}')$. Similarly, if $\boldsymbol{w} \sim_a \boldsymbol{w}'$, there is i_1 satisfying that $d'_{i_1} = d_{i_1} \pm 1$, $d'_i = d_i$ for $i \neq i_1$ and $c'_i = c_i$ for all i. Hence we have $\varphi(\boldsymbol{w}) \sim_a \varphi(\boldsymbol{w}')$, and find that φ is a homomorphism.

Since φ^{-1} is given by $\varphi^{-1}(\boldsymbol{w}) = \boldsymbol{w} + (\boldsymbol{z} - \boldsymbol{z}')$, this also is a homomorphism. Hence φ is an isomorphism. Thus we find the vertex-transitivity of a Kähler graph of m-dimensional complex lattice.

Proposition 2.3. *Every Cayley Kähler graph is vertex-transitive.*

Proof. Let $G = \bigl(\mathcal{G}, E(\mathcal{G}; \mathcal{S}^{(p)}) \cup E(\mathcal{G}; \mathcal{S}^{(a)})\bigr)$ be a Cayley Kähler graph. We take distinct two vertices $g, g' \in \mathcal{G}$ and define a map $\varphi_{g,g'} : \mathcal{G} \to \mathcal{G}$ by $\varphi_{g,g'}(x) = g'g^{-1}x$. As we have

$$\varphi_{g,g'}(x)^{-1}\varphi_{g,g'}(y) = \bigl(g'g^{-1}x\bigr)^{-1}\bigl(g'g^{-1}y\bigr) = x^{-1}g(g')^{-1}g'g^{-1}y = x^{-1}y,$$

we find that $x^{-1}y \in \mathcal{S}^{(p)}$ if and only if $\bigl(\varphi_{g,g'}(x)\bigr)^{-1}\varphi_{g,g'}(y) \in \mathcal{S}^{(p)}$ and that $x^{-1}y \in \mathcal{S}^{(a)}$ if and only if $\bigl(\varphi_{g,g'}(x)\bigr)^{-1}\varphi_{g,g'}(y) \in \mathcal{S}^{(a)}$. Therefore $\varphi_{g,g'}$ is an isomorphism. Since $\varphi_{g,g'}(g) = g'$ we find that G is vertex-transitive. □

Example 2.23. We consider a Kähler graph $G = (V, E)$ given in Fig. 60. That is, $V = \{v_0, \dots, v_6\}$ and

$$E^{(p)} = \Bigl\{\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_0\}\Bigr\},$$
$$E^{(a)} = \Bigl\{\{v_0, v_4\}, \{v_4, v_6\}, \{v_6, v_1\}, \{v_1, v_3\}, \{v_3, v_5\}, \{v_5, v_2\}, \{v_2, v_0\}\Bigr\}.$$

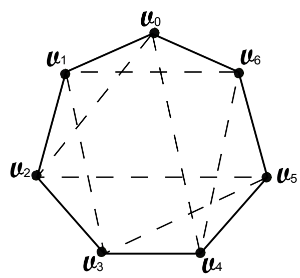


Fig. 60

Since both its principal and auxiliary graphs are 7-circuits, they are vertex transitive as ordinary graphs, in particular it is a regular Kähler graph. As isomorphisms of its principal graph are rotations $f_0 = Id, f_1, \dots, f_6$ which are given by $v_i \mapsto v_{i+j}$, we study how they map auxiliary edges. At a vertex v_0 we have auxiliary edges $\{v_0, v_2\}$ and $\{v_0, v_4\}$. Their differences between indices of vertices are 2 and 4. If we calculate in the same way we have

$$v_1 \mapsto 2, 5, \quad v_2 \mapsto 3, 5, \quad v_3 \mapsto 2, 5, \quad v_4 \mapsto 2, 3, \quad v_5 \mapsto 4, 5, \quad v_5 \mapsto 2, 5.$$

Hence we find that rotations do not preserves auxiliary edges. Therefore G is not vertex-transitive.

We here show the converse of Proposition 2.2.

THEOREM 2.1. *Let $N, d^{(p)}, d^{(a)}$ be positeve integers satisfying $N \geq 5$, $d^{(p)} \geq 2$, $d^{(a)} \geq 2$ and $d^{(p)} + d^{(a)} \leq N - 1$. Then there exists a vertex-transitive finite Kähler graph G satisfying $n_G = N$, $d_G^{(p)} = d^{(p)}$ and $d_G^{(a)} = d^{(a)}$ if and only if one of the following conditions holds*:

1) *N is odd and both $d^{(p)}, d^{(a)}$ are even,*
2) *N in even.*

PROOF. We shall show the assertion step by step. We take $V = \{v_0, v_1, \cdots, v_{N-1}\}$ We shall give principal and auxiliary edges by considering the indices of vertices by modulo N.

(1) The case that N is odd and both $d^{(p)}, d^{(a)}$ are even.

We denote $d^{(p)}, d^{(a)}$ as $d^{(p)} = 2d_1$ and $d^{(a)} = 2d_2$ with positive integers d_1, d_2. We define principal edges so that each vertex v_i is principally adjacent to vertices v_{i+j} with $j = \pm 1, \pm 2, \cdots, \pm d_1$, and define auxiliary edges so that it is auxiliary adjacent to vertices v_{i+j} with $j = \pm(d_1+1), \pm(d_1+2), \cdots, \pm(d_1+d_2)$. Since $d^{(p)}+d^{(a)} \leq N-1$, this graph does not have multiple edges.

We consider rotations $f_k : V \to V \quad (k = 1, 2, \cdots, N-1)$ which are given by $f_k(v_i) = v_{i+k}$. Then they are automorphisms of our Kähler graph $(V, E^{(p)} \cup E^{(a)})$. It is clear that f_k is a bijection. When $v_i \sim_p v_\ell$ then $|i - \ell| \leq d_1$. As $f_k(v_s) = v_{s+k}$ and

$f_k^{-1}(v_s) = v_{s-k}$, and $|(i+k)-(\ell+k)| = |i-\ell| = |(i-k)-(\ell-k)|$, we find that $f_k(v_i) \sim_p f_k(v_\ell)$ and $f_k^{-1}(v_i) \sim_p f_k^{-1}(v_\ell)$. Similarly, when $v_i \sim_a v_\ell$ then $d_1 < |i-\ell| \le d_1+d_2$. As $f_k(v_s) = v_{s+k}$ and $f_k^{-1}(v_s) = v_{s-k}$, and $|(i+k)-(\ell+k)| = |i-\ell| = |(i-k)-(\ell-k)|$, we find that $f_k(v_i) \sim_a f_k(v_\ell)$ and $f_k^{-1}(v_i) \sim_a f_k^{-1}(v_\ell)$. Thus we find that f_k is an isomorphism(see Fig. 61). Therefore our Kähler graph is vertex-transitive.

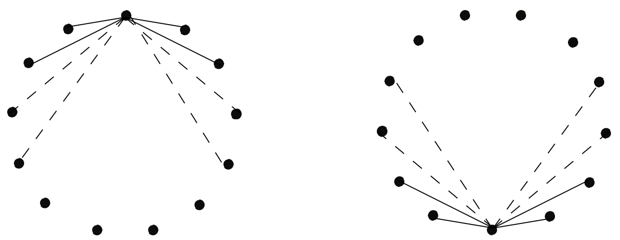


FIG. 61

(2) The case that N and $d^{(p)}$ are even and $d^{(a)}$ is odd.

We denote $N, d^{(p)}, d^{(a)}$ as $N = 2m, d^{(p)} = 2d_1$ and $d^{(a)} = 2d_2 + 1$ with positive integers m, d_1, d_2. We define principal edges so that each vertex v_i is principally adjacent to v_{i+j} for $j = \pm1, \pm2, \cdots, \pm d_1$, and define auxiliary edges so that it is auxiliary adjacent to v_{i+j} for $j = \pm(d_1+1), \pm(d_1+2), \cdots, \pm(d_1+d_2)$. By these, we have $2d_1$ principal edges and $2d_2$ auxiliary edges at each vertex. Since $N-1$ is odd we see $2(d_1+d_2) \le N-2 = 2m-2$, we can join v_i and its antipodal point v_{i+m} by an auxiliary edge (see Fig. 62). We then have $d_G^{(p)} = 2d_1$ and $d_G^{(a)} = 2d_2+1$ and G does not have multiple edges.

We take the rotations $f_k : V \to V \quad (k = 1, 2, \cdots, N-1)$. As $v_i \sim_p v_\ell$ if and only if $|i-\ell| \le d_1$, and as $|(i+k)-(\ell+k)| = |i-\ell| = |(i-k)-(\ell-k)|$, we find that $v_i \sim_p v_\ell$ if and only if $f_k(v_i) \sim_p f_k(v_\ell)$. Similarly, as $v_i \sim_a v_\ell$ if and only if $d_1 < |i-\ell| \le d_2$ or $|i-\ell| = m$, we find that $v_i \sim_a v_\ell$ if and only if $f_k(v_i) \sim_a f_k(v_\ell)$. Thus these rotations f_k $(k = 1, 2, \cdots, N-1)$ are automorphisms of our Kähler graph. We hence find that it is vertex-transitive.

(3) The case that N and $d^{(a)}$ are even and $d^{(p)}$ is odd.

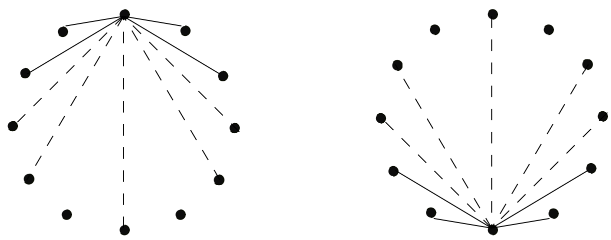


Fig. 62

If we change the roles of the principal and the auxiliary edges in the argument in the case of (2), we can obtain a desirable vertex-transitive Kähler graph. We here give our Kähler graph explicitly. We denote $N, d^{(p)}, d^{(a)}$ as $N = 2m, d^{(p)} = 2d_1 + 1$ and $d^{(a)} = 2d_2$ with positive integers m, d_1, d_2. We define principal edges so that each vertex v_i is principally adjacent to v_{i+j} for $j = \pm 1, \pm 2, \dots, \pm d_1$ and is principally adjacent to v_{i+m}, and define auxiliary edges so that each vertex v_i is auxiliary adjacent to v_{i+j} for $j = \pm(d_1+1), \pm(d_1+2), \dots, \pm(d_1+d_2)$. We then have $d_G^{(p)} = 2d_1 + 1$ and $d_G^{(a)} = 2d_2$ and G does not have multiple edges because $2(d_1 + d_2) \leq N - 2 = 2m - 2$. (see Fig. 63).

We take the rotations $f_k : V \to V \quad (k = 1, 2, \cdots, N-1)$. As $v_i \sim_p v_\ell$ if and only if $|i - \ell| \leq d_1$ or $|i - \ell| = m$, and as $|(i+k) - (\ell+k)| = |i - \ell| = |(i-k) - (\ell-k)|$, we find that $v_i \sim_p v_\ell$ if and only if $f_k(v_i) \sim_p f_k(v_\ell)$. Similarly, as $v_i \sim_a v_\ell$ if and only if $d_1 < |i - \ell| \leq d_2$, we find that $v_i \sim_a v_\ell$ if and only if $f_k(v_i) \sim_a f_k(v_\ell)$. Thus these rotations f_k $(k = 1, 2, \cdots, N-1)$ are automorphisms of our Kähler graph. We hence find that it is vertex-transitive.

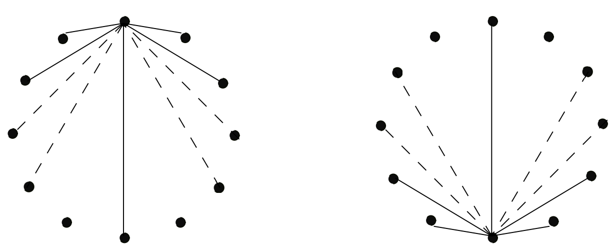


Fig. 63

(4) The case that N is even and both $d^{(p)}, d^{(a)}$ are odd.

We denote $N, d^{(p)}, d^{(a)}$ as $N = 2m,\ d^{(p)} = 2d_1 + 1$ and $d^{(a)} = 2d_2 + 1$ with positive integers m, d_1, d_2. First, we define principal edges so that $v_{2\ell-2}$ and $v_{2\ell-1}$ with $\ell = 1, 2, \cdots, m$ are principally adjacent to each other, and define auxiliary edges so that $v_{2\ell-1}$ and $v_{2\ell}$ are auxiliary adjacent to each other. Next we define principal edges so that each vertex v_i is principally adjacent to vertex v_{i+j} for $j = \pm 2, \pm 3, \ldots, \pm(d_1+1)$, and define auxiliary edges so that it is auxiliary adjacent to vertex v_{i+j} for $j = \pm(d_1+2), \pm(d_1+3), \ldots, \pm(d_1+d_2+1)$. By these we have $d_G^{(p)} = 2d_1+1$ and $d_G^{(a)} = 2d_2+1$. We note that the condition $d^{(p)} + d^{(a)} \le N-1$ guarantees that $2d_1+1+2d_2+1 \le 2m-1$. This shows $2(d_1+d_2) \le 2(m-1)-1$, hence leads us to $d_1+d_2 \le m-2$. Therefore, G does not have multiple edges (see Fig. 64). Moreover, there are no edges joining v_i and v_{i+m}.

We shall show that this Kähler graph is vertex transitive. First, we study transitivity for even $k = 2\hat{k}$. We take the rotation $f_k : V \to V$. As we have $f_k(v_{2\ell-2}) = v_{2(\ell+\hat{k})-2}$, $f_k(v_{2\ell-1}) = v_{2(\ell+\hat{k})-1}$, $f_k(v_{2\ell}) = v_{2(\ell+\hat{k})}$, we see $f_k(v_{2\ell-2}) \sim_p f_k(v_{2\ell-1})$ and $f_k(v_{2\ell-1}) \sim_a f_k(v_{2\ell})$. By a similar argument as in other cases we find that f_k is an isomorphism. (see Fig. 64)

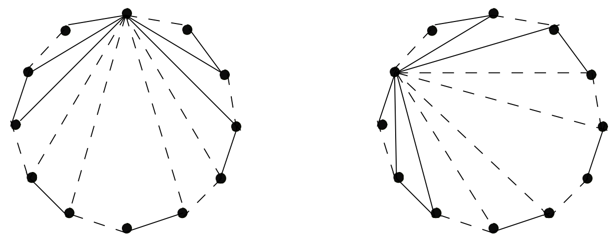


Fig. 64

Next we study transitivity for odd $k = 2\hat{k} - 1$. We define a map $g_k : V \to V$ by $g_k(v_i) = v_{-i+k}$ which is a composition of a reflection given by $v_i \mapsto v_{-i}$ and a rotation f_k. As we have $g_k(v_{2\ell-2}) = v_{2(\hat{k}-\ell+1)-1}$, $g_k(v_{(2\ell-1)}) = v_{2(\hat{k}-\ell+1)-2} = v_{2(\hat{k}-\ell)}$ and $g_k(v_{2\ell}) = v_{2(\hat{k}-\ell)-1}$, we see $g_k(v_{2\ell-2}) \sim_p g_k(v_{2\ell-1})$ and $g_k(v_{2\ell-1}) \sim_a g_k(v_{2\ell})$. Since the sets of principal and auxiliary edges which we secondary took are invariant under the action of reflection $v_i \mapsto v_{-i}$, we find that g_k is an isomorphism of our Kähler graph. We hence find that it is vertex-transitive. This completes the proof. $\square$

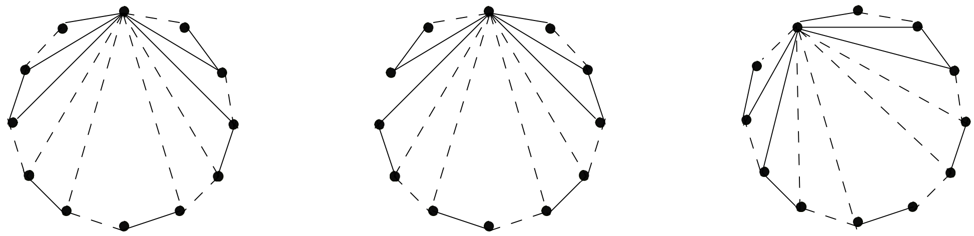


FIG. 65

3.3. Examples of vertex-transitive Kähler graphs. By Theorem 2.1, we see that there are many vertex-transitive Kähler graphs. We here give some more examples. A *Petersen graph* (V, E) is a graph of 10 vertices which is given as follows: We take a set $V = \{v_{1,0}, v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{2,0}, v_{2,1}, v_{2,2}, v_{2,3}, v_{2,4}\}$ of vertices, and set

$$E = \left\{ \begin{array}{l} \{v_{1,0}, v_{1,1}\}, \{v_{1,1}, v_{1,2}\}, \{v_{1,2}, v_{1,3}\}, \{v_{1,3}, v_{1,4}\}, \{v_{1,4}, v_{1,0}\}, \\ \{v_{2,0}, v_{2,2}\}, \{v_{2,2}, v_{2,4}\}, \{v_{2,4}, v_{2,1}\}, \{v_{2,1}, v_{2,3}\}, \{v_{2,3}, v_{2,0}\}, \\ \{v_{1,0}, v_{2,0}\}, \{v_{1,1}, v_{2,1}\}, \{v_{1,2}, v_{2,2}\}, \{v_{1,3}, v_{2,3}\}, \{v_{1,4}, v_{2,4}\} \end{array} \right\}.$$

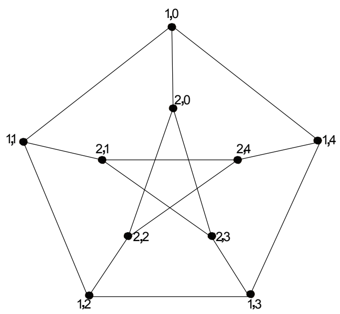


FIG. 66. Petersen graph

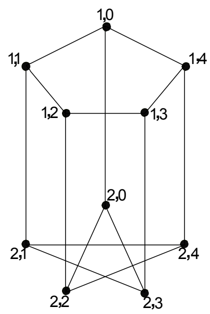


FIG. 67. 3-dim. representation

It is known that a Petersen graph is not a Cayly graph. For $j = 1, 2, 3, 4$ we define a map $f_j : V \to V$ by $f_j(v_{1,i}) = v_{1,i+j}$, $f_j(v_{2,i}) = v_{2,i+j}$, where we consider the second index by modulo 5. We define $g : V \to V$ by

$$g : \begin{array}{l} v_{1,0} \mapsto v_{2,0},\ v_{1,1} \mapsto v_{2,3},\ v_{1,2} \mapsto v_{2,1},\ v_{1,3} \mapsto v_{2,4},\ v_{1,4} \mapsto v_{2,2}, \\ v_{2,0} \mapsto v_{1,0},\ v_{2,2} \mapsto v_{1,1},\ v_{2,4} \mapsto v_{1,2},\ v_{2,1} \mapsto v_{1,3},\ v_{2,3} \mapsto v_{1,4} \end{array}.$$

The maps $f_1, \dots, f_4$ are rotations, and the map g is a reversing of upper and lower in the Fig. 67. Thus these 5 maps are isomorphisms of $G = (V, E)$ as an ordinary graph. Considering $f_j,\ g \circ f_j$ $(j = 1, 2, 3, 4)$ and g we find that a Petersen graph is a vertex-transitive graph.

Example 2.24. Let (V, E) be a Petersen graph. We put $E^{(p)} = E$. We define seven sets $E_j^{(a)}$ $(j = 1, \dots, 7)$ as

$$E_1^{(a)} = \left\{ \begin{array}{l} \{v_{1,0}, v_{1,2}\}, \{v_{1,2}, v_{1,4}\}, \{v_{1,4}, v_{1,1}\}, \{v_{1,1}, v_{1,3}\}, \{v_{1,3}, v_{1,0}\}, \\ \{v_{2,0}, v_{2,1}\}, \{v_{2,1}, v_{2,2}\}, \{v_{2,2}, v_{2,3}\}, \{v_{2,3}, v_{2,4}\}, \{v_{2,4}, v_{2,0}\} \end{array} \right\},$$

$$E_2^{(a)} = \left\{ \begin{array}{l} \{v_{1,0}, v_{2,1}\}, \{v_{1,0}, v_{2,2}\}, \{v_{1,0}, v_{2,3}\}, \{v_{1,0}, v_{2,4}\}, \\ \{v_{1,1}, v_{2,0}\}, \{v_{1,1}, v_{2,2}\}, \{v_{1,1}, v_{2,3}\}, \{v_{1,1}, v_{2,4}\}, \\ \{v_{1,2}, v_{2,0}\}, \{v_{1,2}, v_{2,1}\}, \{v_{1,2}, v_{2,3}\}, \{v_{1,2}, v_{2,4}\}, \\ \{v_{1,3}, v_{2,0}\}, \{v_{1,3}, v_{2,1}\}, \{v_{1,3}, v_{2,2}\}, \{v_{1,3}, v_{2,4}\}, \\ \{v_{1,4}, v_{2,0}\}, \{v_{1,4}, v_{2,1}\}, \{v_{1,4}, v_{2,2}\}, \{v_{1,4}, v_{2,3}\} \end{array} \right\},$$

$$E_3^{(a)} = \left\{ \begin{array}{l} \{v_{1,0}, v_{1,2}\}, \{v_{1,2}, v_{1,4}\}, \{v_{1,4}, v_{1,1}\}, \{v_{1,1}, v_{1,3}\}, \{v_{1,3}, v_{1,0}\}, \\ \{v_{2,0}, v_{2,1}\}, \{v_{2,1}, v_{2,2}\}, \{v_{2,2}, v_{2,3}\}, \{v_{2,3}, v_{2,4}\}, \{v_{2,4}, v_{2,0}\}, \\ \{v_{1,0}, v_{2,1}\}, \{v_{1,1}, v_{2,2}\}, \{v_{1,2}, v_{2,3}\}, \{v_{1,3}, v_{2,4}\}, \{v_{1,4}, v_{2,0}\} \end{array} \right\},$$

$$E_4^{(a)} = \big\{ \{v_{1,j}, v_{2,j+1}\}, \{v_{1,j}, v_{2,j+2}\} \bigm| j = 0, 1, 2, 3, 4 \big\},$$

$$E_5^{(a)} = \big\{ \{v_{1,j}, v_{2,j+1}\}, \{v_{1,j}, v_{2,j-1}\} \bigm| j = 0, 1, 2, 3, 4 \big\},$$

$$E_6^{(a)} = \big\{ \{v_{1,j}, v_{2,j+1}\}, \{v_{1,j}, v_{2,j-1}\}. \{v_{1,j}, v_{2,j+2}\} \bigm| j = 0, 1, 2, 3, 4 \big\},$$

$$E_7^{(a)} = \big\{ \{v_{1,j}, v_{2,j+1}\}, \{v_{1,j}, v_{2,j+2}\}. \{v_{1,j}, v_{2,j-2}\} \bigm| j = 0, 1, 2, 3, 4 \big\},$$

where in the last four sets the second indices of edges are considered by modulo 5. We then get thirteen Kähler graphs

$G_1 = (V, E \cup E_1^{(a)})$ (see Figs. 68, 80), $G_2 = (V, E \cup E_2^{(a)})$ (see Figs. 69, 81),

$G_3 = (V, E \cup E_3^{(a)})$ (see Figs. 71, 83), $G_4 = (V, E \cup E_5^{(a)})$ (see Fig. 72),

$G_5 = (V, E \cup E_6^{(a)})$ (see Figs. 73, 85), $G_6 = (V, E \cup E_7^{(a)})$ (see Fig. 74),

$G_7 = (V, E \cup E_8^{(a)})$ (see Fig. 75), $G_8 = (V, E \cup E_1^{(a)} \cup E_5^{(a)})$ (see Figs. 76, 84),

$G_9 = (V, E \cup E_1^{(a)} \cup E_6^{(a)})$ (see Fig. 77), $G_{10} = (V, E \cup E_1^{(a)} \cup E_7^{(a)})$ (see Fig. 78),

$G_{11} = (V, E \cup E_1^{(a)} \cup E_8^{(a)})$ (see Fig. 79),

and the complement-filled Kähler graph

$$G_{12} = \big(V, E^{(p)} \cup (E_1^{(a)} \cup E_2^{(a)})\big) \quad \text{(see Figs. 70, 82).}$$

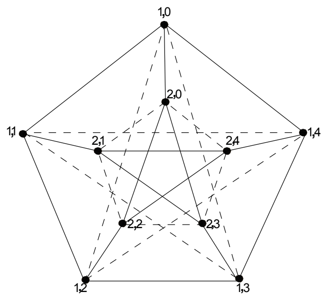


FIG. 68. G_1

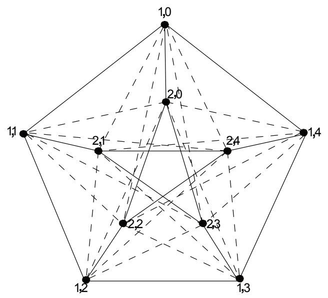


FIG. 69. G_2

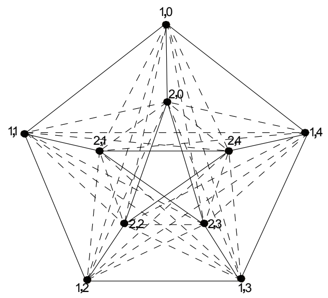


FIG. 70. G_{12}

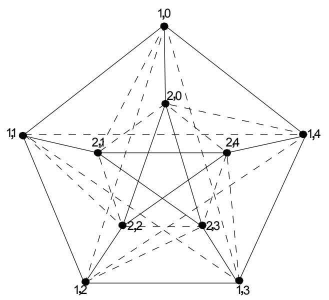


FIG. 71. G_3

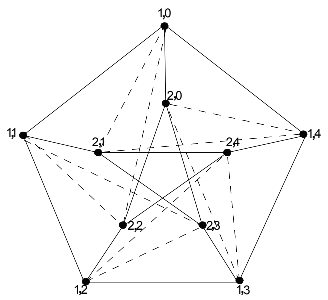


FIG. 72. G_4

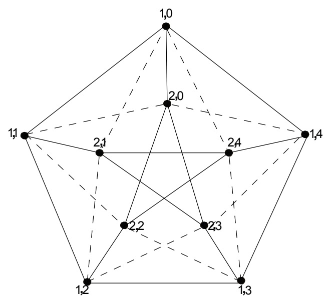


FIG. 73. G_5

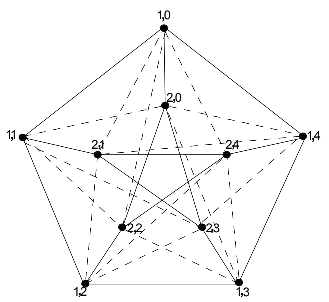


FIG. 74. G_6

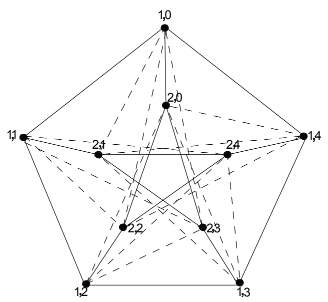


FIG. 75. G_7

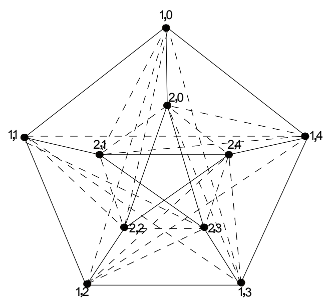


FIG. 76. G_8

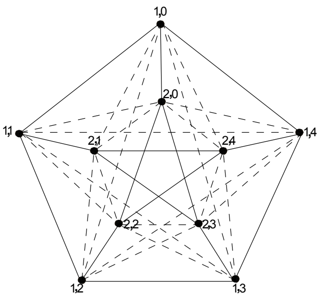


FIG. 77. G_9

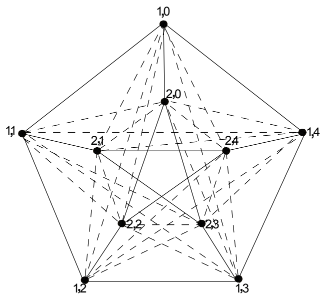


FIG. 78. G_{10}

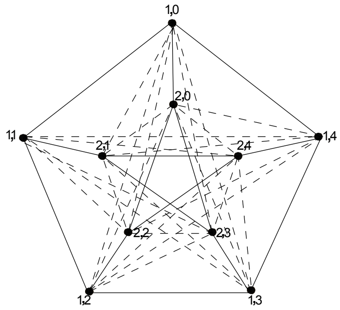


FIG. 79. G_{11}

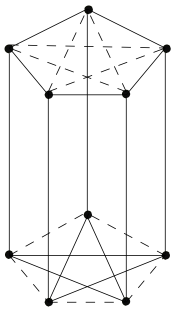


FIG. 80. G_1

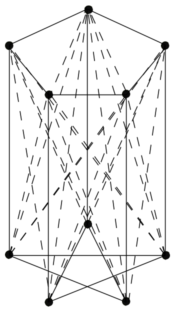


FIG. 81. G_2

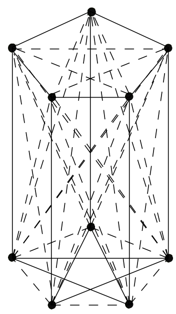


FIG. 82

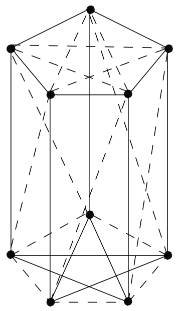


FIG. 83. G_3

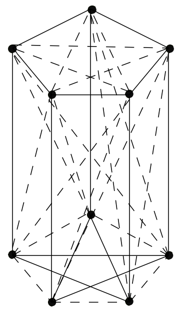


FIG. 84. G_8

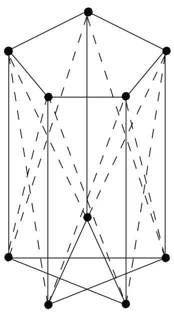


FIG. 85. G_5

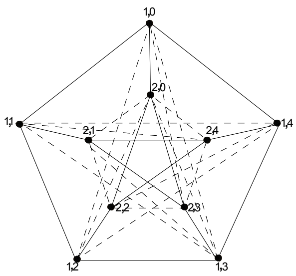


FIG. 86. G'_3

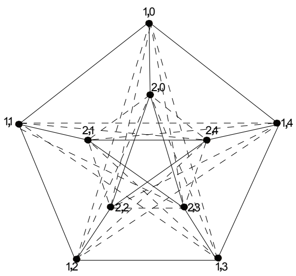


FIG. 87. G'_8

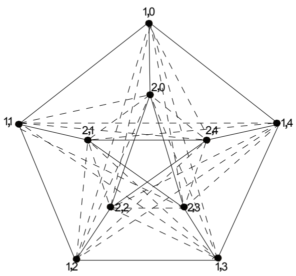


FIG. 88. G'_{10}

These graphs are regular and have

$$d^{(p)}_{G_j} = 3 \quad (j = 1, \dots, 12)$$

$$d^{(a)}_{G_1} = d^{(a)}_{G_4} = d^{(a)}_{G_5} = 2, \quad d^{(a)}_{G_3} = d^{(a)}_{G_6} = d^{(a)}_{G_7} = 3,$$

$$d^{(a)}_{G_2} = d^{(a)}_{G_8} = d^{(a)}_{G_9} = 4, \quad d^{(a)}_{G_{10}} = d^{(a)}_{G_{11}} = 5, \quad d^{(a)}_{G_{12}} = 6.$$

In particular, if these Kähler graphs have different auxiliary degrees they are not isomorphic to each other.

By Figs, 68, 69, 70, we find that G_1, G_2 hence G_{12} are vertex-transitive by the isomorphisms f_j, $g \circ f_j$ $(j = 1, 2, 3, 4)$ and g. But G_3, G_5 and G_9 are not vertex-transitive because g is not an isomorphism between Kähler graphs. As a matter of fact, $v_{1,0} \sim_a v_{2,1}$ but $g(v_{1,0}) = v_{2,0} \not\sim_a v_{1,3} = g(v_{2,1})$. Similarly, G_4, G_7, G_8 and G_{11} are not vertex-transitive because $v_{1,0} \sim_a v_{2,2}$ but $g(v_{1,0}) = v_{2,0} \not\sim_a v_{1,1} = g(v_{2,2})$. Also G_6 and G_{10} are not vertex-transitive because $v_{1,0} \sim_a v_{2,4}$ but $g(v_{1,0}) = v_{2,0} \not\sim_a v_{1,2} = g(v_{2,4})$. In, particular we find that G_1 is not isomorphic to G_4, G_5, and G_2 is not to G_8, G_9. Since g is not an isomorphism between G_4 and G_5, we find they are not isomorphic. Similarly G_6 and G_7 are not isomorphic to each other. By the same reason we see non two of G_8, G_9 are not isomorphic to each other, and nore are G_{10}, G_{11} are.

We note that if we set

$$\hat{E}_3^{(a)} = \big\{\{v_{1,j}, v_{2,j-2}\} \bigm| j = 0, 1, 2, 3, 4\big\},$$

$$\hat{E}_5^{(a)} = \big\{\{v_{1,j}, v_{2,j+2}\},\ \{v_{1,j}, v_{2,j-2}\} \bigm| j = 0, 1, 2, 3, 4\big\},$$

$$\hat{E}_7^{(a)} = \big\{\{v_{1,j}, v_{2,j+1}\},\ \{v_{1,j}, v_{2,j-1}\},\ \{v_{1,j}, v_{2,j-2}\} \bigm| j = 0, 1, 2, 3, 4\big\},$$

we have five Kähler graphs

$$G_3' = (V, E \cup E_1^{(a)} \cup \hat{E}_3^{(a)}) \ \text{(see Figs. 86)},$$

$$G_5' = (V, E \cup \hat{E}_5^{(a)}), \qquad G_8' = (V, E \cup E_1^{(a)} \cup \hat{E}_5^{(a)}) \ \text{(see Fig. 87)},$$

$$G_7' = (V, E \cup \hat{E}_7^{(a)}), \qquad G_{10}' = (V, E \cup E_1^{(a)} \cup \hat{E}_7^{(a)}) \ \text{(see Fig. 88)},$$

but they are isomorphic to $G_3, G_5, G_8, G_7, G_{10}$, respectively.

We call G_1 a *Kähler Petersen graph.* We call G_3 (or G_3') *Petersen Kähler graphs of first kind*, G_8, G_9 *Petersen Kähler graphs of second kind*, and G_{10}, G_{11} *Petersen Kähler graphs of third kind.*

Of course, we have more Kähler graphs obtained from a Petersen graph which are not "symmetric" (in particular which are not regular) by modifying our ways of constructing auxiliary edges. For example, we can set

$$E_{21}^{(a)} = \left\{ \begin{array}{l} \{v_{1,0}, v_{1,2}\}, \{v_{1,2}, v_{1,4}\}, \{v_{1,4}, v_{1,1}\}, \{v_{1,1}, v_{1,3}\}, \{v_{1,3}, v_{1,0}\}, \\ \{v_{1,0}, v_{2,1}\}, \{v_{1,1}, v_{2,2}\}, \{v_{1,2}, v_{2,3}\}, \{v_{1,3}, v_{2,4}\}, \{v_{1,4}, v_{2,0}\}, \\ \{v_{1,0}, v_{2,2}\}, \{v_{1,1}, v_{2,3}\}, \{v_{1,2}, v_{2,4}\}, \{v_{1,3}, v_{2,0}\}, \{v_{1,4}, v_{2,1}\} \end{array} \right\},$$

$$E_{22}^{(a)} = \left\{ \begin{array}{l} \{v_{2,0}, v_{2,1}\}, \{v_{2,1}, v_{2,2}\}, \{v_{2,2}, v_{2,3}\}, \{v_{2,3}, v_{2,4}\}, \{v_{2,4}, v_{2,0}\}, \\ \{v_{1,0}, v_{2,1}\}, \{v_{1,1}, v_{2,2}\}, \{v_{1,2}, v_{2,3}\}, \{v_{1,3}, v_{2,4}\}, \{v_{1,4}, v_{2,0}\}, \\ \{v_{1,0}, v_{2,2}\}, \{v_{1,1}, v_{2,3}\}, \{v_{1,2}, v_{2,4}\}, \{v_{1,3}, v_{2,0}\}, \{v_{1,4}, v_{2,1}\} \end{array} \right\}.$$

A *Heawood graph* is (V, E) is a graph of 14 vertices which is given as follows: We take a set $V = \{v_0, v_1, \cdots, v_{13}\}$ of vertices, and set

$$E = \left\{ \begin{array}{l} \{v_i, v_{i+1}\} \ (0 \le i \le 13), \\ \{v_0, v_5\},\ \{v_2, v_7\},\ \{v_4, v_9\},\ \{v_6, v_{11}\},\ \{v_8, v_{13}\},\ \{v_{10}, v_1\},\ \{v_{12}, v_3\} \end{array} \right\},$$

where we consider the index of vertices by modulo 14 (see Fig. 89). We define $f_j : V \to V$ by $f_{2k}(v_i) = v_{i+2k}$ and $f_{2k-1}(v_i) = v_{2k-1-i}$. That is, f_{2k} is a rotation and f_{2k-1} is a composition of a rotation and reversing $\iota : V \to V$ given by $\iota(v_i) = v_{-i}$. Then we see they are isomorphisms as an ordinary graph.

Example 2.25. Let (V, E) be a Heawood graph. We set $E^{(p)} = E$. If we define the sets of auxiliary edges by

$$E_1^{(a)} = \bigl\{\{v_i, v_{i+2}\} \bigm| 0 \le i \le 13\bigr\}, \qquad E_2^{(a)} = \bigl\{\{v_i, v_{i+3}\} \bigm| 0 \le i \le 13\bigr\},$$

$$E_3^{(a)} = \bigl\{\{v_i, v_{i+4}\} \bigm| 0 \le i \le 13\bigr\}, \qquad E_4^{(a)} = \bigl\{\{v_i, v_{i+6}\} \bigm| 0 \le i \le 13\bigr\},$$

$$E_5^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+7}\} \ (0 \le i \le 6), \\ \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\} \end{array} \right\},$$

$$E_6^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+7}\} \ (0 \le i \le 6), \\ \{v_1, v_4\},\ \{v_3, v_6\},\ \{v_5, v_8\},\ \{v_7, v_{10}\},\ \{v_9, v_{12}\},\ \{v_{11}, v_0\},\ \{v_{13}, v_2\} \end{array} \right\},$$

we obtain 6 vertex-transitive Kähler graphs $H_1, \ldots, H_6$ of auxiliary degree $d^{(a)} = 2$ (see Figs. 90, 91, 92, 93, 94, 95). As a matter of fact, it is clear by definitions of these Kähler graphs that f_{2k} $(k = 1, 2, 3, 4, 5, 6)$ are isomorphisms of Kähler graphs. By the map ι we have $\iota(v_i) = v_{-i}$ and $\iota(v_{i+a}) = v_{-i-a}$, Hence by putting $i' = -i - a$ we see $-i = i' + a$. Thus we find that f_{2k-1} are also isomorphisms.

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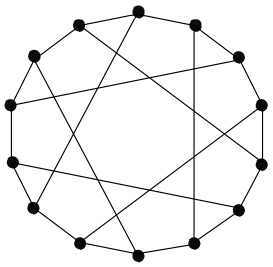

Fig. 89. original Heawood graph

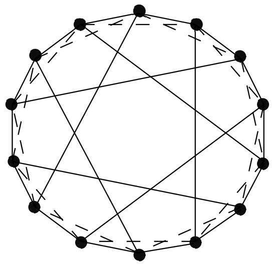

Fig. 90. H_1

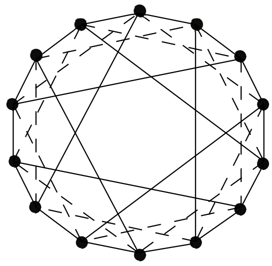

Fig. 91. H_2

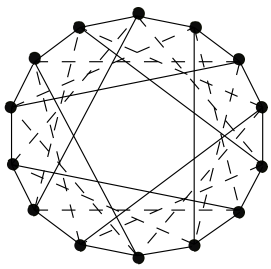

Fig. 92. H_3

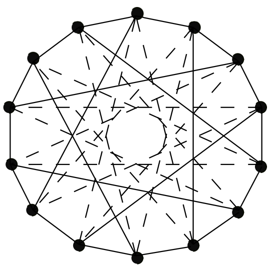

Fig. 93. H_4

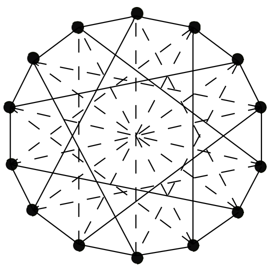

Fig. 94. H_5

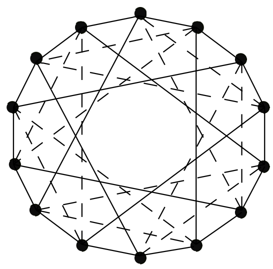

Fig. 95. H_6

Example 2.26. Let (V, E) be a Heawood graph. We set $E^{(p)} = E$. If we define the sets of auxiliary edges by

$$E_{21}^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+2}\}\ (0 \le i \le 13), \\ \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\} \end{array} \right\},$$

$$E_{22}^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+3}\}\ (0 \le i \le 13), \\ \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\} \end{array} \right\},$$

$$E_{23}^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+4}\}\ (0 \le i \le 13), \\ \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\} \end{array} \right\},$$

$$E_{24}^{(a)} = \left\{ \begin{array}{l} \big\{\{v_i, v_{i+6}\} \bigm| 0 \le i \le 13\big\}, \\ \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\} \end{array} \right\},$$

$$E_{25}^{(a)} = \big\{\{v_i, v_{i+2}\}\ (0 \le i \le 13),\ \ \{v_i, v_{i+7}\}\ (0 \le i \le 6)\big\},$$

$$E_{26}^{(a)} = \big\{\{v_i, v_{i+3}\}\ (0 \le i \le 13),\ \ \{v_i, v_{i+7}\}\ (0 \le i \le 6)\big\},$$

$$E_{27}^{(a)} = \big\{\{v_i, v_{i+4}\}\ (0 \le i \le 13),\ \ \{v_i, v_{i+7}\}\ (0 \le i \le 6)\big\},$$

$$E_{28}^{(a)} = \big\{\{v_i, v_{i+6}\}\ (0 \le i \le 13),\ \ \{v_i, v_{i+7}\}\ (0 \le i \le 6)\big\},$$

$$E_{31}^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+2}\} \ (0 \le i \le 13), \\ \{v_1, v_4\},\ \{v_3, v_6\},\ \{v_5, v_8\},\ \{v_7, v_{10}\},\ \{v_9, v_{12}\},\ \{v_{11}, v_0\},\ \{v_{13}, v_2\} \end{array} \right\},$$

$$E_{32}^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+4}\} \ (0 \le i \le 13), \\ \{v_1, v_4\},\ \{v_3, v_6\},\ \{v_5, v_8\},\ \{v_7, v_{10}\},\ \{v_9, v_{12}\},\ \{v_{11}, v_0\},\ \{v_{13}, v_2\} \end{array} \right\},$$

$$E_{33}^{(a)} = \left\{ \begin{array}{l} \big\{\{v_i, v_{i+6}\} \bigm| 0 \le i \le 13\big\}, \\ \{v_1, v_4\},\ \{v_3, v_6\},\ \{v_5, v_8\},\ \{v_7, v_{10}\},\ \{v_9, v_{12}\},\ \{v_{11}, v_0\},\ \{v_{13}, v_2\} \end{array} \right\},$$

$$E_{34}^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+7}\} \ (0 \le i \le 6), \\ \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\}, \\ \{v_1, v_4\},\ \{v_3, v_6\},\ \{v_5, v_8\},\ \{v_7, v_{10}\},\ \{v_9, v_{12}\},\ \{v_{11}, v_0\},\ \{v_{13}, v_2\} \end{array} \right\},$$

we get 12 kinds of Kähler graphs $H_{21}, \ldots, H_{28}, H_{31}, \ldots, H_{34}$ of auxiliary degree $d^{(a)} = 3$ (see Figs. 96, …, 107). In view of their construction we find they are vertex-transitive by f_j $(j = 1, \ldots, 13)$. We shall call $H_{22} = (V, E \cup E_{22}^{(a)})$ a *Heawood Kähler graph*, and $H_{21} = (V, E \cup E_{21}^{(a)}), H_{23} = (V, E \cup E_{23}^{(a)}), H_{24} = (V, E \cup E_{24}^{(a)})$ Kähler Heawood graphs.

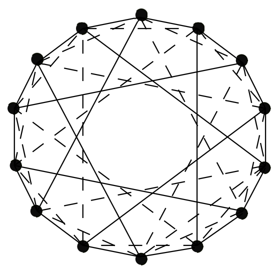

FIG. 96. H_{21}

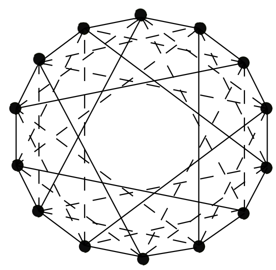

FIG. 97. H_{22}

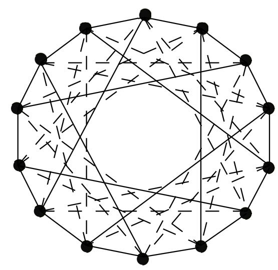

FIG. 98. H_{23}

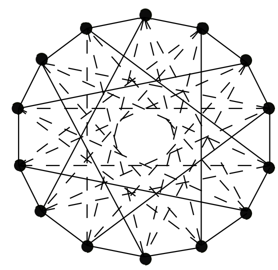

FIG. 99. H_{24}

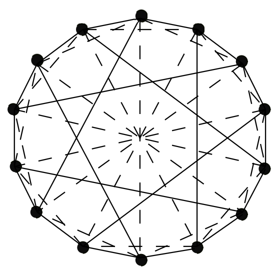

FIG. 100. H_{25}

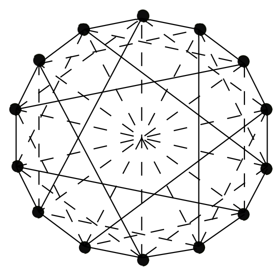

FIG. 101. H_{26}

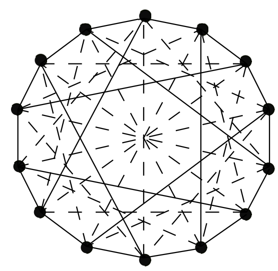

FIG. 102. H_{27}

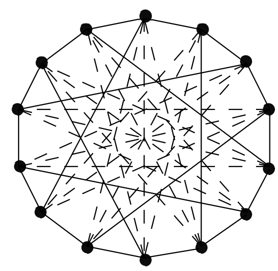

FIG. 103. H_{28}

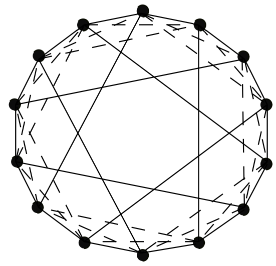

FIG. 104. H_{31}

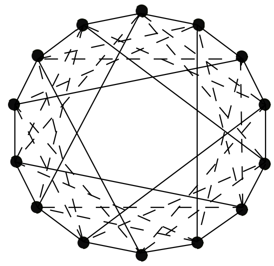

FIG. 105. H_{32}

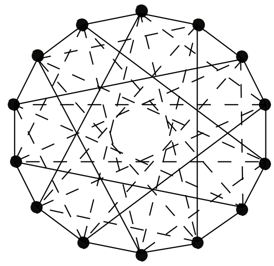

FIG. 106. H_{33}

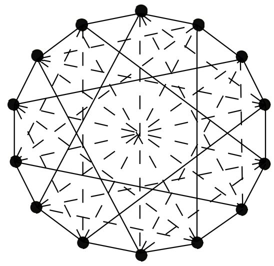

FIG. 107. H_{34}

EXAMPLE 2.27. Let (V, E) be a Heawood graph. We take a Kähler graph given by Fig. 90. If we add it auxiliary edges in the following way

$$E_{41}^{(a)} = \big\{ \{v_i, v_{i+2}\},\ \{v_i, v_{i+3}\} \ \big|\ 0 \le i \le 13 \big\},$$

$$E_{42}^{(a)} = \big\{ \{v_i, v_{i+2}\},\ \{v_i, v_{i+4}\} \ \big|\ 0 \le i \le 13 \big\},$$

$$E_{43}^{(a)} = \big\{ \{v_i, v_{i+2}\},\ \{v_i, v_{i+6}\} \ \big|\ 0 \le i \le 13 \big\},$$

$$E_{44}^{(a)} = \left\{ \begin{matrix} \{v_i, v_{i+2}\}\ (0 \le i \le 13), \quad \{v_i, v_{i+7}\}\ (0 \le i \le 6), \\ \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\} \end{matrix} \right\},$$

$$E_{45}^{(a)} = \left\{ \begin{matrix} \{v_i, v_{i+2}\}\ (0 \le i \le 13), \quad \{v_i, v_{i+7}\}\ (0 \le i \le 6), \\ \{v_1, v_4\},\ \{v_3, v_6\},\ \{v_5, v_8\},\ \{v_7, v_{10}\},\ \{v_9, v_{12}\},\ \{v_{11}, v_0\},\ \{v_{13}, v_2\} \end{matrix} \right\},$$

$$E_{46}^{(a)} = \left\{ \begin{matrix} \{v_i, v_{i+2}\}\ (0 \le i \le 13), \\ \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\}, \\ \{v_1, v_4\},\ \{v_3, v_6\},\ \{v_5, v_8\},\ \{v_7, v_{10}\},\ \{v_9, v_{12}\},\ \{v_{11}, v_0\},\ \{v_{13}, v_2\} \end{matrix} \right\},$$

then we get 6 kinds of Kähler graphs whose auxiliary degree is 4 (see Figs. 108, ..., 113).

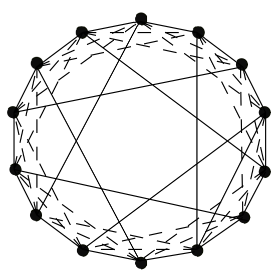

FIG. 108. H_{41}

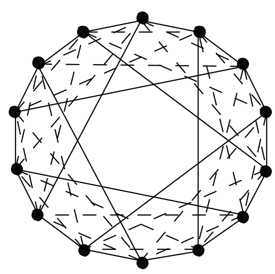

FIG. 109. H_{42}

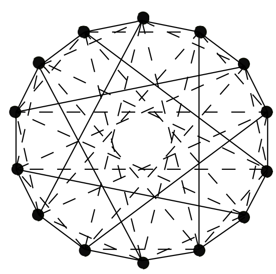

FIG. 110. H_{43}

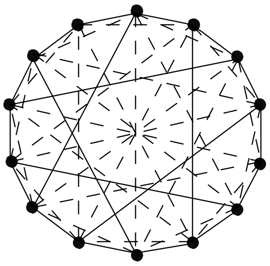

FIG. 111. H_{44}

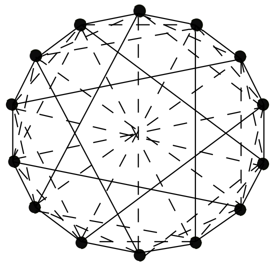

FIG. 112. H_{45}

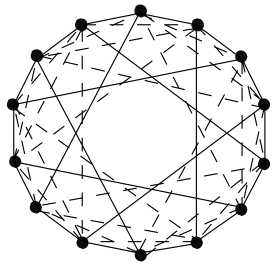

FIG. 113. H_{46}

Similarly, by taking a Kähler graph given by Fig. 91 and adding auxiliary edges as

$$E_{51}^{(a)} = \big\{\{v_i, v_{i+3}\},\ \{v_i, v_{i+4}\} \bigm| 0 \le i \le 13\big\},$$

$$E_{52}^{(a)} = \big\{\{v_i, v_{i+3}\},\ \{v_i, v_{i+6}\} \bigm| 0 \le i \le 13\big\},$$

$$E_{53}^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+3}\}\ (0 \le i \le 13), \quad \{v_i, v_{i+7}\}\ (0 \le i \le 6), \\ \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\} \end{array} \right\},$$

or by taking a Kähler graph given by Fig. 92 and adding auxiliary edges as

$$E_{54}^{(a)} = \big\{\{v_i, v_{i+4}\},\ \{v_i, v_{i+6}\} \bigm| 0 \le i \le 13\big\},$$

$$E_{55}^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+4}\}\ (0 \le i \le 13), \quad \{v_i, v_{i+7}\}\ (0 \le i \le 6), \\ \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\} \end{array} \right\},$$

$$E_{56}^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+4}\}\ (0 \le i \le 13), \quad \{v_i, v_{i+7}\}\ (0 \le i \le 6), \\ \{v_1, v_4\},\ \{v_3, v_6\},\ \{v_5, v_8\},\ \{v_7, v_{10}\},\ \{v_9, v_{12}\},\ \{v_{11}, v_0\},\ \{v_{13}, v_2\} \end{array} \right\},$$

$$E_{57}^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+4}\}\ (0 \le i \le 13), \\ \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\}, \\ \{v_1, v_4\},\ \{v_3, v_6\},\ \{v_5, v_8\},\ \{v_7, v_{10}\},\ \{v_9, v_{12}\},\ \{v_{11}, v_0\},\ \{v_{13}, v_2\} \end{array} \right\},$$

or by taking a Kähler graph given by Fig. 93 and adding auxiliary edges as

$$E_{58}^{(a)} = \left\{ \begin{array}{l} \{v_i, v_{i+6}\}\ (0 \le i \le 13), \quad \{v_i, v_{i+7}\}\ (0 \le i \le 6), \\ \{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\} \end{array} \right\},$$

$$E_{59}^{(a)} = \left\{ \begin{aligned} &\{v_i, v_{i+6}\}\ (0 \le i \le 13), \quad \{v_i, v_{i+7}\}\ (0 \le i \le 6), \\ &\{v_1, v_4\},\ \{v_3, v_6\},\ \{v_5, v_8\},\ \{v_7, v_{10}\},\ \{v_9, v_{12}\},\ \{v_{11}, v_0\},\ \{v_{13}, v_2\} \end{aligned} \right\},$$

$$E_{60}^{(a)} = \left\{ \begin{aligned} &\{v_i, v_{i+6}\}\ (0 \le i \le 13), \\ &\{v_1, v_6\},\ \{v_3, v_8\},\ \{v_5, v_{10}\},\ \{v_7, v_{12}\},\ \{v_9, v_0\},\ \{v_{11}, v_2\},\ \{v_{13}, v_4\}, \\ &\{v_1, v_4\},\ \{v_3, v_6\},\ \{v_5, v_8\},\ \{v_7, v_{10}\},\ \{v_9, v_{12}\},\ \{v_{11}, v_0\},\ \{v_{13}, v_2\} \end{aligned} \right\},$$

we get 10 other kinds of Kähler graphs whose auxiliary degree is 4. By definition, it is clear that all these Kähler graphs are vertex-transitive. We have many other regular Kähler graphs obtained by a Heawood graph.

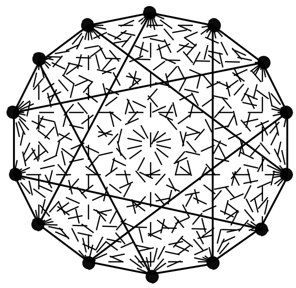

FIG. 114. complement-filled Heawood graph

EXAMPLE 2.28. We set $Q_k = \{(a_1, a_2, \dots, a_k) \mid a_i \in \{0, 1\}\}$ for an integer $k \ge 3$. We define that two vertices $v = (a_1, \dots, a_k)$, $w = (b_1, \dots, b_k) \in Q_k$ are adjacent to each other in the principal graph if and only if there is i_0 $(1 \le i_0 \le k)$ satisfying that $a_{i_0} \ne b_{i_0}$ and $a_i = b_i$ for $i \ne i_0$, and define that they are adjacent to each other in the auxiliary graph if and only if there are i_1, i_2 $(1 \le i_1 < i_2 \le k)$ satisfying that $a_{i_1} \ne b_{i_1}$, $a_{i_2} \ne b_{i_2}$ and $a_i = b_i$ for $i \ne i_1, i_2$, Since the graph $(Q_k, E^{(p)})$ is called a k-cube, we shall call the graph $G = (Q_k, E^{(p)} \cup E^{(a)})$ a *Kähler k-cube*. By definition we have $d_G^{(p)} = k$ and $d_G^{(a)} = k(k-1)/2$.

EXAMPLE 2.29. For the sake of explanation, we here consider a Kähler 3-cube $G = (Q_3, E^{(p)} \cup E^{(a)})$. Six vertices

$$\mathrm{O} = (0,0,0),\ \mathrm{A} = (1,0,0),\ \mathrm{B} = (1,1,0),\ \mathrm{C} = (0,1,0),$$

$$\mathrm{D} = (0,0,1),\ \mathrm{E} = (1,0,1),\ \mathrm{F} = (1,1,1),\ \mathrm{G} = (0,1,1)$$

and principal edges

$$\{\mathrm{O},\mathrm{A}\},\ \{\mathrm{A},\mathrm{B}\},\ \{\mathrm{B},\mathrm{C}\},\ \{\mathrm{C},\mathrm{O}\},\ \{\mathrm{D},\mathrm{E}\},\ \{\mathrm{E},\mathrm{F}\},\ \{\mathrm{F},\mathrm{G}\},\ \{\mathrm{G},\mathrm{D}\},$$

$$\{\mathrm{O},\mathrm{D}\},\ \{\mathrm{A},\mathrm{E}\},\ \{\mathrm{B},\mathrm{F}\},\ \{\mathrm{C},\mathrm{G}\}$$

form a cube in $\mathbb{R}^3$. Auxiliary edges are diagonal lines on six faces:

$$\{\mathrm{O},\mathrm{B}\},\ \{\mathrm{A},\mathrm{C}\},\ \{\mathrm{O},\mathrm{E}\},\ \{\mathrm{A},\mathrm{D}\},\ \{\mathrm{A},\mathrm{F}\},\ \{\mathrm{B},\mathrm{E}\},$$

$$\{\mathrm{B},\mathrm{G}\},\ \{\mathrm{C},\mathrm{F}\},\ \{\mathrm{C},\mathrm{D}\},\ \{\mathrm{O},\mathrm{G}\},\ \{\mathrm{D},\mathrm{F}\},\ \{\mathrm{E},\mathrm{G}\}.$$

Thus Q_3 have 12 principal edges and 12 auxiliary edges, and $d_G^{(p)} = d_{Q_3}^{(a)} = 3$. We note that the auxiliary graph is not connected.

We take a rotation f and reversing upper and lower g which are given as

$$f : \mathrm{O} \mapsto \mathrm{A},\ \mathrm{A} \mapsto \mathrm{B},\ \mathrm{B} \mapsto \mathrm{C},\ \mathrm{C} \mapsto \mathrm{O};\ \mathrm{D} \mapsto \mathrm{E},\ \mathrm{E} \mapsto \mathrm{F},\ \mathrm{F} \mapsto \mathrm{G},\ \mathrm{G} \mapsto \mathrm{D},$$

$$g : \mathrm{O} \mapsto \mathrm{D},\ \mathrm{A} \mapsto \mathrm{E},\ \mathrm{B} \mapsto \mathrm{F},\ \mathrm{C} \mapsto \mathrm{G};\ \mathrm{D} \mapsto \mathrm{O},\ \mathrm{E} \mapsto \mathrm{A},\ \mathrm{F} \mapsto \mathrm{B},\ \mathrm{G} \mapsto \mathrm{C}.$$

Then they are isomorphisms. By using $f, f^2, f^3, g, g \circ f, g \circ f^2, g \circ f^3$ we see G is vertex-transitive.

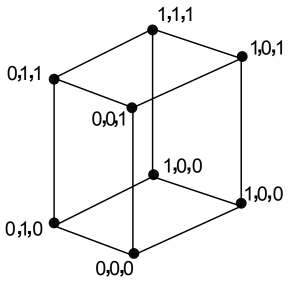

FIG. 115. 3-cube

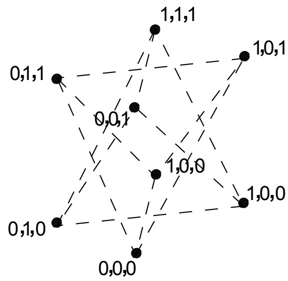

FIG. 116. auxiliary graph

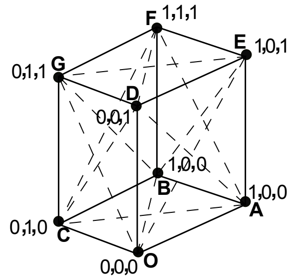

FIG. 117. Kähler 3-cube

PROPOSITION 2.4. *A Kähler k-cube is vertex-transitive.*

PROOF. We prove the assertion by induction with respect to k. When $k = 3$, we see in the above that a Kähler 3-cube is vertex-transitive. We suppose a Kähler k-cube is vertex-transitive. We study a Kähler $(k+1)$-cube. Let $f : Q_k \to Q_k$ be an isomorphism of a Kähler 3-cube. We define $\tilde{f} : Q_{k+1} \to Q_{k+1}$ as

$$\tilde{f}\big((a_1, \dots, a_k, a_{k+1})\big) = \big(f\big((a_1, \dots, a_k)\big), a_{k+1}\big).$$

In oder to show that $\tilde{f}$ is an isomorphism, we only need to check edges of the form $(\boldsymbol{a}, \boldsymbol{b})$ with $\boldsymbol{a} = (a_1, \dots, a_k, 0)$ and $\boldsymbol{b} = (b_1, \dots, b_k, 1)$. When $\boldsymbol{a} \sim_p \boldsymbol{b}$, we have $(a_1, \dots, a_k) =$

$(b_1,\dots,b_k)$. Hence we see $\tilde{f}(\boldsymbol{a}) \sim_p \tilde{f}(\boldsymbol{b})$. When $\boldsymbol{a}\sim_a\boldsymbol{b}$, we have $(a_1,\dots,a_k) \sim_p (b_1,\dots,b_k)$ in a Kähler k-cube. Hence $f\big((a_1,\dots,a_k)\big) \sim_p f\big((b_1,\dots,b_k)\big)$ in this Kähler k-cube, in particular their difference is only one coordinate. Thus we see $\tilde{f}(\boldsymbol{a}) \sim_a \tilde{f}(\boldsymbol{b})$. We define $g : Q_{k+1} \to Q_{k+1}$ by

$$g\big((a_1,\dots,a_k,0)\big) = \big(a_1,\dots,a_k,1\big) \quad \text{and} \quad g\big((a_1,\dots,a_k,1)\big) = \big(a_1,\dots,a_k,0\big).$$

This is also an isomorphism. Thus considering $\tilde{f}, \tilde{f}\circ g$ for all isomorphisms f of a Kähler k-cube, we find that a Kähler $(k+1)$-cube is vertex-transitive. □

4. Complete Kähler graphs

We say a Kähler graph to be a *complete Kähler graph* if it is a complete graph as an ordinary graph and is regular as a Kähler graph. Thus each pair of vertices of a complete Kähler graph is joined by either a principal edge or an auxiliary edge.

One of the most typical way to construct complete Kähler graphs is to take complement-filled Kähler graphs (see §2.1). We take an ordinary regular finite graph $G = (V, E)$ of degree $2 \le d_G \le n_G - 3$, and consider its complement-filled Kähler graph $G^K = (V, E \cup E^c)$. Since the complement graph $G^c = (V, E^c)$ is regular of degree $d_{G^c} = n_G - d_G - 1$, this Kähler graph is a complete Kähler graph whose principal degree is d_G and whose auxiliary degree is $n_G - d_G - 1$.

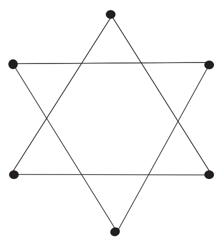

FIG. 118. G

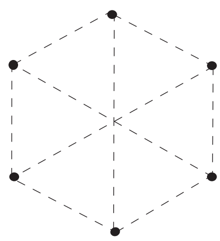

FIG. 119. G^c

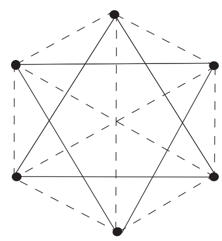

FIG. 120. G^K

We here give a condition that we can construct a complete Kähler graph.

PROPOSITION 2.5. *Let $N, d^{(p)}, d^{(a)}$ be positive integers satisfying $N \ge 5$, $d^{(p)} \ge 2$, $d^{(a)} \ge 2$ and $d^{(p)} + d^{(a)} = N - 1$. Then there exists a vertex-transitive complete Kähler graph G satisfying $n_G = N$ and $d_G^{(p)} = d^{(p)}$, $d_G^{(a)} = d^{(a)}$ if and only if one of the following conditions holds*:

i) *N is odd and both $d^{(p)}, d^{(a)}$ are even,*

ii) *N is even, and one of $d^{(p)}, d^{(a)}$ is even and the other is odd.*

PROOF. Since $d^{(p)} + d^{(a)} = N - 1$, when N is even then $N - 1$ is old, hence one of $d^{(p)}, d^{(a)}$ is even and the other is odd. Thus we find by Theorem 2.1 that the condition on $N, d^{(p)}, d^{(a)}$ is necessary. On the other hand, we can construct a vertex-transitive

Kähler graph G satisfying $n_G = N$ and $d_G^{(p)} = d^{(p)}$, $d_G^{(a)} = d^{(a)}$ by Theorem 2.1, Since the condition $d^{(p)} + d^{(a)} = N - 1$ shows that G is complete, we get the conclusion. □

COROLLARY 2.1. *Let $N \geq 5$ be a positive integer. There exists a vertex-transitive complete Kähler graph G satisfying $n_G = N$ and $d_G^{(p)} = d_G^{(a)}$ if and only if $N \equiv 1 \ (mod\ 4)$.*

PROOF. If we have a complete Kähler graph whose cardinality of the set of vertices is N and whose principal and auxiliary degrees are d, we have $N - 1 = 2d$. Therefore N is odd. By Proposition 2.5 we find d is even, hence find that $N - 1$ is divided by 4. On the other hand, if N satisfies the condition, Proposition 2.5 shows that we have such a complete Kähler graph. □

The above results show that we have many vertex-transitive complete Kähler graphs. We here study whether they are isomorphic. Though complete ordinary graphs of given cardinality of the sets of vertices are isomorphic to each other (Proposition 1.3), as we have two kinds of edges for Kähler graphs, even if we fix the cardinality of the set of vertices there exist non-isomorphic Kähler graphs.

When $N = 5$, as we have $d^{(p)} = d^{(a)} = 2$, we find that the principal and the auxiliary graphs are circuits. Hence we find that complete Kähler graphs of $n_G = 5$ are isomorphic to each other by Proposition 1.5.

EXAMPLE 2.30. Figs. 121, 122 show complete vertex-transitive Kähler graphs with $n_G = 9$, $d_G^{(p)} = d_G^{(a)} = 4$ which are not isomorphic and whose principal and whose auxiliary graphs are connected. For the set of vertices $V = \{v_0, v_1, v_2, \dots, v_8\}$, we define their sets of principal edges by

$$E_1^{(p)} = \big\{\{v_i, v_{i+1}\},\ \{v_i, v_{i+2}\} \ \big|\ 0 \leq i \leq 8\big\},$$
$$E_2^{(p)} = \big\{\{v_i, v_{i+1}\},\ \{v_i, v_{i+3}\} \ \big|\ 0 \leq i \leq 8\big\}.$$

By definition, two graphs $G_1 = (V, E_1^{(p)})$ and $G_2 = (V, E_2^{(p)})$ are vertex-transitive by rotations $f_k : V \to V$ defined by $v_i \mapsto v_{i+k}$ for $k = 1, \dots, 8$. Hence their complement-filled Kähler graphs G_1^K, G_2^K are. We can see that they are not isomorphic by observing

3-step closed principal paths. In the Kähler graph in Fig. 121 we have three 3-step closed principal paths emanating from each vertex. On the other hand, in the Kähler graph in Fig. 122 we have only one 3-step closed principal path emanating from each vertex.

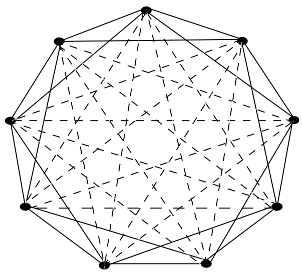

FIG. 121. $n_G = 9, d_G^{(p)} = d_G^{(a)} = 4$

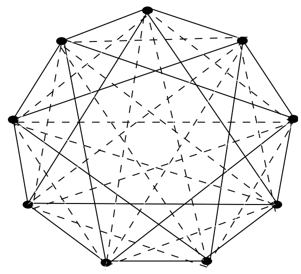

FIG. 122. $n_G = 9, d_G^{(p)} = d_G^{(a)} = 4$

EXAMPLE 2.31. Figs. 123, and 124 show complete vertex-transitive Kähler graphs with $n_G = 6$, $d_G^{(p)} = 2$, $d_G^{(a)} = 3$ which are not isomorphic. The former has a connected principal graph but the latter does not. Their auxiliary graphs, which are principal graphs of their dual Kähler graphs, are connected.

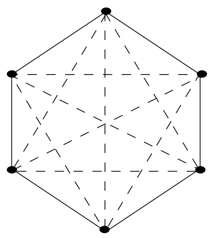

FIG. 123. $n_G = 6, d_G^{(p)} = 2, d_G^{(a)} = 3$
(principally connected)

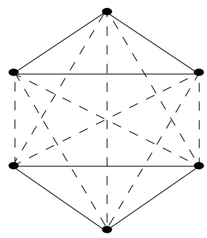

FIG. 124. $n_G = 6, d_G^{(p)} = 2, d_G^{(a)} = 3$
(principally inconnected)

Since a complete Kähler graph is a complement-filled Kähler graph of its principal graph, we obtain the following.

PROPOSITION 2.6. (1) *Two complete Kähler graphs are isomorphic to each other if and only if their principal graphs are congruent to each other.*

(2) *Two complete Kähler graphs are isomorphic to each other if and only if their auxiliary graphs are congruent to each other.*

We now classify complete Kähler graphs whose principal graphs are regular graphs of degree 2 by using Propositions 1.5 and 2.6. We denote by $\mathfrak{p} : \mathbb{N} \to \mathbb{N}$ the partition function. This function is defined as follows. For a positive integer n, we consider its representation as a sum of positive integers. Here, we are allowed to use same integers in the representation, but the order of summing is irrelevant. The (integer) partition $\mathfrak{p}(n)$ is the number of such representations of n. For example, we have

$$\begin{aligned}
&\mathfrak{p}(1) = 1, \\
&\mathfrak{p}(2) = 2, &&\text{because} \quad 2 = 1 + 1, \\
&\mathfrak{p}(3) = 3, &&\text{because} \quad 3 = 2 + 1 = 1 + 1 + 1, \\
&\mathfrak{p}(4) = 5, &&\text{because} \quad 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1, \\
&\mathfrak{p}(5) = 7, &&\text{because} \quad 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 \\
& && \qquad\quad = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1, \\
&\mathfrak{p}(6) = 11, &&\text{because} \quad 6 = 5 + 1 = 4 + 2 = 3 + 3 \\
& && \qquad\quad = 4 + 1 + 1 = 3 + 2 + 1 = 2 + 2 + 2 \\
& && \qquad\quad = 3 + 1 + 1 + 1 = 2 + 2 + 1 + 1 \\
& && \qquad\quad = 2 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1, \\
&\mathfrak{p}(7) = 15, &&\text{because} \quad 7 = 6 + 1 = 5 + 2 = 4 + 3 \\
& && \qquad\quad = 5 + 1 + 1 = 4 + 2 + 1 = 3 + 3 + 1 = 3 + 2 + 2 \\
& && \qquad\quad = 4 + 1 + 1 + 1 = 3 + 2 + 1 + 1 = 2 + 2 + 2 + 1 \\
& && \qquad\quad = 3 + 1 + 1 + 1 + 1 = 2 + 2 + 1 + 1 + 1 \\
& && \qquad\quad = 2 + 1 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1.
\end{aligned}$$

For more detail, see §19 of [**6**].

Proposition 2.7. *For each positive number n (≥ 5) the number of isomorphic classes of complete Kähler graphs whose sets of vertices have the cardinality n and whose auxiliary degrees are 2 is $\mathfrak{p}(n) - \mathfrak{p}(n-1) - \mathfrak{p}(n-2) + \mathfrak{p}(n-3)$.*

PROOF. Let $G = (V, E^{(p)} \cup E^{(a)})$ be a complete finite Kähler graph with $n_G = n$ and $d_G^{(a)} = 2$. If we consider its auxiliary graph, each of its component is a circuit, which is a circle as a 1-dimensional CW-complex. Since G is obtained by considering the complement graph of $(V, E^{(a)})$, we are enough to consider the number of congruence classes of ordinary regular graphs of degree 2 and of $n_G = n$.

By Proposition 1.5, two circuit graphs are isomorphic to each other if and only if they have the same cardinality of their sets of vertices. As our graph does not have multiple edges and loops, each of these circuits has at least three vertices. Thus the number of congruence classes coincides with the number of partition of n using only integers greater than 2.

Let $\mathfrak{R}(n)$ denote the set of all partitions of n. That is, $\mathfrak{R}(3) = \big\{(3),\ (2,1),\ (1,1,1)\big\}$, for example. If $\mathfrak{r} = (a_1, a_2, \dots, a_{k-1}, 1) \in \mathfrak{R}(n)$, then we have $\mathfrak{r}' = (a_1, \dots, a_{k-1}) \in \mathfrak{R}(n-1)$. On the other hand, for each $\mathfrak{r}' \in \mathfrak{R}(n-1)$ we can construct $\mathfrak{r}$ by adding 1 at last. Thus we find that $\big\{(a_1, \dots, a_{k-1}, 1) \in \mathfrak{R}(n)\big\}$ corresponds to $\mathfrak{R}(n-1)$ bijectively. If $\mathfrak{s} = (b_1, \dots, b_{\ell-1}, 2) \in \mathfrak{R}(n)$, then we have $\mathfrak{s}' = (b_1, \dots, b_{\ell-1}) \in \mathfrak{R}(n-2)$. On the other hand, if $\mathfrak{s}' = (b_1, \dots, b_{\ell-1}) \in \mathfrak{R}(n-2)$ satisfies $b_{\ell-1} \geq 2$, we can construct $\mathfrak{s}$ by adding 2 at last. Since the set $\big\{(b_1, \dots, b_{\ell-2}, 1) \in \mathfrak{R}(n-2)\big\}$ corresponds to $\mathfrak{R}(n-3)$ bijectively, We see the cardinality of the set $\big\{(a_1, \dots, a_k) \in \mathfrak{R}(n) \bigm| a_k \geq 3\big\}$ coincides with $\mathfrak{p}(n) - \mathfrak{p}(n-1) - \big\{\mathfrak{p}(n-2) - \mathfrak{p}(n-3)\big\}$. Hence we get the conclusion. □

By considering dual Kähler graphs we have

COROLLARY 2.2. *For each positive n (≥ 5) the number of isomorphic classes of complete Kähler graphs whose sets of vertices have the cardinality n and whose principal degree is 2 is $\mathfrak{p}(n) - \mathfrak{p}(n-1) - \mathfrak{p}(n-2) + \mathfrak{p}(n-3)$.*

Also, if we add a condition of connectivity we get a congruence results.

COROLLARY 2.3. (1) *Two finite complete Kähler graphs whose auxiliary graphs are connected and are of degree 2 are isomorphic to each other if and only if cardinalities of their sets of vertices coincide.*

(2) *Two finite complete Kähler graphs whose principal graphs are connected and are of degree* 2 *are isomorphic to each other if and only if cardinalities of their sets of vertices coincide.*

PROPOSITION 2.8. *For a positive integer* n (≥ 5), *the number of isomorphic classes of complete vertex-transitive Kähler graphs whose sets of vertices have the cardinality* n *and whose auxiliary degree is* 2 *coincides with the number of divisors of* n *which are greater than* 2.

PROOF. We are enough to consider the auxiliary graph. If we have such a vertex-transitive Kähler graph, as a component of the auxiliary graph is transferred to a component, we see every component have the same cardinality of the set of vertices. Thus we get a divisor of n which is greater than 2 as this cardinality.

We construct a Kähler graph corresponding to a given divisor of n. Suppose $n = n_1 n_2$ with some positive integers n_1, n_2 satisfying $n_2 \geq 3$. We prepare n_1 circuit graphs having n_2 vertices. By making them an auxiliary graph we have a complete Kähler graph $(V, E^{(p)} \cup E^{(a)})$ satisfying $\sharp V = n$ and $d_G^{(a)} = 2$. Since all the components of $(V, E^{(a)})$ are circuits having the same numbers of vertices, for arbitrary distinct $v, v' \in V$ we have an isomorphism of $(V, E^{(a)})$ which maps v to v' and maps the component containing v to the component containing v'. It is clear that this induces an isomorphism of $(V, E^{(p)} \cup E^{(a)})$. Thus, we find that this Kähler graph is vertex-transitive, and get the conclusion. □

COROLLARY 2.4. *Let* n (≥ 5) *be a positive prime integer. Two complete vertex-transitive Kähler graphs whose sets of vertices have the cardinality* n *and whose auxiliary degrees are* 2 *are isomorphic to each other.*

CHAPTER 3

Discrete models of trajectories for magnetic fields

1. Trajectories for magnetic fields

A static magnetic field on $\mathbb{R}^3$ is a vector-valued function $\mathbb{B} = (B_1, B_2, B_3) : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying Gauss formula $\mathrm{div}(\mathbb{B}) = \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} = 0$. This gives the Lorentz force $v \times \mathbb{B} = \Omega_{\mathbb{B}} v$ on a unit charged particle when its velocity vector is v. Here $\Omega_{\mathbb{B}}$ is a skew-symmetric matrix given by

$$\begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix}.$$

If we define a 2-form $\boldsymbol{B}$ on $\mathbb{R}^3$ by $\boldsymbol{B}(u, v) = \langle u, \Omega_{\mathbb{B}} v\rangle$ with the standard inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^3$, then this form is represented as

$$\boldsymbol{B} = B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2.$$

Since we have

$$d\boldsymbol{B} = \Big(\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3}\Big) dx_1 \wedge dx_2 \wedge dx_3,$$

we find that the Gauss formula $\mathrm{div}(\boldsymbol{B}) = 0$ is equivalent to $d\boldsymbol{B} = 0$, which means that $\boldsymbol{B}$ is a closed 2-form.

Under this consideration, we call a closed 2-form $\boldsymbol{B}$ on a Riemannian manifold M a *magnetic field*. For a magnetic field $\boldsymbol{B}$ on M, we define a bundle map $\Omega_{\boldsymbol{B}} : TM \to TM$ on the tangent bundle TM of M by $\boldsymbol{B}(u, v) = \langle u, \Omega_{\boldsymbol{B}}(v)\rangle$ for every $u, v \in T_xM$ at an arbitrary point $x \in M$ with Riemannian metric $\langle\ ,\ \rangle$ on M. We then find that $\Omega_{\boldsymbol{B}}$ is skew symmetric, that is $\langle u, \Omega_{\boldsymbol{B}}(v)\rangle = -\langle \Omega_{\boldsymbol{B}}(u), v\rangle$.

When $\Omega_{\boldsymbol{B}}$ is parallel, that is $\nabla \Omega_{\boldsymbol{B}} = 0$, we say that $\boldsymbol{B}$ is an uniform magnetic field. Here, ∇ denotes the Riemannian connection on M. For example, we take a Kähler manifold M with complex structure J. Then its Kähler form $\boldsymbol{B}_J$ which is defined by $\boldsymbol{B}_J(u,v) = \langle u, Jv\rangle$ is a closed 2-form and $\Omega_{\boldsymbol{B}_J} = J$ is parallel. Therefore every constant multiple $\boldsymbol{B}_\kappa = \kappa \boldsymbol{B}_J$ ($\kappa \in \mathbb{R}$) is an uniform magnetic field. This magnetic field is called a Kähler magnetic field (for more detail see [**1**]).

It is needless to say that we have many magnetic fields which are not uniform. Let M be a real hypersurface of a Kähler manifold $\widetilde{M}$. That is, when $\widetilde{M}$ is of complex dimension n then M is a real submanifold of real dimension $2n-1$. For a unit normal vector field $\mathcal{N}_M$ of M in $\widetilde{M}$, we define a vector field ξ on M by $\xi = -J\mathcal{N}_M$, and define a $(1,1)$-tensor $\phi : TM \to TM$ by $\phi(v) = Jv - \langle v, \xi\rangle \mathcal{N}_M$. They are called the characteristic vector field and the characteristic tensor of M. If we define $\boldsymbol{F}_\phi$ by $\boldsymbol{F}_\phi(u,v) = \langle u, \phi(v)\rangle$, then it is a closed 2-form and $\Omega_{\boldsymbol{F}_\phi} = \phi$ (see [**4**]). Generally, it is not uniform. We call a constant multiple $\boldsymbol{F}_\kappa = \kappa \boldsymbol{F}_\phi$ ($\kappa \in \mathbb{R}$) a Sasakian magnetic field.

Under the influence of a static magnetic field, the equation of motions of a unit charged particle of mass m is given as $m\dfrac{dv}{dt} = v \times \mathbb{B}$. As we have $\dfrac{d}{dt}\|v\|^2 = 2\langle v, \dfrac{dv}{dt}\rangle = 2\langle v, \Omega_{\mathbb{B}} v\rangle = 0$, this particle has constant speed. We shall call a smooth curve on M satisfying the differential equation $\nabla_{\gamma'}\gamma' = \Omega_{\boldsymbol{B}}(\gamma')$ a *trajectory* for $\boldsymbol{B}$. Here, $\gamma' = \dfrac{d\gamma}{dt}$, and $\nabla_{\gamma'}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ on M. Since we have

$$\gamma'(\|\gamma'\|^2) = \gamma'\langle \gamma', \gamma'\rangle = \langle \nabla_{\gamma'}\gamma', \gamma'\rangle + \langle \gamma', \nabla_{\gamma'}\gamma'\rangle = \langle \Omega_{\boldsymbol{B}}(\dot\gamma), \dot\gamma\rangle + \langle \dot\gamma, \Omega_{\boldsymbol{B}}(\dot\gamma)\rangle,$$

and $\Omega_{\boldsymbol{B}}$ is skew symmetric, we find $\gamma'(\|\gamma'\|^2) = 0$. This shows that γ has constant speed. We usually call treat trajectories of unit speed.

In the field of geometry, ordinary graphs are considered as discretizations of Riemannian manifolds and paths are considered as correspondences of geodesics. In his paper [**2**] Adachi introduced Kähler graphs as discritizations of Riemannian manifolds admitting uniform magnetic fields. In the next section, following to [**2**] we introduce correspondences of trajectories on Kähler graphs and show why Kähler graphs can be considered as discritizations of Riemannian manifolds with magnetic fields.

2. Bicolored path

Let $G = (V, E^{(p)} \cup E^{(a)})$ be a Kähler graph. For a pair (p, q) of relatively prime positive integers, we say a $(p+q)$-step path $\gamma = (v_0, v_1, \cdots, v_{p+q}) \in V \times \cdots\cdots \times V$ to be a (p, q)-*primitive bicolored path* if it satisfies the following conditions;

i) $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq p+q-1$,

ii) $v_{i-1} \sim_p v_i$ for $1 \leq i \leq p$,

iii) $v_{i-1} \sim_a v_i$ for $p+1 \leq i \leq p+q$.

The first condition shows that this path does not have backtracking, the second shows that the first p-step path is a path in the principal graph and the third shows that the last q-step path is a path in the auxiliary graph. When an $m(p+q)$-step path γ is of the form $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_m$ with $(p+q)$-primitive bicolored paths γ_i $(i = 1, \ldots, m)$, it is called a (p, q)-*bicolored path*.

Example 3.1. On a Heawood Kähler graph of $d^{(p)} = 3, d^{(a)} = 2$ given as Fig. 1, the paths $\gamma_1 = (0, 1, 2, 5)$, $\gamma_2 = (5, 6, 7, 10)$ are $(2, 1)$-primitive bicolored paths (see Fig. 2), and the paths $\gamma_3 = (0, 1, 4), \gamma_4 = (4, 5, 8), \gamma_5 = (8, 9, 12), \gamma_6 = (12, 13, 2), \gamma_7 = (2, 3, 6), \gamma_8 = (6, 7, 10)$ are $(1, 1)$-primitive bicolored path (see Fig. 3). Hence $\gamma_3 \cdot \gamma_4$, $\gamma_4 \cdot \gamma_5$, $\gamma_5 \cdot \gamma_6$, $\gamma_6 \cdot \gamma_7, \gamma_7 \cdot \gamma_8$ are 4-step $(1, 1)$-bicolored paths, and $\gamma_3 \cdot \gamma_4 \cdot \gamma_5 \cdot \gamma_6 \cdot \gamma_7 \cdot \gamma_8$ is a 12-step $(1, 1)$-bicolred path.

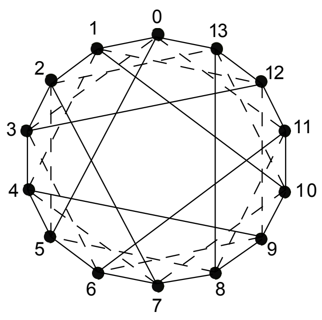

Fig. 1

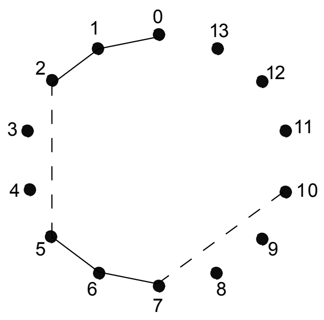

Fig. 2

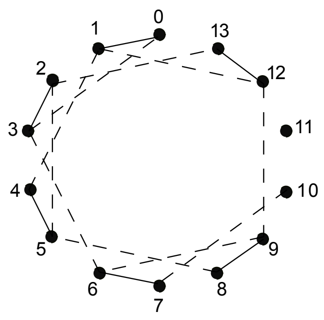

Fig. 3

Since we pose a condition on Kähler graphs that their principal and auxiliary graphs do not have hairs, we have a (p, q)-bicolored path passing through an arbitrary vertex for each pair (p, q). Therefore, if we only study $(1, 1)$-paths we can weaken the condition to the condition that there is at least one principal edge and one auxiliary edge emanating from each vertex, that is, to the condition that there are no isolated vertices in both principal and auxiliary graphs.

Ordinary graphs are usually regarded as discrete models of Riemannian manifolds and paths on graphs are considered as correspondences of geodesics. We therefore regard paths on the principal graph of a Kähler graph as geodesics which are motions of charged particles without the influence of magnetic fields. Considering Kähler graphs as discrete models of complex manifolds, we regard (p, q)-bicolored paths as trajectories for a magnetic field of strength q/p on these graphs. This means that a p-step path on the principal graph of a Kähler graph is bended under the action of a magnetic field and its terminus turns to the terminus of a (p, q)-primitive bicolored path whose first p-step coincides with the given path.

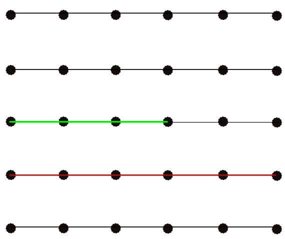

Fig. 4. path on principal edge

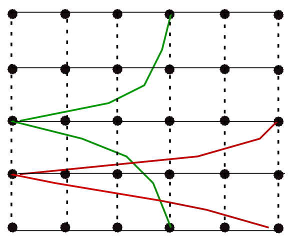

Fig. 5. bicolored path on a Kähler graph

In order to consider correspondences of trajectories for a magnetic field of strength q/p, we define (p, q)-primitive bicolored paths for a pair (p, q) of relatively prime positive integers. But for the sake of interpretation it is easier to extend this notion to all pairs of positive integers. So, if a $(p + q)$-step path satisfies the conditions for (p, q)-primitive bicolored paths, we sometimes call it a (p, q)-bicolored path even if p, q have common divisor. Moreover, we sometimes call a p-step path in the principal graph a $(p, 0)$-primitive bicolored path, and call a q-step path in the auxiliary graph a $(0, q)$-primitive bicolored path. We note that we only use the terminology (p, q)-bicolored paths only for a pair of relatively prime positive integers.

As graphs do not have 2-dimensional objects, we can not show the direction of the action of the magnetic field. Therefore, if there are two and more (p,q)-primitive bicolored paths whose first p-step paths coincide with the given p-step path, we can not determine the terminus of trajectories. In order to get rid of bifurcations of motions of charged particles, we shall consider (p,q)-bicolored paths probabilistically. For a (p,q)-primitive bicolored path $\gamma = (v_0, \cdots, v_{p+q})$, we define its *probabilistic weight* $\omega(\gamma)$ by

$$\omega(\gamma) = \frac{1}{d_G^{(a)}(v_p)\prod_{i=p+1}^{p+q-1}\{d_G^{(a)}(v_i)-1\}}.$$

For a (p,q)-bicolored path $\gamma = (\gamma_1, \gamma_2 \cdots, \gamma_n)$ with (p,q)-primitive bicolored paths γ_i $(i=1,\dots,m)$, we difine its probabilistic weight by $\omega(\gamma) = \prod_{i=1}^{m}\omega(\gamma_i)$.

Example 3.2. Let $G = (V, E^{(p)} \cup E^{(a)})$ be a Kähler graph. A part of it is shown in Fig. 6. We take a $(3,4)$-bicolored path $\gamma = (v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ in this graph. We find that auxiliary degrees at vertices $v_3, v_4, v_5, v_6 \in V$ are

$$d^{(a)}(v_3) = 3,\ d^{(a)}(v_4) = 6,\ d^{(a)}(v_5) = 4,\ d^{(a)}(v_6) = 2.$$

Thus we have the probabilistic weight of γ is

$$\frac{1}{d_G^{(a)}(v_3)\{d_G^{(a)}(v_4)-1\}\{d_G^{(a)}(v_5)-1\}\{d_G^{(a)}(v_6)-1\}} = \frac{1}{45}.$$

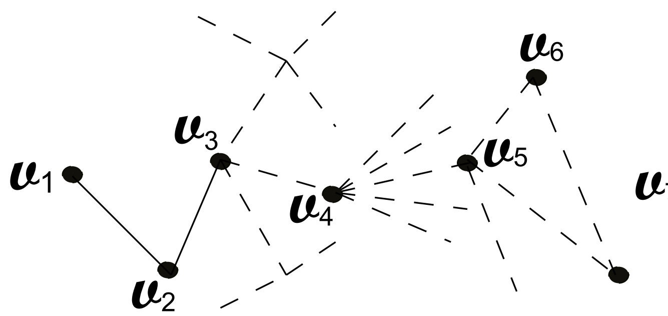

Fig. 6. a part of a Kähler graph G

For a p-step path σ in the principal graph of a Kähler graph G, we denote by $\mathfrak{P}_q(\sigma)$ the set of all (p,q)-primitive bicolored paths whose first p-step coincide with σ. That is, if $\sigma = (v_0, \ldots, v_p)$ then each (p,q)-primitive bicolored path $\gamma \in \mathfrak{P}_q(\sigma)$ is of the form $\gamma = (v_0, \ldots, v_p, w_1, \ldots, w_q)$.

Lemma 3.1. *For each p-step path σ in the principal graph of a Kähler graph G, we have*

$$\sum_{\gamma \in \mathfrak{P}_q(\sigma)} \omega(\gamma) = 1.$$

Proof. Let $\tau = (v_0, \ldots, v_p, w_1, \ldots, w_j)$ with $j \geq 1$ be a (p,j)-primitive bicolored path. Then $\gamma = \tau \cdot (w_j, w)$ is a $(p, j+1)$-primitive bicolored path if and only if $w \sim w_j$ and $w \neq w_{j-1}$. Here, we consider $w_{j-1} = v_p$ when $j = 1$. Therefore we have $d_G^{(a)}(w_j) - 1$ $(p, j+1)$-primitive bicolored paths whose first $p + j$ coincide with τ. We hence have

$$\omega(\tau) = \frac{1}{d_G^{(a)}(v_p) \prod_{i=1}^{j} \bigl\{ d_G^{(a)}(w_i) - 1 \bigr\}} = \sum_{\substack{w \,:\, w \neq w_{j-1}, \\ w \sim w_j \text{ in } G^{(a)}}} \omega\bigl(\tau \cdot (w_j, w)\bigr).$$

As we have $d_G^{(a)}(v_p)$ $(p,1)$-primitive bicolored paths whose first p coincide with σ, we get the conclusion. $\square$

Remark 3.1. For $\gamma \in \mathfrak{P}_q(\sigma)$, its probabilistic weight does not coincides with $1/\sharp\bigl(\mathcal{P}_q(\sigma)\bigr)$, in general. If the auxiliary graph of G is regular, then they coincide with each other.

3. Derived graph of Kähler graphs

In this section we explain how to construct new graphs from a Kähler graph by using paths without backtracking.

3.1. Derived graph. We shall start by using ordinary graphs. Let $G = (V, E)$ be an ordinary (non-oriented) graph. For a positive integer n, we denote by $\mathfrak{P}_n(G)$ the set of all n-step paths without backtracking on V. We shall call the oriented graph $G_{[n]} = \big(V, \mathfrak{P}_n(G)\big)$ the n-step derived graph of G. This means that if we have $\gamma \in \mathfrak{P}_n(G)$ with $o(\gamma) = v$ and $t(\gamma) = w$ then we regard it as an oriented edge from v to w on $G_{[n]}$. Therefore, the oriented graph $G_{[n]}$ may have loops and multiple edges.

As G is non-oriented, for a path $\gamma \in \mathfrak{P}_n(G)$ we can consider its reversed path $\gamma^{-1} \in \mathfrak{P}_n(G)$. For two paths $\gamma_1, \gamma_2 \in \mathfrak{P}_n(G)$, we set $\gamma_1 \approx \gamma_2$ if either $\gamma_1 = \gamma_2$ or $\gamma_1 = \gamma_2^{-1}$ holds. Then it is clear that $\approx$ is an equivalence relation on $\mathfrak{P}_n(G)$. We denote by $\mathfrak{P}_n(G)/\approx$ the set of all equivalence classes of n-step paths without backtracking on G. We shall call the non-oriented graph $\widehat{G}_{[n]} = \big(V, \mathfrak{P}_n(G)/\approx\big)$ the n-step derived non-oriented graph of G. This means that if we have $\gamma \in \mathfrak{P}_n(G)$ with $o(\gamma) = v$ and $t(\gamma) = w$ then we regard its equivalence class $[\gamma]$ as a non-oriented edge between v and w on $\widetilde{G}_{[n]}$.

We set $\mathfrak{P}_n(v) = \mathfrak{P}_n(v; G) = \big\{\gamma \in \mathfrak{P}_n(G) \bigm| o(\gamma) = v\big\}$. Then we see that $d_{G_{[n]}}(v)$ is the cardinality of this set and satisfies $d_{G_{[n]}}(v) \le (n_G - 1)(n_G - 2)^{n-1}$ when G is finite. We call the adjacency and the transition operators of $G_{[n]}$, which are the same as those of $\widehat{G}_{[n]}$ the n-step adjacency and the n-step transition operators, respectively. They are given as

$$\mathcal{A}_{G_{[n]}} f(v) = \sum_{\gamma \in \mathfrak{P}_n(v)} f\big(t(\gamma)\big), \qquad \mathcal{P}_{G_{[n]}} f(v) = \frac{1}{d_{G_{[n]}}(v)} \sum_{\gamma \in \mathfrak{P}_n(v)} f\big(t(\gamma)\big).$$

Derived graphs and derived non-oriented graphs are generally complicated. Even the original graph is connected, its derived graphs are not necessarily connected. To get rid of complexity, we shall reduce edges of derived graphs. We define a non-oriented graph $\widetilde{G}_{[n]} = (V, E_{[n]})$ so that two vertices $v, v' \in V$ are adjacent to each other in this graph if and only if there is $\gamma \in \mathfrak{P}_n(G)$ with $o(\gamma) = v,\ t(\gamma) = v'$. Even if there are two and more paths joining them, we only attach an edge between them. Thus this graph may have loops but does not have multiple edges. For a pair (v, v') of vertices we set $\mathfrak{P}_n(v, v') = \{\gamma \in \mathfrak{P}_n(v) \mid t(\gamma) = v'\}$. When G is locally finite, we define a function $\mathfrak{m} : E_{[n]} \to \mathbb{Z}$ so that $\mathfrak{m}\big((v, v')\big)$ shows the cardinality of the set $\mathfrak{P}_n(v, v')$. We shall call the "weight graph" $\big(\widetilde{G}_{[n]}, \mathfrak{m}\big)$ the reduced n-step derived graph of G.

When $n = 1$, it is clear by definition that $G = \widehat{G}_{[1]} = \widetilde{G}_{[1]}$ and $\mathfrak{m}$ only takes the value 1. We note that these terminologies of derived graphs may not be general. But for the sake of extending these notions to Kähler graphs we use these terminologies.

3.2. Derived graphs of Kähler graphs. Next we construct derived graphs corresponding (p, q)-primitive bicolored paths on Kähler graphs. Let $G = (V, E^{(p)} \cup E^{(a)})$ be a Kähler graph. For a pair (p, q) of relatively prime positive integers, we denote the set of all (p, q)-primitive bicolored paths on G by $\mathfrak{P}_{[p,q]}(G)$. We call the oriented ordinary graph $G_{[p,q]} = \big(V, \mathfrak{P}_{p,q}(G)\big)$ the (p, q)-derived graph of G. This oriented graph may have loops and multiple edges. But it does not have hairs by the condition of Kähler graphs. If we set $\mathfrak{P}_{p,q}(v) = \mathfrak{P}_{p,q}(v; G) = \{\gamma \in \mathfrak{P}_{p,q}(G) \mid o(\gamma) = v\}$ for a vertex $v \in V$, then the adjacency operator of $G_{[p,q]}(G)$ is given as

$$\mathcal{A}_{G_{[p,q]}} f(v) = \sum_{\gamma \in G_{[p,q]}(G)} f\big(t(\gamma)\big).$$

Considering probabilistic weights of (p, q)-primitive bicolored paths, we have a function $\omega : \mathfrak{P}_{p,q}(G) \to \mathbb{R}$. Hence we get a "weighted graph" $\big(G_{[p,q]}, \omega\big)$.

Lemma 3.2. *For a pair (p, q) of relatively prime positive integers, we have*

$$d_{G_{[p]}}(v) = \sum_{\gamma \in \mathfrak{P}_{p,q}(v)} \omega(\gamma)$$

for each vertex $v \in V$.

Proof. This is a direct consequence of Lemma 3.1. We decompose the set $\mathfrak{P}_{p,q}(v;G)$ into a disjoint union of paths as

$$\mathfrak{P}_{p,q}(v;G) = \bigcup_{\sigma\in\mathfrak{P}_p(v;G^{(p)})} \mathfrak{P}_q(\sigma).$$

We then have

$$\sum_{\gamma\in\mathfrak{P}_{p,q}(v)} \omega(\gamma) = \sum_{\sigma\in\mathfrak{P}_p(v;G^{(p)})} \sum_{\gamma\in\mathfrak{P}_q(\sigma)} \omega(\gamma) = \sum_{\sigma\in\mathfrak{P}_p(v;G^{(p)})} 1 = \sharp\bigl(\mathfrak{P}_p(v;G^{(p)})\bigr) = d_{G_{[p]}}(v).$$

□

For vertices $v, v' \in V$ (which may coincide with each other), we set $\mathfrak{P}_{p,q}(v,v') = \bigl\{\gamma \in \mathfrak{P}_{p,q}(v) \bigm| t(\gamma) = v'\bigr\}$. Since the inverse path γ^{-1} of a (p,q)-bicolored path is not a (p,q)-bicolored path, we see $\mathfrak{P}_{p,q}(v,v') \neq \mathfrak{P}_{p,q}(v',v)$, except the case that both of these sets are empty. We here suppose that

i) G is a finite Kähler graph;
ii) for each pair (v,v') of vertices, there is a bijection $\iota_{v,v'} : \mathfrak{P}_{p,q}(v,v') \to \mathfrak{P}_{p,q}(v',v)$ satisfying $\omega(\gamma) = \omega\bigl(\iota_{v,v'}(\gamma)\bigr)$.

Here, we take the bijections in the above conditions as $\iota_{v',v} = \iota_{v,v'}^{-1}$ for each pair (v,v'). For two primitive bicolored paths $\gamma_1, \gamma_2 \in \mathfrak{P}_{p,q}(G)$, we set $\gamma_1 \approx \gamma_2$ if either $\gamma_1 = \gamma_2$ or $\gamma_1 = \iota_{o(\gamma_2),t(\gamma_2)}(\gamma_2)$ holds. Then it is an equivalence relation on $\mathfrak{P}_{p,q}(G)$. We can define an non-oriented graph $\widehat{G}_{[p,q]} = \bigl(V, \mathfrak{P}_{p,q}(G)/\approx\bigr)$. Under the above assumption we define a non-oriented graph $\widetilde{G}_{[p,q]} = (V, E_{[p,q]})$ so that two vertices $v, v' \in V$ are adjacent to each other if there is $\gamma \in \mathfrak{P}_{p,q}(G)$ satisfying $o(\gamma) = v$ and $t(\gamma) = v'$. We define a function $\mathfrak{m} : E_{[p,q]} \to \mathbb{R}$ by $\mathfrak{m}\bigl((v,v')\bigr) = \sum_{\gamma\in\mathfrak{P}_{p,q}(v,v')} \omega(\gamma)$. We shall call the "weighted graph" $\bigl(\widetilde{G}_{[p,q]}, \mathfrak{m}\bigr)$ the reduced (p,q)-derived graph of G.

Example 3.3. We take a complete Kähler graph G of $n_G = 5$ whose principal and auxiliary degrees are $d_G^{(p)} = d_G^{(a)} = 2$ (see Fig. 7). On this graph $(1,1)$-bicolored paths and $(2,1)$-bicolored paths of origin v_1 are

$$\mathfrak{P}_{1,1}(v_1) = \bigl\{(v_1,v_2,v_4),\ (v_1,v_2,v_5),\ (v_1,v_5,v_2),\ (v_1,v_5,v_3)\bigr\}$$

and

$$\mathfrak{P}_{2,1}(v_1) = \big\{(v_1, v_2, v_3, v_1),\, (v_1, v_2, v_3, v_5),\, (v_1, v_5, v_4, v_1),\, (v_1, v_5, v_4, v_2)\big\}.$$

Thus, the directed edges in $G_{[1,1]}$ and $G_{[2,1]}$ at v_1 are like Figs. 8, 9. Since G is vertex-transitive by rotations, we find that $G_{[1,1]}$ and $G_{[2,1]}$ are like Figs. 10, 11. Therefore $\widetilde{G}_{[1,1]}$ is a complete graph (see Fig. 12) and $\widetilde{G}_{[2,1]}$ are like Fig. 13.

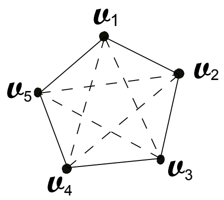

FIG. 7. $G_5^{(2,2)}$

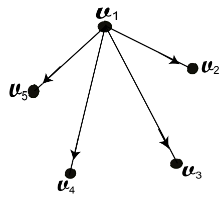

FIG. 8. $E_{[1,1]}$ at v_1

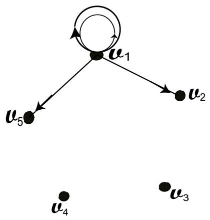

FIG. 9. $E_{[2,1]}$ at v_1

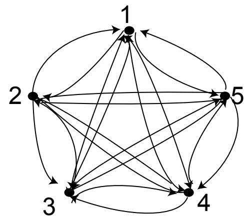

FIG. 10. $G_{[1,1]}$

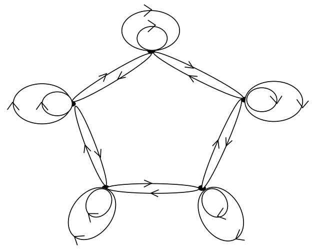

FIG. 11. $G_{[2,1]}$

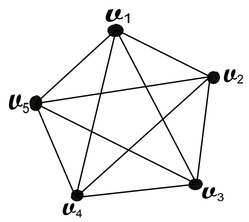

FIG. 12. $\widetilde{G}_{[1,1]}$

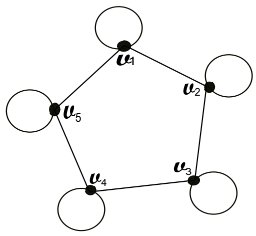

FIG. 13. $\widetilde{G}_{[2,1]}$

We study derived graphs for some Kähler graphs of product types.

EXAMPLE 3.4. When G and H are graphs of real lattice, we consider their Kähler graph of Cartesian product type. Then the edges in its reduced $(1,1)$-derived graph

$(\widetilde{G\widehat{\square}H})_{[1,1]}$ and in the reduced $(2,1)$-derived graph $(\widetilde{G\widehat{\square}H})_{[2,1]}$ at a vertex are like the following figures. They do not have multiple edges.

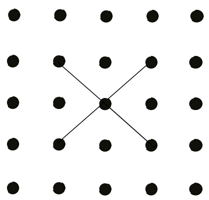

FIG. 14. edges in $(\widetilde{G\widehat{\square}H})_{[1,1]}$ at a vertex

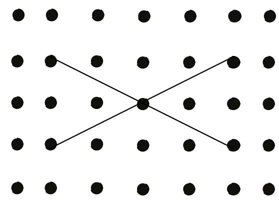

FIG. 15. edges in $(\widetilde{G\widehat{\square}H})_{[2,1]}$ at a vertex

EXAMPLE 3.5. When G and H be graphs of real lattice, we consider their Kähler graphs of strong product type, of semi-tensor product type and of lexicographical product type. Edges in their reduced $(1,1)$-derived graphs $(\widetilde{G\widehat{\boxtimes}H})_{[1,1]}$, $(\widetilde{G\widehat{\otimes}H})_{[1,1]}$ and $(\widetilde{G \triangleright H})_{[1,1]}$ at a vertex are like the following figures. They have multiple edges.

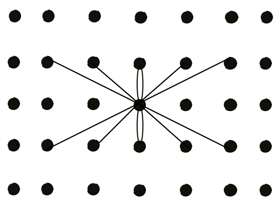

FIG. 16. $(\widetilde{G\widehat{\boxtimes}H})_{[1,1]}$

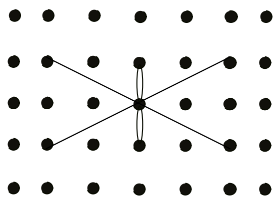

FIG. 17. $(\widetilde{G\widehat{\otimes}H})_{[1,1]}$

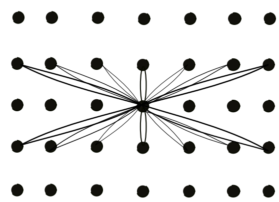

FIG. 18. $(\widetilde{G \triangleright H})_{[1,1]}$

EXAMPLE 3.6. When G and H are graphs of real lattice, edges in the reduced $(1,0)$-derived graph and in the reduced $(1,1)$-derived graph of $G \boxplus H$ at a vertex are like the following figures.

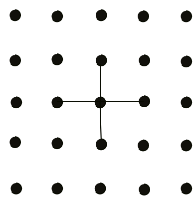

FIG. 19. $(\widetilde{G \boxplus H})_{[1,0]}(v)$

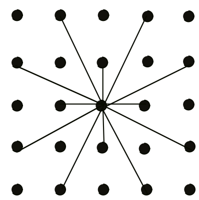

FIG. 20. $(\widetilde{G \boxplus H})_{[1,1]}(v)$

CHAPTER 4

Eigenvalues of $(1,1)$-Laplacians for Kähler graphs

In this chapter we define Laplacians corresponding to bicolored paths on finite Kähler graphs and study their eigenvalues.

1. Definitions of Laplacians for Kähler graphs

Let $G = (V, E^{(p)} \cup E^{(a)})$ be a finite Kähler graph. We denote by $C(V, \mathbb{C})$ the space of all complex valued function on V. As in Chapter 3, for a pair (p, q) of relatively prime positive integers and $v \in V$, we denote by $\mathfrak{P}_{p,q}(v)$ the set of all (p, q)-primitive bicolored paths on G whose origins are v. We define the (p, q)-*adjacency operator* $\mathcal{A}_{(p,q)} = \mathcal{A}_{G(p,q)}$ and the (p, q)-*probabilistic transition operator* $\mathcal{Q}_{(p,q)} = \mathcal{Q}_{G(p,q)}$ acting on $C(V, \mathbb{C})$ are defined as follows:

$$\mathcal{A}_{G(p,q)} f(v) = \sum_{\gamma \in \mathfrak{P}_{p,q}(v)} \omega(\gamma) f\bigl(t(\gamma)\bigr),$$

$$\mathcal{Q}_{G(p,q)} f(v) = \frac{1}{\displaystyle\sum_{\gamma \in \mathfrak{P}_{p,q}(v)} \omega(\gamma)} \sum_{\gamma \in \mathfrak{P}_{p,q}(v)} \omega(\gamma) f\bigl(t(\gamma)\bigr),$$

for each $f \in C(V, \mathbb{C})$. Here, $\omega(\gamma)$ denotes the probabilistic weight of γ (see §3.2). When G is a locally finite Kähler graph, we denote by $L^2(V, \mathbb{C})$ the space of all square summable complex valued function on V. That is,

$$L^2(V, \mathbb{C}) = \Bigl\{ f \in C(V, \mathbb{C}) \Bigm| \sum_{v \in V} |f(v)|^2 < \infty \Bigr\}.$$

We can then define $\mathcal{A}_{G(p,q)}$ and $\mathcal{Q}_{G(p,q)}$ acting on $L^2(V, \mathbb{C})$ by the same way. But in this paper, we only treat the case of finite Kähler graphs.

For a positive p, we denote by $\mathfrak{P}_{p,0}(v)$ the set of all p-step paths on the principal graph $G^{(p)} = (V, E^{(p)})$ whose origins are v and that do not have backtracking. That is we set $\mathfrak{P}_{p,0}(v) = \mathfrak{P}_p(v; G^{(p)})$. We denote the cardinality of this set $\mathfrak{P}_{p,0}(v)$ by $d_{G(p,0)}(v)$, and define the degree operator $\mathcal{D}_{G(p,0)}$ acting on $C(V, \mathbb{C})$ by

$$\mathcal{D}_{G(p,0)} f(v) = d_{G(p,0)}(v) f(v)$$

for each $f \in C(V, \mathbb{C})$. By use of the notation in Chapter 3, we have $\mathcal{D}_{G(p,0)} = \mathcal{D}_{G_{[p]}}$. By using these operators we define the (p, q)-combinatorial Laplacian $\Delta_{\mathcal{A}_{(p,q)}}$ and the (p, q)-probabilistic transitional Laplacian $\Delta_{\mathcal{Q}_{(p,q)}}$ of G acting on $C(V, \mathbb{C})$ by

$$\Delta_{\mathcal{A}_{(p,q)}} = \mathcal{D}_{G(p,0)} - \mathcal{A}_{G(p,q)} \quad \text{and} \quad \Delta_{\mathcal{Q}_{(p,q)}} = \mathcal{I} - \mathcal{Q}_{G(p,q)},$$

where $\mathcal{I}$ denotes the identity operator. We sometimes just call them (p, q)-Laplacians.

Just like we used matrix representations of adjacency and transition operators in §1.2, by using characteristic functions $\delta_v : V \to \mathbb{R}\ (\subset \mathbb{C})$ we use matrix representations $A_{G(p,q)}$ and $Q_{G(p,q)}$ of (p, q)-adjacency and (p, q)-probabilistic transition operators with respect to the basis $\{\delta_v \mid v \in V\}$. Similarly, we use matrix representations $\Delta_{\mathcal{A}_{(p,q)}}$, $\Delta_{\mathcal{Q}_{(p,q)}}$ of $\Delta_{\mathcal{A}_{(p,q)}}$ and $\Delta_{\mathcal{Q}_{(p,q)}}$ with respect to this basis.

1.1. $(1, 1)$**-Laplacians.** First we study the case $(p, q) = (1, 1)$. A $(1, 1)$-bicolored path is a path where principal and auxiliary edges appear alternatively. Just like the fundamental 2-forms on Kähler manifolds and on their real hypersurfaces, which are fundamental magnetic fields of Kähler magnetic fields and Sasakian magnetic fields, $(1, 1)$-bicolored paths show a "fundamental" magnetic structure on a Kähler graphs. We therefore specialize $(1, 1)$-Laplacians of a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$.

We put $\mathcal{A}_G^{(p)} = \mathcal{A}_{G^{(p)}}$, $\mathcal{P}_G^{(p)} = \mathcal{P}_{G^{(p)}}$, which are the adjacency and the transition operators of the principal graph $G^{(p)} = (V, E^{(p)})$, and put $\mathcal{P}_G^{(a)} = \mathcal{P}_{G^{(a)}}$, which is the transition operator of the auxiliary graph $G^{(a)} = (V, E^{(a)})$. Though in §1.2 we define adjacency and transition operators of an ordinary graph as operators acting on the space $C(V, \mathbb{R})$ of real valued functions, we extend them and consider that they act on the space $C(V,\mathbb{C})$. Therefore, we define these three operators as

$$\mathcal{A}_G^{(p)} f(v) = \sum_{v':v'\sim_p v} f(v'), \quad \mathcal{P}_G^{(p)} f(v) = \frac{1}{d_G^{(p)}(v)} \sum_{v':v'\sim_p v} f(v'),$$
$$\mathcal{P}_G^{(a)} f(v) = \frac{1}{d_G^{(a)}(v)} \sum_{v':v'\sim_a v} f(v').$$

First we consider the relationship between $(1,1)$-adjacency and $(1,1)$-probabilistic transition operators and these operators.

LEMMA 4.1. *We have* $\mathcal{A}_{G(1,1)} = \mathcal{A}_G^{(p)}\mathcal{P}_G^{(a)}$ *and* $\mathcal{Q}_{G(1,1)} = \mathcal{P}_G^{(p)}\mathcal{P}_G^{(a)}$.

PROOF. A $(1,1)$-bicolored path $\gamma \in \mathfrak{P}_{1,1}(v)$ of origin v is expressed as $\gamma = (v, v', w)$ with vertices $v', w \in V$ satisfying $v \sim_p v'$ and $v' \sim_a w$. On contrary if we take such vertices then they form a $(1,1)$-bicolored path, because we do not have multiple edges (i.e. $v \neq w$). As we have $\omega(\gamma) = 1/d_G^{(a)}(v')$, we have

$$\begin{aligned}
\mathcal{A}_{(1,1)} f(v) &= \sum_{\substack{(v,v',w)\\ v\sim_p v'\sim_a w}} \frac{1}{d_G^{(a)}(v')} f(w) \\
&= \sum_{v':v'\sim_p v}\ \sum_{w:w\sim_a v'} \frac{1}{d_G^{(a)}(v')} f(w) = \mathcal{A}_G^{(p)}\mathcal{P}_G^{(a)} f(v). \\
\mathcal{Q}_{(1,1)} f(v) &= \frac{1}{d_G^{(p)}(v)} \sum_{\substack{(v,v',w)\\ v\sim_p v'\sim_a w}} \frac{1}{d_G^{(a)}(v')} f(w) \\
&= \frac{1}{d_G^{(p)}(v)} \sum_{v':v'\sim_p v}\ \sum_{w:w\sim_a v'} \frac{1}{d_G^{(a)}(v')} f(w) = \mathcal{P}_G^{(p)}\mathcal{P}_G^{(a)} f(v).
\end{aligned}$$

Hence we get the conclusion. □

By this Lemma, when the principal graph of a Kähler graph is regular as an ordinary graph, we find $\mathcal{A}_{G(1,1)} = d_G^{(p)}\mathcal{P}_{G(1,1)}$. Since $d_{G_{(p,0)}}(v) = d_{G^{(p)}}(v)$, if we denote by $\mathcal{D}_G^{(p)} = \mathcal{D}_{G^{(p)}}$ the degree operator acting on $C(V,\mathbb{C})$ of the principal graph $G^{(p)}$ of G, we have $\Delta_{\mathcal{A}_{(1,1)}} = \mathcal{D}_G^{(p)} - \mathcal{A}_{(1,1)}$. Hence, if the principal graph of a Kähler graph is regular, we have $\Delta_{\mathcal{A}_{(1,1)}} = d_G^{(p)}\Delta_{\mathcal{P}_{(1,1)}}$.

EXAMPLE 4.1. We take a complete Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ of principal and auxiliary degrees $d^{(p)} = d^{(a)} = 2$ and of cardinality of the set of vertices $n_G = 5$. (see Fig. 1). We set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and

$$E^{(p)} = \big\{(v_1, v_2),\ (v_2, v_3),\ (v_3, v_4),\ (v_4, v_5),\ (v_5, v_6)\big\},$$
$$E^{(a)} = \big\{(v_1, v_3),\ (v_3, v_5),\ (v_5, v_2),\ (v_2, v_4),\ (v_4, v_1)\big\}.$$

Its $(1,1)$-adjacency matrix and $(1,1)$-probabilistic transition matrix are

$$A_{G_{(1,1)}} = A_{G^{(p)}} P_{G^{(a)}} = \begin{pmatrix} 0&1&0&0&1\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&1&0&1\\ 1&0&0&1&0 \end{pmatrix} \begin{pmatrix} 0&0&\frac12&\frac12&0\\ 0&0&0&\frac12&\frac12\\ \frac12&0&0&0&\frac12\\ \frac12&\frac12&0&0&0\\ 0&\frac12&\frac12&0&0 \end{pmatrix} = \begin{pmatrix} 0&\frac12&\frac12&\frac12&\frac12\\ \frac12&0&\frac12&\frac12&\frac12\\ \frac12&\frac12&0&\frac12&\frac12\\ \frac12&\frac12&\frac12&0&\frac12\\ \frac12&\frac12&\frac12&\frac12&0 \end{pmatrix},$$

$$Q_{G_{(1,1)}} = P_{G^{(p)}} P_{G^{(a)}} = \begin{pmatrix} 0&\frac12&0&0&\frac12\\ \frac12&0&\frac12&0&0\\ 0&\frac12&0&\frac12&0\\ 0&0&\frac12&0&\frac12\\ \frac12&0&0&\frac12&0 \end{pmatrix} \begin{pmatrix} 0&0&\frac12&\frac12&0\\ 0&0&0&\frac12&\frac12\\ \frac12&0&0&0&\frac12\\ \frac12&\frac12&0&0&0\\ 0&\frac12&\frac12&0&0 \end{pmatrix} = \begin{pmatrix} 0&\frac14&\frac14&\frac14&\frac14\\ \frac14&0&\frac14&\frac14&\frac14\\ \frac14&\frac14&0&\frac14&\frac14\\ \frac14&\frac14&\frac14&0&\frac14\\ \frac14&\frac14&\frac14&\frac14&0 \end{pmatrix}.$$

Therefore, its matrix representations of $(1,1)$-combinatorial and $(1,1)$-probabilistic transitional Laplacians are

$$\Delta_{\mathcal{A}_{(1,1)}} = \begin{pmatrix} 2&0&0&0&0\\ 0&2&0&0&0\\ 0&0&2&0&0\\ 0&0&0&2&0\\ 0&0&0&0&2 \end{pmatrix} - \begin{pmatrix} 0&\frac12&\frac12&\frac12&\frac12\\ \frac12&0&\frac12&\frac12&\frac12\\ \frac12&\frac12&0&\frac12&\frac12\\ \frac12&\frac12&\frac12&0&\frac12\\ \frac12&\frac12&\frac12&\frac12&0 \end{pmatrix} = -\begin{pmatrix} -2&\frac12&\frac12&\frac12&\frac12\\ \frac12&-2&\frac12&\frac12&\frac12\\ \frac12&\frac12&-2&\frac12&\frac12\\ \frac12&\frac12&\frac12&-2&\frac12\\ \frac12&\frac12&\frac12&\frac12&-2 \end{pmatrix},$$

$$\Delta_{\mathcal{Q}_{(1,1)}} = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} - \begin{pmatrix} 0&\frac14&\frac14&\frac14&\frac14\\ \frac14&0&\frac14&\frac14&\frac14\\ \frac14&\frac14&0&\frac14&\frac14\\ \frac14&\frac14&\frac14&0&\frac14\\ \frac14&\frac14&\frac14&\frac14&0 \end{pmatrix} = -\begin{pmatrix} -1&\frac14&\frac14&\frac14&\frac14\\ \frac14&-1&\frac14&\frac14&\frac14\\ \frac14&\frac14&-1&\frac14&\frac14\\ \frac14&\frac14&\frac14&-1&\frac14\\ \frac14&\frac14&\frac14&\frac14&-1 \end{pmatrix}.$$

We note that G is a regular Kähler graph. As we can see, these matrices satisfy $A_{G(1,1)} = 2Q_{G(1,1)}$ and $\Delta_{\mathcal{A}_{(1,1)}} = 2\Delta_{\mathcal{Q}_{(1,1)}}$.

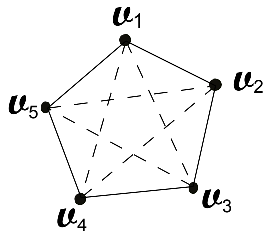

FIG. 1. $G_5^{(2,2)}$

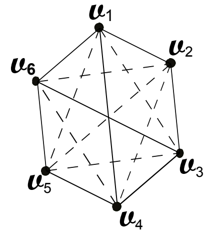

FIG. 2. G_6

EXAMPLE 4.2. We take a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ of $n_G = 6$ which is complete as a graph and that is not regular (see Fig. 2). That is, we set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and

$$E^{(p)} = \big\{(v_1, v_2),\ (v_2, v_3),\ (v_3, v_4),\ (v_4, v_5),\ (v_5, v_6),\ (v_6, v_1),\ (v_1, v_4),\ (v_3, v_6),\big\},$$
$$E^{(a)} = \big\{(v_1, v_3),\ (v_3, v_5),\ (v_5, v_1),\ (v_2, v_4),\ (v_4, v_6),\ (v_6, v_2),\ (v_2, v_5)\big\}.$$

Its $(1,1)$-adjacency matrix and $(1,1)$-probabilistic transition matrix are

$$\begin{aligned} A_{G(1,1)} &= A_{G^{(p)}} P_{G^{(a)}} \\ &= \begin{pmatrix} 0&1&0&1&0&1\\ 1&0&1&0&0&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&0&0&1&0&1\\ 1&0&1&0&1&0 \end{pmatrix} \begin{pmatrix} 0&0&\frac12&0&\frac12&0\\ 0&0&0&\frac13&\frac13&\frac13\\ \frac12&0&0&0&\frac12&0\\ 0&\frac12&0&0&0&\frac12\\ \frac13&\frac13&\frac13&0&0&0\\ 0&\frac12&0&\frac12&0&0 \end{pmatrix} = \begin{pmatrix} 0&1&0&\frac56&\frac13&\frac56\\ \frac12&0&\frac12&0&1&0\\ 0&1&0&\frac56&\frac13&1\\ \frac56&\frac13&\frac56&0&1&0\\ 0&1&0&\frac12&0&\frac12\\ \frac56&\frac13&\frac56&0&1&0 \end{pmatrix}, \\ Q_{G(1,1)} &= P_{G^{(p)}} P_{G^{(a)}} \\ &= \begin{pmatrix} 0&\frac13&0&\frac13&0&\frac13\\ \frac12&0&\frac12&0&0&0\\ 0&\frac13&0&\frac13&0&\frac13\\ \frac13&0&\frac13&0&\frac13&0\\ 0&0&0&\frac12&0&\frac12\\ \frac13&0&\frac13&0&\frac13&0 \end{pmatrix} \begin{pmatrix} 0&0&\frac12&0&\frac12&0\\ 0&0&0&\frac13&\frac13&\frac13\\ \frac12&0&0&0&\frac12&0\\ 0&\frac12&0&0&0&\frac12\\ \frac13&\frac13&\frac13&0&0&0\\ 0&\frac12&0&\frac12&0&0 \end{pmatrix} = \begin{pmatrix} 0&\frac13&0&\frac{5}{18}&\frac19&\frac{5}{18}\\ \frac14&0&\frac12&0&\frac12&0\\ 0&\frac13&0&\frac{5}{18}&\frac19&\frac{5}{18}\\ \frac{5}{18}&\frac19&\frac{5}{18}&0&\frac13&0\\ 0&\frac12&0&\frac14&0&\frac14\\ \frac{5}{18}&\frac19&\frac{5}{18}&0&\frac13&0 \end{pmatrix}. \end{aligned}$$

Therefore, its matrix representations of $(1,1)$-combinatorial and $(1,1)$-probabilistic transitional Laplacians are

$$\Delta_{\mathcal{A}_{(1,1)}} = \begin{pmatrix} 3&0&0&0&0&0\\ 0&2&0&0&0&0\\ 0&0&3&0&0&0\\ 0&0&0&3&0&0\\ 0&0&0&0&2&0\\ 0&0&0&0&0&3 \end{pmatrix} - \begin{pmatrix} 0&1&0&\frac{5}{6}&\frac{1}{3}&\frac{5}{6}\\ \frac{1}{2}&0&\frac{1}{2}&0&1&0\\ 0&1&0&\frac{5}{6}&\frac{1}{3}&1\\ \frac{5}{6}&\frac{1}{3}&\frac{5}{6}&0&1&0\\ 0&1&0&\frac{1}{2}&0&\frac{1}{2}\\ \frac{5}{6}&\frac{1}{3}&\frac{5}{6}&0&1&0 \end{pmatrix} = - \begin{pmatrix} -3&1&0&\frac{5}{6}&\frac{1}{3}&\frac{5}{6}\\ \frac{1}{2}&-2&\frac{1}{2}&0&1&0\\ 0&1&-3&\frac{5}{6}&\frac{1}{3}&1\\ \frac{5}{6}&\frac{1}{3}&\frac{5}{6}&-3&1&0\\ 0&1&0&\frac{1}{2}&-2&\frac{1}{2}\\ \frac{5}{6}&\frac{1}{3}&\frac{5}{6}&0&1&-3 \end{pmatrix},$$

$$\Delta_{\mathcal{Q}_{(1,1)}} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} - \begin{pmatrix} 0&\frac{1}{3}&0&\frac{5}{18}&\frac{1}{9}&\frac{5}{18}\\ \frac{1}{4}&0&\frac{1}{2}&0&\frac{1}{2}&0\\ 0&\frac{1}{3}&0&\frac{5}{18}&\frac{1}{9}&\frac{5}{18}\\ \frac{5}{18}&\frac{1}{9}&\frac{5}{18}&0&\frac{1}{3}&0\\ 0&\frac{1}{2}&0&\frac{1}{4}&0&\frac{1}{4}\\ \frac{5}{18}&\frac{1}{9}&\frac{5}{18}&0&\frac{1}{3}&0 \end{pmatrix} = - \begin{pmatrix} -1&\frac{1}{3}&0&\frac{5}{18}&\frac{1}{9}&\frac{5}{18}\\ \frac{1}{4}&-1&\frac{1}{2}&0&\frac{1}{2}&0\\ 0&\frac{1}{3}&-1&\frac{5}{18}&\frac{1}{9}&\frac{5}{18}\\ \frac{5}{18}&\frac{1}{9}&\frac{5}{18}&-1&\frac{1}{3}&0\\ 0&\frac{1}{2}&0&\frac{1}{4}&-1&\frac{1}{4}\\ \frac{5}{18}&\frac{1}{9}&\frac{5}{18}&0&\frac{1}{3}&-1 \end{pmatrix}.$$

1.2. (p,q)**-step Laplacian.** Next we study general (p,q). For a finite Kähler graph $G=(V,E^{(p)}\cup E^{(a)})$, we denote by $\mathcal{A}^{(p)}_{(p,0)}$ and $\mathcal{P}^{(p)}_{(p,0)}$ the p-step adjacency operator and the p-step transition operator of the principal graph $G^{(p)}=(V,E^{(p)})$ which act on $C(V,\mathbb{C})$, respectively. That is, we define these operators as

$$\mathcal{A}^{(p)}_{(p,0)}f(v) = \sum_{\sigma\in\mathfrak{P}_{p,0}(v)} f\bigl(t(\sigma)\bigr), \qquad \mathcal{P}^{(p)}_{(p,0)}f(v) = \frac{1}{d^{(p)}_{(p,0)}(v)} \sum_{\sigma\in\mathfrak{P}_{p,0}(v)} f\bigl(t(\sigma)\bigr).$$

In other words, we set $\mathcal{A}^{(p)}_{(p,0)} = \mathcal{A}_{G^{(p)}_{[p]}}$ and $\mathcal{P}^{(p)}_{(p,0)} = \mathcal{P}_{G^{(p)}_{[p]}}$. We denote by $\mathfrak{P}_{0,q}(v)$ the set of all q-step paths on the auxiliary graph $G^{(a)}=(V,E^{(a)})$ without backtracking whose origins are v. That is, we set $\mathfrak{P}_{0,q}(v)=\mathfrak{P}_q(v;G^{(a)})$. For each $\rho\in\mathfrak{P}_{0,q}(v)$ we define its probabilistic weight $\omega(\rho)$ by regarding it as $(0,q)$-primitive bicolored path. That is, when $\rho=(w_0,w_1,\ldots,w_q)$ we set

$$\omega(\rho) = \frac{1}{d^{(a)}_G(w_0)\bigl(d^{(a)}_G(w_1)-1\bigr)\cdots\bigl(d^{(a)}_G(w_{q-1})-1\bigr)}.$$

We define q-step probabilistic transition operator $\mathcal{Q}^{(a)}_{(0,q)}$ of $G^{(a)}$ acting on $C(V,\mathbb{C})$ by

$$\mathcal{Q}^{(a)}_{(0,q)}f(v) = \sum_{\rho\in\mathfrak{P}_{0,q}(v)} \omega(\rho)f\big(t(\rho)\big).$$

Here, we define $\mathcal{P}^{(a)}_{(0,q)}$ acting on $C(V,\mathbb{C})$ by

$$\mathcal{P}^{(a)}_{(0,q)}f(v) = \frac{1}{d^{(a)}_{(0,q)}(v)} \sum_{\rho\in\mathfrak{P}_{0,q}(v)} f\big(t(\rho)\big)$$

with the cardinality $d_{G(0,q)}(v)$ of the set of $\mathfrak{P}_{0,q}(v)$. That is, we set $\mathcal{P}^{(a)}_{(0,q)} = \mathcal{P}_{G^{(a)}_{[q]}}$. We should note that $\mathcal{Q}^{(a)}_{(0,q)} \neq \mathcal{P}^{(a)}_{(0,q)}$ in general.

Lemma 4.2. *When $G^{(a)}$ is regular, we have $\mathcal{Q}^{(a)}_{(0,q)} = \mathcal{P}^{(a)}_{(0,q)}$.*

Proof. When $G^{(a)}$ is regular, we have

$$\omega(\rho) = \frac{1}{d^{(a)}_G (d^{(a)}_G - 1)^{q-1}} = d_{G(0,q)}\big(o(\rho)\big).$$

Hence we get the conclusion. □

By using these operators we can decompose the (p,q)-adjacency and (p,q)-probabilistic transition operators as follows.

Lemma 4.3. *We have $\mathcal{A}_{(p,q)} = \mathcal{A}^{(p)}_{(p,0)}\mathcal{Q}^{(a)}_{(0,q)}$ and $\mathcal{Q}_{(p,q)} = \mathcal{P}^{(p)}_{(p,0)}\mathcal{Q}^{(a)}_{(0,q)}$.*

Proof. As we can decompose $\mathfrak{P}_{p,q}(v)$ as $\mathfrak{P}_{p,q}(v) = \bigcup_{\sigma\in\mathfrak{P}_{p,0}(v)} \mathfrak{P}_q(\sigma)$, by direct computation we have

$$\begin{aligned}\mathcal{A}_{(p,q)}f(v) &= \sum_{\gamma\in\mathfrak{P}_{p,q}(v)} \omega(\gamma)f(t(\gamma)) = \sum_{\sigma\in\mathfrak{P}_{p,0}(v)}\sum_{\rho\in\mathfrak{P}_q(\sigma)} \omega(\rho)f(t(\rho))\\ &= \sum_{\sigma\in\mathfrak{P}_{p,0}(v)} \big(\mathcal{Q}^{(a)}_{(0,q)}f\big)(t(\sigma)) = \mathcal{A}^{(p)}_{(p,0)}\mathcal{Q}^{(a)}_{(0,q)}f(v),\end{aligned}$$

$$
\begin{aligned}
\mathcal{P}_{G(p,q)} f(v) &= \frac{1}{\displaystyle\sum_{\gamma\in\mathfrak{P}_{p,q}(v)} \omega(\gamma)} \sum_{\gamma\in\mathfrak{P}_{p,q}(v)} \omega(\gamma) f(t(\gamma)) \\
&= \frac{1}{d_{G(p,0)}} \sum_{\sigma\in\mathfrak{P}_{p,0}(v)} \sum_{\rho\in\mathfrak{P}_q(\sigma)} \omega(\rho) f(t(\rho)) \\
&= \frac{1}{d_{G(p,0)}} \sum_{\sigma\in\mathfrak{P}_{p,0}(v)} \big(\mathcal{Q}_q^{(a)} f\big)(t(\sigma)) \\
&= \mathcal{P}^{(p)}_{(p,0)} \mathcal{Q}^{(a)}_{(0,q)} f(v)
\end{aligned}
$$

with Lemma 3.2. We hence get the conclusion. □

Example 4.3. For the Kähler graph G in Example 4.1, the $(1,2)$-adjacency matrix and $(1,2)$-probabilistic transition matrix are

$$
A_{G_{(1,2)}} = A_{G^{(p)}} P^{(a)}_{(0,2)} = \begin{pmatrix} 0&1&0&0&1\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&1&0&1\\ 1&0&0&1&0 \end{pmatrix} \begin{pmatrix} 0&\frac12&0&0&\frac12\\ \frac12&0&\frac12&0&0\\ 0&\frac12&0&\frac12&0\\ 0&0&\frac12&0&\frac12\\ \frac12&0&0&\frac12&0 \end{pmatrix} = \begin{pmatrix} 1&0&\frac12&\frac12&0\\ 0&1&0&\frac12&\frac12\\ \frac12&0&1&0&\frac12\\ \frac12&\frac12&0&1&0\\ 0&\frac12&\frac12&0&1 \end{pmatrix},
$$

$$
Q_{G_{(1,2)}} = P_{G^{(p)}} P^{(a)}_{(0,2)} = \begin{pmatrix} 0&\frac12&0&0&\frac12\\ \frac12&0&\frac12&0&0\\ 0&\frac12&0&\frac12&0\\ 0&0&\frac12&0&\frac12\\ \frac12&0&0&\frac12&0 \end{pmatrix} \begin{pmatrix} 0&\frac12&0&0&\frac12\\ \frac12&0&\frac12&0&0\\ 0&\frac12&0&\frac12&0\\ 0&0&\frac12&0&\frac12\\ \frac12&0&0&\frac12&0 \end{pmatrix} = \begin{pmatrix} \frac12&0&\frac14&\frac14&0\\ 0&\frac12&0&\frac14&\frac14\\ \frac14&0&\frac12&0&\frac14\\ \frac14&\frac14&0&\frac12&0\\ 0&\frac14&\frac14&0&\frac12 \end{pmatrix}.
$$

Its $(2,1)$-adjacency matrix and $(1,2)$-probabilistic transition matrix are

$$
A_{G_{(2,1)}} = A^{(p)}_{(2,0)} P_{G^{(a)}} = \begin{pmatrix} 0&0&1&1&0\\ 0&0&0&1&1\\ 1&0&0&0&1\\ 1&1&0&0&0\\ 0&1&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&\frac12&\frac12&0\\ 0&0&0&\frac12&\frac12\\ \frac12&0&0&0&\frac12\\ \frac12&\frac12&0&0&0\\ 0&\frac12&\frac12&0&0 \end{pmatrix} = \begin{pmatrix} 1&\frac12&0&0&\frac12\\ \frac12&1&\frac12&0&0\\ 0&\frac12&1&\frac12&0\\ 0&0&\frac12&1&\frac12\\ \frac12&0&0&\frac12&1 \end{pmatrix},
$$

$$
Q_{G_{(2,1)}} = P^{(p)}_{(2,0)} P_{G^{(a)}} = \begin{pmatrix} 0&0&\frac12&\frac12&0\\ 0&0&0&\frac12&\frac12\\ \frac12&0&0&0&\frac12\\ \frac12&\frac12&0&0&0\\ 0&\frac12&\frac12&0&0 \end{pmatrix} \begin{pmatrix} 0&0&\frac12&\frac12&0\\ 0&0&0&\frac12&\frac12\\ \frac12&0&0&0&\frac12\\ \frac12&\frac12&0&0&0\\ 0&\frac12&\frac12&0&0 \end{pmatrix} = \begin{pmatrix} \frac12&\frac14&0&0&\frac14\\ \frac14&\frac12&\frac14&0&0\\ 0&\frac14&\frac12&\frac14&0\\ 0&0&\frac14&\frac12&\frac14\\ \frac14&0&0&\frac14&\frac12 \end{pmatrix}.
$$

Therefore, its matrix representations of $(1,2)$-combinatorial and $(1,2)$-probabilistic transitional Laplacians are

$$\Delta_{\mathcal{A}_{(1,2)}} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix} = - \begin{pmatrix} -1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -1 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & -1 \end{pmatrix},$$

$$\Delta_{\mathcal{Q}_{(1,2)}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix} = - \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{2} \end{pmatrix}.$$

Its matrix representations of $(2,1)$-combinatorial and $(2,1)$-probabilistic transitional Laplacians are given as

$$\Delta_{\mathcal{A}_{(2,1)}} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 \end{pmatrix} = - \begin{pmatrix} -1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & -1 \end{pmatrix},$$

$$\Delta_{\mathcal{Q}_{(2,1)}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix} = - \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{2} \end{pmatrix}.$$

Example 4.4. For the Kähler graph G in Example 4.2, the $(1,2)$-adjacency matrix and $(1,2)$-probabilistic transition matrix are

$$A_{G_{(1,2)}} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \frac{7}{12} & \frac{1}{2} & \frac{7}{12} & \frac{1}{6} & 1 & \frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{7}{12} & \frac{1}{2} & \frac{7}{12} & \frac{1}{6} & 1 & \frac{1}{6} \end{pmatrix},$$

$$Q_{G_{(1,2)}} = \begin{pmatrix} 0 & \frac13 & 0 & \frac13 & 0 & \frac13 \\ \frac12 & 0 & \frac12 & 0 & 0 & 0 \\ 0 & \frac13 & 0 & \frac13 & 0 & \frac13 \\ \frac13 & 0 & \frac13 & 0 & \frac13 & 0 \\ 0 & 0 & 0 & \frac12 & 0 & \frac12 \\ \frac13 & 0 & \frac13 & 0 & \frac13 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac14 & \frac14 & 0 & \frac12 & 0 \\ 0 & 0 & 0 & \frac12 & 0 & \frac12 \\ \frac14 & \frac14 & 0 & 0 & \frac12 & 0 \\ 0 & \frac12 & 0 & 0 & 0 & \frac12 \\ \frac13 & 0 & \frac13 & \frac16 & 0 & \frac16 \\ 0 & \frac12 & 0 & \frac12 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac13 & 0 & \frac13 & 0 & \frac13 \\ \frac18 & \frac14 & \frac18 & 0 & \frac12 & 0 \\ 0 & \frac13 & 0 & \frac13 & 0 & \frac13 \\ \frac{7}{36} & \frac16 & \frac{7}{36} & \frac{1}{18} & \frac13 & \frac{1}{18} \\ 0 & \frac12 & 0 & \frac14 & 0 & \frac14 \\ \frac{7}{36} & \frac16 & \frac{7}{36} & \frac{1}{18} & \frac13 & \frac{1}{18} \end{pmatrix}.$$

Its $(2,1)$-adjacency matrix and $(1,2)$-probabilistic transition matrix are

$$A_{G_{(2,1)}} = \begin{pmatrix} 0 & 0 & 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 3 \\ 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac12 & 0 & \frac12 & 0 \\ 0 & 0 & 0 & \frac13 & \frac13 & \frac13 \\ \frac12 & 0 & 0 & 0 & \frac12 & 0 \\ 0 & \frac12 & 0 & 0 & 0 & \frac12 \\ \frac13 & \frac13 & \frac13 & 0 & 0 & 0 \\ 0 & \frac12 & 0 & \frac12 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{13}{6} & \frac23 & \frac23 & 0 & \frac32 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ \frac23 & \frac23 & \frac53 & 0 & 1 & 0 \\ 0 & \frac32 & 0 & \frac{13}{6} & \frac23 & \frac23 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & \frac32 & 0 & \frac32 & \frac32 & \frac{13}{6} \end{pmatrix},$$

$$Q_{G_{(2,1)}} = \begin{pmatrix} 0 & 0 & \frac35 & 0 & \frac25 & 0 \\ 0 & 0 & 0 & \frac12 & 0 & \frac12 \\ \frac12 & 0 & 0 & 0 & \frac12 & 0 \\ 0 & \frac25 & 0 & 0 & 0 & \frac35 \\ \frac12 & 0 & \frac12 & 0 & 0 & 0 \\ 0 & \frac25 & 0 & \frac35 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac12 & 0 & \frac12 & 0 \\ 0 & 0 & 0 & \frac13 & \frac13 & \frac13 \\ \frac12 & 0 & 0 & 0 & \frac12 & 0 \\ 0 & \frac12 & 0 & 0 & 0 & \frac12 \\ \frac13 & \frac13 & \frac13 & 0 & 0 & 0 \\ 0 & \frac12 & 0 & \frac12 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{13}{30} & \frac{2}{15} & \frac{2}{15} & 0 & \frac{3}{10} & 0 \\ 0 & \frac12 & 0 & \frac14 & 0 & \frac14 \\ \frac16 & \frac16 & \frac{5}{12} & 0 & \frac14 & 0 \\ 0 & \frac{3}{10} & 0 & \frac{13}{30} & \frac{2}{15} & \frac{2}{15} \\ \frac14 & 0 & \frac14 & 0 & \frac12 & 0 \\ 0 & \frac{3}{10} & 0 & \frac{2}{15} & \frac{2}{15} & \frac{13}{30} \end{pmatrix}.$$

2. $(1,1)$-Laplacians of complement-filled Kähler graphs

In this section and the following four sections, we study eigenvalues of $(1,1)$-Laplacians for typical series of Kähler graphs.

2.1. Eigenvaleus of $(1,1)$-Laplacians of complement-filled Kähler graphs. First we study complement-filled Kähler graphs. For an ordinary graph $G = (V, E)$ we define operators $\mathcal{M}$ and $\mathcal{N}$ acting on $C(V, \mathbb{R})$ by

$$\mathcal{M}f(v) = \sum_{w \in V} f(w) \qquad \text{and} \qquad \mathcal{N} = \mathcal{M} - \mathcal{I}.$$

The matrix representation M of $\mathcal{M}$ with respect to the canonical basis $\{\delta_v \mid v \in V\}$ is a square matrix of degree n_G all of whose components are 1, that is

$$M = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix},$$

and the matrix representation of $\mathcal{N}$ is $N = M - I$ with an identity matrix I. Hence we have

$$N = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}.$$

THEOREM 4.1. *Let $G = (V, E)$ be a connected regular finite ordinary graph whose degree satisfies $2 \le d_G \le n_G - 3$. We denote the eigenvalues of $\Delta_{\mathcal{A}_G}$ of G as $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_{n_G}$. Then the eigenvalues of the $(1,1)$-adjacency Laplacian $\Delta_{\mathcal{A}_{(1,1)}}$ of its complement-filled Kähler graph G^K are*

$$\widehat{\lambda}_1 = 0,\ \widehat{\lambda}_i = \{\lambda_i^2 - \lambda_i(2d_G + 1) + n_G d_G\}(n_G - d_G - 1)^{-1} \quad (i = 2, \cdots, n_G).$$

Moreover if $f_i : V \to \mathbb{R}$ is an eigenfunction corresponding to λ_i, then it is an eigenfunction corresponding to $\widehat{\lambda}_i$.

PROOF. The adjacency matrix A_{G^c} of the complement graph G^c of G is given by $A_G^c = N - A_G = M - I - A_G$. In particular, we have $d_{G^c}(v) = n_G - 1 - d_G(v)$ at each $v \in V$.

We take an eigenfunction $f_i : V \to \mathbb{R}$ corresponding to the eigenvalue λ_i. We then have

$$\lambda_i f_i = \Delta_{A_G} f_i = (D_G - A_G) f_i = d_G f_i - A_G f_i,$$

hence get $A_G f_i = (d_G - \lambda_i) f_i$. Since our graph $G = (V, E)$ is regular, its complement graph G^c is also regular. We hence have

$$\mathcal{A}_{G^K(1,1)} = \mathcal{A}_G \frac{1}{d_G^c} A_G^c = \frac{1}{d_{G^c}} \mathcal{A}_G \mathcal{A}_{G^c} = \frac{1}{d_{G^c}} \mathcal{A}_G \big(\mathcal{M} - \mathcal{I} - \mathcal{A}_G\big).$$

Since G is connected regular, the multiplicity of null eigenvalues is one and corresponding eigenfunctions are constant (see Proposition 1.8).

1) For $\lambda_1 = 0$, we take a corresponding eigenfunction f_1 which is non-zero constant. We then have

$$A_{G^c} f_1 = \mathcal{N} f_1 - A_G f_1 = (n_G - 1 - d_G) f_1.$$

Therefore, we find that

$$\begin{aligned}\Delta_{\mathcal{A}_{G^K(1,1)}} f_1 &= (\mathcal{D}_G - \mathcal{A}_{G^K(1,1)}) f_1 = d_G f_1 - \frac{1}{n_G - d_G - 1} \mathcal{A}_G \mathcal{A}_{G^c} f_1 \\ &= d_G f_1 - \mathcal{A}_G f_1 = d_G f_1 - d_G f_1 = 0.\end{aligned}$$

2) For λ_i $(i \geq 2)$, as G is connected, we have $\lambda_i \neq 0$. Thus by Note 1.1, f_i is orthogonal to f_1. That is, as f_1 is a constant function, we have

$$0 = \langle f_1, f_i \rangle = \sum_{v \in V} f_1(v) f_i(v) = f_1(*) \sum_{v \in V} f_i(v),$$

where $*$ denotes an arbitrary vertex, we hence have $\sum_{v \in V} f_i(v) = 0$. Therefore we get

$$\mathcal{A}_{G^c} f_i(v) = (\mathcal{M} - \mathcal{I} - \mathcal{A}_G) f_i(v) = \sum_{w \in V} f_i(w) - f_i(v) - (d_G - \lambda_i) f_i(v) = (\lambda_i - d_G - 1)) f_i(v).$$

So that we have

$$\begin{aligned}\Delta_{\mathcal{A}_{G^K_{(1,1)}}} f_i &= (\mathcal{D}_G - \frac{1}{d_G^c} \mathcal{A}_G \mathcal{A}_{G^c}) f_i = d_G f_i - \frac{1}{(n_G - d_G - 1)} (\lambda_i - d_G - 1) \mathcal{A}_G f_i \\ &= \left(d_G - \frac{(d_G - \lambda_i)(\lambda_i - d_G - 1)}{n_G - 1 - d_G} \right) f_i = \frac{\lambda_i^2 - \lambda_i (2 d_G + 1) + n_G d_G}{n_G - d_G - 1} f_i.\end{aligned}$$

This completes the proof. □

Remark 4.1. Since we have $\Delta_{\mathcal{Q}_{G^K_{(1,1)}}} = \frac{1}{d_G} \Delta_{\mathcal{A}_{G^K_{(1,1)}}}$ for a regular graph G, we have

1) When $\lambda_1 = 0$,

$$\Delta_{\mathcal{P}_{G^K(1,1)}} f_1 = f_1 - \frac{1}{d_G} A_G \frac{1}{d_G^c} A_G^c f_1 = f_1 - f_1 = 0.$$

2) When $\lambda_i \quad (i \geq 2)$,

$$\Delta_{\mathcal{P}_{G^K(1,1)}} f_i = \frac{\lambda_i^2 - \lambda_i(2d_G + 1) + n_G d_G}{d_G(n_G - d_G - 1)}, \qquad (i = 2, \cdots, n_G).$$

Let G be a finite Kähler graph. If the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ are $\lambda_1, \lambda_2, \ldots, \lambda_{n_G}$, we denote as $\mathrm{Spec}(\Delta_{\mathcal{A}_{(1,1)}}) = \{\lambda_1, \lambda_2, \ldots, \lambda_{n_G}\}$ according to convention. Though we use the notation of sets, we write the same eigenvalues according to their multiplicities. Similarly, we use $\mathrm{Spec}(\Delta_{\mathcal{P}_{(1,1)}})$ for the eigenvalues of $\Delta_{\mathcal{P}_{(1,1)}}$.

EXAMPLE 4.5. We take a 5-circuit G (see Fig. 3) and consider its complement-filled Kähler graph G^K. It is a regular Kähler graph (see Fig. 4).

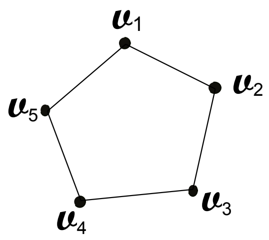

FIG. 3. 5-circuit

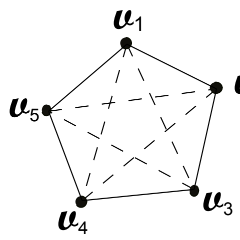

FIG. 4. G^K

The adjacency Laplacian of 5-circuit is represented as

$$\Delta_{\mathcal{A}_G} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} = - \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix},$$

hence its eigenvalues are

$$\mathrm{Spec}(\Delta_{\mathcal{A}_G}) = \Big\{0,\ \frac{1}{2}(5-\sqrt{5}),\ \frac{1}{2}(5-\sqrt{5}),\ \frac{1}{2}(5+\sqrt{5}),\ \frac{1}{2}(5+\sqrt{5})\Big\}.$$

Since we have

$$A_{G^K_{(1,1)}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

the eigenvalues of $(1,1)$-adjacency Laplacian and of $(1,1)$-probabilistic transition Laplacian of G^K are

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^K_{(1,1)}}}) = \left\{0,\ \frac{5}{2},\ \frac{5}{2},\ \frac{5}{2},\ \frac{5}{2}\right\} \quad \text{and} \quad \mathrm{Spec}(\Delta_{\mathcal{Q}_{G^K_{(1,1)}}}) = \left\{0,\ \frac{5}{4},\ \frac{5}{4},\ \frac{5}{4},\ \frac{5}{4}\right\}.$$

When an ordinary graph is not regular, the eigenvalues of its complement-filled Kähler graph are not necessarily real in general.

Proposition 4.1. *Let G be a non-regular ordinary graph. Then $\mathcal{A}_G$ and $\mathcal{P}_{G^c}$ are not simultaneously diagonalizable.*

Proof. Since we have $\mathcal{A}_{G^c} = \mathcal{M} - \mathcal{I} - \mathcal{A}_G$, we find that $\mathcal{A}_G \circ \mathcal{A}_{G^c} = \mathcal{A}_{G^c} \circ \mathcal{A}_G$ if and only if $\mathcal{A}_G \circ \mathcal{M} = \mathcal{M} \circ \mathcal{A}_G$.

For an arbitrary $f \in C(V, \mathbb{C})$ we have

$$\mathcal{M}_G\mathcal{A}_G f(v) = \sum_{w\in V}\ \sum_{w':w'\sim w} f(w') = \sum_{w'\in V}\ \sum_{w:w\sim w'} f(w') = \sum_{w'\in V} d_G(w')f(w').$$

Hence $\mathcal{M}\mathcal{A}_G f$ is a constant function. On the other hand, we have

$$\mathcal{A}_G\mathcal{M}f(v) = d_G(v)\sum_{w\in V} f(w),$$

which is not constant, because G is not regular. Thus we get the conclusion. □

Example 4.6. We take a complement-filled Kähler graph G of $n_G = 6$ like Fig. 5. Its $(1,1)$-adjacency matrix $\mathcal{A}_{G_{(1,1)}}$ is given as

$$A_{G_{(1,1)}} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{5}{6} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & \frac{5}{6} \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 & \frac{5}{6} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{5}{6} \\ \frac{2}{3} & \frac{1}{3} & \frac{5}{6} & 0 & \frac{5}{6} & \frac{1}{3} \\ 0 & \frac{5}{6} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{5}{6} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

and its (1,1)-probabilistic transition matrix $\mathcal{Q}_{G_{(1,1)}}$ is given as

$$Q_{G_{(1,1)}} = \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{5}{18} & \frac{1}{9} & \frac{2}{9} & \frac{1}{9} & \frac{5}{18} \\ \frac{1}{6} & 0 & \frac{1}{4} & 0 & \frac{5}{12} & \frac{1}{6} \\ 0 & \frac{1}{4} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{5}{12} \\ \frac{2}{9} & \frac{1}{9} & \frac{5}{18} & 0 & \frac{5}{18} & \frac{1}{9} \\ 0 & \frac{5}{12} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{6} & \frac{5}{12} & 0 & \frac{1}{4} & 0 \end{pmatrix}.$$

The eigenvalues of the combinatorial Laplacian of the principal graph and those of (1,1)-combinatorial and (1,1)-probabilistic transitional Laplacians are as follows;

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^{(p)}}}) = \{0, 1, 2, 3, 3, 5\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(1,1)}}}) = \Big\{0, 2, 4, \frac{8}{3}, \frac{8}{3}, \frac{8}{3}\Big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{Q}_{G_{(1,1)}}}) = \Big\{0, 1, \frac{4}{3}, \frac{17}{18}, \frac{(49+\sqrt{97})}{36}, \frac{(49-\sqrt{97})}{36}\Big\}.$$

Those eigenvalues do not satisfy Theorem 4.1 because $G^{(p)}$ is not regular.

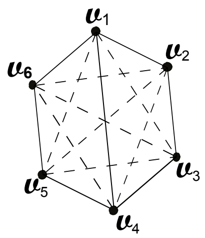

Fig. 5

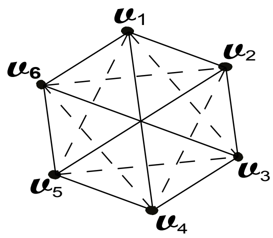

Fig. 6

We note the difference of the principal degree and the auxiliary degree of a regular Kähler graph does not give influence in Theorem 4.1.

Example 4.7. We take a complement-filled Kähler graph of an ordinary regular graph of degree 3 (see Fig. 6). Its (1,1)-adjacency matrix is given as

$$A_{G_{(1,1)}} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

and $Q_{G_{(1,1)}} = \frac{1}{3}A_{G_{(1,1)}}$. Eigenvalues of $(1,1)$-combinatorial and $(1,1)$-probabilistic transitional Laplacians are

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,1)}}) = \{0,3,3,3,3,6\} \quad \text{and} \quad \mathrm{Spec}(\Delta_{\mathcal{Q}_{G(1,1)}}) = \{0,1,1,1,1,2\}.$$

Next we extend the previous result to non-connected regular graphs.

THEOREM 4.2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected regular ordinary graphs. We denote the eigenvalues $\Delta_{\mathcal{A}_{G_1}}$ by $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_{n_{G_1}}$ and those of $\Delta_{\mathcal{A}_{G_2}}$ by $0 = \eta_1 < \eta_2 \le \cdots \le \eta_{n_{G_2}}$. We set $G = (V_1 + V_2, E_1 + E_2)$ the disjoint union of G_1 and G_2. Then the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ of G^K are*

$$\begin{aligned} &0,\ d_{G_1} - \frac{(d_{G_1}-\lambda_j)(\lambda_j - d_{G_1} - 1)}{\widehat{d}_{G_1}}, \\ &d_{G_2} - \frac{(d_{G_2}-\eta_k)(\eta_k - d_{G_1} - 1)}{\widehat{d}_{G_2}}, \quad \frac{d_{G_1} n_{G_2}}{\widehat{d}_{G_1}} + \frac{d_{G_2} n_{G_1}}{\widehat{d}_{G_2}} \\ &\qquad\qquad j = 2, \dots, n_{G_1},\ k = 2, \dots, n_{G_2}, \end{aligned}$$

where $\widehat{d}_{G_i} = n_{G_1} + n_{G_2} - d_{G_i} - 1$ for $i = 1, 2$.

PROOF. We denote by $M_{i\ell}$ a $n_{G_i} \times n_{G_\ell}$-matrix all of whose components are 1. By using the adjacency matrices $A_{G_j}, A_{G_j^c}$ of G_j and its complement graph G_j^c, we can express the adjacency matrix $A^{(p)}_{G^K}$ and the transition matrix $P^{(a)}_{G^K}$ as

$$A^{(p)}_{G^K} = \begin{pmatrix} A_{G_1} & \vdots & O \\ \cdots & & \cdots \\ O & \vdots & A_{G_2} \end{pmatrix}, \qquad P^{(a)}_{G^K} = \begin{pmatrix} \frac{1}{\widehat{d}_{G_1}} A_{G_1^c} & \vdots & \frac{1}{\widehat{d}_{G_1}} M_{12} \\ \cdots & & \cdots \\ \frac{1}{\widehat{d}_{G_2}} M_{21} & \vdots & \frac{1}{\widehat{d}_{G_2}} A_{G_2^c} \end{pmatrix}.$$

We take an eigenfunction $f_j : V_1 \to \mathbb{R}$ corresponding to the eigenvalue λ_j and an eigenfunction $g_k : V_2 \to \mathbb{R}$ associated with η_k. We define $\widehat{f}_j, \widehat{g}_k : V \to \mathbb{R}$ by

$$\widehat{f}_j(v) = \begin{cases} f_j(v), & \text{when } v \in V_1, \\ 0, & \text{when } v \in V_2, \end{cases} \qquad \widehat{g}_k(v) = \begin{cases} 0, & \text{when } v \in V_1, \\ g_k(v), & \text{when } v \in V_2. \end{cases}$$

Since G_1, G_2 are connected, we have

$$\sum_{v \in V} \widehat{f}_j(v) = \sum_{v \in V_1} f_j(v) = 0, \qquad \sum_{v \in V} \widehat{g}_k(v) = \sum_{v \in V_2} g_k(v) = 0$$

for $j \geq 2$ and $k \geq 2$. We define $\widehat{\mathcal{A}}_{G_i}$ acting on $C(V, \mathbb{C})$ by

$$\widehat{\mathcal{A}}_{G_i} h(v) = \begin{cases} \mathcal{A}_{G_i} h|_{V_i}(v), & \text{when } v \in V_i, \\ 0, & \text{when } v \notin V_i, \end{cases}$$

where $h|_{V_i}$ denotes the restriction of h onto V_i. Then, as $\mathcal{A}_{G_1} f_j = (d_{G_1} - \lambda_j) f_j$ and $\mathcal{A}_{G_2} g_k = (d_{G_2} - \eta_k) g_k$, we have $\widehat{\mathcal{A}}_{G_1} \widehat{f_j} = (d_{G_1} - \lambda_j) \widehat{f_j}$ and $\widehat{\mathcal{A}}_{G_2} \widehat{g}_k = (d_{G_2} - \eta_k) \widehat{g}_k$. if $f : V_1 \to \mathbb{R}$ and $g : V_2 \to \mathbb{R}$ correspond to

$$f \leftrightarrow \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{n_{G_1}} \end{pmatrix}, \qquad g \leftrightarrow \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{n_{G_2}} \end{pmatrix},$$

then $\widehat{f}$, $\widehat{g} : V \to \mathbb{R}$ correspond to

$$\widehat{f} \leftrightarrow \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{n_{G_1}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \widehat{g} \leftrightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \xi_1 \\ \vdots \\ \xi_{n_{G_2}} \end{pmatrix}.$$

Since $A_{G_i^c} = M_{n_{G_i} n_{G_i}} - I - A_{G_i}$, we find for $j \geq 2$ and $k \geq 2$ that

$$\begin{aligned} \mathcal{A}^{(p)}_{G^K} \mathcal{P}^{(a)}_{G^K} \widehat{f_j} &= \frac{1}{\widehat{d}_{G_1}} \mathcal{A}^{(p)}_{G^K} (\mathcal{M}_{11} - \mathcal{I} - \widehat{\mathcal{A}}_{G_1}) \widehat{f_j} = \frac{\lambda_j - d_{G_1} - 1}{\widehat{d}_{G_1}} \mathcal{A}^{(p)}_{G^K} \widehat{f_j} \\ &= \frac{1}{\widehat{d}_{G_1}} (d_{G_1} - \lambda_j)(\lambda_j - d_{G_1} - 1) \widehat{f_j}, \\ \mathcal{A}^{(p)}_{G^K} \mathcal{P}^{(a)}_{G^K} \widehat{g}_k &= \frac{1}{\widehat{d}_{G_2}} \mathcal{A}^{(p)}_{G^K} (\mathcal{M}_{22} - \mathcal{I} - \widehat{\mathcal{A}}_{G_2}) \widehat{g}_k = \frac{\eta_k - d_{G_2} - 1}{\widehat{d}_{G_2}} \mathcal{A}^{(p)}_{G^K} \widehat{g}_k \\ &= \frac{1}{\widehat{d}_{G_2}} (d_{G_2} - \eta_k)(\eta_k - d_{G_2} - 1) \widehat{g}_k. \end{aligned}$$

Hence we have

$$\begin{aligned} \Delta_{\mathcal{A}_{(1,1)}} \widehat{f_j} &= \Big(d_{G_1} - \frac{1}{\widehat{d}_{G_1}} (d_{G_1} - \lambda_j)(\lambda_j - d_{G_1} - 1) \Big) \widehat{f_j}, \\ \Delta_{\mathcal{A}_{(1,1)}} \widehat{g}_k &= \Big(d_{G_2} - \frac{1}{\widehat{d}_{G_2}} (d_{G_2} - \eta_k)(\eta_k - d_{G_2} - 1) \Big) \widehat{g}_k. \end{aligned}$$

Next we consider a function $\phi[\alpha] : V \to \mathbb{R}$ for a constant α defined by

$$\phi[\alpha](v) = \begin{cases} 1, & \text{when } v \in V_1, \\ \alpha, & \text{when } v \in V_2. \end{cases}$$

This function corresponds to

$$\phi[\alpha] \leftrightarrow \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \alpha \\ \vdots \\ \alpha \end{pmatrix}.$$

We express this vector as $\begin{pmatrix} 1 \\ \alpha \end{pmatrix}$. Then we have

$$\begin{aligned} A^{(p)}_{G^K} P^{(a)}_{G^K} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} &= \begin{pmatrix} A_{G_1} & \vdots & O \\ \cdots & & \cdots \\ O & \vdots & A_{G_2} \end{pmatrix} \begin{pmatrix} \dfrac{1}{\widehat{d}_{G_1}}\{d_{G_1^c} + n_{G_2}\alpha\} \\ \dfrac{1}{\widehat{d}_{G_2}}\{n_{G_1} + d_{G_2^c}\alpha\} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{d_{G_1}}{\widehat{d}_{G_1}}\{n_{G_1} - d_{G_1} - 1 + n_{G_2}\alpha\} \\ \dfrac{d_{G_2}}{\widehat{d}_{G_2}}\{n_{G_1} + (n_{G_2} - d_{G_2} - 1)\alpha\} \end{pmatrix}. \end{aligned}$$

Therefore, in order that $\phi[\alpha]$ is an eigenfunction associated with an eigenvalue Λ, as

$$(D_G - A^{(p)}_{G^K} P^{(a)}_{G^K}) \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \Lambda \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \qquad \text{shows} \qquad A^{(p)} P^{(a)} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} d_{G_1} - \Lambda \\ (d_{G_2} - \Lambda)\alpha \end{pmatrix},$$

the following system of equations holds:

$$\begin{cases} \dfrac{d_{G_1}}{\widehat{d}_{G_1}}\{n_{G_1} - d_{G_1} - 1 + n_{G_2}\alpha\} = d_{G_1} - \Lambda, \\ \dfrac{d_{G_2}}{\widehat{d}_{G_2}}\{n_{G_1} + (n_{G_2} - d_{G_2} - 1)\alpha\} = (d_{G_2} - \Lambda)\alpha. \end{cases}$$

By the first equality we have

$$\Lambda = \frac{n_{G_2} d_{G_1}}{\widehat{d}_{G_1}}(1 - \alpha).$$

Substituting this into the second equality, we have

$$\frac{d_{G_2}}{\widehat{d}_{G_2}}\{n_{G_1} + (n_{G_2} - d_{G_2} - 1)\alpha\} = \Big(d_{G_2} + \frac{n_{G_2} d_{G_1}}{\widehat{d}_{G_1}}(\alpha - 1)\Big)\alpha.$$

Thus

$$\frac{n_{G_2} d_{G_1}}{\widehat{d}_{G_1}}\alpha^2 + \Big(\frac{n_{G_1} d_{G_2}}{\widehat{d}_{G_2}} - \frac{n_{G_2} d_{G_1}}{\widehat{d}_{G_1}}\Big)\alpha - \frac{n_{G_1} d_{G_2}}{\widehat{d}_{G_2}} = 0,$$

hence

$$(\alpha - 1)\Big(\frac{n_{G_2} d_{G_1}}{\widehat{d}_{G_1}}\alpha + \frac{n_{G_1} d_{G_2}}{\widehat{d}_{G_2}}\Big) = 0.$$

Therefore we have $\alpha = 1$ and $\Lambda = 0$, or $\alpha = -\dfrac{d_{G_2}\widehat{d}_{G_1}n_{G_1}}{d_{G_1}\widehat{d}_{G_2}n_{G_2}}$ and

$$\Lambda = \frac{d_{G_1}n_{G_2}}{\widehat{d}_{G_1}}\left(1 + \frac{d_{G_2}\widehat{d}_{G_1}n_{G_1}}{d_{G_1}\widehat{d}_{G_2}n_{G_2}}\right) = \frac{d_{G_1}n_{G_2}}{\widehat{d}_{G_1}} + \frac{d_{G_2}n_{G_1}}{\widehat{d}_{G_2}}.$$

Thus we get the conclusion. □

THEOREM 4.3. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected regular ordinary graphs. We denote the eigenvalues $\Delta_{\mathcal{A}_{G_1}}$ by $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_{nG_1}$ and those of $\Delta_{\mathcal{A}_{G_2}}$ by $0 = \eta_1 < \eta_2 \le \cdots \le \eta_{n_{G_2}}$. We set $G = (V_1 + V_2, E_1 + E_2)$ the disjoint union of G_1 and G_2. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ of G^K are*

$$0,\ 1 - \frac{(d_{G_1} - \lambda_j)(\lambda_j - d_{G_1} - 1)}{d_{G_1}\widehat{d}_{G_1}}, \quad 1 - \frac{(d_{G_2} - \eta_k)(\eta_k - d_{G_1} - 1)}{d_{G_2}\widehat{d}_{G_2}}, \quad \frac{n_{G_2}}{\widehat{d}_{G_1}} + \frac{n_{G_1}}{\widehat{d}_{G_2}},$$
$$j = 2, \dots, n_{G_1},\ k = 2, \dots, n_{G_2},$$

where $\widehat{d}_{G_i} = n_{G_1} + n_{G_2} - d_{G_i} - 1$ for $i = 1, 2$.

PROOF. We use the same notations as in the proof of Theorem 4.2. By using the adjacency matrices $A_{G_j}, A_{G_j^c}$ of G_j and its complement graph G_j^c, we can express the transition matrices $A^{(p)}_{G^K}$ and $P^{(a)}_{G^K}$ as

$$P^{(p)}_{G^K} = \begin{pmatrix} \dfrac{1}{d_{G_1}}A_{G_1} & \vdots & O \\ \cdots & & \cdots \\ O & \vdots & \dfrac{1}{d_{G_2}}A_{G_2} \end{pmatrix}, \qquad P^{(a)}_{G^K} = \begin{pmatrix} \dfrac{1}{\widehat{d}_{G_1}}A_{G_1^c} & \vdots & \dfrac{1}{\widehat{d}_{G_1}}M_{12} \\ \cdots & & \cdots \\ \dfrac{1}{\widehat{d}_{G_2}}M_{21} & \vdots & \dfrac{1}{\widehat{d}_{G_2}}A_{G_2^c} \end{pmatrix}.$$

We take an eigenfunction $f_j : V_1 \to \mathbb{R}$ corresponding to the eigenvalue λ_j and an eigenfunction $g_k : V_2 \to \mathbb{R}$ associated with η_k. We define $\widehat{f}_j, \widehat{g}_k : V \to \mathbb{R}$ by

$$\widehat{f}_j(v) = \begin{cases} f_j(v), & \text{when } v \in V_1, \\ 0, & \text{when } v \in V_2, \end{cases} \qquad \widehat{g}_k(v) = \begin{cases} 0, & \text{when } v \in V_1, \\ g_k(v), & \text{when } v \in V_2. \end{cases}$$

Since $A_{G_i^c} = M_{n_{G_i} n_{G_i}} - I - A_{G_i}$, we find for $j \geq 2$ and $k \geq 2$ that

$$\begin{aligned}
\mathcal{P}^{(p)}_{G^K}\mathcal{P}^{(a)}_{G^K}\widehat{f}_j &= \frac{1}{\widehat{d}_{G_1}}\mathcal{P}^{(p)}_{G^K}(\mathcal{M}_{11}-\mathcal{I}-\widehat{\mathcal{A}}_{G_1})\widehat{f}_j = \frac{\lambda_j - d_{G_1}-1}{\widehat{d}_{G_1}}\mathcal{P}^{(p)}_{G^K}\widehat{f}_j \\
&= \frac{1}{d_{G_1}\widehat{d}_{G_1}}(d_{G_1}-\lambda_j)(\lambda_j - d_{G_1}-1)\widehat{f}_j, \\
\mathcal{P}^{(p)}_{G^K}\mathcal{P}^{(a)}_{G^K}\widehat{g}_k &= \frac{1}{\widehat{d}_{G_2}}\mathcal{A}^{(p)}_{G^K}(\mathcal{M}_{22}-\mathcal{I}-\widehat{\mathcal{A}}_{G_2})\widehat{g}_k = \frac{\eta_k - d_{G_2}-1}{\widehat{d}_{G_2}}\mathcal{P}^{(p)}_{G^K}\widehat{g}_k \\
&= \frac{1}{d_{G_2}\widehat{d}_{G_2}}(d_{G_2}-\eta_k)(\eta_k - d_{G_2}-1)\widehat{g}_k.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\Delta_{\mathcal{Q}_{(1,1)}}\widehat{f}_j &= \Big(1 - \frac{1}{d_{G_1}\widehat{d}_{G_1}}(d_{G_1}-\lambda_j)(\lambda_j - d_{G_1}-1)\Big)\widehat{f}_j, \\
\Delta_{\mathcal{Q}_{(1,1)}}\widehat{g}_k &= \Big(1 - \frac{1}{d_{G_2}\widehat{d}_{G_2}}(d_{G_2}-\eta_k)(\eta_k - d_{G_2}-1)\Big)\widehat{g}_k.
\end{aligned}$$

Next we consider a function $\phi[\alpha] : V \to \mathbb{R}$ for a constant α defined by

$$\phi[\alpha](v) = \begin{cases} 1, & \text{when } v \in V_1, \\ \alpha, & \text{when } v \in V_2. \end{cases}$$

This function corresponds to a vector $\begin{pmatrix} 1 \\ \alpha \end{pmatrix}$. Thus we have

$$\begin{aligned}
P^{(p)}_{G^K}P^{(a)}_{G^K}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} &= \begin{pmatrix} \frac{1}{d_{G_1}}A_{G_1} & \vdots & O \\ \cdots & & \cdots \\ O & \vdots & \frac{1}{d_{G_2}}A_{G_2} \end{pmatrix} \begin{pmatrix} \frac{1}{\widehat{d}_{G_1}}\{d_{G_1^c} + n_{G_2}\alpha\} \\ \frac{1}{\widehat{d}_{G_2}}\{n_{G_1} + d_{G_2^c}\alpha\} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\widehat{d}_{G_1}}\{n_{G_1} - d_{G_1} - 1 + n_{G_2}\alpha\} \\ \frac{1}{\widehat{d}_{G_2}}\{n_{G_1} + (n_{G_2} - d_{G_2} - 1)\alpha\} \end{pmatrix}.
\end{aligned}$$

Therefore, in order that $\phi[\alpha]$ is an eigenfunction associated with an eigenvalue Θ, as

$$(I - P^{(p)}_{G^K}P^{(a)}_{G^K})\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \Theta\begin{pmatrix} 1 \\ \alpha \end{pmatrix} \qquad \text{shows} \qquad P^{(p)}P^{(a)}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 1-\Theta \\ (1-\Theta)\alpha \end{pmatrix},$$

the following system of equations holds:

$$\begin{cases} \dfrac{1}{\widehat{d}_{G_1}}\{n_{G_1} - d_{G_1} - 1 + n_{G_2}\alpha\} = 1 - \Theta, \\ \dfrac{1}{\widehat{d}_{G_2}}\{n_{G_1} + (n_{G_2} - d_{G_2} - 1)\alpha\} = (1-\Theta)\alpha. \end{cases}$$

By the first equality we have

$$\Theta = \frac{n_{G_2}}{\widehat{d}_{G_1}}(1-\alpha).$$

Substituting this into the second equality, we have

$$\frac{n_{G_2}}{\widehat{d}_{G_1}}\alpha^2 + \Big(\frac{n_{G_1} - d_{G_1} - 1}{\widehat{d}_{G_1}} - \frac{n_{G_2} - d_{G_2} - 1}{\widehat{d}_{G_2}}\Big)\alpha - \frac{n_{G_1}}{\widehat{d}_{G_2}} = 0,$$

which is equivalent to

$$\frac{n_{G_2}}{\widehat{d}_{G_1}}\alpha^2 + \Big(\frac{n_{G_1}}{\widehat{d}_{G_2}} - \frac{n_{G_2}}{\widehat{d}_{G_1}}\Big)\alpha - \frac{n_{G_1}}{\widehat{d}_{G_2}} = 0.$$

Hence

$$(\alpha - 1)\Big(\frac{n_{G_2}}{\widehat{d}_{G_1}}\alpha + \frac{n_{G_1}}{\widehat{d}_{G_2}}\Big) = 0.$$

Therefore we have $\alpha = 1$ and $\Theta = 0$, or $\alpha = -\dfrac{\widehat{d}_{G_1} n_{G_1}}{\widehat{d}_{G_2} n_{G_2}}$ and

$$\Theta = \frac{n_{G_2}}{\widehat{d}_{G_1}}\Big(1 + \frac{\widehat{d}_{G_1} n_{G_1}}{\widehat{d}_{G_2} n_{G_2}}\Big) = \frac{n_{G_2}}{\widehat{d}_{G_1}} + \frac{n_{G_1}}{\widehat{d}_{G_2}}.$$

Thus we get the conclusion. □

EXAMPLE 4.8. We take a 3-circuit $G_1 = (V_1, E_1)$ and a 4-circuit $G_2 = (V_2, E_2)$ which are regular of degree 2. We consider the union $G = (V_1 + V_2, E_1 + E_2)$ and take its complement-filled Kähler graph G^K (see Fig. 7).

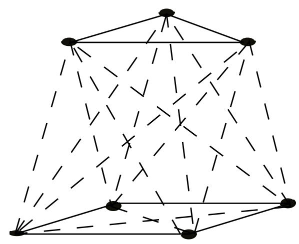

FIG. 7

The adjacency matrix of G and $(1,1)$-adjacency matrix and $(1,1)$-probabilistic transition matrix are given as

$$A_G = \begin{pmatrix} A_{G_1} & O \\ O & A_{G_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

$$A_{G^K_{(1,1)}} = \begin{pmatrix} A_{G_1} & O \\ O & A_{G_2} \end{pmatrix} \begin{pmatrix} O & \frac{1}{4}M_{12} \\ \frac{1}{4}M_{21} & \frac{1}{4}A_{G_2^c} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix},$$

$$Q_{G^K_{(1,1)}} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{8} & 0 & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & 0 & \frac{1}{8} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{8} & 0 & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & 0 & \frac{1}{8} & 0 \end{pmatrix}$$

The eigenvalues of the combinatorial Laplacian of the principal graph and those of $(1,1)$-combinatorial and $(1,1)$-probabilistic transitional Laplacians are as follows;

$$\begin{aligned} \mathrm{Spec}(\Delta_{\mathcal{A}_G}) &= \{0,0,2,2,3,3,4\}, \\ \mathrm{Spec}(\Delta_{\mathcal{A}_{G^K_{(1,1)}}}) &= \Big\{0,2,2,2,2,\frac{7}{2},\frac{7}{2}\Big\}, \\ \mathrm{Spec}(\Delta_{\mathcal{Q}_{G^K_{(1,1)}}}) &= \Big\{0,1,1,1,\frac{5}{4},\frac{7}{4}\Big\}. \end{aligned}$$

2.2. Isospectral Kähler graphs. Two ordinary finite graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be *combinatorically isospectral* if they are not isomorphic to each other and if their combinatorial Laplacians have the same eigenvalues by taking account of their multiplicities. Also, we say that two ordinary finite graph

$G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *transitionally isospectral* if they are not isomorphic to each other and if their transitional Laplacian have the same eigenvalues by taking account of their multiplicities. When these graphs are regular, as their combinatorial and transitional Laplacians are related to each other by multiplying their degrees, those notions on isospectrality are equivalent. So in this case, we just say that these graphs are isospectral. It is known that there exist many pairs of isospectral graphs (see [**5**]).

We extend the notion of isospectrality to Kähler graphs. We say that two Kähler graphs $G_1 = (V_1, E_1^{(p)} \cup E_1^{(a)})$ and $G_1 = (V_1, E_2^{(p)} \cup E_1^{(a)})$ are $(1,1)$-*combinatorial isospectral* if they satisfy the following conditions:

i) their principal graphs $G_1^{(p)} = (V, E^{(p)})$ and $G_2^{(p)} = (V, E_2^{(p)})$ are combinatorially isospectral;
ii) their $(1,1)$-combinatorial Laplacians have the same eigenvalues by taking account of their multiplicities.

Also, we say that those Kähler graphs G_1 and G_2 are $(1,1)$-*probabilistic transitional isospectral* if they satisfy the following conditions:

i) their principal graphs $G_1^{(p)} = (V, E^{(p)})$ and $G_2^{(p)} = (V, E_2^{(p)})$ are transitionally isospectral;
ii) their $(1,1)$-transitional Laplacians have the same eigenvalues by taking account of their multiplicities.

When the principal graphs of these two Kähler graphs are regular, they are $(1,1)$-combinatorial isospectral if and only if they are $(1,1)$-probabilistic transitional isospectral. Hence in this case we just call them $(1,1)$-isospectral.

As a direct consequence of Theorem 4.1, we have the following.

PROPOSITION 4.2. *If two finite connected regular ordinary graphs G_1, G_2 have the same degree with $2 \le d_{G_1} = d_{G_2} \le n_{G_1} - 3$ $(= n_{G_2} - 3)$ and are isospectral, then their complement-filled Kähler graphs G_1^K, G_2^K are $(1,1)$-isospectral.*

Example 4.9. We take two ordinary regular graphs G_1, G_2 having ten vertices as in Figs. 8, 9. It in known that they are isospectral. We take their complement-filled Kähler graphs G_1^K, G_2^K. We show their principal and auxiliary graphs separately in figures to get their feature clearly. They are $(1,1)$-isospectral Kähler graphs. The adjacency matrices of G_1 and G_2 are

$$A_{G_1} = \begin{pmatrix} 0&1&0&0&0&1&1&1&0&0\\ 1&0&1&0&0&0&1&0&1&0\\ 0&1&0&1&0&0&0&1&1&0\\ 0&0&1&0&1&0&0&1&0&1\\ 0&0&0&1&0&1&0&0&1&1\\ 1&0&0&0&1&0&1&0&0&1\\ 1&1&0&0&0&1&0&1&0&0\\ 1&0&1&1&0&0&1&0&0&0\\ 0&1&1&0&1&0&0&0&0&1\\ 0&0&0&1&1&1&0&0&1&0 \end{pmatrix}, \quad A_{G_2} = \begin{pmatrix} 0&1&0&0&0&1&1&0&1&0\\ 1&0&1&0&0&0&1&1&0&0\\ 0&1&0&1&0&0&0&1&1&0\\ 0&0&1&0&1&0&0&0&1&1\\ 0&0&0&1&0&1&0&1&0&1\\ 1&0&0&0&1&0&1&0&0&1\\ 1&1&0&0&0&1&0&1&0&0\\ 0&1&1&0&1&0&1&0&0&0\\ 1&0&1&1&0&0&0&0&0&1\\ 0&0&0&1&1&1&0&0&1&0 \end{pmatrix}.$$

Eigenvalues of combinatorial Laplacians of their principal graphs and those of $(1,1)$-combinatorial Laplacians are

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^{(p)}}}) = \{0,3,5,5,5,5,4-\sqrt{5},4+\sqrt{5},(9-\sqrt{17})/2,(9+\sqrt{17})/2\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^K(1,1)}}) = \{0,4,4,4,4,22/5,24/5,24/5,(25-\sqrt{5})/5,(25+\sqrt{5})/5\}.$$

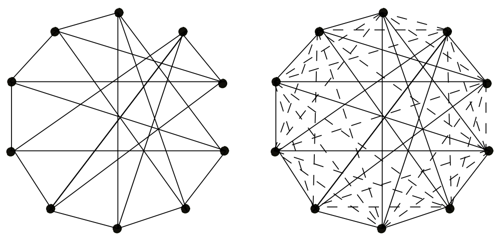

Fig. 8

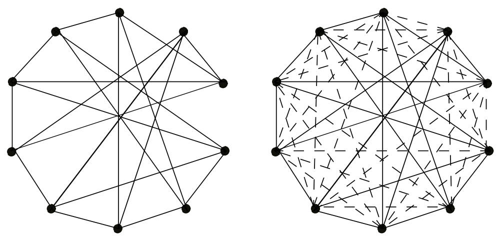

Fig. 9

We note that their $(1,1)$-adjacency matrices are different:

$$A_{G_1^K{}_{(1,1)}} = \frac{1}{5}\begin{pmatrix} 0&2&2&3&3&2&0&2&3&3\\ 2&0&2&3&3&2&2&1&2&3\\ 2&2&0&2&2&4&2&2&2&2\\ 3&3&2&0&2&2&3&2&1&2\\ 3&3&2&2&0&2&3&3&2&0\\ 2&2&4&2&2&0&2&2&2&2\\ 0&2&2&3&3&2&0&2&3&3\\ 2&1&2&2&3&2&2&0&3&3\\ 3&2&2&1&2&2&3&3&0&2\\ 3&3&2&2&0&2&3&3&2&0 \end{pmatrix}, \quad A_{G_2^K{}_{(1,1)}} = \frac{1}{5}\begin{pmatrix} 0&2&2&3&3&2&1&2&3&2\\ 2&0&2&3&3&2&1&1&2&4\\ 2&2&0&2&2&4&2&2&2&2\\ 3&3&2&0&2&2&4&2&1&1\\ 3&3&2&2&0&2&2&3&2&1\\ 2&2&4&2&2&0&2&2&2&2\\ 1&1&2&4&2&2&0&2&3&3\\ 2&1&2&2&3&2&2&0&3&3\\ 3&2&2&1&2&2&3&3&0&2\\ 2&4&2&1&1&2&3&3&2&0 \end{pmatrix}.$$

We here recall eigenvalues of complement graphs.

LEMMA 4.4. *We denote the eigenvalues of* $\Delta_{\mathcal{A}_G}$ *of a connected regular ordinary graph* G *by* $0=\lambda_1<\lambda_2\le\cdots\le\lambda_{n_G}$. *Then the eigenvalues of* $\Delta_{\mathcal{A}_{G^c}}$ *of the complement graph* G^c *of* G *are* 0 *and* $n_G-\lambda_j$ $(j=2,\ldots,n_G)$.

PROOF. Let f be an eigenfunction associated with an eigenvalue λ of $\Delta_{\mathcal{A}_G}$. We have $\mathcal{A}_G f=(d_G-\lambda)f$. Since $G=(V,E)$ is connected regular graph, when $\lambda=0$ we have f is a constant function and $\sum_{v\in V} f(v)=n_G f(v)$, and when $\lambda\neq 0$ we have $\sum_{v\in V} f(v)=0$. Thus we see

$$\mathcal{A}_{G^c}f=(\mathcal{M}-\mathcal{I}-\mathcal{A}_G)f=\begin{cases}(n_G-1-d_G)f, & \text{when } \lambda=0,\\ (-1-d_G+\lambda)f, & \text{when } \lambda\neq 0.\end{cases}$$

As $d_{G^c}=n_G-1-d_G$ we have

$$\Delta_{\mathcal{A}_{G^c}}f=\begin{cases}\{(n_G-1-d_G)-(n_G-1-d_G)\}f=0 & \text{when } \lambda=0,\\ \{(n_G-1-d_G)-(-1-d_G+\lambda)\}f=(n_G-\lambda)f, & \text{when } \lambda\neq 0.\end{cases}$$

Hence we get the conclusion. □

As a consequence of this Lemma, we have the following.

COROLLARY 4.1. *If two connected regular ordinary graphs* $G_1,\ G_2$ *are isospectral, then their complement graphs* G_1^c, G_2^c *are also isospectral.*

For a Kähler graph $G=(V,E^{(p)}\cup E^{(a)})$, we define a new Kähler graph $G^*=(V,F^{(p)}\cup F^{(a)})$ by putting $F^{(p)}=E^{(a)}$ and $F^{(a)}=E^{(p)}$. We call this the *dual* Kähler graph of G. By Corollary 4.1 and by Proposition 4.2 we have the following.

COROLLARY 4.2. *If two finite connected regular ordinary graphs* $G_1,\ G_2$ *have the same degree with* $2\le d_{G_1}=d_{G_2}\le n_{G_1}-3$ $(=n_{G_2}-3)$ *and are isospectral, then the dual Kähler graphs* $(G_1^K)^*,\ (G_2^K)^*$ *of their complement-filled Kähler graphs* $G_1^K,\ G_2^K$ *are* $(1,1)$*-isospectral.*

EXAMPLE 4.10. The dual Kähler graphs $(G_1^K)^*$, $(G_2^K)^*$ of the complement-filled Kähler graphs in Example 4.9 are also $(1,1)$- isospectral. Their eigenvalues of principal graphs and of (1,1)-combinatorial Laplacians are as follows:

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^*}}) = \Big\{0,5,5,5,5,7,6-\sqrt{5},6+\sqrt{5},\frac{(11-\sqrt{17})}{2},\frac{(11+\sqrt{17})}{2}\Big\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{A}_{(G^K)^*_{(1,1)}}}) = \Big\{0,5,5,5,5,\frac{11}{2},6,6,\frac{(25-\sqrt{5})}{4},\frac{(25+\sqrt{5})}{4}\Big\}.$$

If we give their (1,1)-adjacency matrices, they are

$$A_{(G_1^K)^*_{(1,1)}} = \frac{1}{4}\begin{pmatrix} 0&2&2&3&3&2&0&2&3&3\\ 2&0&2&3&3&2&2&1&2&3\\ 2&2&0&2&2&4&2&2&2&2\\ 3&3&2&0&2&2&3&2&1&2\\ 3&3&2&2&0&2&3&3&2&0\\ 2&2&4&2&2&0&2&2&2&2\\ 0&2&2&3&3&2&0&2&3&3\\ 2&1&2&2&3&2&2&0&3&3\\ 3&2&2&1&2&2&3&3&0&2\\ 3&3&2&2&0&2&3&3&2&0 \end{pmatrix},$$

$$A_{(G_2^K)^*_{(1,1)}} = \frac{1}{4}\begin{pmatrix} 0&2&2&3&3&2&1&2&3&2\\ 2&0&2&3&3&2&1&1&2&4\\ 2&2&0&2&2&4&2&2&2&2\\ 3&3&2&0&2&2&4&2&1&1\\ 3&3&2&2&0&2&2&3&2&1\\ 2&2&4&2&2&0&2&2&2&2\\ 1&1&2&4&2&2&0&2&3&3\\ 2&1&2&2&3&2&2&0&3&3\\ 3&2&2&1&2&2&3&3&0&2\\ 2&4&2&1&1&2&3&3&2&0 \end{pmatrix}.$$

Thus we can verify that $Q_{(G_j^K)^*_{(1,1)}} = Q_{(G_j^K)_{(1,1)}}$ $(j=1,2)$.

Example 4.11. We take another pair of isospectral ordinary regular graphs G_1, G_2 having ten vertices like Figs. 10, 11. Then their complement filled Kähler graphs are (1,1)-isospectral. The adjacency matrices of G_1 and G_2 are

$$A_{G_1} = \begin{pmatrix} 0&1&1&0&0&0&0&1&1&0\\ 1&0&1&0&0&1&0&0&0&1\\ 1&1&0&1&1&0&0&0&0&0\\ 0&0&1&0&1&1&0&0&0&1\\ 0&0&1&1&0&1&1&0&0&0\\ 0&1&0&1&1&0&1&0&0&0\\ 0&0&0&0&1&1&0&1&1&0\\ 1&0&0&0&0&0&1&0&1&1\\ 1&0&0&0&0&0&1&1&0&1\\ 0&1&0&1&0&0&0&1&1&0 \end{pmatrix}, \quad A_{G_2} = \begin{pmatrix} 0&1&1&1&0&0&0&0&1&0\\ 1&0&1&0&0&0&1&0&0&1\\ 1&1&0&1&0&0&0&1&0&0\\ 1&0&1&0&1&0&0&1&0&0\\ 0&0&0&1&0&1&0&1&0&1\\ 0&0&0&0&1&0&1&0&1&1\\ 0&1&0&0&0&1&0&1&1&0\\ 0&0&1&1&1&0&1&0&0&0\\ 1&0&0&0&0&1&1&0&0&1\\ 0&1&0&0&1&1&0&0&1&0 \end{pmatrix}.$$

Since we have

$$A_{G^K_{1(1,1)}} = \frac{1}{5}\begin{pmatrix} 0&2&2&3&3&3&2&2&2&1\\ 2&0&2&1&2&3&3&2&2&3\\ 2&2&0&2&2&1&3&3&3&2\\ 3&1&2&0&1&2&2&3&3&3\\ 3&2&2&1&0&1&2&3&3&3\\ 3&3&1&2&1&0&2&3&3&2\\ 2&3&3&2&2&2&0&2&2&2\\ 2&2&3&3&3&3&2&0&0&2\\ 2&2&3&3&3&3&2&0&0&2\\ 1&3&2&3&3&2&2&2&2&0 \end{pmatrix}, \quad A_{G^K_{2(1,1)}} = \frac{1}{5}\begin{pmatrix} 0&2&1&2&3&3&2&2&3&2\\ 2&0&2&2&3&2&3&2&1&3\\ 1&2&0&1&2&4&2&2&3&3\\ 2&2&1&0&2&3&3&1&3&3\\ 3&3&2&2&0&2&2&2&2&2\\ 3&2&4&3&2&0&2&2&1&1\\ 2&3&2&3&2&2&0&3&2&1\\ 2&2&2&1&2&2&3&0&3&3\\ 3&1&3&3&2&1&2&3&0&2\\ 2&3&3&3&2&1&1&3&2&0 \end{pmatrix},$$

their eigenvalues of principal graphs and of $(1, 1)$-combinatorial Laplacians are as follows:

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^*}}) = \left\{ \begin{array}{l} 0, 5, 5, \dfrac{9-\sqrt{5}}{2}, \dfrac{9+\sqrt{5}}{2}, \\ \text{solutions of the equation } t^5 - 21t^4 + 167t^3 - 624t^2 + 1092t - 716 = 0 \end{array} \right\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{(G^K)^*_{(1,1)}}}) = \left\{ \begin{array}{l} 0, 4, 4, \dfrac{21}{5}, \dfrac{21}{5}, \\ \text{solutions of the equation} \\ \quad 5^5t^5 - 5^4\cdot 118t^4 + 5^3\cdot 5557t^3 - 5^2\cdot 130552t^2 + 5\cdot 1530052t - 7156316 = 0 \end{array} \right\}.$$

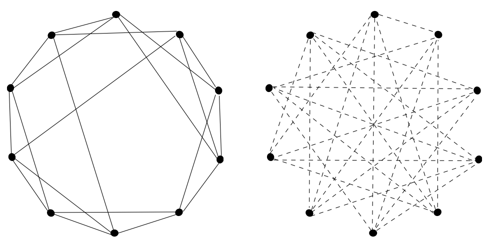

Fig. 10

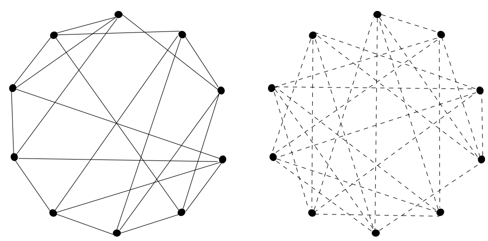

Fig. 11

Their dual Kähler graphs $(G^K_1)^*$, $(G^K_2)^*$ are also $(1, 1)$-isospectral.

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(p)}}) = \left\{ \begin{array}{l} 0, 5, 5, \dfrac{11-\sqrt{5}}{2}, \dfrac{11+\sqrt{5}}{2}, \\ \text{solutions of the equation } t^5 - 29t^4 + 327t^3 - 1786t^2 + 4712t - 4804 = 0 \end{array} \right\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(1,1)}}}) = \left\{ \begin{array}{l} 0, 5, 5, \frac{21}{4}, \frac{21}{4}, \\ \text{solutions of the equation} \\ \quad 4^5 t^5 - 4^4 \cdot 118 t^4 + 4^3 \cdot 5557 t^3 - 4^2 \cdot 130552 t^2 + 4 \cdot 1530052 t - 7156316 = 0 \end{array} \right\}.$$

It is known that there do not exist pairs of regular graphs whose cardinalities of the set of vertices are less than ten, and that those pairs in Examples 4.9, 4.11 are the only examples of isospectral pairs whose cardinalities of the set of vertices are ten. Therefore the above examples are the examples of $(1,1)$-isospectral pairs of Kähler graphs whose cardinalities of the set of vertices are the smallest.

When we study isospectral pairs of Kähler graphs, the condition on their principal graphs is important. If we drop the condition, we include pairs of Kähler graphs which are not isomorphic but their $(1,1)$-derived graphs are isomorphic.

Example 4.12. We take two vertex-transitive Kähler graphs having nine vertices like Figs. 12, 13. By observing 3-step closed paths, we find that they are not isomorphic but thier $(1,1)$-derived graphs are isomorphic. Their adjacency matrices are given as

$$A_{G_1} = \begin{pmatrix} 0&1&1&0&0&0&0&1&1 \\ 1&0&1&1&0&0&0&0&1 \\ 1&1&0&1&1&0&0&0&0 \\ 0&1&1&0&1&1&0&0&0 \\ 0&0&1&1&0&1&1&0&0 \\ 0&0&0&1&1&0&1&1&0 \\ 0&0&0&0&1&1&0&1&1 \\ 1&0&0&0&0&1&1&0&1 \\ 1&1&0&0&0&0&1&1&0 \end{pmatrix}, \qquad A_{G_2} = \begin{pmatrix} 0&1&0&1&0&0&1&0&1 \\ 1&0&1&0&1&0&0&1&0 \\ 0&1&0&1&0&1&0&0&1 \\ 1&0&1&0&1&0&1&0&0 \\ 0&1&0&1&0&1&0&1&0 \\ 0&0&1&0&1&0&1&0&1 \\ 1&0&0&1&0&1&0&1&0 \\ 0&1&0&0&1&0&1&0&1 \\ 1&0&1&0&0&1&0&1&0 \end{pmatrix}.$$

As we have

$$A_{G^K_{1(1,1)}} = \frac{1}{4} \begin{pmatrix} 0&1&2&2&3&3&2&2&1 \\ 1&0&1&2&2&3&3&2&2 \\ 2&1&0&1&2&2&3&3&2 \\ 2&2&1&0&1&2&2&3&3 \\ 3&2&2&1&0&1&2&2&3 \\ 3&3&2&2&1&0&1&2&2 \\ 2&3&3&2&2&1&0&1&2 \\ 2&2&3&3&2&2&1&0&1 \\ 1&2&2&3&3&2&2&1&0 \end{pmatrix}, \quad A_{G^K_{2(1,1)}} = \frac{1}{4} \begin{pmatrix} 0&3&1&2&2&2&2&1&3 \\ 3&0&3&1&2&2&2&2&1 \\ 1&3&0&3&1&2&2&2&2 \\ 2&1&3&0&3&1&2&2&2 \\ 2&2&1&3&0&3&1&2&2 \\ 2&2&2&1&3&0&3&1&2 \\ 2&2&2&2&1&3&0&3&1 \\ 1&2&2&2&2&1&3&0&3 \\ 3&1&2&2&2&2&1&3&0 \end{pmatrix}.$$

the eigenvalues of their $(1,1)$-combinatorial Laplacians are as follows:

$$\mathrm{Spec}(\varDelta_{\mathcal{A}_{G^K_{(1,1)}}}) = \left\{ \begin{array}{l} 0, \dfrac{9}{2}, \dfrac{9}{2}, \dfrac{(9+\sqrt{3}\cos\frac{\pi}{18})}{2}, \dfrac{(9+\sqrt{3}\cos\frac{\pi}{18})}{2}, \\ \quad \dfrac{(9-\sqrt{3}\cos\frac{5}{18}\pi)}{2}, \dfrac{(9-\sqrt{3}\cos\frac{5}{18}\pi)}{2}, \\ \quad \dfrac{(9-\sqrt{3}\cos\frac{7}{18}\pi)}{2}, \dfrac{(9-\sqrt{3}\cos\frac{7}{18}\pi)}{2} \end{array} \right\}.$$

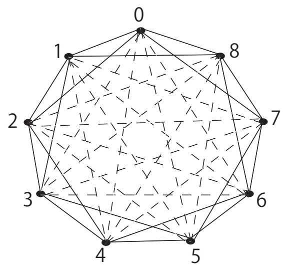

FIG. 12. $K^1(9;4,4)$

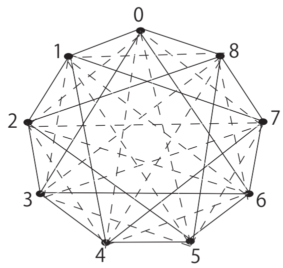

FIG. 13. $K^2(9;4,4)$

We note that the eigenvalues of these G_1, G_2 are

$$\mathrm{Spec}(\varDelta_{\mathcal{A}_{G_1}}) = \left\{ \begin{array}{l} 0, 6, 6 \\ \text{solutions of } t^3 - 12t^2 + 45t - 51 = 0 \\ \quad \text{(multiplicity of each solution is 2)} \end{array} \right\},$$

$$\mathrm{Spec}(\varDelta_{\mathcal{A}_{G_2}}) = \left\{ \begin{array}{l} 0, 3, 3 \\ \text{solutions of } t^3 - 15t^2 + 72t^{-}111 = 0 \\ \quad \text{(multiplicity of each solution is 2)} \end{array} \right\}.$$

3. $(1,1)$-Laplacians of Kähler graphs of product type whose principal graphs are unions of copies of original graphs

In this section and following three sections, we study eigenvalues of $(1,1)$-Laplacians for finite Kähler graphs of product type given in §2.2.

First we study $(1,1)$-Laplacians of Kähler graphs of Cartesian, strong, semi-tensor, lexicographical product types and their related graphs. For functions $f : V \to \mathbb{R}$ and $g : W \to \mathbb{R}$, we define a function $\varphi_{f,g} : V \times W \to \mathbb{R}$ by $\varphi_{f,g}(v,w) = f(v)g(w)$.

3.1. $(1,1)$-Laplacians of Kähler graphs of Cartesian product type.

THEOREM 4.4. *Let $G = (V,E), H = (W,F)$ be finite ordinary graphs. We denote by $0 = \mu_1 \leq \cdots \leq \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 \leq \cdots \leq \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G\widehat{\square}H$ of Cartesian product type are*

$$\mu_i + \nu_\alpha - \mu_i\nu_\alpha \quad (1 \leq i \leq n_G, 1 \leq \alpha \leq n_H).$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V \times W$ is an eigenfunction associated with $\mu_i + \nu_\alpha - \mu_i\nu_\alpha$.

PROOF. We denote by $A_G = (a^G_{ij})$ the adjacency matrix of the graph G and by $P_H = (p^H_{\alpha\beta})$ the transition matrix of the graph H. Then by definition of $G\widehat{\square}H$ we have

$$A^{(p)}_{G\widehat{\square}H} = \begin{pmatrix} 0 & a^G_{12}I & \cdots & a^G_{1n_G}I \\ a^G_{21}I & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a^G_{n_G-1n_G}I \\ a^G_{n_G1}I & \cdots & a^G_{n_Gn_G-1}I & 0 \end{pmatrix}, \quad P^{(a)}_{G\widehat{\square}H} = \begin{pmatrix} P_H & 0 & \cdots & 0 \\ 0 & P_H & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & P_H \end{pmatrix},$$

where I denotes the unit matrix (identify) and the components a^G of $A^{(p)}$ and $P^{(a)}$ are expressed according to lexicographical order. In other way of expressions, the adjacency matrix $A^{(p)} = (a^{(p)}_{(i,\alpha),(j,\beta)})$ of the principal graph of $G\widehat{\square}H$ and the transition matrix $P^{(a)} = (p^{(a)}_{(i,\alpha),(j,\beta)})$ of the auxiliary graph of $G\widehat{\square}H$ are given as

$$a^{(p)}_{(i,\alpha),(j,\beta)} = a^G_{ij}\delta_{\alpha\beta}, \qquad p^{(a)}_{(i,\alpha),(j,\beta)} = \delta_{ij}p^H_{\alpha\beta}$$

with Kronecker delta.

For functions $f : V \to \mathbb{R}$, $g : W \to \mathbb{R}$ we express them by canonical basis of $C(V,\mathbb{R})$ and $C(W,\mathbb{R})$ as

$$f \leftrightarrow \zeta = \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{n_G} \end{pmatrix}, \qquad g \leftrightarrow \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{n_H} \end{pmatrix}$$

Then $\varphi_{f,g}$ is expressed by the canonical basis $\{\varphi_{\delta_v,\delta_w} \mid v \in V, w \in W\}$ of $C(V \times W, \mathbb{R})$ as

$$\varphi_{f,g} \leftrightarrow \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_1\eta_{n_H} \\ \vdots \\ \zeta_{n_G}\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_H} \end{pmatrix}.$$

If functions f and g satisfy $\Delta_{\mathcal{P}_G} f = \mu f$ and $\Delta_{\mathcal{P}_H} g = \nu g$, then we have $\mathcal{P}_G f = (1-\mu) f$ and $\mathcal{P}_H g = (1-\nu) g$. These mean that $P_G \zeta = (1-\mu)\zeta$ and $P_H \eta = (1-\nu)\eta$. Therefore we obtain

$$A^{(p)}_{G\hat{\square}H} P^{(a)}_{G\hat{\square}H} \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_H} \end{pmatrix} = A^{(p)}_{G\hat{\square}H} \begin{pmatrix} \zeta_1 P_H \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{n_H} \end{pmatrix} \\ \vdots \\ \zeta_{n_G} P_H \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{n_H} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^m a^G_{1j}\zeta_j P_H \eta \\ \vdots \\ \sum_{j=1}^m a^G_{mj}\zeta_j P_H \eta \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^m a^G_{1j}\zeta_j (1-\nu)\eta \\ \vdots \\ \sum_{j=1}^m a^G_{mj}\zeta_j (1-\nu)\eta \end{pmatrix} = (1-\nu) \begin{pmatrix} d_G(v_1)(1-\mu)\zeta_1\eta \\ \vdots \\ d_G(v_{n_G})(1-\mu)\zeta_{n_G}\eta \end{pmatrix}.$$

Thus we have

$$\Delta_{\mathcal{Q}_{(1,1)}} \varphi_{f,g} = (\mathcal{I} - \mathcal{P}^{(p)}_{G\hat{\square}H} \mathcal{P}^{(a)}_{G\hat{\square}H}) \varphi_{f,g} = \{1 - (1-\mu)(1-\nu)\}\varphi_{f,g} = (\mu + \nu - \mu\nu)\varphi_{f,g}.$$

This completes the proof. □

THEOREM 4.5. *Let G be a regular finite graph and H be a finite graph. We denote by $0 = \mu_1 \le \cdots \le \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 \le \cdots \le \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ for their Kähler graph $G\widehat{\Box}H$ of Cartesian product type are*

$$d_G(\mu_i + \nu_\alpha - \mu_i\nu_\alpha) \quad (1 \le i \le n_G, 1 \le \alpha \le n_H).$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V \times W$ is an eigenfunction associated with $d_G(\mu_i + \nu_\alpha - \mu_i\nu_\alpha)$.

PROOF. Since G is regular graph, by the proof of Theorem 4.4 we have

$$A^{(p)}_{G\widehat{\Box}H} P^{(a)}_{G\widehat{\Box}H} \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_H} \end{pmatrix} = (1-\mu)(1-\nu)d_G \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_H} \end{pmatrix}.$$

Thus we have

$$\begin{aligned} \Delta_{\mathcal{A}_{(1,1)}}\varphi_{f,g} &= (\mathcal{D} - \mathcal{A}^{(p)}_{G\widehat{\Box}H}\mathcal{P}^{(a)}_{G\widehat{\Box}H})\varphi_{f,g} \\ &= d_G\{1-(1-\mu)(1-\nu)\}\varphi_{f,g} = d_G(v)(\mu+\nu-\mu\nu)\varphi_{f,g}. \end{aligned}$$

We get the conclusion. □

EXAMPLE 4.13. Let G and H be non-regular ordinary graphs given in Figs. 14 and 15, respectively. Their transition matrices are given as

$$P_G = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix} \quad \text{and} \quad P_H = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

and the eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are

$$\Big\{0, 1, \frac{4}{3}, \frac{5}{3}\Big\}, \qquad \Big\{0, \frac{7-\sqrt{7}}{6}, \frac{7}{6}, \frac{3}{2}, \frac{7+\sqrt{7}}{6}\Big\}.$$

The $(1,1)$-probabilistic transition matrix $Q_{(G\widehat{\Box}H)_{(1,1)}}$ is given as

$$Q_{(G\widehat{\Box}H)_{(1,1)}} = \begin{pmatrix} O & \frac{1}{2}I & O & \frac{1}{2}I \\ \frac{1}{3}I & O & \frac{1}{3}I & \frac{1}{3}I \\ O & \frac{1}{2}I & O & \frac{1}{2}I \\ \frac{1}{3}I & \frac{1}{3}I & \frac{1}{3}I & O \end{pmatrix} \begin{pmatrix} P_H & O & O & O \\ O & P_H & O & O \\ O & O & P_H & O \\ O & O & O & P_H \end{pmatrix} = \begin{pmatrix} O & \frac{1}{2}P_H & O & \frac{1}{2}P_H \\ \frac{1}{3}P_H & O & \frac{1}{3}P_H & \frac{1}{3}P_H \\ O & \frac{1}{2}P_H & O & \frac{1}{2}P_H \\ \frac{1}{3}P_H & \frac{1}{3}P_H & \frac{1}{3}P_H & O \end{pmatrix}$$

$$= \left(\begin{array}{cccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\
0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\
\frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 \\
\frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 \\
\frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} \\
\frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\
0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

The eigenvalues of $\Delta_{\mathcal{Q}_{(G\widehat{\Box}H)_{(1,1)}}}$ are

$$\left\{ \begin{array}{l} 0, \frac{1}{9}\bigl(8-\sqrt{7}\bigr),\ \frac{2}{3},\ \frac{1}{6}\bigl(7-\sqrt{7}\bigr),\ \frac{1}{18}\bigl(17-\sqrt{7}\bigr),\ \frac{5}{6},\ \frac{8}{9},\ \frac{17}{18}, \\ 1,1,1,1,1,\ \frac{1}{18}\bigl(17+\sqrt{7}\bigr),\ \frac{7}{6},\ \frac{1}{9}\bigl(8+\sqrt{7}\bigr),\ \frac{4}{3},\ \frac{3}{2},\ \frac{1}{6}\bigl(7+\sqrt{7}\bigr),\ \frac{5}{3} \end{array} \right\}.$$

The $(1,1)$-adjacency matrix $A_{(G\widehat{\square}H)_{(1,1)}}$ is given as

$$A_{(G\widehat{\square}H)_{(1,1)}} = \begin{pmatrix} O & I & O & I \\ I & O & I & I \\ O & I & O & I \\ I & I & I & O \end{pmatrix} \begin{pmatrix} P_H & O & O & O \\ O & P_H & O & O \\ O & O & P_H & O \\ O & O & O & P_H \end{pmatrix} = \begin{pmatrix} O & P_H & O & P_H \\ P_H & O & P_H & P_H \\ O & P_H & O & P_H \\ P_H & P_H & P_H & O \end{pmatrix}$$

$$= \begin{pmatrix}
0&0&0&0&0&0&\frac14&\frac14&\frac14&\frac14&0&0&0&0&0&0&\frac14&\frac14&\frac14&\frac14\\
0&0&0&0&0&\frac12&0&\frac12&0&0&0&0&0&0&0&\frac12&0&\frac12&0&0\\
0&0&0&0&0&\frac13&\frac13&0&\frac13&0&0&0&0&0&0&\frac13&\frac13&0&\frac13&0\\
0&0&0&0&0&\frac13&0&\frac13&0&\frac13&0&0&0&0&0&\frac13&0&\frac13&0&\frac13\\
0&0&0&0&0&\frac12&0&0&\frac12&0&0&0&0&0&0&\frac12&0&0&\frac12&0\\
0&\frac14&\frac14&\frac14&\frac14&0&0&0&0&0&0&\frac14&\frac14&\frac14&\frac14&0&\frac14&\frac14&\frac14&\frac14\\
\frac12&0&\frac12&0&0&0&0&0&0&0&\frac12&0&\frac12&0&0&\frac12&0&\frac12&0&0\\
\frac13&\frac13&0&\frac13&0&0&0&0&0&0&\frac13&\frac13&0&\frac13&0&\frac13&\frac13&0&\frac13&0\\
\frac13&0&\frac13&0&\frac13&0&0&0&0&0&\frac13&0&\frac13&0&\frac13&\frac13&0&\frac13&0&\frac13\\
\frac12&0&0&\frac12&0&0&0&0&0&0&\frac12&0&0&\frac12&0&\frac12&0&0&\frac12&0\\
0&0&0&0&0&0&\frac14&\frac14&\frac14&\frac14&0&0&0&0&0&0&\frac14&\frac14&\frac14&\frac14\\
0&0&0&0&0&\frac12&0&\frac12&0&0&0&0&0&0&0&\frac12&0&\frac12&0&0\\
0&0&0&0&0&\frac13&\frac13&0&\frac13&0&0&0&0&0&0&\frac13&\frac13&0&\frac13&0\\
0&0&0&0&0&\frac13&0&\frac13&0&\frac13&0&0&0&0&0&\frac13&0&\frac13&0&\frac13\\
0&0&0&0&0&\frac12&0&0&\frac12&0&0&0&0&0&0&\frac12&0&0&\frac12&0\\
0&\frac14&\frac14&\frac14&\frac14&0&\frac14&\frac14&\frac14&\frac14&0&\frac14&\frac14&\frac14&\frac14&0&0&0&0&0\\
\frac12&0&\frac12&0&0&\frac12&0&\frac12&0&0&\frac12&0&\frac12&0&0&0&0&0&0&0\\
\frac13&\frac13&0&\frac13&0&\frac13&\frac13&0&\frac13&0&\frac13&\frac13&0&\frac13&0&0&0&0&0&0\\
\frac13&0&\frac13&0&\frac13&\frac13&0&\frac13&0&\frac13&\frac13&0&\frac13&0&\frac13&0&0&0&0&0\\
\frac12&0&0&\frac12&0&\frac12&0&0&\frac12&0&\frac12&0&0&\frac12&0&0&0&0&0&0
\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{A}_{(G\widehat{\square}H)_{(1,1)}}}$ are

$$\left\{\begin{array}{l}
0,\ \frac{1}{12}\big(31+\sqrt{7}-\sqrt{23}-\sqrt{161}\big),\ \frac14\big(11-\sqrt{23}\big),\ \frac{1}{12}\big(31-\sqrt{7}+\sqrt{23}-\sqrt{161}\big),\\
\quad \frac{1}{12}\big(31-\sqrt{65}\big),\ 2,2,2,2,2,\ \frac16\big(17-\sqrt{7}\big),\ \frac52,\ \frac{17}{6},\\
\quad \frac{1}{12}\big(31-\sqrt{7}-\sqrt{23}+\sqrt{161}\big),\ \frac{1}{12}\big(31+\sqrt{65}\big),\ \frac16\big(17+\sqrt{7}\big),\\
\quad \frac14\big(11+\sqrt{23}\big),\ 4,4,\ \frac{1}{12}\big(31+\sqrt{7}+\sqrt{23}+\sqrt{161}\big)
\end{array}\right\}.$$

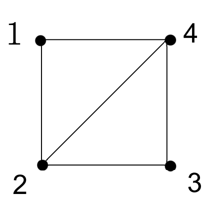

FIG. 14

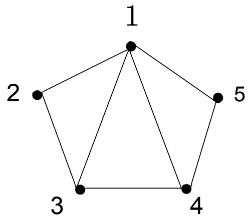

FIG. 15

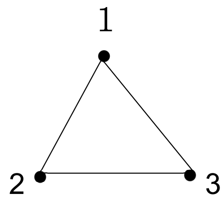

FIG. 16

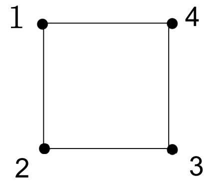

FIG. 17

EXAMPLE 4.14. Let G be a 3-circuit (Fig. 16) and H be the graph given in Fig. 14. The adjacency matrix A_G of G and the transition matrix P_H of H are given as

$$A_G = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad P_H = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0, \frac{3}{2}, \frac{3}{2}\}$ and $\{0, 1, \frac{4}{3}, \frac{5}{3}\}$. The $(1,1)$-adjacency matrix $A_{(G\widehat{\square}H)_{(1,1)}}$ is given as

$$A_{(G\widehat{\square}H)_{(1,1)}} = \begin{pmatrix} O & I & I \\ I & O & I \\ I & I & O \end{pmatrix} \begin{pmatrix} P_H & O & O \\ O & P_H & O \\ O & O & P_H \end{pmatrix} = \begin{pmatrix} O & P_H & P_H \\ P_H & O & P_H \\ P_H & P_H & O \end{pmatrix}$$

$$= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{A}_{(G\widehat{\square}H)_{(1,1)}}}$ are

$$\Big\{0, \frac{4}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{3}, 2, 2, 2, \frac{8}{3}, 3, 3, \frac{10}{3}\Big\}.$$

3.2. $(1,1)$-Laplacians of Kähler graphs of strong product type.

THEOREM 4.6. *Let G be a regular finite graph and H be a finite graph. We denote by $0=\mu_1\le\cdots\le\mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0=\nu_1\le\cdots\le\nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G\widehat{\boxtimes}H$ of strong product type are*

$$\frac{1}{d_G+1}\{(1+d_G-d_G\mu_i)(\mu_i+\nu_\alpha-\mu_i\nu_\alpha)+d_G\mu_i\}\quad(1\le i\le n_G, 1\le\alpha\le n_H).$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V\times W$ is an eigenfunction associated with $\{(1+d_G-d_G\mu_i)(\mu_i+\nu_\alpha-\mu_i\nu_\alpha)+d_G\mu_i\}/(d_G+1)$.

PROOF. We use the same notations as in the proof of Theorem 4.4. Since the principal graph of $G\widehat{\boxtimes}H$ and that of $G\widehat{\square}H$ are the same, we have

$$A^{(p)}_{G\widehat{\boxtimes}H}=(a^{(p)}_{(i,\alpha),(j,\beta)})=(a^G_{ij}\delta_{\alpha\beta}).$$

The transition patrix $P^{(a)}_{G\widehat{\boxtimes}H}$ of the auxiliary graph of $G\widehat{\boxtimes}H$ (for general G) is given by

$$P^{(a)}_{G\widehat{\boxtimes}H}=\begin{pmatrix}\frac{1}{(d_G(v_1)+1)}P_H & \frac{a^G_{12}}{(d_G(v_1)+1)}P_H & \cdots & \frac{a^G_{1n_G}}{(d_G(v_1)+1)}P_H\\ \frac{a^G_{21}}{(d_G(v_2)+1)}P_H & \frac{1}{(d_G(v_1)+1)}P_H & \ddots & \vdots\\ \vdots & \ddots & \ddots & \frac{a^G_{n_G-1n_G}}{(d_G(v_{n_G-1})+1)}P_H\\ \frac{a^G_{n_G1}}{(d_G(v_{n_G})+1)}P_H & \cdots & \frac{a^G_{n_Gn_G-1}}{(d_G(v_{n_G})+1)}P_H & \frac{1}{(d_G(v_{n_G})+1)}P_H\end{pmatrix},$$

hence we have

$$P^{(a)}_{G\widehat{\boxtimes}H}=(p^{(a)}_{(i,\alpha),(j,\beta)})=\left(\frac{p^H_{\alpha\beta}(\delta_{ij}+a^G_{ij})}{d_G(v)+1}\right).$$

We therefore obtain

$$A^{(p)}_{G\widehat{\boxtimes}H}P^{(a)}_{G\widehat{\boxtimes}H}=\left(\frac{1}{d_G(v_i)+1}\{p^H_{\alpha\beta}(a^G_{ij}+\sum_{k=1}^{n_G}a^G_{ik}a^G_{kj})\}\right).$$

When G is regular, we have

$$A^{(p)}_{G\widehat{\boxtimes}H}P^{(a)}_{G\widehat{\boxtimes}H}\begin{pmatrix}\zeta_1\eta_1\\ \vdots\\ \zeta_{n_G}\eta_{n_H}\end{pmatrix} = \left(\frac{1}{d_G+1}\Big\{\Big(\sum_{\beta=1}^{n_H} p^H_{\alpha\beta}\eta_\beta\Big)\Big(\sum_{j=1}^{n_G} a^G_{ij}\zeta_j + \sum_{j=1}^{n_G}\sum_{k=1}^{n_G} a^G_{ik}a^G_{kj}\zeta_j\Big)\Big\}\right)$$
$$= \left(\frac{d_G(1-\mu)(1-\nu)}{d_G+1}\Big\{\eta_\alpha\Big(\zeta_i + \sum_{k=1}^{n_G} a^G_{ik}\zeta_k\Big)\Big\}\right)$$
$$= \frac{d_G(1-\mu)(1-\nu)\{1+d_G(1-\mu)\}}{d_G+1}\begin{pmatrix}\zeta_1\eta_1\\ \vdots\\ \zeta_{n_G}\eta_{n_H}\end{pmatrix}$$

We therefore get

$$\begin{aligned}\varDelta_{\mathcal{P}_{(1,1)}}\varphi_{f_i,g_\alpha} &= (\mathcal{I} - \mathcal{P}^{(p)}_{G\widehat{\boxtimes}H}\mathcal{P}^{(a)}_{G\widehat{\boxtimes}H})\varphi_{f_i,g_\alpha} = \varphi_{f_i,g_\alpha} - \frac{1}{d_G}\mathcal{A}^{(p)}_{G\widehat{\boxtimes}H}\mathcal{P}^{(a)}_{G\widehat{\boxtimes}H}\varphi_{f_i,g_\alpha}\\ &= \varphi_{f_i,g_\alpha} - \frac{(1-\mu_i)(1-\nu_\alpha)\{1+d_G(1-\mu_i)\}}{d_G+1}\varphi_{f_i,g_\alpha}\\ &= \frac{1}{d_G+1}\{(1+d_G-d_G\mu_i)(\mu_i+\nu_\alpha-\mu_i\nu_\alpha)+d_G\mu_i\}\varphi_{f_i,g_\alpha}\end{aligned}$$

This completes the proof. □

Proposition 4.3. *Let G be a regular finite graph and H be a finite graph. We denote by $0=\mu_1\le\cdots\le\mu_{n_G}$ the eigenvalues of $\varDelta_{\mathcal{P}_G}$, and by $0=\nu_1\le\cdots\le\nu_{n_H}$ that of $\varDelta_{\mathcal{P}_H}$. Then the eigenvalues of $\varDelta_{\mathcal{A}_{(1,1)}}$ for their Kähler graph $G\widehat{\boxtimes}H$ of strong product type are*

$$\frac{d_G}{d_G+1}\{(1+d_G-d_G\mu_i)(\mu_i+\nu_\alpha-\mu_i\nu_\alpha)+d_G\mu_i\}\quad (1\le i\le n_G, 1\le\alpha\le n_H).$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V\times W$ is an eigenfunction associated with $d_G\{(1+d_G-d_G\mu_i)(\mu_i+\nu_\alpha-\mu_i\nu_\alpha)+d_G\mu_i\}/(d_G+1)$.

Proof. Since G is regular, we obtain our conclusion directly by Theorem 4.6. □

EXAMPLE 4.15. Let G be a 3-circuit (Fig. 16) and H be the graph given in Fig. 14. The adjacency matrix A_G of G and the transition matrix P_H of H are given as

$$P_G = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad P_H = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0, \frac{3}{2}, \frac{3}{2}\}$ and $\{0, 1, \frac{4}{3}, \frac{5}{3}\}$, respectively. The $(1,1)$-adjacency matrix $A_{(G\widehat{\boxtimes}H)_{(1,1)}}$ is given as

$$A_{(G\widehat{\boxtimes}H)_{(1,1)}} = \begin{pmatrix} O & I & I \\ I & O & I \\ I & I & O \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} P_H & P_H & P_H \\ P_H & P_H & P_H \\ P_H & P_H & P_H \end{pmatrix} = \frac{2}{3} \begin{pmatrix} P_H & P_H & P_H \\ P_H & P_H & P_H \\ P_H & P_H & P_H \end{pmatrix}$$

$$= \begin{pmatrix}
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} \\
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 \\
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} \\
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 \\
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} \\
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0
\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{A}_{(G\widehat{\boxtimes}H)_{(1,1)}}}$ are

$$\Big\{0, 2, 2, 2, 2, 2, 2, 2, 2, 2, \frac{8}{3}, \frac{10}{3}\Big\},$$

and the eigenvalues of $\Delta_{\mathcal{Q}_{(G\widehat{\boxtimes}H)_{(1,1)}}}$ are

$$\Big\{0, 1, 1, 1, 1, 1, 1, 1, 1, 1, \frac{4}{3}, \frac{5}{3}\Big\}.$$

Example 4.16. Let G be a 4-circuit (Fig. 17) and H be the graph given in Fig. 14. The adjacency matrix A_G of G and the transition matrix P_H are given as

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad P_H = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0,1,1,2\}$ and $\{0,1,\frac{4}{3},\frac{5}{3}\}$, respectively. The $(1,1)$-adjacency matrix $A_{(G\widehat{\boxtimes}H)_{(1,1)}}$ is given as

$$A_{(G\widehat{\boxtimes}H)_{(1,1)}} = \begin{pmatrix} O & I & O & I \\ I & O & I & O \\ O & I & O & I \\ I & O & I & O \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} P_H & P_H & O & P_H \\ P_H & P_H & P_H & O \\ O & P_H & P_H & P_H \\ P_H & O & P_H & P_H \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2P_H & P_H & 2P_H & P_H \\ P_H & 2P_H & P_H & 2P_H \\ 2P_H & P_H & 2P_H & P_H \\ P_H & 2P_H & P_H & 2P_H \end{pmatrix}$$

$$= \begin{pmatrix}
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\
\frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} \\
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\
\frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} \\
0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 0 & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} \\
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\
\frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} \\
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\
\frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & 0 & \frac{2}{9} & \frac{2}{9} \\
0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & 0
\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{A}_{(G\widehat{\boxtimes}H)_{(1,1)}}}$ are

$$\left\{0,\ \frac{4}{3},\ 2,2,2,2,2,2,2,2,2,2,\ \frac{20}{9},\ \frac{22}{9},\ \frac{8}{3},\ \frac{10}{3}\right\}.$$

For comparison we here give an example of the case that G is not regular.

EXAMPLE 4.17. Let G be a non-regular graph given in Fig. 14 and H be a 3-circuit. The adjacency matrix A_G of G and the transition matrix P_H of H are given as

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad P_H = \begin{pmatrix} 0 & \frac12 & \frac12 \\ \frac12 & 0 & \frac12 \\ \frac12 & \frac12 & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{A}_G}$, $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0, 1, \frac12(\sqrt{17}-1), \frac12(\sqrt{17}-1)\}$, $\{0, 1, \frac43, \frac53\}$ and $\{0, \frac32, \frac32\}$, respectively. The $(1,1)$-adjacency matrix $A_{(G\widehat{\boxtimes}H)_{(1,1)}}$ is given as

$$A_{(G\widehat{\boxtimes}H)_{(1,1)}} = \begin{pmatrix} O & I & O & I \\ I & O & I & I \\ O & I & O & I \\ I & I & I & O \end{pmatrix} \cdot \begin{pmatrix} \frac13 P_H & \frac13 P_H & O & \frac13 P_H \\ \frac14 P_H & \frac14 P_H & \frac14 P_H & \frac14 P_H \\ O & \frac13 P_H & \frac13 P_H & \frac13 P_H \\ \frac14 P_H & \frac14 P_H & \frac14 P_H & \frac14 P_H \end{pmatrix}$$

$$= \frac{1}{12} \begin{pmatrix} 6P_H & 6P_H & 6P_H & 6P_H \\ 7P_H & 11P_H & 7P_H & 11P_H \\ 6P_H & 6P_H & 6P_H & 6P_H \\ 7P_H & 11P_H & 7P_H & 11P_H \end{pmatrix}$$

$$= \begin{pmatrix}
0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 \\
\frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 \\
\frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 \\
0 & \frac{7}{24} & \frac{7}{24} & 0 & \frac{11}{24} & \frac{11}{24} & 0 & \frac{7}{24} & \frac{7}{24} & 0 & \frac{11}{24} & \frac{11}{24} \\
\frac{7}{24} & 0 & \frac{7}{24} & \frac{11}{24} & 0 & \frac{11}{24} & \frac{7}{24} & 0 & \frac{7}{24} & \frac{11}{24} & 0 & \frac{11}{24} \\
\frac{7}{24} & \frac{7}{24} & 0 & \frac{11}{24} & \frac{11}{24} & 0 & \frac{7}{24} & \frac{7}{24} & 0 & \frac{11}{24} & \frac{11}{24} & 0 \\
0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 \\
\frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 \\
\frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 \\
0 & \frac{7}{24} & \frac{7}{24} & 0 & \frac{11}{24} & \frac{11}{24} & 0 & \frac{7}{24} & \frac{7}{24} & 0 & \frac{11}{24} & \frac{11}{24} \\
\frac{7}{24} & 0 & \frac{7}{24} & \frac{11}{24} & 0 & \frac{11}{24} & \frac{7}{24} & 0 & \frac{7}{24} & \frac{11}{24} & 0 & \frac{11}{24} \\
\frac{7}{24} & \frac{7}{24} & 0 & \frac{11}{24} & \frac{11}{24} & 0 & \frac{7}{24} & \frac{7}{24} & 0 & \frac{11}{24} & \frac{11}{24} & 0
\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{A}_{(G\widehat{\boxtimes}H)_{(1,1)}}}$ are

$$\left\{ \begin{matrix} 0, 2, 2, 2, \frac{13}{6}, \frac{1}{24}\left(77-\sqrt{457}\right), \frac{1}{24}\left(77-\sqrt{457}\right), \\ 3, 3, 3, \frac{1}{24}\left(77+\sqrt{457}\right), \frac{1}{24}\left(77+\sqrt{457}\right) \end{matrix} \right\}.$$

The $(1,1)$-probabilistic transition matrix $Q_{(G\widehat{\boxtimes}H)_{(1,1)}}$ is given as

$$Q_{(G\widehat{\boxtimes}H)_{(1,1)}} = \begin{pmatrix} O & \frac{1}{2}I & O & \frac{1}{2}I \\ \frac{1}{3}I & O & \frac{1}{3}I & \frac{1}{3}I \\ O & \frac{1}{2}I & O & \frac{1}{2}I \\ \frac{1}{3}I & \frac{1}{3}I & \frac{1}{3}I & O \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3}P_H & \frac{1}{3}P_H & O & \frac{1}{3}P_H \\ \frac{1}{4}P_H & \frac{1}{4}P_H & \frac{1}{4}P_H & \frac{1}{4}P_H \\ O & \frac{1}{3}P_H & \frac{1}{3}P_H & \frac{1}{3}P_H \\ \frac{1}{4}P_H & \frac{1}{4}P_H & \frac{1}{4}P_H & \frac{1}{4}P_H \end{pmatrix}$$

$$= \frac{1}{36}\begin{pmatrix} 9P_H & 9P_H & 9P_H & 9P_H \\ 7P_H & 11P_H & 7P_H & 11P_H \\ 9P_H & 9P_H & 9P_H & 9P_H \\ 7P_H & 11P_H & 7P_H & 11P_H \end{pmatrix}$$

$$= \begin{pmatrix}
0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} \\
\frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 \\
0 & \frac{7}{24} & \frac{7}{24} & 0 & \frac{11}{24} & \frac{11}{24} & 0 & \frac{7}{24} & \frac{7}{24} & 0 & \frac{11}{24} & \frac{11}{24} \\
\frac{7}{72} & 0 & \frac{7}{72} & \frac{11}{72} & 0 & \frac{11}{72} & \frac{7}{72} & 0 & \frac{7}{72} & \frac{11}{72} & 0 & \frac{11}{72} \\
\frac{7}{72} & \frac{7}{72} & 0 & \frac{11}{72} & \frac{11}{72} & 0 & \frac{7}{72} & \frac{7}{72} & 0 & \frac{11}{72} & \frac{11}{72} & 0 \\
0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} \\
\frac{1}{8} & \frac{1}{4} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{4} & 0 & \frac{1}{8} & \frac{1}{8} & 0 \\
0 & \frac{7}{72} & \frac{7}{72} & 0 & \frac{11}{72} & \frac{11}{72} & 0 & \frac{7}{72} & \frac{7}{72} & 0 & \frac{11}{72} & \frac{11}{72} \\
\frac{7}{72} & 0 & \frac{7}{72} & \frac{11}{72} & 0 & \frac{11}{72} & \frac{7}{72} & 0 & \frac{7}{72} & \frac{11}{72} & 0 & \frac{11}{72} \\
\frac{7}{72} & \frac{7}{72} & 0 & \frac{11}{72} & \frac{11}{72} & 0 & \frac{7}{72} & \frac{7}{72} & 0 & \frac{11}{72} & \frac{11}{72} & 0
\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{Q}_{(G\widehat{\boxtimes}H)_{(1,1)}}}$ are

$$\Big\{0,\ \frac{8}{9},\ 1,1,1,1,1,1,\ \frac{19}{18},\ \frac{19}{18},\ \frac{3}{2},\ \frac{3}{2}\Big\}.$$

3.3. $(1,1)$-Laplacians of Kähler graphs of semi-tensor product type.

THEOREM 4.7. *Let $G = (V, E), H = (W, F)$ be finite ordinary graphs. We denote by $0 = \mu_1 \le \cdots \le \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 \le \cdots \le \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G \widehat{\otimes} H$ of semi-tensor product type are*

$$1 - (1-\nu_\alpha)(1-\mu_i)^2 \quad (1 \le i \le n_G, 1 \le \alpha \le n_H).$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V \times W$ is an eigenfunction associated with $1-(1-\nu_\alpha)(1-\mu_i)^2$.

PROOF. We use the same notations as in the proof of Theorem 4.4. The adjacency matrix $A^{(p)}_{G\widehat{\otimes}H}$ of the principal graph of $G\widehat{\otimes}H$ is the same as the adjacency matrix of $G\widehat{\square}H$. Thus we have

$$A^{(p)}_{G\widehat{\otimes}H} = (a^{(p)}_{(i,\alpha),(j,\beta)}) = \big(a^G_{ij}\delta_{\alpha\beta}\big).$$

The transition matrix $P^{(a)}_{G\widehat{\otimes}H}$ of the auxiliary graph of $G\widehat{\otimes}H$ is given as

$$P^{(a)}_{G\widehat{\otimes}H} = \begin{pmatrix} 0 & \dfrac{a^G_{12}}{d_G(v_1)}P_H & \cdots & \dfrac{a^G_{1n_G}}{d_G(v_1)}P_H \\ \dfrac{a^G_{21}}{d_G(v_2)}P_H & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \dfrac{a^G_{n_G-1n_G}}{d_G(v_{n_G-1})}P_H \\ \dfrac{a^G_{n_G1}}{d_G(v_{n_G})}P_H & \cdots & \dfrac{a^G_{n_Gn_G-1}}{d_G(v_{n_G})}P_H & 0 \end{pmatrix},$$

hence we have

$$P^{(a)}_{G\widehat{\otimes}H} = (p^{(a)}_{(i,\alpha),(j,\beta)}) = \left(\frac{a^G_{ij}p^H_{\alpha\beta}}{d_G(v_i)}\right).$$

We denote by $P_G = (p^G_{ij})$ the transition matrix of G. Then we have $p^G_{ij} = a^G_{ij}/d_G(v_i)$. Thus we have

$$A^{(p)}_{G\widehat{\otimes}H}P^{(a)}_{G\widehat{\otimes}H} = \left(p^H_{\alpha\beta}\sum_{k=1}^{n_G} a^G_{ik}p^G_{kj}\right).$$

We hence get

$$A^{(p)}_{G\widehat{\otimes}H}P^{(a)}_{G\widehat{\otimes}H}\begin{pmatrix}\zeta_1\eta_1\\ \vdots\\ \zeta_{n_G}\eta_{n_H}\end{pmatrix} = \Bigl(\Bigl(\sum_{\alpha=1}^{n_H} p^H_{\alpha\beta}\eta_\beta\Bigr)\Bigl(\sum_{j=1}^{n_G}\sum_{k=1}^{n_G} a^G_{ik}p^G_{kj}\zeta_j\Bigr)\Bigr)$$

$$= \Bigl((1-\nu)\eta_\alpha\Bigl((1-\mu)\sum_{k=1}^{n_G} a^G_{ik}\zeta_k\Bigr)\Bigr) = \Bigl((1-\mu)^2(1-\nu)d_G(v_i)\zeta_i\eta_\alpha\Bigr).$$

Therefore we get

$$\Delta_{\mathcal{Q}_{(1,1)}}\varphi_{f_i,g_\alpha} = (\mathcal{I}-\mathcal{P}^{(p)}_{G\widehat{\otimes}H}\mathcal{P}^{(a)}_{G\widehat{\otimes}H})\varphi_{f_i,g_\alpha} = \varphi_{f_i,g_\alpha} - (1-\nu_\alpha)(1-\mu_i)^2\}\varphi_{f_i,g_\alpha}$$

$$= \{1-(1-\nu_\alpha)(1-\mu_i)^2\}\varphi_{f_i,g_\alpha}.$$

Hence the eigenvalues are

$$1-(1-\nu_\alpha)(1-\mu_i)^2 = \mu_i(\mu_i-2)(\nu_\alpha-1)+\nu_\alpha,$$

and we get the conclusion. □

THEOREM 4.8. *Let G be a regular finite graph and H be a finite graph. We denote by $0=\mu_1\le\cdots\le\mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0=\nu_1\le\cdots\le\nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ for their Kähler graph $G\widehat{\otimes}H$ of semi-tensor product type are*

$$d_G\{1-(1-\nu_\alpha)(1-\mu_i)^2\} \quad \{(1\le i\le n_G, 1\le\alpha\le n_H)\}.$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V\times W$ is an eigenfunction associated with $d_G\{1-(1-\nu_\alpha)(1-\mu_i)^2\}$.

PROOF. Since G is regular, we obtain our conclusion directly by Theorem 4.7. □

EXAMPLE 4.18. Let G be a 3-circuit (Fig. 16) and H be the graph given in Fig. 14. The adjacency matrix A_G of G and the transition matrix P_H of H are given as

$$A_G = \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix} \quad\text{and}\quad P_H = \begin{pmatrix}0&\frac12&0&\frac12\\ \frac13&0&\frac13&\frac13\\ 0&\frac12&0&\frac12\\ \frac13&\frac13&\frac13&0\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0, \frac{3}{2}, \frac{3}{2}\}$ and $\{0, 1, \frac{4}{3}, \frac{5}{3}\}$, respectively. The $(1,1)$-adjacency matrix $A_{(G\widehat{\otimes}H)_{(1,1)}}$ is given as

$$A_{(G\widehat{\otimes}H)_{(1,1)}} = \begin{pmatrix} O & I & I \\ I & O & I \\ I & I & O \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} O & P_H & P_H \\ P_H & O & P_H \\ P_H & P_H & O \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2P_H & P_H & P_H \\ P_H & 2P_H & P_H \\ P_H & P_H & 2P_H \end{pmatrix}$$

$$= \begin{pmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\
\frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0
\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{A}_{(G\widehat{\otimes}H)_{(1,1)}}}$ are

$$\left\{0, \frac{3}{2}, \frac{3}{2}, 2, 2, 2, \frac{13}{6}, \frac{13}{6}, \frac{7}{3}, \frac{7}{3}, \frac{8}{3}, \frac{10}{3}\right\}.$$

EXAMPLE 4.19. Let G and H be non-regular ordinary graphs given in Figs. 14 and 15, respectively. Their transition matrices are given as

$$P_G = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}, \qquad P_H = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

and the eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are

$$\left\{0, 1, \frac{4}{3}, \frac{5}{3}\right\} \quad \text{and} \quad \left\{0, \frac{7-\sqrt{7}}{6}, \frac{7}{6}, \frac{3}{2}, \frac{7+\sqrt{7}}{6}\right\}.$$

The $(1,1)$-probabilistic transition matrix $Q_{(G\widehat{\otimes}H)_{(1,1)}}$ is given as

$$Q_{(G\widehat{\otimes}H)_{(1,1)}} = \begin{pmatrix} O & \frac12 I & O & \frac12 I \\ \frac13 I & O & \frac13 I & \frac13 I \\ O & \frac12 I & O & \frac12 I \\ \frac13 I & \frac13 I & \frac13 I & O \end{pmatrix} \cdot \begin{pmatrix} O & \frac12 P_H & O & \frac12 P_H \\ \frac13 P_H & O & \frac13 P_H & \frac13 P_H \\ O & \frac12 P_H & O & \frac12 P_H \\ \frac13 P_H & \frac13 P_H & \frac13 P_H & O \end{pmatrix}$$

$$= \frac{1}{18} \begin{pmatrix} 6P_H & 3P_H & 6P_H & 3P_H \\ 2P_H & 8P_H & 2P_H & 6P_H \\ 6P_H & 3P_H & 6P_H & 3P_H \\ 2P_H & 6P_H & 2P_H & 8P_H \end{pmatrix}$$

$$= \left(\begin{array}{cccccccccccccccccccc}
0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} \\
\frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{12} & 0 & \frac{1}{12} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{12} & 0 & \frac{1}{12} & 0 & 0 \\
\frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{18} & \frac{1}{18} & 0 & \frac{1}{18} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{18} & \frac{1}{18} & 0 & \frac{1}{18} & 0 \\
\frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{18} & 0 & \frac{1}{18} & 0 & \frac{1}{18} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{18} & 0 & \frac{1}{18} & 0 & \frac{1}{18} \\
\frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{12} & 0 & 0 & \frac{1}{12} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{12} & 0 & 0 & \frac{1}{12} & 0 \\
0 & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\
\frac{1}{18} & 0 & \frac{1}{18} & 0 & 0 & \frac{2}{9} & 0 & \frac{2}{9} & 0 & 0 & \frac{1}{18} & 0 & \frac{1}{18} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 \\
\frac{1}{27} & \frac{1}{27} & 0 & \frac{1}{27} & 0 & \frac{4}{27} & \frac{4}{27} & 0 & \frac{4}{27} & 0 & \frac{1}{27} & \frac{1}{27} & 0 & \frac{1}{27} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 \\
\frac{1}{27} & 0 & \frac{1}{27} & 0 & \frac{1}{27} & \frac{4}{27} & 0 & \frac{4}{27} & 0 & \frac{4}{27} & \frac{1}{27} & 0 & \frac{1}{27} & 0 & \frac{1}{27} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} \\
\frac{1}{18} & 0 & 0 & \frac{1}{18} & 0 & \frac{2}{9} & 0 & 0 & \frac{2}{9} & 0 & \frac{1}{18} & 0 & 0 & \frac{1}{18} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\
0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} \\
\frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{12} & 0 & \frac{1}{12} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{12} & 0 & \frac{1}{12} & 0 & 0 \\
\frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{18} & \frac{1}{18} & 0 & \frac{1}{18} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{18} & \frac{1}{18} & 0 & \frac{1}{18} & 0 \\
\frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{18} & 0 & \frac{1}{18} & 0 & \frac{1}{18} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{18} & 0 & \frac{1}{18} & 0 & \frac{1}{18} \\
\frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{12} & 0 & 0 & \frac{1}{12} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{12} & 0 & 0 & \frac{1}{12} & 0 \\
0 & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & 0 & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\
\frac{1}{18} & 0 & \frac{1}{18} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{18} & 0 & \frac{1}{18} & 0 & 0 & \frac{2}{9} & 0 & \frac{2}{9} & 0 & 0 \\
\frac{1}{27} & \frac{1}{27} & 0 & \frac{1}{27} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{27} & \frac{1}{27} & 0 & \frac{1}{27} & 0 & \frac{4}{27} & \frac{4}{27} & 0 & \frac{4}{27} & 0 \\
\frac{1}{27} & 0 & \frac{1}{27} & 0 & \frac{1}{27} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{27} & 0 & \frac{1}{27} & 0 & \frac{1}{27} & \frac{4}{27} & 0 & \frac{4}{27} & 0 & \frac{4}{27} \\
\frac{1}{18} & 0 & 0 & \frac{1}{18} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{18} & 0 & 0 & \frac{1}{18} & 0 & \frac{2}{9} & 0 & 0 & \frac{2}{9} & 0
\end{array}\right).$$

The eigenvalues of $\Delta_{\mathcal{Q}_{(G\widehat{\otimes}H)_{(1,1)}}}$ are

$$\left\{ \begin{array}{l} 0, \frac{5}{9}, \frac{1}{6}\bigl(7-\sqrt{7}\bigr), \frac{1}{27}\bigl(29-2\sqrt{7}\bigr), \frac{8}{9}, \frac{1}{54}\bigl(55-\sqrt{7}\bigr), 1, 1, 1, 1, 1, \\ \frac{55}{56}, \frac{19}{18}, \frac{1}{54}\bigl(55+\sqrt{7}\bigr), \frac{29}{27}, \frac{7}{6}, \frac{11}{9}, \frac{1}{27}\bigl(29+2\sqrt{7}\bigr), \frac{3}{2}, \frac{1}{6}\bigl(7+\sqrt{7}\bigr) \end{array} \right\}.$$

3.4. $(1,1)$**-Laplacians of Kähler graphs of lexicographical product type.** In order to study eigenvalues of a Kähler graph $G \triangleright H$ of lexicographical product type obtained by $G = (V, E)$ and $H = (W, F)$, we use the operator $\mathcal{M}$ acting on $C(V, \mathbb{R})$ given by $\mathcal{M}f(v) = \sum_{u\in V} f(u)$ given in §4.2. The eigenvalues of $\mathcal{M}$ are $0, \cdots, 0, n_G$. We define a function $\epsilon_1 : V \to \mathbb{R}$ by $e_1(u) = 1$ for all $u \in V$, and define a function $\epsilon_2, \cdots, \epsilon_{n_G}$ by $\epsilon_k = \delta_{v_1} - \delta_{v_k}$ with characteristic functions δ_v $(v \in V)$.

THEOREM 4.9. *Let $G = (V, E), H = (W, F)$ be finite ordinary graphs. We denote by $0 = \mu_1 \le \cdots \le \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 \le \cdots \le \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G \triangleright H$ of lexicographical product type are $0, 1, \cdots, 1, \nu_2, \cdots, \nu_{n_H}$, where 1 appears $(n_G - 1)n_H$ times.*

Moreover, if g_α is an eigenfunction associated with ν_α, then the function $\varphi_{\epsilon_1, g_\alpha}$ is an eigenfunction for $\Delta_{\mathcal{Q}_{(1,1)}}$ associated with ν_α and the function $\varphi_{\epsilon_k, g_\alpha}$ $(k = 2, \dots, n_G)$ are eigenfunctions for $\Delta_{\mathcal{Q}_{(1,1)}}$ associated with 1.

PROOF. We use the same notations as in the proof of Theorem 4.4. The adjacency matrix $A^{(p)}_{G\triangleright H}$ for the principal graph of $G \triangleright H$ is the same as that of $G\widehat{\square}H$. Hence we have

$$A^{(p)}_{G\triangleright H} = (a^{(p)}_{(i,\alpha),(j,\beta)}) = \bigl(a^G_{ij}\delta_{\alpha\beta}\bigr).$$

The transition matrix $P^{(a)}_{G\triangleright H}$ of the auxiliary graph of $G \triangleright H$ is given as

$$P^{(a)}_{G\triangleright H} = \begin{pmatrix} \dfrac{1}{n_G}P_H & \cdots & \dfrac{1}{n_G}P_H \\ \vdots & & \vdots \\ \dfrac{1}{n_G}P_H & \cdots & \dfrac{1}{n_G}P_H \end{pmatrix},$$

hence we have

$$P^{(a)}_{G\triangleright H} = \bigl(p^{(a)}_{(i,\alpha),(j,\beta)}\bigr) = \Bigl(\frac{1}{n_G}p^H_{\alpha\beta}\Bigr).$$

Thus we have

$$A^{(p)}_{G\triangleright H}P^{(a)}_{G\triangleright H} = \Bigl(\frac{1}{n_G}p^H_{\alpha\beta}\sum_{k=1}^{n_G} a^G_{ik}\Bigr) = \Bigl(\frac{1}{n_G}p^H_{\alpha\beta}d_G(v_i)\Bigr).$$

This shows that

$$A^{(p)}_{G\rhd H}P^{(a)}_{G\rhd H}\begin{pmatrix}\zeta_1\eta_1\\ \vdots \\ \zeta_{n_G}\eta_{n_H}\end{pmatrix} = \left(\frac{d_G(v_i)}{n_G}\Bigl(\sum_{\beta=1}^{n_H} p^H_{\alpha\beta}\eta_\beta\Bigr)\Bigl(\sum_{j=1}^{n_G}\zeta_j\Bigr)\right)$$
$$= \left(\frac{d_G(v_i)(1-\nu)}{n_G}\eta_\beta\Bigl(\sum_{j=1}^{n_G}\zeta_j\Bigr)\right).$$

When $k=1$, we have

$$\Delta_{\mathcal{Q}_{(1,1)}}\varphi_{\epsilon_1,g_\alpha} = \varphi_{\epsilon_1,g_\alpha} - \frac{1-\nu_\alpha}{n_G}n_G\varphi_{\epsilon_1,g_\alpha} = \nu_\alpha\varphi_{\epsilon_1,g_\alpha},$$

and when $k\neq 1$, we have

$$\Delta_{\mathcal{Q}_{(1,1)}}\varphi_{\epsilon_k,g_\alpha} = \varphi_{\epsilon_k,g_\alpha},$$

because $\sum_{j=1}^{n_G}\epsilon_k(v_j)=0$. This completes the proof. □

Proposition 4.4. *Let G be a regular finite graph and H be a finite graph. We denote by $0=\nu_1\le\cdots\le\nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ for their Kähler graph $G\rhd H$ of lexicographical product type are $0, d_G,\dots,d_G, d_G\nu_2,\dots,d_G\nu_{n_H}$ where d_G appears $(n_G-1)n_H$ times.*

Moreover, if g_α is an eigenfunction associated with ν_α, then the function $\varphi_{\epsilon_1,g_\alpha}$ is an eigenfunction for $\Delta_{\mathcal{A}_{(1,1)}}$ associated with $d_G\nu_\alpha$ and the function $\varphi_{\epsilon_k,g_\alpha}$ $(k=2,\dots,n_G)$ are eigenfunctions for $\Delta_{\mathcal{A}_{(1,1)}}$ associated with d_G.

Example 4.20. Let G be a 4-circuit and H be the graph given in Fig. 14. The adjacency matrix A_G of G and the transition matrix P_H of H are given as

$$A_G = \begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix} \quad\text{and}\quad P_H = \begin{pmatrix}0&\frac12&0&\frac12\\ \frac13&0&\frac13&\frac13\\ 0&\frac12&0&\frac12\\ \frac13&\frac13&\frac13&0\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0,1,1,2\}$ and $\{0,1,\frac{4}{3},\frac{5}{3}\}$, respectively. The $(1,1)$-adjacency matrix $A_{(G \triangleright H)_{(1,1)}}$ is given as

$$
\begin{aligned}
A_{(G \triangleright H)_{(1,1)}} &= \begin{pmatrix} O & I & O & I \\ I & O & I & O \\ O & I & O & I \\ I & O & I & O \end{pmatrix} \cdot \frac{1}{4} \begin{pmatrix} P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \end{pmatrix} = \frac{1}{2} \begin{pmatrix} P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \end{pmatrix} \\
&= \begin{pmatrix}
0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 \\
\frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 \\
0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 \\
\frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 \\
0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 \\
\frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 \\
0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 \\
\frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 \\
0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 \\
\frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 \\
0 & \frac13 & 0 & \frac13 & 0 & \frac13 & 0 & \frac13 & 0 & \frac13 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 \\
\frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0 & \frac16 & \frac16 & \frac16 & 0
\end{pmatrix}.
\end{aligned}
$$

The eigenvalues of $\Delta_{\mathcal{A}_{(G \triangleright H)_{(1,1)}}}$ are

$$\Big\{0,2,2,2,2,2,2,2,2,2,2,2,2,2,\frac{8}{3},\frac{10}{3}\Big\}.$$

Example 4.21. Let G and H be non-regular ordinary graphs given in Figs. 14 and 15, respectively. Their transition matrices are given as

$$P_G = \begin{pmatrix} 0 & \frac12 & 0 & \frac12 \\ \frac13 & 0 & \frac13 & \frac13 \\ 0 & \frac12 & 0 & \frac12 \\ \frac13 & \frac13 & \frac13 & 0 \end{pmatrix} \quad \text{and} \quad P_H = \begin{pmatrix} 0 & \frac14 & \frac14 & \frac14 & \frac14 \\ \frac12 & 0 & \frac12 & 0 & 0 \\ \frac13 & \frac13 & 0 & \frac13 & 0 \\ \frac13 & 0 & \frac13 & 0 & \frac13 \\ \frac12 & 0 & 0 & \frac12 & 0 \end{pmatrix},$$

and the eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are

$$\Big\{0,1,\frac{4}{3},\frac{5}{3}\Big\} \quad \text{and} \quad \Big\{0,\frac{7-\sqrt{7}}{6},\frac{7}{6},\frac{3}{2},\frac{7+\sqrt{7}}{6}\Big\}.$$

The $(1,1)$-probabilistic transition matrix $Q_{(G\triangleright H)_{(1,1)}}$ is given as

$$Q_{(G\triangleright H)_{(1,1)}} = \begin{pmatrix} O & \frac{1}{2}I & O & \frac{1}{2}I \\ \frac{1}{3}I & O & \frac{1}{3}I & \frac{1}{3}I \\ O & \frac{1}{2}I & O & \frac{1}{2}I \\ \frac{1}{3}I & \frac{1}{3}I & \frac{1}{3}I & O \end{pmatrix} \cdot \frac{1}{4}\begin{pmatrix} P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \end{pmatrix} = \frac{1}{4}\begin{pmatrix} P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{Q}_{(G\triangleright H)_{(1,1)}}}$ are

$$\Big\{0,\ \frac{1}{6}\big(7-\sqrt{7}\big),\ 1,1,1,1,1,1,1,1,1,1,1,1,1,1,\ \frac{7}{6},\ \frac{3}{2},\ \frac{1}{6}\big(7+\sqrt{7}\big)\Big\}.$$

The $(1,1)$-adjacency matrix $A_{(G\triangleright H)_{(1,1)}}$ is given as

$$A_{(G\triangleright H)_{(1,1)}} = \begin{pmatrix} O & I & O & I \\ I & O & I & I \\ O & I & O & I \\ I & I & I & O \end{pmatrix} \cdot \frac{1}{4}\begin{pmatrix} P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2P_H & 2P_H & 2P_H & 2P_H \\ 3P_H & 3P_H & 3P_H & 3P_H \\ 2P_H & 2P_H & 2P_H & 2P_H \\ 3P_H & 3P_H & 3P_H & 3P_H \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{A}_{(G\triangleright H)_{(1,1)}}}$ are

$$\left\{\begin{array}{l} 0,\ \frac{1}{6}\Big(65-5\sqrt{7}-\sqrt{368-74\sqrt{7}}\,\Big),\ 2,2,2,2,2,\ \frac{1}{6}\big(65-\sqrt{193}\big),\ \frac{9}{4}, \\ \frac{1}{6}\Big(65-5\sqrt{7}+\sqrt{368-74\sqrt{7}}\,\Big),\ \frac{5}{2},\ \frac{1}{6}\Big(65+5\sqrt{7}-\sqrt{368+74\sqrt{7}}\,\Big), \\ 3,3,3,3,3,\ \frac{1}{6}\big(65-\sqrt{193}\big),\ 4,\ \frac{1}{6}\Big(65+5\sqrt{7}+\sqrt{368+74\sqrt{7}}\,\Big), \end{array}\right\}.$$

The above example shows that when G is not regular even for Kähler graphs of lexicographical product type the eigenvalues of their $(1,1)$-combinatorial Laplacian are complicated.

Example 4.22. Let G be a union of two 3-circuit and H be a 4-circuit. The eigenvalues of $\Delta_{\mathcal{P}_H}$ are $\{0,1,1,2\}$. The adjacency matrices of the principal and the auxiliary graphs of $G\triangleright H$ are given as

$$A_{(G\triangleright H)^{(p)}} = \begin{pmatrix} O & I & I & O & O & O \\ I & O & I & O & O & O \\ I & I & O & O & O & O \\ O & O & O & O & I & I \\ O & O & O & I & O & I \\ O & O & O & I & I & O \end{pmatrix}, \quad A_{(G\triangleright H)^{(a)}} = \begin{pmatrix} P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H \end{pmatrix},$$

with

$$P_H = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Hence we have

$$A_{(G\rhd H)_{(1,1)}} = A_{(G\rhd H)^{(p)}} \cdot \frac{1}{12} A_{(G\rhd H)^{(a)}} = \frac{1}{6}\begin{pmatrix} P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H \end{pmatrix}.$$

Thus the eigenvalues of the $(1,1)$-probabilistic transition Laplacian and those of the $(1,1)$-adjacency Laplacian are

$$\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(1,1)}}}\big) = \{0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2\},$$
$$\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(1,1)}}}\big) = \{0,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,4\}.$$

EXAMPLE 4.23. Let G be a union of a 3-circuit and a 4-circuit, and H be a 4-circuit. The eigenvalues of $\Delta_{\mathcal{P}_H}$ are $\{0,1,1,2\}$. The adjacency matrices of the principal and the auxiliary graphs of $G \rhd H$ are given as

$$A_{(G\rhd H)^{(p)}} = \begin{pmatrix} O & I & I & O & O & O & O \\ I & O & I & O & O & O & O \\ I & I & O & O & O & O & O \\ O & O & O & O & I & O & I \\ O & O & O & I & O & I & O \\ O & O & O & O & I & I & O \\ O & O & O & I & O & I & O \end{pmatrix},$$

$$A_{(G\rhd H)^{(a)}} = \begin{pmatrix} P_H & P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H & P_H \end{pmatrix} \quad \text{with} \quad P_H = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Hence we have

$$A_{(G\triangleright H)_{(1,1)}} = A_{(G\triangleright H)^{(p)}} \cdot \frac{1}{14} A_{(G\triangleright H)^{(a)}} = \frac{1}{7} \begin{pmatrix} P_H & P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H & P_H \\ P_H & P_H & P_H & P_H & P_H & P_H & P_H \end{pmatrix}.$$

Thus the eigenvalues of the $(1,1)$-probabilistic transition Laplacian are

$$\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(1,1)}}}\big) = \{0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2\}.$$

4. Eigenvalues of $(1,1)$-Laplacians of Kähler graphs of product type added complement-filling operations

Next we calculate eigenvalues of $(1,1)$-Laplacians of Kähler graph of product type with complement-filling operations step by step, which are $G\,\widehat{\square}^K H$, $G\,\widehat{\boxtimes}^K H$, $G\widehat{\otimes}^K H$ and $G \rhd^K H$. In this section also, for functions $f : V \to \mathbb{C}$ and $g : W \to \mathbb{C}$ we denote by $\varphi_{f,g} : V \times W \to \mathbb{C}$ the function defined by $\varphi_{f,g}(v,w) = f(v)g(w)$.

4.1. $(1,1)$-Laplacians of Kähler graphs of complement-filling Cartesian product type.

Theorem 4.10. *Let $G = (V,E), H = (W,F)$ be finite regular ordinary graphs. We suppose G is connected. We denote by $0 = \mu_1 < \mu_2 \le \cdots \le \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 \le \cdots \le \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. We put $\mathfrak{D} = n_G - d_G - 1 + d_H$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G\,\widehat{\square}^K H$ of complement-filling Cartesian product type are*

$$\frac{1}{\mathfrak{D}} d_H \nu_\alpha, \quad 1 - \frac{1}{\mathfrak{D}}(1-\mu_i)(d_G\mu_i - d_H\nu_\alpha - d_G + d_H - 1) \quad (2 \le i \le n_G,\ 1 \le \alpha \le n_H),$$

and the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ for $G\,\widehat{\square}^K H$ are

$$\frac{1}{\mathfrak{D}} d_G d_H \nu_\alpha, \quad d_G - \frac{d_G}{\mathfrak{D}}(1-\mu_i)(d_G\mu_i - d_H\nu_\alpha - d_G + d_H - 1) \quad (2 \le i \le n_G,\ 1 \le \alpha \le n_H).$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V \times W$ is an eigenfunction for both $\Delta_{\mathcal{Q}_{(1,1)}}$ and $\Delta_{\mathcal{A}_{(1,1)}}$.

Proof. We use the same notations as in the proof of Theorem 4.4. Since the principal graph of $G\,\widehat{\square}^K H$ is the same as that of $G\,\widehat{\square} H$, the adjacency matrix $A^{(p)}_{G\,\widehat{\square}^K H}$ for the principal graph $G\,\widehat{\square}^K H$ is given by

$$A^{(p)}_{G\,\widehat{\square}^K H} = (a^{(p)}_{(i,\alpha),(j,\beta)}) = \bigl(a^G_{ij}\delta_{\alpha\beta}\bigr).$$

We denote by $A_{G^c} = (a^{G^c}_{ij})$ the adjacency matrix of the complement graph G^c. We then have $a^{G^c}_{ij} = 1 - a^G_{ij} - \delta_{ij}$. Since G and H are regular, the Kähler graph $G\,\widehat{\square}^K H$ is also regular, and its auxiliary degree is $d_H + d_{G^c} = d_H + n_G - d_G - 1 = \mathfrak{D}$. Therefore we find that the transition matrix $P^{(a)}_{G\widehat{\square}^K H}$ of $G\widehat{\square}^K H$ is

$$P^{(a)}_{G\widehat{\square}^K H} = \begin{pmatrix} \frac{1}{\mathfrak{D}}A_H & \frac{a^{G^c}_{12}}{\mathfrak{D}}I & \cdots & \frac{a^{G^c}_{1n_G}}{\mathfrak{D}}I \\ \frac{a^{G^c}_{21}}{\mathfrak{D}}I & \frac{1}{\mathfrak{D}}A_H & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{a^{G^c}_{n_G-1n_G}}{\mathfrak{D}}I \\ \frac{a^{G^c}_{n_G1}}{\mathfrak{D}}I & \cdots & \frac{a^{G^c}_{n_Gn_G-1}}{\mathfrak{D}}I & \frac{1}{\mathfrak{D}}A_H \end{pmatrix}.$$

That is,

$$P^{(a)}_{G\widehat{\square}^K H} = (p^{(a)}_{(i,\alpha),(j,\beta)}) = \Big(\frac{1}{\mathfrak{D}}(\delta_{ij}a^H_{\alpha\beta} + a^{G^c}_{ij}\delta_{\alpha\beta})\Big).$$

Therefore, we have

$$A^{(p)}_{G\widehat{\square}^K H}P^{(a)}_{G\widehat{\square}^K H} = \Big(\frac{1}{\mathfrak{D}}\Big\{a^G_{ij}a^H_{\alpha\beta} + \sum_{k=1}^{n_G} a^G_{ik}a^{G^c}_{kj}\delta_{\alpha\beta}\Big\}\Big).$$

For functions f and g satisfying $\Delta_{\mathcal{P}_G} f = \mu f$ and $\Delta_{\mathcal{P}_H} g = \nu g$, we have $\mathcal{A}_G f = d_G(1-\mu)f$, $\mathcal{A}_H g = d_H(1-\nu)g$ and

$$\mathcal{A}_{G^c} f = (\mathcal{M} - \mathcal{I} - \mathcal{A}_G)f = \begin{cases} (n_G - 1 - d_G)f, & \text{when } \mu = 0, \\ \{d_G(\mu - 1) - 1)\}f, & \text{when } \mu \neq 0. \end{cases}$$

We take $\varphi_{f,g}$. Then we have

$$\begin{aligned} &A^{(p)}_{G\widehat{\square}^K H}P^{(a)}_{G\widehat{\square}^K H}\begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_H} \end{pmatrix} \\ &\quad = \bigg(\frac{1}{\mathfrak{D}}\Big\{d_G d_H(1-\nu)\zeta_i\eta_\alpha + (n_G - 1 - d_G)\eta_\alpha\Big(\sum_{k=1}^{n_G} a^G_{ik}\zeta_k\Big)\Big\}\bigg) \\ &\quad = \bigg(\frac{d_G}{\mathfrak{D}}\{d_H(1-\nu) + n_G - 1 - d_G\}\zeta_i\eta_\alpha\bigg) \end{aligned}$$

when $\mu = 0$, and have

$$
\begin{aligned}
&A^{(p)}_{G\hat{\square}^K H} P^{(a)}_{G\hat{\square}^K H} \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_H} \end{pmatrix} \\
&\quad = \left(\frac{1}{\mathfrak{D}} \Big\{ d_G d_H (1-\mu)(1-\nu)\zeta_i\eta_\alpha + (d_G\mu - d_G - 1)\eta_\alpha \Big(\sum_{k=1}^{n_G} a^G_{ik}\zeta_k \Big) \Big\} \right) \\
&\quad = \left(\frac{d_G(1-\mu)}{\mathfrak{D}} \{ d_H(1-\nu) + d_G\mu - d_G - 1\} \zeta_i\eta_\alpha \right)
\end{aligned}
$$

when $\mu \neq 0$. Thus we obtain

$$
\begin{aligned}
\Delta_{\mathcal{P}_{(1,1)}} \varphi_{f,g} &= (\mathcal{I} - \frac{1}{d_G} \mathcal{A}^{(p)}_{G\hat{\square}^K H} \mathcal{P}^{(a)}_{G\hat{\square}^K H}) \varphi_{f,g} \\
&= \begin{cases} \big(1 - \frac{(1-\mu)}{\mathfrak{D}} \{ d_H(1-\nu) + d_G\mu - d_G - 1\}\big) \varphi_{f,g}, & \text{when } \mu \neq 0, \\ \big(1 - \frac{1}{\mathfrak{D}} \{ d_H - d_H\nu + n_G - 1 - d_G\}\big) \varphi_{f,g}, & \text{when } \mu = 0. \end{cases}
\end{aligned}
$$

We hence get the conclusion. □

Example 4.24. Let G be a 4-circuit and H be a 3-circuit. The adjacency matrices of G and H are given as

$$
A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.
$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0,1,1,2\}$ and $\{0,\ \frac{3}{2},\ \frac{3}{2}\}$. We have $\mathfrak{D} = 3$. The $(1,1)$-probabilistic transition matrix $P_{(G\hat{\square}^K H)_{(1,1)}}$ is given as

$$
\begin{aligned}
P_{(G\hat{\square}^K H)_{(1,1)}} &= \frac{1}{2} \begin{pmatrix} O & I & O & I \\ I & O & I & O \\ O & I & O & I \\ I & O & I & O \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} A_H & O & I & O \\ O & A_H & O & I \\ I & O & A_H & O \\ O & I & O & A_H \end{pmatrix} \\
&= \frac{1}{6} \begin{pmatrix} O & A_H + I & O & A_H + I \\ A_H + I & O & A_H + I & O \\ O & A_H + I & O & A_H + I \\ A_H + I & O & A_H + I & O \end{pmatrix}
\end{aligned}
$$

$$
= \frac{1}{6}\begin{pmatrix}
0&0&0&1&1&1&0&0&0&1&1&1\\
0&0&0&1&1&1&0&0&0&1&1&1\\
0&0&0&1&1&1&0&0&0&1&1&1\\
1&1&1&0&0&0&1&1&1&0&0&0\\
1&1&1&0&0&0&1&1&1&0&0&0\\
1&1&1&0&0&0&1&1&1&0&0&0\\
0&0&0&1&1&1&0&0&0&1&1&1\\
0&0&0&1&1&1&0&0&0&1&1&1\\
0&0&0&1&1&1&0&0&0&1&1&1\\
1&1&1&0&0&0&1&1&1&0&0&0\\
1&1&1&0&0&0&1&1&1&0&0&0\\
1&1&1&0&0&0&1&1&1&0&0&0
\end{pmatrix}.
$$

The eigenvalues of $\Delta_{\mathcal{P}_{(G\widehat{\square}^K H)_{(1,1)}}}$ are $\big\{0,1,1,1,1,1,1,1,1,1,1,2\big\}$.

4.2. $(1,1)$**-Laplacians of Kähler graphs of compliment-filling strong product type.**

THEOREM 4.11. *Let $G=(V,E), H=(W,F)$ be finite regular ordinary graphs. We suppose G is connected. We denote by $0=\mu_1<\mu_2\le\cdots\le\mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0=\nu_1\le\cdots\le\nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. We put $\mathfrak{D}=n_G+d_Gd_H-d_G+d_H-1$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G\widehat{\boxtimes}^K H$ of compliment-filling strong product type are*

$$\frac{1}{\mathfrak{D}}d_G\nu_\alpha,\quad 1-\frac{1}{\mathfrak{D}}(1-\mu_i)(d_G\mu_i-d_H\nu_\alpha-d_G+d_H-1)\quad (2\le i\le n_G,\ 1\le\alpha\le n_H),$$

and the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ for $G\widehat{\boxtimes}^K H$ are

$$\frac{1}{\mathfrak{D}}d_G^2\nu_\alpha,\quad d_G-\frac{d_G}{\mathfrak{D}}(1-\mu_i)(d_G\mu_i-d_H\nu_\alpha-d_G+d_H-1)\quad (2\le i\le n_G,\ 1\le\alpha\le n_H).$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V\times W$ is an eigenfunction for both $\Delta_{\mathcal{Q}_{(1,1)}}$ and $\Delta_{\mathcal{A}_{(1,1)}}$.

PROOF. We use the same notations as in the proof of Theorems 4.4 and 4.10. The adjacency matrix $A^{(p)}_{G\widehat{\boxtimes}^K H}$ for the principal graph of $G\widehat{\boxtimes}^K H$ is the same as that of

$G\widehat{\boxtimes}H$. Hence we have

$$A^{(p)}_{G\widehat{\boxtimes}^K H} = (a^{(p)}_{(i,\alpha),(j,\beta)}) = \big(a^G_{ij}\delta_{\alpha\beta}\big).$$

The auxiliary degree of $G\,\widehat{\boxtimes}^K H$ is $d_{G^c} + d_H(d_G+1) = n_G + d_G d_H - d_G + d_H - 1 = \mathfrak{D}$. Therefore the transition matrix $P^{(a)}_{G\widehat{\boxtimes}^K H}$ of the auxiliary graph of $G\,\widehat{\boxtimes}^K H$ is

$$P^{(a)}_{G\widehat{\boxtimes}^K H} = \frac{1}{\mathfrak{D}}\begin{pmatrix} A_H & a^G_{12}A_H + a^{G^c}_{12}I & \cdots & a^G_{1n_G}A_H + a^{G^c}_{1n_G}I \\ a^G_{21}A_H + a^{G^c}_{21}I & A_H & \ddots & \vdots \\ \vdots & \ddots & \ddots & a^G_{n_G-1n_G}A_H + a^{G^c}_{n_G-1n_G}I \\ a^G_{n_G1}A_H + a^{G^c}_{n_G1}I & \cdots & a^G_{n_Gn_G-1}A_H + a^{G^c}_{n_Gn_G-1}I & A_H \end{pmatrix}.$$

Hence we have

$$P^{(a)}_{G\widehat{\boxtimes}^K H} = (p^{(a)}_{(i,a),(j,b)}) = \left(\frac{(a^G_{ij} + \delta_{ij})a^H_{\alpha\beta} + \delta_{\alpha\beta}a^{G^c}_{ij}}{d_H(d_G+1) + d_G{}^c}\right).$$

Therefore, we have

$$A^{(p)}_{G\widehat{\boxtimes}^K H}P^{(a)}_{G\widehat{\boxtimes}^K H} = \frac{1}{\mathfrak{D}}\left(\Big(a^G_{ij} + \sum_{k=1}^{n_G} a^G_{ik}a^G_{ik}\Big)a^H_{\alpha\beta} + \Big(\sum_{k=1}^{n_G} a^G_{ik}a^{G^c}_{kj}\Big)\delta_{\alpha\beta})\right).$$

For functions f and g satisfying $\Delta_{\mathcal{P}_G} f = \mu f$ and $\Delta_{\mathcal{P}_H} g = \nu g$, we take $\varphi_{f,g}$. Then we have

$$\begin{aligned} & A^{(p)}_{G\widehat{\boxtimes}^K H}P^{(a)}_{G\widehat{\boxtimes}^K H}\begin{pmatrix}\zeta_1\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_H}\end{pmatrix} \\ &= \Bigg(\frac{1}{\mathfrak{D}}\Big\{d_G d_H(1-\nu)\zeta_i\eta_\alpha + d_G d_H(1-\nu)\eta_\alpha\Big(\sum_{k=1}^{n_G} a^G_{ik}\zeta_k\Big) \\ &\qquad + (n_G - 1 - d_G)\eta_\alpha\Big(\sum_{k=1}^{n_G} a^G_{ik}\zeta_k\Big)\Big\}\Bigg) \\ &= \left(\frac{d_G}{\mathfrak{D}}\{d_H(1-\nu) + d_G d_H(1-\nu) + n_G - 1 - d_G\}\zeta_i\eta_\alpha\right) \end{aligned}$$

when $\mu = 0$, and have

$$\begin{aligned}
&A^{(p)}_{G\widehat{\boxtimes}^K H} P^{(a)}_{G\widehat{\boxtimes}^K H} \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_H} \end{pmatrix} \\
&= \left(\frac{1}{\mathfrak{D}}\Big\{d_G d_H(1-\mu)(1-\nu)\zeta_i\eta_\alpha + d_G d_H(1-\mu)(1-\nu)\eta_\alpha\Big(\sum_{k=1}^{n_G} a^G_{ik}\zeta_k\Big)\right. \\
&\qquad\qquad \left. + (d_G\mu - d_G - 1)\eta_\alpha\Big(\sum_{k=1}^{n_G} a^G_{ik}\zeta_k\Big)\Big\}\right) \\
&= \left(\frac{d_G(1-\mu)}{\mathfrak{D}}\{d_H(1-\nu) + d_G d_H(1-\mu)(1-\nu) + d_G\mu - d_G - 1\}\zeta_i\eta_\alpha\right)
\end{aligned}$$

when $\mu \neq 0$. Thus we obtain

$$\begin{aligned}
\Delta_{\mathcal{P}_{(1,1)}}\varphi_{f,g} &= \Big(\mathcal{I} - \frac{1}{d_G}\mathcal{A}^{(p)}_{G\widehat{\boxtimes}^K H}\mathcal{P}^{(a)}_{G\widehat{\boxtimes}^K H}\Big)\varphi_{f,g} \\
&= \begin{cases} \big(1 - \frac{1-\mu}{\mathfrak{D}}\{d_H(1-\nu) + d_G d_H(1-\mu)(1-\nu) + d_G\mu - d_G - 1\}\big)\varphi_{f,g}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{when } \mu \neq 0, \\ \big(1 - \frac{1}{\mathfrak{D}}\{d_H(1-\nu) + d_G d_H(1-\nu) + n_G - 1 - d_G\}\big)\varphi_{f,g}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{when } \mu = 0. \end{cases}
\end{aligned}$$

We hence get the conclusion. □

Example 4.25. Let G be a 4-circuit and H be a 3-circuit. The adjacency matrices of G and H are given as

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0, 1, 1, 2\}$ and $\{0, \frac{3}{2}, \frac{3}{2}\}$. We have $\mathfrak{D} = 7$. The $(1,1)$-probabilistic transition matrix $P_{(G\widehat{\boxtimes}^K H)_{(1,1)}}$ is given as

$$P_{(G\widehat{\boxtimes}^K H)_{(1,1)}} = \frac{1}{2}\begin{pmatrix} O & I & O & I \\ I & O & I & O \\ O & I & O & I \\ I & O & I & O \end{pmatrix} \cdot \frac{1}{7}\begin{pmatrix} A_H & A_H & I & A_H \\ A_H & A_H & A_H & I \\ I & A_H & A_H & A_H \\ A_H & I & A_H & A_H \end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix} 2A & A_H+I & 2A_H & A_H+I \\ A_H+I & 2A & A_H+I & 2A_H \\ 2A & A_H+I & 2A_H & A_H+I \\ A_H+I & 2A & A_H+I & 2A_H \end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix} 0&2&2&1&1&1&0&2&2&1&1&1 \\ 2&0&2&1&1&1&2&0&2&1&1&1 \\ 2&2&0&1&1&1&2&2&0&1&1&1 \\ 1&1&1&0&2&2&1&1&1&0&2&2 \\ 1&1&1&2&0&2&1&1&1&2&0&2 \\ 1&1&1&2&2&0&1&1&1&2&2&0 \\ 0&2&2&1&1&1&0&2&2&1&1&1 \\ 2&0&2&1&1&1&2&0&2&1&1&1 \\ 2&2&0&1&1&1&2&2&0&1&1&1 \\ 1&1&1&0&2&2&1&1&1&0&2&2 \\ 1&1&1&2&0&2&1&1&1&2&0&2 \\ 1&1&1&2&2&0&1&1&1&2&2&0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_{(G\widehat{\boxtimes}^K H)_{(1,1)}}}$ are $\left\{0,\ \frac{6}{7},\ 1,1,1,1,1,1,\ \frac{9}{7},\ \frac{9}{7},\ \frac{9}{7},\ \frac{9}{7}\right\}$.

4.3. $(1,1)$**-Laplacians of Kähler graphs of compliment-filling semi-tensor product type.**

THEOREM 4.12. *Let $G = (V,E), H = (W,F)$ be finite regular ordinary graphs. We suppose G is connected. We denote by $0 = \mu_1 < \mu_2 \le \cdots \le \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 \le \cdots \le \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. We put $\mathfrak{D} = n_G + d_G d_H - d_G - 1$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G\widehat{\otimes}^K H$ of compliment-filling semi-tensor product type are*

$$\frac{d_G d_H}{\mathfrak{D}}\nu_\alpha, \quad 1 - \frac{1}{\mathfrak{D}}(1-\mu_i)\{d_G d_H(1-\mu_i)(1-\nu_\alpha) + d_G\mu_i - d_G - 1\}$$
$$(2 \le i \le n_G,\ 1 \le \alpha \le n_H),$$

and the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ for $G\widehat{\otimes}^K H$ are

$$\frac{1}{\mathfrak{D}} d_G^2 d_H \nu_\alpha, \quad d_G - \frac{d_G}{\mathfrak{D}}(1-\mu_i)\{d_G d_H(1-\mu_i)(1-\nu_\alpha d_G \mu_i - d_G - 1\}$$
$$(2 \le i \le n_G,\ 1 \le \alpha \le n_H).$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V \times W$ is an eigenfunction for both $\Delta_{\mathcal{Q}_{(1,1)}}$ and $\Delta_{\mathcal{A}_{(1,1)}}$.

Proof. We use the same notations as in the proof of Theorems 4.4 and 4.10. The adjacency matrix $A^{(p)}_{G\widehat{\otimes}^K H}$ for the principal graph of $G\widehat{\otimes}^K H$ is the same as that of $G\widehat{\otimes}H$. Hence we have

$$A^{(p)}_{G\widehat{\otimes}^K H} = (a^{(p)}_{(i,\alpha),(j,\beta)}) = \big(a^G_{ij}\delta_{\alpha\beta}\big).$$

The auxiliary degree of $G\widehat{\otimes}^K H$ is $d_{G^c} + d_H d_G = n_G + d_G d_H - d_G - 1 = \mathfrak{D}$. Therefore the transition matrix $P^{(a)}_{G\widehat{\otimes}^K H}$ of the auxiliary graph of $G\widehat{\otimes}^K H$ is

$$P^{(a)}_{G\widehat{\otimes}^K H} = \frac{1}{\mathfrak{D}}\begin{pmatrix} O & a^G_{12}A_H + a^{G^c}_{12}I & \cdots & a^G_{1n_G}A_H + a^{G^c}_{1n_G}I \\ a^G_{21}A_H + a^{G^c}_{21}I & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & a^G_{n_G-1n_G}A_H + a^{G^c}_{n_G-1n_G}I \\ a^G_{n_G1}A_H + a^{G^c}_{n_G1}I & \cdots & a^G_{n_Gn_G-1}A_H + a^{G^c}_{n_Gn_G-1}I & O \end{pmatrix}.$$

That is, we have

$$P^{(a)}_{G\widehat{\otimes}^K H} = (p^{(a)}_{(i,\alpha),(j,\beta)}) = \frac{1}{\mathfrak{D}}\Big(a^G_{ij}a^H_{\alpha\beta} + a^{G^c}_{ij}\delta_{\alpha\beta}\Big).$$

Therefore, we find

$$A^{(p)}_{G\widehat{\otimes}^K H}P^{(a)}_{G\widehat{\otimes}^K H} = \frac{1}{\mathfrak{D}}\left(\Big(\sum_{k=1}^{n_G} a^G_{ik}a^G_{kj}\Big)a^H_{\alpha\beta} + \Big(\sum_{k=1}^{n_G} a^G_{ik}a^{G^c}_{kj}\Big)\delta_{\alpha\beta}\right).$$

For functions f and g satisfying $\Delta_{\mathcal{P}_G} f = \mu f$ and $\Delta_{\mathcal{P}_H} g = \nu g$, we take $\varphi_{f,g}$. Then we have

$$\begin{aligned} &A^{(p)}_{G\widehat{\otimes}^K H}P^{(a)}_{G\widehat{\otimes}^K H}\begin{pmatrix}\zeta_1\eta_1\\ \vdots \\ \zeta_{n_G}\eta_{n_H}\end{pmatrix} \\ &= \left(\frac{1}{\mathfrak{D}}\Big\{d_G d_H(1-\nu)\eta_\alpha\Big(\sum_{k=1}^{n_G} a^G_{ik}\zeta_k\Big) + (n_G - 1 - d_G)\eta_\alpha\Big(\sum_{k=1}^{n_G} a^G_{ik}\zeta_k\Big)\Big\}\right) \\ &= \left(\frac{d_G}{\mathfrak{D}}\{d_G d_H(1-\nu) + n_G - 1 - d_G\}\zeta_i\eta_\alpha\right) \end{aligned}$$

when $\mu = 0$, and have

$$A^{(p)}_{G\widehat{\otimes}^K H} P^{(a)}_{G\widehat{\otimes}^K H} \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_H} \end{pmatrix}$$

$$= \left(\frac{1}{\mathfrak{D}} \Big\{ d_G d_H (1-\mu)(1-\nu)\eta_\alpha \Big(\sum_{k=1}^{n_G} a^G_{ik}\zeta_k \Big) + (d_G\mu - d_G - 1)\eta_\alpha \Big(\sum_{k=1}^{n_G} a^G_{ik}\zeta_k \Big) \Big\} \right)$$

$$= \left(\frac{d_G(1-\mu)}{\mathfrak{D}} \{ d_G d_H (1-\mu)(1-\nu) + d_G\mu - d_G - 1 \} \zeta_i \eta_\alpha \right)$$

when $\mu \neq 0$. Thus we have

$$\Delta_{\mathcal{Q}_{(1,1)}} \varphi_{f,g} = \big(\mathcal{I} - \frac{1}{d_G} \mathcal{A}^{(p)}_{G\widehat{\otimes}^K H} \mathcal{P}^{(a)}_{G\widehat{\otimes}^K H} \big) \varphi_{f,g}$$

$$= \begin{cases} \Big(1 - \frac{1-\mu}{\mathfrak{D}} \{ d_G d_H (1-\mu)(1-\nu) + d_G\mu - d_G - 1 \}\Big) \varphi_{f,g}, & \text{when } \mu \neq 0, \\ \Big(1 - \frac{1}{\mathfrak{D}} \{ d_G d_H (1-\nu) + n_G - 1 - d_G \}\Big) \varphi_{f,g}, & \text{when } \mu = 0. \end{cases}$$

We hence get the conclusion. □

EXAMPLE 4.26. Let G be a 4-circuit and H be a 3-circuit. The adjacency matrices of G and H are given as

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0, 1, 1, 2\}$ and $\{0, \frac{3}{2}, \frac{3}{2}\}$. We have $\mathfrak{D} = 5$. The $(1,1)$-probabilistic transition matrix $P_{(G\widehat{\otimes}^K H)_{(1,1)}}$ is given as

$$P_{(G\widehat{\otimes}^K H)_{(1,1)}} = \frac{1}{2} \begin{pmatrix} O & I & O & I \\ I & O & I & O \\ O & I & O & I \\ I & O & I & O \end{pmatrix} \cdot \frac{1}{5} \begin{pmatrix} O & A_H & I & A_H \\ A_H & O & A_H & I \\ I & A_H & O & A_H \\ A_H & I & A_H & O \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 2A_H & I & 2A_H & I \\ I & 2A_H & I & 2A_H \\ 2A_H & I & 2A_H & I \\ I & 2A_H & I & 2A_H \end{pmatrix}$$

$$
= \frac{1}{10}\begin{pmatrix}
0&2&2&1&0&0&0&2&2&1&0&0\\
2&0&2&0&1&0&2&0&2&0&1&0\\
2&2&0&0&0&1&2&2&0&0&0&1\\
1&0&0&0&2&2&1&0&0&0&2&2\\
0&1&0&2&0&2&0&1&0&2&0&2\\
0&0&1&2&2&0&0&0&1&2&2&0\\
0&2&2&1&0&0&0&2&2&1&0&0\\
2&0&2&0&1&0&2&0&2&0&1&0\\
2&2&0&0&0&1&2&2&0&0&0&1\\
1&0&0&0&2&2&1&0&0&0&2&2\\
0&1&0&2&0&2&0&1&0&2&0&2\\
0&0&1&2&2&0&0&0&1&2&2&0
\end{pmatrix}.
$$

The eigenvalues of $\Delta_{\mathcal{P}_{(G\,\widehat{\otimes}^K H)_{(1,1)}}}$ are $\Big\{0,\ \frac{2}{5},\ 1,1,1,1,1,1,\ \frac{6}{5},\ \frac{6}{5},\ \frac{8}{5},\ \frac{8}{5}\Big\}$.

4.4. $(1,1)$-Laplacians of Kähler graphs of compliment-filling lexicographical product type.

THEOREM 4.13. *Let $G = (V, E), H = (W, F)$ be finite regular ordinary graphs. We suppose G is connected. We denote by $0 = \mu_1 < \mu_2 \le \cdots \le \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 \le \cdots \le \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. We put $\mathfrak{D} = n_G(d_H + 1) - d_G - 1$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G \rhd^K H$ of compliment-filling lexicographical product type are*

$$\frac{n_G d_H}{\mathfrak{D}}\nu_\alpha, \quad 1 - \frac{1}{\mathfrak{D}}\{n_G - 1 - d_G + n_G d_H(1-\nu)\} \quad (2 \le i \le n_G,\ 1 \le \alpha \le n_H),$$

and the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ for $G \rhd^K H$ are

$$\frac{n_G d_G d_H}{\mathfrak{D}}\nu_\alpha, \quad d_G - \frac{d_G}{\mathfrak{D}}\{n_G - 1 - d_G + n_G d_H(1-\nu)\} \quad (2 \le i \le n_G,\ 1 \le \alpha \le n_H).$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V \times W$ is an eigenfunction for both $\Delta_{\mathcal{Q}_{(1,1)}}$ and $\Delta_{\mathcal{A}_{(1,1)}}$.

PROOF. We use the same notations as in the proof of Theorems 4.4 and 4.10. The adjacency matrix $A^{(p)}_{G \rhd^K H}$ for the principal graph $G \rhd^K H$ is the same as that of $G \rhd H$.

Hence we have

$$A^{(p)}_{G\rhd^K H} = \big(a^{(p)}_{(i,\alpha),(j,\beta)}\big) = \big(a^G_{ij}\delta_{\alpha\beta}\big).$$

The auxiliary degree of $G \rhd^K H$ is $n_G d_H + d_{G^c} = n_G d_H + n_G - 1 - d_G = \mathfrak{D}$. Therefore the transition matrix $P^{(a)}_{G\rhd^K H}$ of the auxiliary graph of $G \rhd^K H$ is

$$P^{(a)}_{G\rhd^K H} = \frac{1}{\mathfrak{D}} \begin{pmatrix} A_H & A_H + a^{G^c}_{12} I & \cdots & A_H + a^{G^c}_{1n_G} I \\ A_H + a^{G^c}_{21} I & A_H & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_H + a^{G^c}_{n_G n_G-1} I \\ A_H + a^{G^c}_{n_G 1} I & \cdots & A_H + a^{G^c}_{n_G n_G-1} I & A_H \end{pmatrix},$$

That is, we have

$$P^{(a)}_{G\rhd^K H} = \big(p^{(a)}_{(i,\alpha),(j,\beta)}\big) = \frac{1}{\mathfrak{D}}\Big(a^{G^c}_{ij}\delta_{\alpha\beta} + a^H_{\alpha\beta}\Big).$$

Therefore, we have

$$\begin{aligned} A^{(p)}_{G\rhd^K H} P^{(a)}_{G\rhd^K H} &= \frac{1}{\mathfrak{D}}\left(\Big(\sum_{k=1}^{n_G} a^G_{ik} a^{G^c}_{kj}\Big)\delta^H_{\alpha\beta} + \Big(\sum_{k=1}^{n_G} a^G_{ik}\Big) a^H_{\alpha\beta}\right) \\ &= \frac{1}{\mathfrak{D}}\left(\Big(\sum_{k=1}^{n_G} a^G_{ik} a^{G^c}_{kj}\Big)\delta^H_{\alpha\beta} + d_G a^H_{\alpha\beta}\right). \end{aligned}$$

For functions f and g satisfying $\Delta_{\mathcal{P}_G} f = \mu f$ and $\Delta_{\mathcal{P}_H} g = \nu g$, we take $\varphi_{f,g}$. Then we have

$$\begin{aligned} &A^{(p)}_{G\rhd^K H} P^{(a)}_{G\rhd^K H} \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_H} \end{pmatrix} \\ &= \left(\frac{1}{\mathfrak{D}}\Big\{(n_G - d_G - 1)\Big(\sum_{k=1}^{n_G} a^G_{ik}\zeta_k\Big)\eta_\alpha + d_G d_H(1-\nu)\Big(\sum_{j=1}^{n_G}\zeta_j\Big)\eta_\alpha\Big\}\right) \\ &= \left(\frac{d_G}{\mathfrak{D}}\{n_G - 1 - d_G + n_G d_H(1-\nu)\}\zeta_i\eta_\alpha\right) \end{aligned}$$

when $\mu = 0$, and have

$$\begin{aligned} &A^{(p)}_{G\rhd^K H} P^{(a)}_{G\rhd^K H} \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_H} \end{pmatrix} \\ &= \left(\frac{1}{\mathfrak{D}}\Big\{(d_G\mu - d_G - 1)\Big(\sum_{k=1}^{n_G} a^G_{ik}\zeta_k\Big)\eta_\alpha + d_H(1-\nu)\Big(\sum_{k=1}^{n_G}\zeta_j\Big)\eta_\alpha\Big\}\right) \\ &= \left(\frac{d_G(1-\mu)(d_G\mu - d_G - 1)}{\mathfrak{D}}\zeta_i\eta_\alpha\right) \end{aligned}$$

when $\mu \neq 0$. Thus we have

$$\begin{aligned}\mathit{\Delta}_{\mathcal{Q}_{(1,1)}}\varphi_{f,g} &= \Big(\mathcal{I} - \frac{1}{d_G}\mathcal{A}^{(p)}_{G\triangleright^K H}\mathcal{P}^{(a)}_{G\triangleright^K H}\Big)\varphi_{f,g} \\ &= \begin{cases} \big(1 - \frac{1}{\mathfrak{D}}\{n_G d_H(1-\nu) + n_G - 1 - d_G\}\big)\varphi_{f,g}, & \text{when } \mu = 0, \\ \big(1 - \frac{1}{\mathfrak{D}}(1-\mu)(d_G\mu - d_G - 1)\big)\varphi_{f,g}, & \text{when } \mu \neq 0. \end{cases}\end{aligned}$$

We get the conclusion. □

Example 4.27. Let G be a 4-circuit and H be a 3-circuit. The adjacency matrices of G and H are given as

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $\mathit{\Delta}_{\mathcal{P}_G}$ and $\mathit{\Delta}_{\mathcal{P}_H}$ are $\{0,1,1,2\}$ and $\{0, \frac{3}{2}, \frac{3}{2}\}$. We have $\mathfrak{D} = 9$. The $(1,1)$-probabilistic transition matrix $P_{(G\triangleright^K H)_{(1,1)}}$ is given as

$$\begin{aligned} P_{(G\triangleright^K H)_{(1,1)}} &= \frac{1}{2}\begin{pmatrix} O & I & O & I \\ I & O & I & O \\ O & I & O & I \\ I & O & I & O \end{pmatrix} \cdot \frac{1}{9}\begin{pmatrix} A_H & A_H & A_H{+}I & A_H \\ A_H & A_H & A_H & A_H{+}I \\ A_H{+}I & A_H & A_H & A_H \\ A_H & A_H{+}I & A_H & A_H \end{pmatrix} \\ &= \frac{1}{18}\begin{pmatrix} 2A_H & 2A_H{+}I & 2A_H & 2A_H{+}I \\ 2A_H{+}I & 2A_H & 2A_H{+}I & 2A_H \\ 2A_H & 2A_H{+}I & 2A_H & 2A_H{+}I \\ 2A_H{+}I & 2A_H & 2A_H{+}I & 2A_H \end{pmatrix} \\ &= \frac{1}{18}\begin{pmatrix} 0&2&2&1&2&2&0&2&2&1&2&2 \\ 2&0&2&2&1&2&2&0&2&2&1&2 \\ 2&2&0&2&2&1&2&2&0&2&2&1 \\ 1&2&2&0&2&2&1&2&2&0&2&2 \\ 2&1&2&2&0&2&2&1&2&2&0&2 \\ 2&2&1&2&2&0&2&2&1&2&2&0 \\ 0&2&2&1&2&2&0&2&2&1&2&2 \\ 2&0&2&2&1&2&2&0&2&2&1&2 \\ 2&2&0&2&2&1&2&2&0&2&2&1 \\ 1&2&2&0&2&2&1&2&2&0&2&2 \\ 2&1&2&2&0&2&2&1&2&2&0&2 \\ 2&2&1&2&2&0&2&2&1&2&2&0 \end{pmatrix}. \end{aligned}$$

The eigenvalues of $\mathit{\Delta}_{\mathcal{P}_{(G\triangleright^K H)_{(1,1)}}}$ are $\Big\{0,1,1,1,1,1,1, \dfrac{10}{9}, \dfrac{10}{9}, \dfrac{10}{9}, \dfrac{4}{3}, \dfrac{4}{3}\Big\}$.

5. Eigenvalues of $(1,1)$-Laplacians of joined Kähler graphs

In this section we study eigenvalues of $(1,1)$-Laplacians of joined Kähler graphs. Though we defined joined Kähler graphs in §2.2 as examples of Kähler extensions, we here give their definitions more explicitly. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two ordinary graphs. We set $V = V_1 \cup V_2$ and $E^{(p)} = E_1 \cup E_2$ which are disjoint unions. We define $E^{(a)}$ so that arbitrary $v \in V_1$ and $w \in V_2$ are auxiliary adjacent to each other but any two vertices in V_1 are not auxiliary adjacent to each other, and nor are two vertices in V_2. We denote this Kähler graph $(V, E^{(p)} \cup E^{(a)})$ by $G\widehat{+}G_2$ and call it the joined Kähler graph of G_1 and G_2.

THEOREM 4.14. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are ordinary finite graphs. The eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ of the joined Kähler graph $G_1\widehat{+}G_2$ are $0, 1 \ldots, 1, 2$, where the multiplicity of 1 is $n_{G_1} + n_{G_2} - 2$.*

PROOF. We denote by M_{ij} a $n_{G_i} \times n_{G_j}$-matrix all of whose complements are 1. The adjacency matrix $A^{(p)}_{G_1\widehat{+}G_2}$ for the principal graph and the transition matrix $P^{(a)}_{G_1\widehat{+}G_2}$ for the auxiliary graph of the Kähler graph $G_1\widehat{+}G_2$ are

$$A^{(p)}_{G_1\widehat{+}G_2} = \begin{pmatrix} A_{G_1} & \vdots & O \\ \cdots & & \cdots \\ O & \vdots & A_{G_1} \end{pmatrix}, \quad P^{(a)}_{G_1\widehat{+}G_2} = \begin{pmatrix} O & \vdots & \dfrac{1}{n_{G_2}} M_{12} \\ \cdots & & \cdots \\ \dfrac{1}{n_{G_1}} M_{21} & \vdots & O \end{pmatrix}.$$

We denote as $V_1 = \{v_1, \ldots, v_{n_{G_1}}\}$ and $V_2 = \{w_1, \ldots, w_{n_{G_2}}\}$. We take functions $\phi, \psi : V_1 \cup V_2 \to \mathbb{R}$ defined by $\phi \equiv 1$, and $\psi(v) = 1$ for $v \in V_1$ and $\psi(w) = -1$ for $w \in V_2$. With canonical basis $\{\delta_{v_1}, \ldots, \delta_{v_{n_{G_1}}}, \delta_{w_1}, \ldots, \delta_{w_{n_{G_2}}}\}$ these correspond to

$$\phi \leftrightarrow \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \cdots \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad \psi \leftrightarrow \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \cdots \\ -1 \\ \vdots \\ -1 \end{pmatrix}.$$

For these functions we have

$$A^{(p)}_{G_1\widehat{+}G_2}P^{(a)}_{G_1\widehat{+}G_2}\begin{pmatrix}1\\ \cdots\\ 1\end{pmatrix}=\begin{pmatrix}A_{G_1} & \vdots & O\\ \cdots & & \cdots\\ O & \vdots & A_{G_2}\end{pmatrix}\begin{pmatrix}1\\ \cdots\\ 1\end{pmatrix}=\begin{pmatrix}d_{G_1}(v_1)\\ \vdots\\ d_{G_1}(v_{n_{G_1}})\\ \cdots\\ d_{G_2}(w_1)\\ \vdots\\ d_{G_2}(w_{n_{G_2}})\end{pmatrix},$$

$$A^{(p)}_{G_1\widehat{+}G_2}P^{(a)}_{G_1\widehat{+}G_2}\begin{pmatrix}1\\ \cdots\\ -1\end{pmatrix}=\begin{pmatrix}A_{G_1} & \vdots & O\\ \cdots & & \cdots\\ O & \vdots & A_{G_2}\end{pmatrix}\begin{pmatrix}-1\\ \cdots\\ 1\end{pmatrix}=\begin{pmatrix}-d_{G_1}(v_1)\\ \vdots\\ -d_{G_1}(v_{n_{G_1}})\\ \cdots\\ d_{G_2}(w_1)\\ \vdots\\ d_{G_2}(w_{n_{G_2}})\end{pmatrix}.$$

If we take $\delta_{v_1}-\delta_{v_i}$ $(i=2,\dots,n_{G_1})$ and $\delta_{w_1}-\delta_{w_j}$ $(j=2,\dots,n_{G_2})$, which correspond to

$$\delta_{v_1}-\delta_{v_i}\leftrightarrow x_i=\begin{pmatrix}1\\0\\ \vdots\\0\\-1\\0\\ \vdots\\0\\ \cdots\\0\\ \vdots\\0\end{pmatrix}\begin{matrix}\\ \\ \\ \\ \langle i\\ \\ \\ \\ \\ \\ \\ \\ \end{matrix},\qquad \delta_{w_1}-\delta_{w_j}\leftrightarrow y_j=\begin{pmatrix}0\\ \vdots\\0\\ \cdots\\1\\0\\ \vdots\\0\\-1\\0\\ \vdots\\0\end{pmatrix}\begin{matrix}\\ \\ \\ \\ \\ \\ \\ \\ \langle n_{G_1}+j\\ \\ \\ \\ \end{matrix}.$$

For these functions we have

$$A^{(p)}_{G_1\widehat{+}G_2}P^{(a)}_{G_1\widehat{+}G_2}x_i=\begin{pmatrix}A_{G_1} & \vdots & O\\ \cdots & & \cdots\\ O & \vdots & A_{G_2}\end{pmatrix}\begin{pmatrix}0\\ \cdots\\ 0\end{pmatrix}=\begin{pmatrix}0\\ \cdots\\ 0\end{pmatrix},$$

$$A^{(p)}_{G_1\widehat{+}G_2}P^{(a)}_{G_1\widehat{+}G_2}y_j=\begin{pmatrix}A_{G_1} & \vdots & O\\ \cdots & & \cdots\\ O & \vdots & A_{G_2}\end{pmatrix}\begin{pmatrix}0\\ \cdots\\ 0\end{pmatrix}=\begin{pmatrix}0\\ \cdots\\ 0\end{pmatrix}.$$

Therefore we obtain that

$$\begin{aligned}
&\Delta_{\mathcal{Q}_{(1,1)}}\phi = \phi - \phi = 0, \qquad \Delta_{\mathcal{Q}_{(1,1)}}\psi = \psi - (-\psi) = 2\psi,\\
&\Delta_{\mathcal{Q}_{(1,1)}}(\delta_{v_1} - \delta_{v_i}) = (\delta_{v_1} - \delta_{v_i}) - 0 = (\delta_{v_1} - \delta_{v_i}),\\
&\Delta_{\mathcal{Q}_{(1,1)}}(\delta_{w_1} - \delta_{w_j}) = (\delta_{w_1} - \delta_{w_j}) - 0 = (\delta_{w_1} - \delta_{w_j}).
\end{aligned}$$

We hence get the conclusion. □

THEOREM 4.15. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be ordinary finite regular graphs. The eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ of the joined Kähler graph $G_1 \widehat{+} G_2$ are*

$$0, d_{G_1}, \dots, d_{G_1}, d_{G_2}, \dots, d_{G_2}, d_{G_1} + d_{G_2},$$

where the multiplicity of d_{G_i} is $n_{G_i} - 1$ for $i = 1, 2$.

PROOF. We use the same notations as in the proof of Theorem 4.14. We take a function $\tilde{\psi} : V \to \mathbb{R}$ given by $\tilde{\psi}(v) = d_{G_1}$ for $v \in V_1$ and $\tilde{\psi}(w) = -d_{G_2}$ for $w \in V_2$, which corresponds to

$$\tilde{\psi} \leftrightarrow \begin{pmatrix} d_{G_1} \\ \vdots \\ d_{G_1} \\ \cdots \\ -d_{G_2} \\ \vdots \\ -d_{G_2} \end{pmatrix}.$$

We have

$$A^{(p)}_{G_1 \widehat{+} G_2} P^{(a)}_{G_1 \widehat{+} G_2} \begin{pmatrix} d_{G_1} \\ \cdots \\ -d_{G_2} \end{pmatrix} = \begin{pmatrix} A_{G_1} & \vdots & O \\ \cdots & & \cdots \\ O & \vdots & A_{G_2} \end{pmatrix} \begin{pmatrix} -d_{G_2} \\ \cdots \\ d_{G_1} \end{pmatrix} = \begin{pmatrix} -d_{G_1} d_{G_2} \\ \cdots \\ d_{G_1} d_{G_2} \end{pmatrix}.$$

Therefore we obtain that

$$(D - A^{(p)}_{G_1 \widehat{+} G_2} P^{(a)}_{G_1 \widehat{+} G_2}) \begin{pmatrix} 1 \\ \cdots \\ 1 \end{pmatrix} = \begin{pmatrix} d_{G_1}(v_1) \\ \vdots \\ d_{G_1}(v_{n_{G_1}}) \\ \cdots \\ d_{G_2}(w_1) \\ \vdots \\ d_{G_2}(w_{n_{G_2}}) \end{pmatrix} - \begin{pmatrix} d_{G_1}(v_1) \\ \vdots \\ d_{G_1}(v_{n_{G_1}}) \\ \cdots \\ d_{G_2}(w_1) \\ \vdots \\ d_{G_2}(w_{n_{G_2}}) \end{pmatrix} = 0,$$

$$(D - A^{(p)}_{G_1 \widehat{+} G_2} P^{(a)}_{G_1 \widehat{+} G_2}) \begin{pmatrix} d_{G_1} \\ \cdots \\ -d_{G_2} \end{pmatrix} = \begin{pmatrix} d^2_{G_1} \\ \cdots \\ -d^2_{G_2} \end{pmatrix} - \begin{pmatrix} -d_{G_1} d_{G_2} \\ \cdots \\ d_{G_1} d_{G_2} \end{pmatrix} = (d_{G_1} + d_{G_2}) \begin{pmatrix} d_{G_1} \\ \cdots \\ -d_{G_2} \end{pmatrix}.$$

These lead us to

$$\Delta_{\mathcal{A}_{(1,1)}} \phi = 0, \qquad \Delta_{\mathcal{A}_{(1,1)}} \tilde{\psi} = (d_{G_1} + d_{G_2}) \tilde{\psi},$$

$$\Delta_{\mathcal{A}_{(1,1)}} (\delta_{v_1} - \delta_{v_i}) = d_{G_1}(\delta_{v_1} - \delta_{v_i}) - 0 = d_{G_1}(\delta_{v_1} - \delta_{v_i}),$$

$$\Delta_{\mathcal{A}_{(1,1)}} (\delta_{w_1} - \delta_{w_j}) = d_{G_2}(\delta_{w_1} - \delta_{w_j}) - 0 = d_{G_2}(\delta_{w_1} - \delta_{w_j}),$$

which show the conclusion. □

EXAMPLE 4.28. Let G and H be non-regular ordinary graphs given in Fig. 14 and 15, respectively. The transition and adjacency matrices of G and H are given as

$$P_G = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix} \quad \text{and} \quad P_H = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad P_H = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0, 1, \frac{4}{3}, \frac{5}{3}\}$ and $\{0, \frac{7-\sqrt{7}}{6}, \frac{7}{6}, \frac{3}{2}, \frac{7+\sqrt{7}}{6}\}$. The eigenvalues of $\Delta_{\mathcal{A}_G}$ and $\Delta_{\mathcal{A}_H}$ are $\{0, 2, 4, 4\}$ and $\{0,\ 3 - \sqrt{2},\ 3,\ 3 + \sqrt{2},\ 5\}$. The $(1,1)$-probabilistic transition matrix $Q_{(G \widehat{+} H)_{(1,1)}}$ is given as

$$Q_{(G\hat{+}H)_{(1,1)}} = \begin{pmatrix} P_G & O \\ O & P_H \end{pmatrix} \begin{pmatrix} O & \frac{1}{5}M_{12} \\ \frac{1}{4}M_{21} & O \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{Q_{(G\hat{+}H)_{(1,1)}}}$ are $\{0,1,1,1,1,1,1,1,2\}$. The $(1,1)$-adjacency matrix $A_{(G\hat{+}H)_{(1,1)}}$ is given as

$$A_{(G\hat{+}H)_{(1,1)}} = \begin{pmatrix} A_G & O \\ O & A_H \end{pmatrix} \begin{pmatrix} O & \frac{1}{5}M_{12} \\ \frac{1}{4}M_{21} & O \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 0 & 0 & \frac{3}{5} & \frac{3}{5} & \frac{3}{5} & \frac{3}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 0 & 0 & \frac{3}{5} & \frac{3}{5} & \frac{3}{5} & \frac{3}{5} & \frac{3}{5} \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues of $\Delta_{Q_{(G\hat{+}H)_{(1,1)}}}$ are

$$\left\{ \begin{array}{l} 0, 2, 2, 3, 3, \\ \text{solutions of } 5t^4 - 70t^3 + 350t^2 - 746t + 576 = 0 \end{array} \right\}.$$

Example 4.29. Let G be a 3-circuit and H be a 4-circuit. Their adjacency matrices are given as

$$A_G = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_H = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0,1,1,2\}$ and $\{0,\ \frac{3}{2},\ \frac{3}{2}\}$. The $(1,1)$-adjacency matrix $A_{(G\hat{+}H)_{(1,1)}}$ is given as

$$A_{(G\hat{+}H)_{(1,1)}} = \begin{pmatrix} A_G & O \\ O & A_H \end{pmatrix} \begin{pmatrix} O & \frac{1}{4}M_{12} \\ \frac{1}{3}M_{21} & O \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{Q}_{(G\hat{+}H)_{(1,1)}}}$ are $\{0,2,2,2,2,2,4\}$.

6. Eigenvalues of $(1,1)$-Laplacians of Kähler graphs of product type obtained by commutative operations

In this section we study eigenvalues of $(1,1)$-Laplacians of Kähler graphs of product type obtained by commutative operations which are $G \boxplus H$, $G \boxdot H$, $G\Diamond H$, $G \circledast H$, $G\spadesuit H$ and $G\clubsuit H$. In this section also, for functions $f : V \to \mathbb{C}$ and $g : W \to \mathbb{C}$ we denote by $\varphi_{f,g} : V \times W \to \mathbb{C}$ the function defined by $\varphi_{f,g}(v,w) = f(v)g(w)$.

6.1. $(1,1)$-Laplacians of Kähler graphs of Cartesian-tensor product type.

THEOREM 4.16. *Let $G = (V,E), H = (W,F)$ be finite regular ordinary graphs. We denote by $0 = \mu_1 \le \cdots \le \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 \le \cdots \le \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G \boxplus H$ of Cartesian-tensor product type are*

$$1 - \frac{(1-\mu_i)(1-\nu_\alpha)\{d_G(1-\mu_i) + d_H(1-\nu_\alpha)\}}{d_G + d_H} \quad (1 \le i \le n_G,\ 1 \le \alpha \le n_H),$$

and the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ for $G \boxplus H$ are

$$d_G + d_H - (1-\mu_i)(1-\nu_\alpha)\{d_G(1-\mu_i) + d_H(1-\nu_\alpha)\} \quad (1 \le i \le n_G,\ 1 \le \alpha \le n_H).$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V \times W$ is an eigenfunction for both $\Delta_{\mathcal{Q}_{(1,1)}}$ and $\Delta_{\mathcal{A}_{(1,1)}}$.

PROOF. We denote by $A_G = (a^G_{ij})$ and $A_H = (a^H_{\alpha\beta})$ the adjacency matrices of the graphs G and H, respectively. By definition of $G \boxplus H$, the adjacency matrices of the principal and the auxiliary graphs of $G \boxplus H$ are given as

$$A^{(p)}_{G\boxplus H} = \begin{pmatrix} A_H & a^G_{12}I & \cdots & a^G_{1n_G}I \\ a^G_{21}I & A_H & \ddots & \vdots \\ \vdots & \ddots & \ddots & a^G_{n_G-1n_G}I \\ a^G_{n_G1}I & \cdots & a^G_{n_Gn_G-1} & A_H \end{pmatrix},$$

$$A^{(a)}_{G\boxplus H} = \begin{pmatrix} O & a^G_{12}A_H & \cdots & a^G_{1n_G}A_H \\ a^G_{21}A_H & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & a^G_{n_G-1n_G}A_H \\ a^G_{n_G1}A_H & \cdots & a^G_{n_Gn_G-1}A_H & O \end{pmatrix},$$

where I denotes the unit matrix (identify) and the components are expressed according to lexicographical order. That is, the adjacency matrices $A^{(p)}_{G\boxplus H} = (a^{(p)}_{(i,\alpha),(j,\beta)})$ and $A^{(a)}_{G\boxplus H} = (p^{(a)}_{(i,\alpha),(j,\beta)})$ of the principal and the auxiliary graphs of $G \boxplus H$ are given as

$$a^{(p)}_{(i,\alpha),(j,\beta)} = a^G_{ij}\delta_{\alpha\beta} + \delta_{ij}a^H_{\alpha\beta}, \qquad a^{(a)}_{(i,\alpha),(j,\beta)} = a^G_{ij}a^H_{\alpha\beta}.$$

Hence we have

$$A^{(p)}_{G\boxplus H}P^{(a)}_{G\boxplus H} = \frac{1}{d_Gd_H}A^{(p)}_{G\boxplus H}A^{(a)}_{G\boxplus H} = \frac{1}{d_Gd_H}\bigg(\Big(\sum_{k=1}^{n_G} a^G_{ik}a^G_{kj}\Big)a^H_{\alpha\beta} + a^G_{ij}\Big(\sum_{\gamma=1}^{n_H} a^H_{\alpha\gamma}a^H_{\gamma\beta}\Big)\bigg).$$

For functions $f : V \to \mathbb{R}$, $g : W \to \mathbb{R}$ we express them by canonical basis of $C(V,\mathbb{R})$ and $C(W,\mathbb{R})$ as

$$f \leftrightarrow \zeta = \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{n_G} \end{pmatrix}, \qquad g \leftrightarrow \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{n_H} \end{pmatrix}$$

Then $\varphi_{f,g}$ is expressed by the canonical basis $\{\varphi_{\delta_v,\delta_w} \mid v \in V, w \in W\}$ of $C(V \times W,\mathbb{R})$ as

$$\varphi_{f,g} \leftrightarrow \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_1\eta_{n_H} \\ \vdots \\ \zeta_{n_G}\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_H} \end{pmatrix}.$$

If functions f and g satisfy $\Delta_{\mathcal{P}_G}f = \mu f$ and $\Delta_{\mathcal{P}_H}g = \nu g$, then we have $\mathcal{A}_Gf = d_G(1-\mu)f$ and $\mathcal{A}_Hg = d_H(1-\nu)g$ because G and H are regular. Therefore we get

$$\begin{aligned} A^{(p)}_{G\boxplus H}P^{(a)}_{G\boxplus H}\begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_H} \end{pmatrix} &= (1-\mu)(1-\nu)\bigg(\Big(\sum_{k=1}^{n_G} a^G_{ij}\zeta_k\Big)\eta_\alpha + \zeta_i\Big(\sum_{\gamma=1}^{n_H} a^H_{\alpha\gamma}\eta_\gamma\Big) \\ &= (1-\mu)(1-\nu)\{d_G(1-\mu) + d_H(1-\nu)\}\big(\zeta_i\eta_\alpha\big), \end{aligned}$$

and obtain the conclusion. □

Example 4.30. Let G be a 4-circuit and H be a 5-circuit. The adjacency matrices of G and H are given as

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_H = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $\mathit{\Delta}_{\mathcal{P}_G}$ and $\mathit{\Delta}_{\mathcal{P}_H}$ are

$$\bigl\{0,1,1,2\bigr\} \quad \text{and} \quad \Bigl\{0,\ \frac{1}{4}\bigl(5-\sqrt{5}\bigr),\ \frac{1}{4}\bigl(5-\sqrt{5}\bigr),\ \frac{1}{4}\bigl(5+\sqrt{5}\bigr),\ \frac{1}{4}\bigl(5+\sqrt{5}\bigr)\Bigr\}.$$

As $d^{(p)}_{G\boxplus H} = 2+2 = 4$, $d^{(a)}_{G\boxplus H} = 2/2 = 4$, the $(1,1)$-probabilistic transition matrix $QP_{(G\boxplus H)_{(1,1)}}$ is given as

$$\begin{aligned}
Q_{(G\boxplus H)_{(1,1)}} &= \frac{1}{4}\begin{pmatrix} A_H & I & O & I \\ I & A_H & I & O \\ O & I & A_H & I \\ I & O & I & A_H \end{pmatrix} \cdot \frac{1}{4}\begin{pmatrix} O & A_H & O & A_H \\ A_H & O & A_H & O \\ O & A_H & O & A_H \\ A_H & O & A_H & O \end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix} 2A_H & A_H^2 & 2A_H & A_H^2 \\ A_H^2 & 2A_H & A_H^2 & 2A_H \\ 2A_H & A_H^2 & 2A_H & A_H^2 \\ A_H^2 & 2A_H & A_H^2 & 2A_H \end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix}
0 & 2 & 0 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 1 & 1 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 1 \\
0 & 2 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 \\
2 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 2 \\
2 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 2 \\
0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 2 & 0 \\
1 & 1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 \\
0 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 0 \\
0 & 2 & 0 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 1 & 1 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 1 \\
0 & 2 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 \\
2 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 2 \\
2 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 2 \\
0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 2 & 0 \\
1 & 1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 \\
0 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 0
\end{pmatrix}.
\end{aligned}$$

The eigenvalues of $\Delta_{\mathcal{Q}_{(G \boxplus H)_{(1,1)}}}$ are

$$\left\{\begin{array}{l} 0,\ \frac{1}{16}(15-\sqrt{5}),\ \frac{1}{16}(15-\sqrt{5}),\ \frac{3}{16}(7-\sqrt{5}),\ \frac{3}{16}(7-\sqrt{5}),\ 1,1,1,1,1,1,1,1,1,1,1, \\ \qquad \frac{1}{16}(15+\sqrt{5}),\ \frac{1}{16}(15+\sqrt{5}),\ \frac{3}{16}(7+\sqrt{5}),\ \frac{3}{16}(7+\sqrt{5}), \end{array}\right\}.$$

6.2. $(1,1)$**-Laplacians of Kähler graphs of Cartesian-complement product type.**

THEOREM 4.17. *Let $G=(V,E), H=(W,F)$ be finite regular ordinary graphs. Suppose G and H are connected. We denote by $0=\mu_1<\mu_2\le\cdots\le\mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0=\nu_1<\nu_2\le\cdots\le\nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G\boxdot H$ of Cartesian-complement product type are*

$$0, \qquad 1-\frac{\{d_G+d_H(1-\nu_\alpha)\}\{d_H(1-\nu_\alpha)(n_G-1-2d_G)-d_G\}}{(d_G+d_H)\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\}},$$

$$1-\frac{\{d_G(1-\mu_i)+d_H\}\{d_G(1-\mu_i)(n_H-1-2d_H)-d_H\}}{(d_G+d_H)\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\}},$$

$$1-\frac{\{d_G(1-\mu_i)+d_H(1-\nu_\alpha)\}\{-2d_Gd_H(1-\mu_i)(1-\nu_\alpha)-d_G(1-\mu_i)-d_H(1-\nu_\alpha)\}}{(d_G+d_H)\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\}},$$

$$(2\le i\le n_G,\ 2\le\alpha\le n_H),$$

and the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ for $G\boxdot H$ are

$$0, \qquad d_G+d_H-\frac{\{d_G+d_H(1-\nu_\alpha)\}\{d_H(1-\nu_\alpha)(n_G-1-2d_G)-d_G\}}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)},$$

$$d_G+d_H-\frac{\{d_G(1-\mu_i)+d_H\}\{d_G(1-\mu_i)(n_H-1-2d_H)-d_H\}}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)},$$

$$d_G+d_H-\frac{\{d_G(1-\mu_i)+d_H(1-\nu_\alpha)\}\{-2d_Gd_H(1-\mu_i)(1-\nu_\alpha)-d_G(1-\mu_i)-d_H(1-\nu_\alpha)\}}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)},$$

$$(2\le i\le n_G,\ 2\le\alpha\le n_H),$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V\times W$ is an eigenfunction for both $\Delta_{\mathcal{Q}_{(1,1)}}$ and $\Delta_{\mathcal{A}_{(1,1)}}$ corresponding to these eigenvalues.

Proof. We use the same notations as in the proof of Theorem 4.16. We denote by $A_{G^c} = (a^{G^c}_{ij})$ and $A_{H^c} = (a^{H^c}_{\alpha\beta})$ the adjacency matrices of the complement graphs G^c and H^c, respectively. The adjacency matrix $A^{(p)}_{G\boxdot H}$ of the principal graph of $G \boxdot H$ is the same as that of $G \boxplus H$. Hence we have

$$A^{(p)}_{G\boxdot H} = (a^{(p)}_{(i,\alpha),(j,\beta)}) = \big(a^G_{ij}\delta_{\alpha\beta} + \delta_{ij}a^H_{\alpha\beta}\big).$$

The adjacency matrix $A^{(a)}_{G\boxdot H}$ of the auxiliary graph of $G \boxdot H$ is given as

$$A^{(a)}_{G\boxdot H} = (a^{(a)}_{(i,\alpha),(j,\beta)}) = \big(a^G_{ij}a^{H^c}_{\alpha\beta} + a^H_{\alpha\beta}a^{G^c}_{ij}\big).$$

That is,

$$A^{(a)}_{G\boxdot H} = \begin{pmatrix} O & a^G_{12}A_{H^c}+a^{G^c}_{12}A_H & \cdots & a^G_{1n_G}A_{H^c}+a^{G^c}_{1n_G}A_H \\ a^G_{21}A_{H^c}+a^{G^c}_{21}A_H & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & a^G_{n_G-1n_G}A_{H^c}+a^{G^c}_{n_G-1n_G}A_H \\ a^G_{n_G1}A_{H^c}+a^{G^c}_{n_G1}A_H & \cdots & a^G_{n_Gn_G-1}A_{H^c}+a^{G^c}_{n_Gn_G-1}A_H & O \end{pmatrix}.$$

(We note that either $a^G_{ij} = 0$ or $a^{G^c}_{ij} = 0$ holds.) We hence have

$$\begin{aligned} A^{(p)}_{G\boxdot H}P^{(a)}_{G\boxdot H} = {} & \frac{1}{d_G(n_H - d_H - 1) + d_H(n_G - d_G - 1)} \\ & \Bigg(\Big(\sum_{k=1}^{n_G} a^G_{ik}a^G_{kj}\Big)a^{H^c}_{\alpha\beta} + a^G_{ij}\Big(\sum_{\gamma=1}^{n_H} a^H_{\alpha\gamma}a^{H^c}_{\gamma\beta}\Big) \\ & + \Big(\sum_{k=1}^{n_G} a^G_{ik}a^{G^c}_{kj}\Big)a^H_{\alpha\beta} + a^{G^c}_{ij}\Big(\sum_{\gamma=1}^{n_H} a^H_{\alpha\gamma}a^H_{\gamma\beta}\Big)\Bigg). \end{aligned}$$

We take functions f and g satisfying $\Delta_{\mathcal{P}_G} f = \mu f$ and $\Delta_{\mathcal{P}_H} g = \nu g$. As we have $A_{G^c} = M - I - A_G$ and $A_{H^c} = M - I - A_H$ and as G, H are connected, we see

$$A_{G^c} f = \begin{cases} (n_G - 1 - d_G)f, & \text{when } \mu = 0, \\ (d_G\mu - d_G - 1)f, & \text{when } \mu \neq 0, \end{cases}$$

$$A_{H^c} g = \begin{cases} (n_H - 1 - d_H)g, & \text{when } \nu = 0, \\ (d_H\nu - d_H - 1)g, & \text{when } \nu \neq 0. \end{cases}$$

Therefore we obtain

$$\mathcal{A}^{(p)}_{G\square H}\mathcal{P}^{(a)}_{G\square H}\varphi_{f.g}$$

$$= \begin{cases} (d_G+d_H)\varphi_{f.g}, & \text{when } \mu=\nu=0, \\ \dfrac{\{d_G+d_H(1-\nu)\}\{d_G(d_H\nu-d_H-1)+d_H(1-\nu)(n_G-1-d_G)\}}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)}\varphi_{f.g}, & \text{when } \mu=0,\ \nu\neq 0, \\ \dfrac{\{d_G(1-\mu)+d_H\}\{d_G(1-\mu)(n_H-1-d_H)+d_H(d_G\mu-d_G-1)\}}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)}\varphi_{f.g}, & \text{when } \mu\neq 0,\ \nu=0, \\ \dfrac{\{d_G(1-\mu)+d_H(1-\nu)\}\{d_G(1-\mu)(d_H\nu-d_H-1)+d_H(1-\nu)(d_G\mu-d_G-1)\}}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)}\varphi_{f.g}, & \text{when } \mu\neq 0,\ \nu\neq 0. \end{cases}$$

Since

$$d_G(d_H\nu-d_H-1)+d_H(1-\nu)(n_G-1-d_G)=d_H(1-\nu)(n_G-1-2d_G)-d_G$$

$$d_G(1-\mu)(n_H-1-d_H)+d_H(d_G\mu-d_G-1)=d_G(1-\mu)(n_H-1-2d_H)-d_H,$$

$$d_G(1-\mu)(d_H\nu-d_H-1)+d_H(1-\nu)(d_G\mu-d_G-1)$$
$$=-2d_Gd_H(1-\mu)(1-\nu)-d_G(1-\mu)-d_H(1-\nu),$$

we get the conclusion. □

EXAMPLE 4.31. Let G be a 4-circuit and H be a 5-circuit. Their adjacency matrices are given as

$$A_G=\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix} \quad\text{and}\quad A_H=\begin{pmatrix}0&1&0&0&1\\1&0&1&0&0\\0&1&0&1&0\\0&0&1&0&1\\1&0&0&1&0\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are

$$\{0,1,1,2\} \quad\text{and}\quad \Big\{0,\ \frac{1}{4}(5-\sqrt{5}),\ \frac{1}{4}(5-\sqrt{5}),\ \frac{1}{4}(5+\sqrt{5}),\ \frac{1}{4}(5+\sqrt{5})\Big\}.$$

Since $d^{(p)}_{G\square H} = 4$ and $d^{(a)}_{G\square H} = 6$, the $(1,1)$-probabilistic transition matrix $Q_{(G\square H)_{(1,1)}}$ is given as

$$Q_{(G\square H)_{(1,1)}} = \frac{1}{4}\begin{pmatrix} A_H & I & O & I \\ I & A_H & I & O \\ O & I & A_H & I \\ I & O & I & A_H \end{pmatrix} \cdot \frac{1}{6}\begin{pmatrix} O & A_{H^c} & A_H & A_{H^c} \\ A_{H^c} & O & A_{H^c} & A_H \\ A_H & A_{H^c} & O & A_{H^c} \\ A_{H^c} & A_H & A_{H^c} & O \end{pmatrix}$$

$$= \frac{1}{24}\begin{pmatrix} 2A_{H^c} & A_HA_{H^c}+A_H & A_H^2+2A_{H^c} & A_HA_{H^c}+A_H \\ A_HA_{H^c}+A_H & 2A_H & A_HA_{H^c}+A_H & A_H^2+2A_{H^c} \\ A_H^2+2A_{H^c} & A_HA_{H^c}+A_H & 2A_H & A_HA_{H^c}+A_H \\ A_HA_{H^c}+A_H & A_H^2+2A_{H^c} & A_HA_{H^c}+A_H & 2A_H \end{pmatrix}$$

$$= \frac{1}{24}\begin{pmatrix}
0 & 0 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 2 & 2 & 0 & 3 & 3 & 0 & 0 & 2 & 1 & 1 & 2 \\
0 & 0 & 0 & 2 & 2 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 3 & 3 & 2 & 0 & 2 & 1 & 1 \\
2 & 0 & 0 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 3 & 0 & 2 & 0 & 3 & 1 & 2 & 0 & 2 & 1 \\
2 & 2 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 2 & 3 & 3 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 \\
0 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 3 & 3 & 0 & 2 & 2 & 1 & 1 & 2 & 0 \\
0 & 2 & 1 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 2 & 2 & 0 & 3 & 3 & 0 \\
2 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 3 & 3 \\
1 & 2 & 0 & 2 & 1 & 2 & 0 & 0 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 3 & 0 & 2 & 0 & 3 \\
1 & 1 & 2 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 2 & 3 & 3 & 0 & 2 & 0 \\
2 & 1 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 3 & 3 & 0 & 2 \\
2 & 0 & 3 & 3 & 0 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 2 \\
0 & 2 & 0 & 3 & 3 & 2 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 2 & 1 & 1 \\
3 & 0 & 2 & 0 & 3 & 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 0 & 2 & 1 \\
3 & 3 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 2 \\
0 & 3 & 3 & 0 & 2 & 2 & 1 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 2 & 0 \\
2 & 0 & 2 & 2 & 0 & 0 & 3 & 1 & 1 & 3 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 2 & 2 & 0 \\
0 & 2 & 0 & 2 & 2 & 3 & 0 & 3 & 1 & 1 & 2 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 2 & 2 \\
2 & 0 & 2 & 0 & 2 & 1 & 3 & 0 & 3 & 1 & 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 2 \\
2 & 2 & 0 & 2 & 0 & 1 & 1 & 3 & 0 & 3 & 1 & 1 & 2 & 0 & 2 & 2 & 2 & 0 & 0 & 0 \\
0 & 2 & 2 & 0 & 2 & 3 & 1 & 1 & 3 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 0
\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{Q}_{(G\square H)_{(1,1)}}}$ are

$$\left\{ \begin{array}{l} 0,\ \frac{1}{48}(43-7\sqrt{5}),\ \frac{1}{48}(43-7\sqrt{5}),\ 1,\ \frac{1}{48}(55-3\sqrt{5}),\ \frac{1}{48}(55-3\sqrt{5}), \\ \frac{1}{48}(51-\sqrt{5}),\ \frac{1}{48}(51-\sqrt{5}),\ \frac{1}{48}(51-\sqrt{5}),\ \frac{1}{48}(51-\sqrt{5}), \\ \frac{1}{48}(51+\sqrt{5}),\ \frac{1}{48}(51+\sqrt{5}),\ \frac{1}{48}(51+\sqrt{5}),\ \frac{1}{48}(51+\sqrt{5}),\ \frac{1}{48}(51+\sqrt{5}), \\ \frac{7}{6},\ \frac{7}{6},\ \frac{1}{48}(43+7\sqrt{5}),\ \frac{1}{48}(43+7\sqrt{5}),\ \frac{1}{48}(55+3\sqrt{5}),\ \frac{1}{48}(55+3\sqrt{5}) \end{array} \right\}.$$

6.3. $(1,1)$**-Laplacians of Kähler graphs of Cartesian-lexicographic product type.**

THEOREM 4.18. *Let $G=(V,E), H=(W,F)$ be finite regular ordinary graphs. Suppose G and H are connected. We denote by $0=\mu_1<\mu_2\le\cdots\le\mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0=\nu_1<\nu_2\le\cdots\le\nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G\Diamond H$ of Cartesian-lexicographic product type are*

$$0, \qquad 1-\frac{\{d_G+d_H(1-\nu_\alpha)\}\{d_H(1-\nu_\alpha)(n_G-1)-d_G\}}{(d_G+d_H)\{d_G(n_H-1)+d_H(n_G-1)\}},$$

$$1-\frac{\{d_G(1-\mu_i)+d_H\}\{d_G(1-\mu_i)(n_H-1)-d_H\}}{(d_G+d_H)\{d_G(n_H-1)+d_H(n_G-1)\}},$$

$$1+\frac{\{d_G(1-\mu_i)+d_H(1-\nu_\alpha)\}^2}{(d_G+d_H)\{d_G(n_H-1)+d_H(n_G-1)\}}, \qquad (2\le i\le n_G,\ 2\le\alpha\le n_H),$$

and the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ for $G\Diamond H$ are

$$0, \qquad d_G+d_H-\frac{\{d_G+d_H(1-\nu_\alpha)\}\{d_H(1-\nu_\alpha)(n_G-1)-d_G\}}{d_G(n_H-1)+d_H(n_G-1)},$$

$$d_G+d_H-\frac{\{d_G(1-\mu_i)+d_H\}\{d_G(1-\mu_i)(n_H-1)-d_H\}}{d_G(n_H-1)+d_H(n_G-1)},$$

$$d_G+d_H+\frac{\{d_G(1-\mu_i)+d_H(1-\nu_\alpha)\}^2}{d_G(n_H-1)+d_H(n_G-1)}, \qquad (2\le i\le n_G,\ 2\le\alpha\le n_H),$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V\times W$ is an eigenfunction for both $\Delta_{\mathcal{Q}_{(1,1)}}$ and $\Delta_{\mathcal{A}_{(1,1)}}$ corresponding to these eigenvalues.

PROOF. We use the same notations as in the proof of Theorem 4.16. The adjacency matrix $A^{(p)}_{G\Diamond H}$ of the principal graph of $G\Diamond H$ is the same as that of $G\boxplus H$. Hence we have

$$A^{(p)}_{G\Diamond H}=(a^{(p)}_{(i,\alpha),(j,\beta)})=\big(a^G_{ij}\delta_{\alpha\beta}+\delta_{ij}a^H_{\alpha\beta}\big).$$

The adjacency matrix $A^{(a)}_{G\Diamond H}$ of the auxiliary graph of $G\Diamond H$ is given as

$$A^{(a)}_{G\Diamond H}=(a^{(a)}_{(i,\alpha),(j,\beta)})=\big(a^G_{ij}(1-\delta_{\alpha\beta})+a^H_{\alpha\beta}(1-\delta_{ij})\big).$$

That is,

$$A^{(a)}_{G\Diamond H} = \begin{pmatrix} O & A_H + a^G_{12}(M-I) & \cdots & A_H + a^G_{1n_G}(M-I) \\ A_H + a^G_{21}(M-I) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_H + a^G_{n_G-1n_G}(M-I) \\ A_H + a^G_{n_G1}(M-I) & \cdots & A_H + a^G_{n_Gn_G-1}(M-I) & O \end{pmatrix}.$$

with an $n_H \times n_H$ matrix M all of whose entries are 1. We hence have

$$\begin{aligned} & A^{(p)}_{G\Diamond H} P^{(a)}_{G\Diamond H} \\ &= \frac{1}{d_H(n_G-1)+d_G(n_H-1)} \\ &\quad \biggl(\Bigl(\sum_{k=1}^{n_G} a^G_{ik}a^G_{kj}\Bigr)\Bigl(\sum_{\gamma\neq\beta}\delta_{\alpha\gamma}\Bigr) + \Bigl(\sum_{k\neq j}\delta_{ik}\Bigr)\Bigl(\sum_{\gamma=1}^{n_H} a^H_{\alpha\gamma}a^H_{\gamma\beta}\Bigr) + a^H_{\alpha\beta}\Bigl(\sum_{k\neq j} a^G_{ik}\Bigr) + a^G_{ij}\Bigl(\sum_{\gamma\neq\beta} a^H_{\alpha\gamma}\Bigr)\biggr) \\ &= \frac{1}{d_H(n_G-1)+d_G(n_H-1)} \\ &\quad \biggl(\Bigl(\sum_{k=1}^{n_G} a^G_{ik}a^G_{kj}\Bigr)(1-\delta_{\alpha\beta}) + (1-\delta_{ij})\Bigl(\sum_{\gamma=1}^{n_H} a^H_{\alpha\gamma}a^H_{\gamma\beta}\Bigr) + (d_G - a^G_{ij})a^H_{\alpha\beta} + a^G_{ij}(d_H - a^H_{\alpha\beta})\biggr). \end{aligned}$$

We take functions f and g satisfying $\Delta_{\mathcal{P}_G} f = \mu f$ and $\Delta_{\mathcal{P}_H} g = \nu g$. We then obtain

$$\begin{aligned} A^{(p)}_{G\Diamond H} P^{(a)}_{G\Diamond H}\varphi_{f.g} &= \frac{d_G^2(n_H-1) + (n_H-1)d_H^2 + (d_Gn_G - d_G)d_H + d_G(d_Hn_H - d_H)}{d_H(n_G-1)+d_G(n_H-1)}\varphi_{f.g} \\ &= (d_G + d_H)\varphi_{f.g} \end{aligned}$$

when $\mu = \nu = 0$,

$$\begin{aligned} A^{(p)}_{G\Diamond H} P^{(a)}_{G\Diamond H}\varphi_{f.g} &= \frac{-d_G^2 + (n_G-1)d_H^2(1-\nu)^2 + d_G(n_G-1)d_H(1-\nu) - d_Gd_H(1-\nu)}{d_H(n_G-1)+d_G(n_H-1)}\varphi_{f.g} \\ &= \frac{\{d_G + d_H(1-\nu)\}\{d_H(n_G-1)(1-\nu) - d_G\}}{d_H(n_G-1)+d_G(n_H-1)}\varphi_{f.g} \end{aligned}$$

when $\mu = 0$ and $\nu \neq 0$,

$$A^{(p)}_{G\Diamond H} P^{(a)}_{G\Diamond H}\varphi_{f.g} = \frac{\{d_G(1-\mu) + d_H\}\{d_G(n_H-1)(1-\mu) - d_H\}}{d_H(n_G-1)+d_G(n_H-1)}\varphi_{f.g}$$

when $\mu \neq 0$ and $\nu = 0$, and

$$\begin{aligned} A^{(p)}_{G\Diamond H} P^{(a)}_{G\Diamond H}\varphi_{f.g} &= \frac{-d_G^2(1-\mu)^2 - d_H^2(1-\nu)^2 - 2d_Gd_H(1-\mu)(1-\nu)}{d_H(n_G-1)+d_G(n_H-1)}\varphi_{f.g} \\ &= -\frac{\{d_G(1-\mu) + d_H(1-\nu)\}^2}{d_H(n_G-1)+d_G(n_H-1)}\varphi_{f.g} \end{aligned}$$

when $\mu \neq 0$ and $\nu \neq 0$. We hence get the conclusion. □

EXAMPLE 4.32. Let G be a 4-circuit and H be a 5-circuit. Their adjacency matrices are given as

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_H = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are

$$\{0, 1, 1, 2\} \quad \text{and} \quad \Big\{0,\ \frac{1}{4}\big(5-\sqrt{5}\big),\ \frac{1}{4}\big(5-\sqrt{5}\big),\ \frac{1}{4}\big(5+\sqrt{5}\big),\ \frac{1}{4}\big(5+\sqrt{5}\big)\Big\}.$$

Since $d^{(p)}_{G\Diamond H} = 4,\ d^{(a)}_{G\Diamond H} = 14$ and $A_H M = 2M$, the $(1,1)$-probabilistic transition matrix $Q_{(G\Diamond H)_{(1,1)}}$ is given as

$$Q_{(G\Diamond H)_{(1,1)}} = \frac{1}{4}\begin{pmatrix} A_H & I & O & I \\ I & A_H & I & O \\ O & I & A_H & I \\ I & O & I & A_H \end{pmatrix} \cdot \frac{1}{14}\begin{pmatrix} O & A_H+M-I & A_H & A_H+M-I \\ A_H+M-I & O & A_H+M-I & A_H \\ A_H & A_H+M-I & O & A_H+M-I \\ A_H+M-I & A_H & A_H+M-I & O \end{pmatrix}$$

$$= \frac{1}{56}\begin{pmatrix} 2(A_H+M-I) & A_H^2+2M & A_H^2+2(A_H+M-I) & A_H^2+2M \\ A_H^2+2M & 2(A_H+M-I) & A_H^2+2M & A_H^2+2(A_H+M-I) \\ A_H^2+2(A_H+M-I) & A_H^2+2M & 2(A_H+M-I) & A_H^2+2M \\ A_H^2+2M & A_H^2+2(A_H+M-I) & A_H^2+2M & 2(A_H+M-I) \end{pmatrix}$$

$$
= \frac{1}{56}\begin{pmatrix}
0 & 4 & 2 & 2 & 4 & 4 & 2 & 3 & 3 & 2 & 2 & 4 & 3 & 3 & 4 & 4 & 2 & 3 & 3 & 2 \\
4 & 0 & 4 & 2 & 2 & 2 & 4 & 2 & 3 & 3 & 4 & 2 & 4 & 3 & 3 & 2 & 4 & 2 & 3 & 3 \\
2 & 4 & 0 & 4 & 2 & 3 & 2 & 4 & 2 & 3 & 3 & 4 & 2 & 4 & 3 & 3 & 2 & 4 & 2 & 3 \\
2 & 2 & 4 & 0 & 4 & 3 & 3 & 2 & 4 & 2 & 3 & 3 & 4 & 2 & 4 & 3 & 3 & 2 & 4 & 2 \\
4 & 2 & 2 & 4 & 0 & 2 & 3 & 3 & 2 & 4 & 4 & 3 & 3 & 4 & 2 & 2 & 3 & 3 & 2 & 4 \\
4 & 2 & 3 & 3 & 2 & 0 & 4 & 2 & 2 & 4 & 4 & 2 & 3 & 3 & 2 & 2 & 4 & 3 & 3 & 4 \\
2 & 4 & 2 & 3 & 3 & 4 & 0 & 4 & 2 & 2 & 2 & 4 & 2 & 3 & 3 & 4 & 2 & 4 & 3 & 3 \\
3 & 2 & 4 & 2 & 3 & 2 & 4 & 0 & 4 & 2 & 3 & 2 & 4 & 2 & 3 & 3 & 4 & 2 & 4 & 3 \\
3 & 3 & 2 & 4 & 2 & 2 & 2 & 4 & 0 & 4 & 3 & 3 & 2 & 4 & 2 & 3 & 3 & 4 & 2 & 4 \\
2 & 3 & 3 & 2 & 4 & 4 & 2 & 2 & 4 & 0 & 2 & 3 & 3 & 2 & 4 & 4 & 3 & 3 & 4 & 2 \\
2 & 4 & 3 & 3 & 4 & 4 & 2 & 3 & 3 & 2 & 0 & 4 & 2 & 2 & 4 & 4 & 2 & 3 & 3 & 2 \\
4 & 2 & 4 & 3 & 3 & 2 & 4 & 2 & 3 & 3 & 4 & 0 & 4 & 2 & 2 & 2 & 4 & 2 & 3 & 3 \\
3 & 4 & 2 & 4 & 3 & 3 & 2 & 4 & 2 & 3 & 2 & 4 & 0 & 4 & 2 & 3 & 2 & 4 & 2 & 3 \\
3 & 3 & 4 & 2 & 4 & 3 & 3 & 2 & 4 & 2 & 2 & 2 & 4 & 0 & 4 & 3 & 3 & 2 & 4 & 2 \\
4 & 3 & 3 & 4 & 2 & 2 & 3 & 3 & 2 & 4 & 4 & 2 & 2 & 4 & 0 & 2 & 3 & 3 & 2 & 4 \\
4 & 2 & 3 & 3 & 2 & 2 & 4 & 3 & 3 & 4 & 4 & 2 & 3 & 3 & 2 & 0 & 4 & 2 & 2 & 4 \\
2 & 4 & 2 & 3 & 3 & 4 & 2 & 4 & 3 & 3 & 2 & 4 & 2 & 3 & 3 & 4 & 0 & 4 & 2 & 2 \\
3 & 2 & 4 & 2 & 3 & 3 & 4 & 2 & 4 & 3 & 3 & 2 & 4 & 2 & 3 & 2 & 4 & 0 & 4 & 2 \\
3 & 3 & 2 & 4 & 2 & 3 & 3 & 4 & 2 & 4 & 3 & 3 & 2 & 4 & 2 & 2 & 2 & 4 & 0 & 4 \\
2 & 3 & 3 & 2 & 4 & 4 & 3 & 3 & 4 & 2 & 2 & 3 & 3 & 2 & 4 & 4 & 2 & 2 & 4 & 0
\end{pmatrix}.
$$

The eigenvalues of $\Delta_{\mathcal{P}_{(G\diamond H)_{(1,1)}}}$ are

$$
\left\{
\begin{array}{l}
0,\, 1,\, \frac{1}{112}(115-\sqrt{5}),\ \frac{1}{112}(115-\sqrt{5}),\ \frac{1}{112}(115-\sqrt{5}),\ \frac{1}{112}(115-\sqrt{5}),\\
\quad \frac{1}{112}(115-\sqrt{5}),\ \frac{1}{112}(115-\sqrt{5}),\ \frac{1}{112}(127-5\sqrt{5}),\ \frac{1}{112}(127-5\sqrt{5}),\\
\quad \frac{1}{112}(115+\sqrt{5}),\ \frac{1}{112}(127+5\sqrt{5}),\ \frac{1}{112}(127+5\sqrt{5}),\ \frac{1}{112}(127+5\sqrt{5}),\\
\quad \frac{1}{112}(127+5\sqrt{5}),\ \frac{1}{112}(127+5\sqrt{5}),\ \frac{1}{48}(51+\sqrt{5}),\ \frac{15}{14},\ \frac{15}{14},\\
\quad \frac{1}{112}(127+5\sqrt{5}),\ \frac{1}{112}(127+5\sqrt{5})
\end{array}
\right\}.
$$

6.4. $(1,1)$**-Laplacians of Kähler graphs of strong-complement product type.**

THEOREM 4.19. *Let $G = (V,E), H = (W,F)$ be finite regular ordinary graphs. Suppose G and H are connected. We denote by $0 = \mu_1 < \mu_2 \le \cdots \le \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 < \nu_2 \le \cdots \le \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues*

of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G \circledast H$ of strong-complement product type are

$$0, \qquad 1-\frac{\{d_G+d_H(d_G+1)(1-\nu_\alpha)\}\{d_H(1-\nu_\alpha)(n_G-2d_G-1)-d_G\}}{(d_G+d_H+d_Gd_H)\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\}},$$

$$1-\frac{\{d_G(d_H+1)(1-\mu_i)+d_H\}\{d_G(1-\mu_i)(n_H-2d_H-1)-d_H\}}{(d_G+d_H+d_Gd_H)\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\}},$$

$$1-\frac{\{d_G(1-\mu)+d_H(1-\nu)+d_Gd_H(1-\mu)(1-\nu)\}\{d_G(1-\mu)(d_H\nu-d_H-1)+d_H(1-\nu)(d_G\mu-d_G-1)\}}{(d_G+d_H+d_Gd_H)\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\}},$$

$$(2\le i\le n_G,\ 2\le\alpha\le n_H),$$

and the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ for $G \circledast H$ are

$$0, \qquad d_G+d_H+d_Gd_H-\frac{\{d_G+d_H(d_G+1)(1-\nu_\alpha)\}\{d_H(1-\nu_\alpha)(n_G-2d_G-1)-d_G\}}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)},$$

$$d_G+d_H+d_Gd_H-\frac{\{d_G(d_H+1)(1-\mu_i)+d_H\}\{d_G(1-\mu_i)(n_H-2d_H-1)-d_H\}}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)},$$

$$d_G+d_H+d_Gd_H \\ -\frac{\{d_G(1-\mu)+d_H(1-\nu)+d_Gd_H(1-\mu)(1-\nu)\}\{d_G(1-\mu)(d_H\nu-d_H-1)+d_H(1-\nu)(d_G\mu-d_G-1)\}}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)},$$

$$(2\le i\le n_G,\ 2\le\alpha\le n_H),$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V\times W$ is an eigenfunction for both $\Delta_{\mathcal{Q}_{(1,1)}}$ and $\Delta_{\mathcal{A}_{(1,1)}}$ corresponding to these eigenvalues.

Proof. We use the same notations as in the proofs of Theorems 4.16 and 4.17. By definition the adjacency matrices $A^{(p)}_{G\circledast H}$ and $A^{(a)}_{G\circledast H}$ of the principal and the auxiliary graphs of $G \circledast H$ are

$$A^{(p)}_{G\circledast H}=(a^{(p)}_{(i,\alpha),(j,\beta)})=\Big(a^G_{ij}\delta_{\alpha\beta}+\delta_{ij}a^H_{\alpha\beta}+a^G_{ij}a^H_{\alpha\beta}\Big),$$
$$A^{(a)}_{G\circledast H}=(a^{(a)}_{(i,\alpha),(j,\beta)})=\Big(a^G_{ij}a^{H^c}_{\alpha\beta}+a^{G^c}_{ij}\,a^H_{\alpha\beta}\Big).$$

That is,

$$A^{(p)}_{G\circledast H} = \begin{pmatrix} A_H & a^G_{12}(A_H+I) & \cdots & a^G_{1n_G}(A_H+I) \\ a^G_{21}(A_H+I) & A_H & \ddots & \vdots \\ \vdots & \ddots & \ddots & a^G_{n_G-1n_G}(A_H+I) \\ a^G_{n_G1}(A_H+I) & \cdots & a^G_{n_Gn_G-1}(A_H+I) & A_H \end{pmatrix}$$

and $A^{(a)}_{G\circledast H} = A^{(a)}_{G\square H}$. Hence we have

$$\begin{aligned} &A^{(p)}_{G\circledast H}P^{(a)}_{G\circledast H} \\ &\quad = \frac{1}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)} \\ &\qquad \Big\{\Big(\sum_{k=1}^{n_G} a^G_{ik}a^G_{kj}\Big)a^{H^c}_{\alpha\beta} + a^G_{ij}\Big(\sum_{\gamma=1}^{n_H} a^H_{\alpha\gamma}a^{H^c}_{\gamma\beta}\Big) + \Big(\sum_{k=1}^{n_G} a^G_{ik}a^G_{kj}\Big)\Big(\sum_{\gamma=1}^{n_H} a^H_{\alpha\gamma}a^{H^c}_{\gamma\beta}\Big) \\ &\qquad\quad + \Big(\sum_{k=1}^{n_G} a^{G^c}_{ik}a^G_{kj}\Big)a^H_{\alpha\beta} + a^{G^c}_{ij}\Big(\sum_{\gamma=1}^{n_H} a^H_{\alpha\gamma}a^H_{\gamma\beta}\Big) + \Big(\sum_{k=1}^{n_G} a^{G^c}_{ik}a^G_{kj}\Big)\Big(\sum_{\gamma=1}^{n_H} a^H_{\alpha\gamma}a^H_{\gamma\beta}\Big)\Big\}. \end{aligned}$$

We take functions f and g satisfying $\varDelta_{\mathcal{P}_G} f = \mu f$ and $\varDelta_{\mathcal{P}_H} g = \nu g$ and consider $\varphi_{f.g}$. We then find that

$$\begin{aligned} &\mathcal{A}^{(p)}_{G\circledast H}\mathcal{P}^{(a)}_{G\circledast H}\varphi_{f.g} \\ &\qquad = \frac{(d_G^2+d_Gd_H+d_G^2d_H)(n_H-1-d_H)+(d_Gd_H+d_H^2+d_Gd_H^2)(n_G-1-d_G)}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)}\varphi_{f.g} \\ &\qquad = (d_G+d_H+d_Gd_H)\varphi_{f.g} \end{aligned}$$

when $\mu=\nu=0$,

$$\begin{aligned} &\mathcal{A}^{(p)}_{G\circledast H}\mathcal{P}^{(a)}_{G\circledast H}\varphi_{f.g} \\ &\ = \Big\{\frac{\big\{d_G^2+d_Gd_H(1-\nu)+d_G^2d_H(1-\nu)\big\}(n_H-1-d_H)}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)} \\ &\qquad\quad + \frac{\big\{d_Gd_H(1-\nu)+d_H^2(1-\nu)^2+d_Gd_H^2(1-\nu)\big\}(n_G-1-d_G)}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)}\Big\}\varphi_{f.g} \\ &\ = \frac{\{d_G+d_H(1-\nu)+d_Gd_H(1-\nu)\}\{d_G(d_H\nu-d_H-1)+d_H(1-\nu)(n_G-1-d_G)\}}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)}\varphi_{f.g} \end{aligned}$$

when $\mu = 0,\ \nu \neq 0$,

$$\mathcal{A}^{(p)}_{G\circledast H}\mathcal{P}^{(a)}_{G\circledast H}\varphi_{f.g}$$
$$= \Bigg\{ \frac{\big\{d_G^2(1-\mu)^2+d_Gd_H(1-\mu)+d_G^2(1-\mu)^2d_H\big\}(n_H-1-d_H)}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)}$$
$$+ \frac{\big\{d_Gd_H(1-\mu)+d_H^2+d_G(1-\mu)d_H^2\big\}(n_G-1-d_G)}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)} \Bigg\}\varphi_{f.g}$$
$$= \frac{\{d_G(1-\mu)+d_H+d_Gd_H(1-\mu)\}\{d_G(1-\mu)(n_H-1-d_H)+d_H(d_G\mu-d_G-1)\}}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)}\varphi_{f.g}$$

when $\mu \neq 0,\ \nu = 0$, and

$$\mathcal{A}^{(p)}_{G\circledast H}\mathcal{P}^{(a)}_{G\circledast H}\varphi_{f.g}$$
$$= \Bigg\{ \frac{\big\{d_G^2(1-\mu)^2+d_Gd_H(1-\mu)(1-\nu)+d_G^2(1-\mu)^2d_H(1-\nu)\big\}(n_H-1-d_H)}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)}$$
$$+ \frac{\big\{d_Gd_H(1-\mu)(1-\nu)+d_H^2(1-\nu)^2+d_G(1-\mu)d_H^2(1-\nu)^2\big\}(n_G-1-d_G)}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)} \Bigg\}\varphi_{f.g}$$
$$= \frac{\{d_G(1-\mu)+d_H(1-\nu)+d_Gd_H(1-\mu)(1-\nu)\}\{d_G(1-\mu)(d_H\nu-d_H-1)+d_H(1-\nu)(d_G\mu-d_G-1)\}}{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)}\varphi_{f.g}$$

when $\mu \neq 0,\ \nu \neq 0$. Thus we get the conclusion. □

Example 4.33. Let G be a 4-circuit and H be a 5-circuit. Their adjacency matrices are given as

$$A_G = \begin{pmatrix} 0&1&0&1\\ 1&0&1&0\\ 0&1&0&1\\ 1&0&1&0 \end{pmatrix} \quad \text{and} \quad A_H = \begin{pmatrix} 0&1&0&0&1\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&1&0&1\\ 1&0&0&1&0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are

$$\big\{0,1,1,2\big\} \quad \text{and} \quad \Big\{0,\ \tfrac{1}{4}\big(5-\sqrt{5}\big),\ \tfrac{1}{4}\big(5-\sqrt{5}\big),\ \tfrac{1}{4}\big(5+\sqrt{5}\big),\ \tfrac{1}{4}\big(5+\sqrt{5}\big)\Big\}.$$

Since $d^{(p)}_{G\circledast H} = 8$ and $d^{(a)}_{G\circledast H} = 6$, the $(1,1)$-probabilistic transition matrix $Q_{(G\circledast H)_{(1,1)}}$ is given as

$$Q_{(G\circledast H)_{(1,1)}} = \frac{1}{8}\begin{pmatrix} A_H & A_H+I & O & A_H+I \\ A_H+I & A_H & A_H+I & O \\ O & A_H+I & A_H & A_H+I \\ A_H+I & O & A_H+I & A_H \end{pmatrix} \cdot \frac{1}{6}\begin{pmatrix} O & A_{H^c} & A_H & A_{H^c} \\ A_{H^c} & O & A_{H^c} & A_H \\ A_H & A_{H^c} & O & A_{H^c} \\ A_{H^c} & A_H & A_{H^c} & O \end{pmatrix}$$

$$= \frac{1}{48}\begin{pmatrix} 2(A_HA_{H^c}+A_{H^c}) & A_H^2+A_H+A_HA_{H^c} & A_H^2+2(A_HA_{H^c}+A_{H^c}) & A_H^2+A_H+A_HA_{H^c} \\ A_H^2+A_H+A_HA_{H^c} & 2(A_HA_{H^c}+A_{H^c}) & A_H^2+A_H+A_HA_{H^c} & A_H^2+2(A_HA_{H^c}+A_{H^c}) \\ A_H^2+2(A_HA_{H^c}+A_{H^c}) & A_H^2+A_H+A_HA_{H^c} & 2(A_HA_{H^c}+A_{H^c}) & A_H^2+A_H+A_HA_{H^c} \\ A_H^2+A_H+A_HA_{H^c} & A_H^2+2(A_HA_{H^c}+A_{H^c}) & A_H^2+A_H+A_HA_{H^c} & 2(A_HA_{H^c}+A_{H^c}) \end{pmatrix}$$

$$= \frac{1}{48}\begin{pmatrix}
0&2&4&4&2&2&2&2&2&2&2&2&5&5&2&2&2&2&2&2\\
2&0&2&4&4&2&2&2&2&2&2&2&2&5&5&2&2&2&2&2\\
4&2&0&2&4&2&2&2&2&2&5&2&2&2&5&2&2&2&2&2\\
4&4&2&0&2&2&2&2&2&2&5&5&2&2&2&2&2&2&2&2\\
2&4&4&2&0&2&2&2&2&2&2&5&5&2&2&2&2&2&2&2\\
2&2&2&2&2&0&2&4&4&2&2&2&2&2&2&2&2&5&5&2\\
2&2&2&2&2&2&0&2&4&4&2&2&2&2&2&2&2&2&5&5\\
2&2&2&2&2&4&2&0&2&4&2&2&2&2&2&5&2&2&2&5\\
2&2&2&2&2&4&4&2&0&2&2&2&2&2&2&5&5&2&2&2\\
2&2&2&2&2&2&4&4&2&0&2&2&2&2&2&2&5&5&2&2\\
2&2&5&5&2&2&2&2&2&2&0&2&4&4&2&2&2&2&2&2\\
2&2&2&5&5&2&2&2&2&2&2&0&2&4&4&2&2&2&2&2\\
5&2&2&2&5&2&2&2&2&2&4&2&0&2&4&2&2&2&2&2\\
5&5&2&2&2&2&2&2&2&2&4&4&2&0&2&2&2&2&2&2\\
2&5&5&2&2&2&2&2&2&2&2&4&4&2&0&2&2&2&2&2\\
2&2&2&2&2&2&2&5&5&2&2&2&2&2&2&0&2&4&4&2\\
2&2&2&2&2&2&2&2&5&5&2&2&2&2&2&2&0&2&4&4\\
2&2&2&2&2&5&2&2&2&5&2&2&2&2&2&4&2&0&2&4\\
2&2&2&2&2&5&5&2&2&2&2&2&2&2&2&4&4&2&0&2\\
2&2&2&2&2&2&5&5&2&2&2&2&2&2&2&2&4&4&2&0
\end{pmatrix}.$$

The eigenvalues of $\varDelta_{\mathcal{Q}_{(G\circledast H)_{(1,1)}}}$ are

$$\left\{\begin{array}{l} 0,\ \frac{6}{5},\frac{5}{96}(21-\sqrt{5}),\ \frac{5}{96}(21-\sqrt{5}),\ \frac{5}{96}(21-\sqrt{5}),\ \frac{5}{96}(21-\sqrt{5}), \\ \quad \frac{1}{96}(99-\sqrt{5}),\ \frac{1}{96}(99-\sqrt{5}),\ \frac{1}{96}(99-\sqrt{5}),\ \frac{1}{96}(99-\sqrt{5}) \\ \quad \frac{1}{96}(99-\sqrt{5}),\ \frac{1}{96}(99-\sqrt{5}),\ \frac{1}{96}(99-\sqrt{5}),\ \frac{1}{96}(99-\sqrt{5}), \\ \quad \frac{13}{12},\ \frac{13}{12},\ \frac{5}{96}(21+\sqrt{5}),\ \frac{5}{96}(21+\sqrt{5}),\ \frac{5}{96}(21+\sqrt{5}),\ \frac{5}{96}(21+\sqrt{5}) \end{array}\right\}.$$

6.5. $(1,1)$-Laplacians of Kähler graphs of complement-tensor product type.

THEOREM 4.20. *Let $G = (V,E), H = (W,F)$ be finite regular ordinary graphs. Suppose G and H are connected. We denote by $0 = \mu_1 < \mu_2 \le \cdots \le \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 < \nu_2 \le \cdots \le \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G♠H$ of complement-tensor product type are*

$$0, \qquad 1 - \frac{d_H(1-\nu_\alpha)^2(n_G - 2d_G)}{d_G(n_H - d_H) + d_H(n_G - d_G)},$$

$$1 - \frac{d_G(1-\mu_i)^2(n_H - 2d_H)}{d_G(n_H - d_H) + d_H(n_G - d_G)}, \qquad 1 + \frac{2d_Gd_H(1-\mu_i)^2(1-\nu_\alpha)^2}{d_G(n_H - d_H) + d_H(n_G - d_G)},$$

$$(2 \le i \le n_G,\ 2 \le \alpha \le n_H),$$

and the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ for $G♠H$ are

$$0, \qquad d_G(n_H - d_H) + d_H(n_G - d_G) - d_H(1-\nu_\alpha)^2(n_G - 2d_G),$$

$$d_G(n_H - d_H) + d_H(n_G - d_G) - d_G(1-\mu_i)^2(n_H - 2d_H),$$

$$d_G(n_H - d_H) + d_H(n_G - d_G) + 2d_Gd_H(1-\mu_i)^2(1-\nu_\alpha)^2,$$

$$(2 \le i \le n_G,\ 2 \le \alpha \le n_H),$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V \times W$ is an eigenfunction for both $\Delta_{\mathcal{Q}_{(1,1)}}$ and $\Delta_{\mathcal{A}_{(1,1)}}$ corresponding to these eigenvalues.

PROOF. We use the same notations as in the proof of Theorems 4.16 and 4.17. By definition the adjacency matrices $A^{(p)}_{G♠H}$ and $A^{(a)}_{G♠H}$ of the principal and the auxiliary graphs of $G♠H$ are

$$\begin{aligned} A^{(p)}_{G♠H} = \big(a^{(p)}_{(i,\alpha)(j,\beta)}\big) &= \big(a^G_{ij}(a^{H^c}_{\alpha\beta} + \delta_{\alpha\beta}) + (a^{G^c}_{ij} + \delta_{ij})a^H_{\alpha\beta}\big) \\ &= \big(a^G_{ij}(1 - a^H_{\alpha\beta}) + (1 - a^G_{ij})a^H_{\alpha\beta}\big), \\ A^{(a)}_{G♠H} = \big(a^{(a)}_{(i,\alpha)(j,\beta)}\big) &= \big(a^G_{ij}a^H_{\alpha\beta}\big). \end{aligned}$$

That is,

$$A^{(p)}_{G\spadesuit H} = \begin{pmatrix} A_H & a^G_{12}(A_H+I)+a^{G^c}_{12}A_H & \cdots & a^G_{1n_G}(A_H+I)+a^{G^c}_{1n_G}A_H \\ a^G_{21}(A_H+I)+a^{G^c}_{21}A_H & A_H & \ddots & \vdots \\ \vdots & \ddots & \ddots & a^G_{n_G-1n_G}(A_H+I)+a^{G^c}_{n_G-1n_G}A_H \\ a^G_{n_G1}(A_H+I)+a^{G^c}_{n_G1}A_H & \cdots\ a^G_{n_Gn_G-1}(A_H+I)+a^{G^c}_{n_Gn_G-1}A_H & & A_H \end{pmatrix}$$

and $A^{(a)}_{G\spadesuit H} = A^{(a)}_{G\boxplus H}$. We hence have

$$A^{(p)}_{G\spadesuit H}P^{(a)}_{G\spadesuit H} = \frac{1}{d_Gd_H}\biggl(\Bigl(\sum_{k=1}^{n_G} a^G_{ik}a^G_{kj}\Bigr)\Bigl(\sum_{\gamma=1}^{n_H} a^{H^c}_{\alpha\gamma}a^H_{\gamma\beta} + a^H_{\alpha\beta}\Bigr) + \Bigl(\sum_{k=1}^{n_G} a^{G^c}_{ik}a^G_{kj} + a^G_{ij}\Bigr)\Bigl(\sum_{\gamma=1}^{n_H} a^H_{\alpha\gamma}a^H_{\gamma\beta}\Bigr)\biggr).$$

We take functions f and g satisfying $\Delta_{\mathcal{P}_G}f = \mu f$ and $\Delta_{\mathcal{P}_H}g = \nu g$ and consider $\varphi_{f.g}$. We then find that

$$\begin{aligned}\mathcal{A}^{(p)}_{G\spadesuit H}\mathcal{P}^{(a)}_{G\spadesuit H}\varphi_{f.g} &= \frac{d_G^2(n_H-d_H)d_H + d_G(n_G-d_G)d_H^2}{d_Gd_H}\varphi_{f.g} \\ &= \{d_G(n_H-d_H) + (n_G-d_G)d_H\}\varphi_{f.g}\end{aligned}$$

when $\mu = \nu = 0$,

$$\begin{aligned}\mathcal{A}^{(p)}_{G\spadesuit H}\mathcal{P}^{(a)}_{G\spadesuit H}\varphi_{f.g} &= \frac{d_G^2(d_H\nu-d_H)d_H(1-\nu) + d_G(n_G-d_G)d_H^2(1-\nu)^2}{d_Gd_H}\varphi_{f.g} \\ &= d_H(1-\nu)^2(n_G-2d_G)\varphi_{f.g}\end{aligned}$$

when $\mu = 0$, $\nu \neq 0$,

$$\begin{aligned}\mathcal{A}^{(p)}_{G\spadesuit H}\mathcal{P}^{(a)}_{G\spadesuit H}\varphi_{f.g} &= \frac{d_G^2(1-\mu)^2(n_H-d_H)d_H + d_G(1-\mu)(d_G\mu-d_G)d_H^2}{d_Gd_H}\varphi_{f.g} \\ &= d_G(1-\mu)^2(n_H-2d_H)\varphi_{f.g}\end{aligned}$$

when $\mu \neq 0$, $\nu = 0$, and

$$\begin{aligned}&\mathcal{A}^{(p)}_{G\spadesuit H}\mathcal{P}^{(a)}_{G\spadesuit H}\varphi_{f.g} \\ &\quad = \frac{d_G^2(1-\mu)^2(d_H\nu-d_H)d_H(1-\nu) + d_G(1-\mu)(d_G\mu-d_G)d_H^2(1-\nu)^2}{d_Gd_H}\varphi_{f.g} \\ &\quad = -2d_Gd_H(1-\mu)^2(1-\nu)^2\varphi_{f.g}\end{aligned}$$

when $\mu \neq 0$, $\nu \neq 0$. These show the conclusion. □

EXAMPLE 4.34. Let G be a 4-circuit and H be a 5-circuit. Their adjacency matrices are given as

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_H = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are

$$\big\{0, 1, 1, 2\big\} \quad \text{and} \quad \Big\{0,\ \frac{1}{4}\big(5-\sqrt{5}\big),\ \frac{1}{4}\big(5-\sqrt{5}\big),\ \frac{1}{4}\big(5+\sqrt{5}\big),\ \frac{1}{4}\big(5+\sqrt{5}\big)\Big\}.$$

Since $d^{(p)}_{G \circledast H} = 10$ and $d^{(a)}_{G\spadesuit H} = 4$, the $(1,1)$-probabilistic transition matrix $Q_{(G\spadesuit H)_{(1,1)}}$ is given as

$$Q_{(G\spadesuit H)_{(1,1)}} = \frac{1}{10}\begin{pmatrix} A_H & A_{A^c}+I & A_H & A_{A^c}+I \\ A_{A^c}+I & A_H & A_{A^c}+I & A_H \\ A_H & A_{A^c}+I & A_H & A_{A^c}+I \\ A_{A^c}+I & A_H & A_{A^c}+I & A_H \end{pmatrix} \cdot \frac{1}{4}\begin{pmatrix} O & A_H & O & A_H \\ A_H & O & A_H & O \\ O & A_H & O & A_H \\ A_H & O & A_H & O \end{pmatrix}$$

$$= \frac{1}{40}\begin{pmatrix} 2(A_{H^c}A_H+A_H) & 2A_H^2 & 2(A_{H^c}A_H+A_H) & 2A_H^2 \\ 2A_H^2 & 2(A_{H^c}A_H+A_H) & 2A_H^2 & 2(A_{H^c}A_H+A_H) \\ 2(A_{H^c}A_H+A_H) & 2A_H^2 & 2(A_{H^c}A_H+A_H) & 2A_H^2 \\ 2A_H^2 & 2(A_{H^c}A_H+A_H) & 2A_H^2 & 2(A_{H^c}A_H+A_H) \end{pmatrix}$$

$$= \frac{1}{40}\begin{pmatrix}
0 & 4 & 2 & 2 & 4 & 4 & 0 & 2 & 2 & 0 & 0 & 4 & 2 & 2 & 4 & 4 & 0 & 2 & 2 & 0 \\
4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 \\
2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 \\
2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 \\
4 & 2 & 2 & 4 & 0 & 0 & 2 & 2 & 0 & 4 & 4 & 2 & 2 & 4 & 0 & 0 & 2 & 2 & 0 & 4 \\
4 & 0 & 2 & 2 & 0 & 0 & 4 & 2 & 2 & 4 & 4 & 0 & 2 & 2 & 0 & 0 & 4 & 2 & 2 & 4 \\
0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 \\
2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 \\
2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 \\
0 & 2 & 2 & 0 & 4 & 4 & 2 & 2 & 4 & 0 & 0 & 2 & 2 & 0 & 4 & 4 & 2 & 2 & 4 & 0 \\
0 & 4 & 2 & 2 & 4 & 4 & 0 & 2 & 2 & 0 & 0 & 4 & 2 & 2 & 4 & 4 & 0 & 2 & 2 & 0 \\
4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 \\
2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 \\
2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 \\
4 & 2 & 2 & 4 & 0 & 0 & 2 & 2 & 0 & 4 & 4 & 2 & 2 & 4 & 0 & 0 & 2 & 2 & 0 & 4 \\
4 & 0 & 2 & 2 & 0 & 0 & 4 & 2 & 2 & 4 & 4 & 0 & 2 & 2 & 0 & 0 & 4 & 2 & 2 & 4 \\
0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 \\
2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 \\
2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 & 2 & 2 & 0 & 4 & 0 & 2 & 2 & 4 & 0 & 4 \\
0 & 2 & 2 & 0 & 4 & 4 & 2 & 2 & 4 & 0 & 0 & 2 & 2 & 0 & 4 & 4 & 2 & 2 & 4 & 0
\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{Q}_{(G\circledast H)_{(1,1)}}}$ are

$$\left\{\begin{array}{l} 0,\ \dfrac{4}{5}, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \\ \quad \dfrac{1}{10}\big(13-\sqrt{5}\big),\ \dfrac{1}{10}\big(13-\sqrt{5}\big),\ \dfrac{1}{10}\big(13+\sqrt{5}\big),\ \dfrac{1}{10}\big(13;\sqrt{5}\big) \end{array}\right\}.$$

6.6. $(1,1)$**-Laplacians of Kähler graphs of tensor-complement product type.**

THEOREM 4.21. *Let $G = (V, E), H = (W, F)$ be finite regular ordinary graphs. Suppose G and H are connected. We denote by $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 < \nu_2 \leq \cdots \leq \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(1,1)}}$ for their Kähler graph $G♣H$ of complement-tensor product type are*

$$0, \qquad \nu - \frac{(1-\nu)\{d_G(d_H\nu - n_H) - d_H\nu(n_G-1-d_G)\}}{d_H(n_G-1-d_G) + d_G(n_H-1-d_H)},$$

$$\mu - \frac{(1-\mu)\{-d_G\mu((n_H-1-d_H) + d_H(d_G\mu - n_G)\}}{d_H(n_G-1-d_G) + d_G(n_H-1-d_H)},$$

$$\mu + \nu - \mu\nu - \frac{(1-\mu)(1-\nu)\{2d_Gd_H(\mu+\nu-\mu\nu) + d_G(\mu - n_H) + d_H(\nu - n_G)\}}{d_H(n_G-1-d_G) + d_G(n_H-1-d_H)},$$

$$(2 \leq i \leq n_G,\ 2 \leq \alpha \leq n_H),$$

and the eigenvalues of $\Delta_{\mathcal{A}_{(1,1)}}$ for $G♣H$ are

$$0, \qquad d_Gd_H\nu - \frac{d_Gd_H(1-\nu)\{d_G(d_H\nu - n_H) - d_H\nu(n_G-1-d_G)\}}{d_H(n_G-1-d_G) + d_G(n_H-1-d_H)},$$

$$d_Gd_H\mu - \frac{d_Gd_H(1-\mu)\{-d_G\mu((n_H-1-d_H) + d_H(d_G\mu - n_G)\}}{d_H(n_G-1-d_G) + d_G(n_H-1-d_H)},$$

$$d_Gd_H(\mu+\nu-\mu\nu) - \frac{d_Gd_H(1-\mu)(1-\nu)\{2d_Gd_H(\mu+\nu-\mu\nu) + d_G(\mu - n_H) + d_H(\nu - n_G)\}}{d_H(n_G-1-d_G) + d_G(n_H-1-d_H)},$$

$$(2 \leq i \leq n_G,\ 2 \leq \alpha \leq n_H),$$

Moreover, if f_i and g_α are eigenfunctions associated with μ_i and ν_α, respectively, then the function φ_{f_i,g_α} on the sets $V \times W$ is an eigenfunction for both $\Delta_{\mathcal{Q}_{(1,1)}}$ and $\Delta_{\mathcal{A}_{(1,1)}}$ corresponding to these eigenvalues.

Proof. We use the same notations as in the proofs of Theorems 4.16 and 4.17. By definition the adjacency matrices $A^{(p)}_{G\clubsuit H}$ and $A^{(a)}_{G\clubsuit H}$ of the principal and the auxiliary graphs of $G\clubsuit H$ are

$$A^{(p)}_{G\clubsuit H} = \big(a^{(a)}_{(i,\alpha)(j,\beta)}\big) = \big(a^G_{ij}a^H_{\alpha\beta}\big),$$
$$A^{(a)}_{G\clubsuit H} = \big(a^{(p)}_{(i,\alpha)(j,\beta)}\big) = \big(a^G_{ij}a^{H^c}_{\alpha\beta} + a^{G^c}_{ij}a^H_{\alpha\beta}\big).$$

That is, $A^{(p)}_{G\clubsuit H} = A^{(a)}_{G\boxplus H}$ and $A^{(a)}_{G\clubsuit H} = A^{(a)}_{G\boxdot H}$. We hence have

$$A^{(p)}_{G\clubsuit H}P^{(a)}_{G\clubsuit H} = \frac{1}{d_H(n_G-1-d_G)+d_G(n_H-1-d_H)} \Bigg(\Big(\sum_{k=1}^{n_G} a^G_{ik}a^G_{kj}\Big)\Big(\sum_{\gamma=1}^{n_H} a^H_{\alpha\gamma}a^{H^c}_{\gamma\beta}\Big) + \Big(\sum_{k=1}^{n_G} a^G_{ik}a^{G^c}_{kj}\Big)\Big(\sum_{\gamma=1}^{n_H} a^H_{\alpha\gamma}a^H_{\gamma\beta}\Big)\Bigg).$$

We take functions f and g satisfying $\Delta_{\mathcal{P}_G}f = \mu f$ and $\Delta_{\mathcal{P}_H}g = \nu g$ and consider $\varphi_{f.g}$. We then find that

$$A^{(p)}_{G\clubsuit H}P^{(a)}_{G\clubsuit H}\varphi_{f,g} = \frac{d_G^2d_H(n_H-1-d_H)+d_G(n_G-1-d_G)d_H^2}{d_H(n_G-1-d_G)+d_G(n_H-1-d_H)}\varphi_{f,g} = d_Gd_H\varphi_{f,g}$$

when $\mu=\nu=0$,

$$\begin{aligned}A^{(p)}_{G\clubsuit H}P^{(a)}_{G\clubsuit H}\varphi_{f,g} &= \frac{d_G^2d_H(1-\nu)(d_H\nu-d_H-1)+d_G(n_G-1-d_G)d_H^2(1-\nu)^2}{d_H(n_G-1-d_G)+d_G(n_H-1-d_H)}\varphi_{f,g}\\ &= d_Gd_H(1-\nu)\Big\{1+\frac{d_G(d_H\nu-n_H)-d_H\nu(n_G-1-d_G)}{d_H(n_G-1-d_G)+d_G(n_H-1-d_H)}\Big\}\varphi_{f,g}\end{aligned}$$

when $\mu=0$ and $\nu\neq 0$,

$$\begin{aligned}A^{(p)}_{G\clubsuit H}P^{(a)}_{G\clubsuit H}\varphi_{f,g} &= \frac{d_G^2(1-\mu)^2d_H(n_H-1-d_H)+d_G(1-\mu)(d_G\mu-d_G-1)d_H^2}{d_H(n_G-1-d_G)+d_G(n_H-1-d_H)}\varphi_{f,g}\\ &= d_Gd_H(1-\mu)\Big\{1+\frac{-d_G\mu((n_H-1-d_H)+d_H(d_G\mu-n_G)}{d_H(n_G-1-d_G)+d_G(n_H-1-d_H)}\Big\}\varphi_{f,g}\end{aligned}$$

when $\mu\neq 0$ and $\nu=0$, and

$$\begin{aligned}&A^{(p)}_{G\clubsuit H}P^{(a)}_{G\clubsuit H}\varphi_{f,g}\\ &\quad= \frac{d_G^2(1-\mu)^2d_H(1-\nu)(d_H\nu-d_H-1)+d_G(1-\mu)(d_G\mu-d_G-1)d_H^2(1-\nu)^2}{d_H(n_G-1-d_G)+d_G(n_H-1-d_H)}\varphi_{f,g}\\ &\quad= d_Gd_H(1-\mu)(1-\nu)\Big\{1+\frac{2d_Gd_H(\mu+\nu-\mu\nu)+d_G(\mu-n_H)+d_H(\nu-n_G)}{d_H(n_G-1-d_G)+d_G(n_H-1-d_H)}\Big\}\varphi_{f,g}\end{aligned}$$

when $\mu\neq 0$ and $\nu\neq 0$. Hence we get the conclusion. □

Example 4.35. Let G be a 4-circuit and H be a 5-circuit. Their adjacency matrices are given as

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_H = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are

$$\{0, 1, 1, 2\} \quad \text{and} \quad \Big\{0,\ \frac{1}{4}\big(5-\sqrt{5}\big),\ \frac{1}{4}\big(5-\sqrt{5}\big),\ \frac{1}{4}\big(5+\sqrt{5}\big),\ \frac{1}{4}\big(5+\sqrt{5}\big)\Big\}.$$

Since $d^{(p)}_{G\clubsuit H} = 10$ and $d^{(a)}_{G\clubsuit H} = 4$, the $(1,1)$-probabilistic transition matrix $Q_{(G\clubsuit H)_{(1,1)}}$ is given as

$$Q_{(G\spadesuit H)_{(1,1)}} = \frac{1}{10}\begin{pmatrix} O & A_H & O & A_H \\ A_H & O & A_H & O \\ O & A_H & O & A_H \\ A_H & O & A_H & O \end{pmatrix} \cdot \frac{1}{4}\begin{pmatrix} O & A_{H^c} & A_H & A_{H^c} \\ A_{H^c} & O & A_{H^c} & A_H \\ A_H & A_{H^c} & O & A_{H^c} \\ A_{H^c} & A_H & A_{H^c} & O \end{pmatrix}$$

$$= \frac{1}{40}\begin{pmatrix} 2A_HA_{H^c} & A_H^2 & 2A_HA_{H^c} & A_H^2 \\ A_H^2 & 2A_HA_{H^c} & A_H^2 & 2A_HA_{H^c} \\ 2A_HA_{H^c} & A_H^2 & 2A_HA_{H^c} & A_H^2 \\ A_H^2 & 2A_HA_{H^c} & A_H^2 & 2A_HA_{H^c} \end{pmatrix}$$

$$= \frac{1}{40}\begin{pmatrix}
0 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 0 \\
2 & 0 & 2 & 2 & 2 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 2 & 2 & 0 & 2 & 0 & 1 & 1 \\
2 & 2 & 0 & 2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 0 & 2 & 2 & 1 & 0 & 2 & 0 & 1 \\
2 & 2 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 2 & 2 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 0 & 2 \\
2 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & 2 & 2 \\
0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 2 & 2 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 2 & 2 \\
1 & 0 & 2 & 0 & 1 & 2 & 2 & 0 & 2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 0 & 2 & 2 \\
1 & 1 & 0 & 2 & 0 & 2 & 2 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 2 & 2 & 2 & 0 & 2 \\
0 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 0 \\
0 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 0 \\
2 & 0 & 2 & 2 & 2 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 2 & 2 & 0 & 2 & 0 & 1 & 1 \\
2 & 2 & 0 & 2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 0 & 2 & 2 & 1 & 0 & 2 & 0 & 1 \\
2 & 2 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 2 & 2 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 0 & 2 \\
2 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & 2 & 2 \\
0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 2 & 2 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 2 & 2 \\
1 & 0 & 2 & 0 & 1 & 2 & 2 & 0 & 2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 0 & 2 & 2 \\
1 & 1 & 0 & 2 & 0 & 2 & 2 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 2 & 2 & 2 & 0 & 2 \\
0 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 0
\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{Q}_{(G \circledast H)_{(1,1)}}}$ are

$$\left\{ \begin{array}{l} 0,\ \frac{2}{3},\ \frac{1}{24}\big(25-\sqrt{5}\big),\ \frac{1}{24}\big(25-\sqrt{5}\big),\ 1,1,1,1,1,1,1,1,1,1,\ \frac{1}{24}\big(25+\sqrt{5}\big), \\ \quad \frac{1}{24}\big(25+\sqrt{5}\big),\ \frac{1}{24}\big(31-\sqrt{5}\big),\ \frac{1}{24}\big(31-\sqrt{5}\big),\ \frac{1}{24}\big(31+\sqrt{5}\big),\ \frac{1}{24}\big(31+\sqrt{5}\big) \end{array} \right\}.$$

7. $(1,1)$-Isospectral Kähler graphs of product type

In this section we study conditions that two Kähler graphs obtained by product operations are isospectral.

7.1. Isospectral Kähler graphs of product type whose principal graphs are unions of copies of original graphs. When two ordinary graphs G_1, G_2 are not isomorphic to each other, then their Kähler graphs of product types studied in §4.3 and §4.4 with ordinary graphs H_1, H_2 are not isomorphic to each other, because their principal graphs are disjoint unions of n_{H_i}-copies of G_i. Moreover, this property shows that the eigenvalues of principal graphs are n_{H_i}-copies of those of the eigenvalues of G_i. Thus our product operations provide many isospectral pairs of Kähler graphs.

[1] Kähler graphs of Cartesian product type

By Theorems 4.4 and 4.5, we have the following.

Proposition 4.5. *Let G_1, G_2 and H_1, H_2 be two pairs of transitionary isospectral ordinary graphs. Then their Kähler graphs $G_1\widehat{\Box}H_1,\ G_2\widehat{\Box}H_2$ of Cartesian product type are $(1,1)$-probabilistically transitionary isospectral.*

Proposition 4.6. *Let G_1, G_2 be a pair of isospectral regular ordinary graphs satisfying $d_{G_1} = d_{G_2}$, and H_1, H_2 be a pair of transitionary isospectral ordinary graphs. Then their Kähler graphs $G_1\widehat{\Box}H_1,\ G_2\widehat{\Box}H_2$ of Cartesian product type are $(1,1)$-isospectral.*

Example 4.36. Let G_1, G_2 be the pair of isospectral regular graphs of $n_{G_1} = n_{G_2} = 10$ given in Example 4.9 (Figs. 18, 19). Let H be a 3-circuit. Then $G_1\widehat{\Box}H,\ G_2\widehat{\Box}H$ are $(1,1)$-isospectral. The eigenvalues of their principal graphs and those of $(1,1)$-combinatorial Laplacians are

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{(G_i\widehat{\Box}H)^{(p)}}}) = \left\{\begin{array}{l} 0,0,0,3,3,3,5,5,5,5,5,5,5,5,5,5,5,5, \\ 4-\sqrt{5},4-\sqrt{5},4-\sqrt{5},4+\sqrt{5},4+\sqrt{5},4+\sqrt{5}, \\ (9-\sqrt{17})/2,\ (9-\sqrt{17})/2,\ (9-\sqrt{17})/2, \\ (9+\sqrt{17})/2,\ (9+\sqrt{17})/2,\ (9+\sqrt{17})/2 \end{array}\right\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_i \widehat{\square} H_{(1,1)}}}) = \left\{ \begin{array}{l} 0, 6, 6, 3,\ \frac{9}{2},\ \frac{9}{2},\ 5, 5, 5, 5,\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2}, \\ 4-\sqrt{5},\ \frac{1}{2}(8+\sqrt{5}),\ \frac{1}{2}(8+\sqrt{5}),\ 4+\sqrt{5},\ \frac{1}{2}(8-\sqrt{5}), \\ \frac{1}{2}(8-\sqrt{5}),\ \frac{1}{2}(9-\sqrt{17}),\ \frac{1}{4}(15+\sqrt{17}),\ \frac{1}{4}(15+\sqrt{17}), \\ \frac{1}{2}(9+\sqrt{17}),\ \frac{1}{4}(15-\sqrt{17}),\ \frac{1}{4}(15-\sqrt{17}) \end{array} \right\}.$$

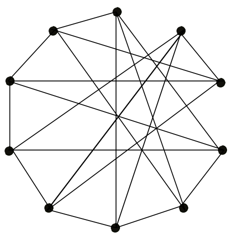

FIG. 18

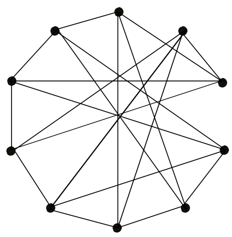

FIG. 19

[2] Kähler graphs of strong product type

By Theorem 4.6 and Proposition 4.3

PROPOSITION 4.7. *Let G_1, G_2 be a pair of isospectral regular ordinary graphs satisfying $d_{G_1} = d_{G_2}$, and H_1, H_2 be a pair of transitionary isospectral ordinary graphs. Then their Kähler graphs $G_1 \widehat{\boxtimes} H_1,\ G_2 \widehat{\boxtimes} H_2$ of strong product type are* $(1,1)$*-isospectral.*

EXAMPLE 4.37. Let G_1, G_2 be the pair of isospectral regular graphs of $n_{G_1} = n_{G_2} = 10$ given in Example 4.9 (Figs. 18, 19). Let H be a complete graph of $n_H = 2$ (Fig. 20). Then $G_1 \widehat{\boxtimes} H,\ G_2 \widehat{\boxtimes} H$ are $(1,1)$-isospectral. The eigenvalues of $\mathcal{P}_H$ are $\{0, 1\}$. The eigenvalues of their principal graphs and those of $(1,1)$-combinatorial Laplacians are

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{(G_i \widehat{\boxtimes} H)^{(p)}}}) = \left\{ \begin{array}{l} 0, 0, 3, 3, 5, 5, 5, 5, 5, 5, 5, 5, 4-\sqrt{5},\ 4-\sqrt{5},\ 4+\sqrt{5}, \\ 4+\sqrt{5},\ (9-\sqrt{17})/2,\ (9-\sqrt{17})/2,\ (9+\sqrt{17})/2,\ (9+\sqrt{17})/2 \end{array} \right\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_i \widehat{\boxtimes} H_{(1,1)}}}) = \left\{ \begin{array}{l} 0, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4,\ \frac{66}{5},\ \frac{68}{5},\ \frac{1}{5}(15-\sqrt{5}),\ \frac{1}{5}(15+7\sqrt{5}) \\ \frac{33}{10},\ \frac{1}{10}(23-\sqrt{17}),\ \frac{1}{10}(5-3\sqrt{17}),\ \frac{1}{10}(13-3\sqrt{17}) \end{array} \right\}.$$

FIG. 20

As a pair of Kähler graphs of product type whose principal graphs are unions of original graphs, this pair consists of graphs having least cardinality of the set of vertices. We should note that when H is a complete graph of $n_H = 2$ then $G\widehat{\Box}H$ does not satisfies the condition of Kähler graphs because its auxiliary degree is 1.

[3] Kähler graphs of semi-tensor product type

By Theorems 4.7 and 4.8, we have the following.

PROPOSITION 4.8. *Let G_1, G_2 and H_1, H_2 be two pairs of transitionary isospectral ordinary graphs. Then their Kähler graphs $G_1\widehat{\otimes}H_1,\ G_2\widehat{\otimes}H_2$ of semi-tensor product type are $(1,1)$-probabilistically transitionary isospectral.*

PROPOSITION 4.9. *Let G_1, G_2 be a pair of isospectral regular ordinary graphs and H_1, H_2 be a pair of transitionary isospectral ordinary graphs. Then their Kähler graphs $G_1\widehat{\otimes}H_1,\ G_2\widehat{\otimes}H_2$ of semi-tensor product type are $(1,1)$-isospectral.*

EXAMPLE 4.38. We take the same G_1, G_2 and H as in Example 4.37. Then $G_1\widehat{\otimes}H$ and $G_2\widehat{\otimes}H$ are $(1,1)$-isospectral. Since their principal graphs are the same as of graphs in Example 4.37, we here give the eigenvalues of $(1,1)$-combinatorial Laplacians

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_i\widehat{\otimes}H_{(1,1)}}}) = \left\{ \begin{array}{l} 0, 1, \frac{15}{4}, \frac{15}{4}, \frac{15}{4}, \frac{15}{4}, \frac{15}{4}, \frac{15}{4}, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, \\ \quad \frac{11}{4},\ \frac{1}{8}(23-\sqrt{17}),\ \frac{1}{8}(23+\sqrt{17}) \end{array} \right\}.$$

[4] Kähler graphs of lexicographical product type

By Theorem 4.9 we have the following.

PROPOSITION 4.10. *Let G_1, G_2 and H_1, H_2 be pairs of transitionary isospectral ordinary graphs. Then their Kähler graphs $G_1 \rhd H_1,\ G_2 \rhd H_2$ of lexicographical product type are $(1,1)$-probabilistically transitionary isospectral.*

Remark 4.2. Let G_1, G_2 be a pair of ordinary graphs satisfying $n_{G_1} = n_{G_2}$,and H_1, H_2 be a pair of transitionary isospectral ordinary graphs. Then their Kähler graphs $G_1 \rhd H_1,\ G_2 \rhd H_2$ of lexicographical product type have the same eigenvalues of $(1,1)$-probabilistically transition Laplacians. But if their principal graphs are not combinatorial isospectral (resp. transitional isospectral), they are not $(1,1)$-combinatorial isospectral (resp. $(1,1)$-probabilistic transitional isospectral).

By Proposition 4.4 we have the following.

Proposition 4.11. *Let G_1, G_2 be a pair of isospectral ordinary regular graphs satisfying $d_{G_1} = d_{G_2}$, and H_1, H_2 be a pair of transitionary isospectral ordinary graphs. Then their Kähler graphs $G_1 \rhd H_1,\ G_2 \rhd H_2$ of lexicographical product type are* $(1,1)$-*isospectral.*

Remark 4.3. Let G_1, G_2 be a pair of ordinary regular graphs satisfying $n_{G_1} = n_{G_2},\ d_{G_1} = d_{G_2}$, and H_1, H_2 be a pair of transitionary isospectral ordinary graphs. Then their Kähler graphs $G_1 \rhd H_1,\ G_2 \rhd H_2$ of lexicographical product type have the same eigenvalues of $(1,1)$-adjacency Laplacians and of $(1,1)$-probabilistic transition Laplacians.

Example 4.39. We take the same G_1, G_2 and H as in Example 4.37. Then $G_1 \rhd H$ and $G_2 \rhd H$ are $(1,1)$-isospectral. Since their principal graphs are the same as of graphs in Example 4.37, we here give the eigenvalues of $(1,1)$-combinatorial Laplacians:

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_i \rhd H_{(1,1)}}}) = \big\{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,8\big\}.$$

Example 4.40. Let G_1 be a 4-circuit, G_2 be a non-regular graph of $n_{G_2} = 4$ given in Fig. 21, and H be a 3-circuit. Then $G_1 \rhd H$ and $G_2 \rhd H$ have the same eigenvalues of $(1,1)$-probabilistic transition Laplacians:

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_i \rhd H_{(1,1)}}}) = \Big\{0,1,1,1,1,1,1,1,1,1,\frac{3}{2},\frac{3}{2}\Big\}.$$

We note that G_1 and G_2 are not combinatorially and transitionally isospectral.

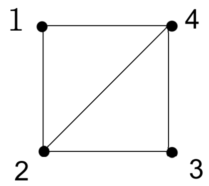

FIG. 21

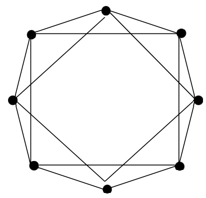

FIG. 22

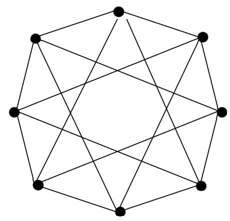

FIG. 23

EXAMPLE 4.41. Let G_1 and G_2 be a regular graph of $n_{G_i} = 8$ given in Figs. 22 and 23, respectively, and H be a 3-circuit. Then $G_1 \triangleright H$ and $G_2 \triangleright H$ have the same eigenvalues of $(1,1)$-combinatorial Laplacians:

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_i \triangleright H_{(1,1)}}}) = \{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,12,12\}.$$

We note that G_1 and G_2 are not isospectral.

[5] Kähler graphs of product type added complement-filling operations

PROPOSITION 4.12. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular ordinary graphs satisfying $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. If G_1, G_2 are connected, then their Kähler graphs $G_1 \widehat{\Box}^K H_1$, $G_2 \widehat{\Box}^K H_2$ of complement-filling Cartesian product type are $(1,1)$-isospectral.*

PROPOSITION 4.13. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular ordinary graphs satisfying $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. If G_1, G_2 are connected, then their Kähler graphs $G_1 \widehat{\boxtimes}^K H_1$, $G_2 \widehat{\boxtimes}^K H_2$ of complement-filling strong product type are $(1,1)$-isospectral.*

PROPOSITION 4.14. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular ordinary graphs satisfying $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. If G_1, G_2 are connected, then their Kähler graphs $G_1 \widehat{\otimes}^K H_1$, $G_2 \widehat{\otimes}^K H_2$ of complement-filling semi-tensor product type are $(1,1)$-isospectral.*

PROPOSITION 4.15. *Let G_1, G_2 and H_1, H_2 be pairs of isospectral ordinary regular graphs satisfying $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. Then their Kähler graphs $G_1 \rhd^K H_1,\ G_2 \rhd^K H_2$ of complement-filling lexicographical product type are $(1,1)$-isospectral.*

REMARK 4.4. Let G_1, G_2 be a pair of ordinary regular graphs satisfying $n_{G_1} = n_{G_2}$, $d_{G_1} = d_{G_2}$, and H_1, H_2 be a pair of isospectral regular ordinary graphs satisfying $d_{H_1} = d_{H_2}$. If $G_1$and G_2 are connected, then their Kähler graphs $G_1 \rhd^K H_1,\ G_2 \rhd^K H_2$ of complement-filling lexicographical product type have the same eigenvalues of $(1,1)$-adjacency Laplacians and of $(1,1)$-probabilistic transition Laplacians.

7.2. Isospectral joined Kähler graphs. We note that $(1,1)$-probabilistic transition operators of joined Kähler graphs do not inherit the structures of original graphs, and that $(1,1)$-adjacency operators only inherit property of degrees on original graphs. Therefore eigenvalues of $(1,1)$-Laplacians do not show the structure of original graphs. The eigenvalues of the principal graph of a joined Kähler graph $G\widehat{+}H$ of graphs G, H are given as

$$\mathrm{Spec}(\varDelta_{\mathcal{A}_{(G\widehat{+}H)^{(p)}}}) = \mathrm{Spec}(\varDelta_{\mathcal{A}_G}) \cup \mathrm{Spec}(\varDelta_{\mathcal{A}_H}),$$
$$\mathrm{Spec}(\varDelta_{\mathcal{P}_{(G\widehat{+}H)^{(p)}}}) = \mathrm{Spec}(\varDelta_{\mathcal{P}_G}) \cup \mathrm{Spec}(\varDelta_{\mathcal{P}_H}).$$

Therefore, if we take two pairs of combinatorially (resp. transitionally) isospectral graphs, then their principal graph of their joined Kähler graphs are trivially combinatorially (resp. transitionally) isospectral. By Theorem 4.14 we have the following.

PROPOSITION 4.16. *Let G_1, G_2 and H_1, H_2 be pairs of isospectral ordinary graphs. We suppose that one of these pairs are not isomorphic, and suppose that G_1 is not isomorphic to H_2 and G_2 is not isomorphic to H_1. Then their joined Kähler graphs $G_1\widehat{+}H_1,\ G_2\widehat{+}H_2$ are $(1,1)$-probabilistic transitionary isospectral.*

REMARK 4.5. Let G_1, G_2 and H_1, H_2 be two pairs of ordinary graphs satisfying $n_{G_1} = n_{G_2}$ and $n_{H_1} = n_{H_2}$. Then their joined Kähler graphs $G_1\widehat{+}H_1,\ G_2\widehat{+}H_2$ have the same eigenvalues of $(1,1)$-probabilistic transition Laplacians.

PROPOSITION 4.17. *Let G_1, G_2 and H_1, H_2 be pairs of isospectral ordinary regular graphs satisfying $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. We suppose that one of these pairs are not isomorphic, and suppose that G_1 is not isomorphic to H_2 and G_2 is not isomorphic to H_1. Then their joined Kähler graphs $G_1 \widehat{+} H_1$, $G_2 \widehat{+} H_2$ are $(1,1)$-isospectral.*

REMARK 4.6. Let G_1, G_2 and H_1, H_2 be two pairs of ordinary regular graphs satisfying $n_{G_1} = n_{G_2}$, $n_{H_1} = n_{H_2}$ and $d_{G_1} = d_{G_2}$, $d_{H_1} = d_{H_2}$. Then their joined Kähler graphs $G_1 \widehat{+} H_1$, $G_2 \widehat{+} H_2$ have the same eigenvalues of $(1,1)$-adjacency Laplacians and of $(1,1)$-probabilistic transition Laplacians.

EXAMPLE 4.42. Let G_1, G_2 be the pair of isospectral regular graphs of $n_{G_1} = n_{G_2} = 10$ given in Example 4.9. Let H be a 3-circuit. Then $G_1 \widehat{+} H$, $G_2 \widehat{+} H$ are $(1,1)$-isospectral. The eigenvalues of their principal graphs and those of $(1,1)$-combinatorial Laplacians are

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{(G_i \widehat{+} H)^{(p)}}}) = \{0,0,3,3,3,5,5,5,5,4-\sqrt{5},4+\sqrt{5},(9-\sqrt{17})/2,(9+\sqrt{17})/2\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_i \widehat{+} H_{(1,1)}}}) = \{0,2,2,4,4,4,4,4,4,4,4,4,6\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{Q}_{G_i \widehat{+} H_{(1,1)}}}) = \{0,1,1,1,1,1,1,1,1,1,1,1,2\}.$$

EXAMPLE 4.43. Let G_1 be a 4-circuit, G_2 be the graph in Fig. 24, and H be a 3-circuit. Their transition matrices are given as

$$P_{G_1} = \begin{pmatrix} 0 & \frac12 & 0 & \frac12 \\ \frac12 & 0 & \frac12 & 0 \\ 0 & \frac12 & 0 & \frac12 \\ \frac12 & 0 & \frac12 & 0 \end{pmatrix}, \quad P_{G_2} = \begin{pmatrix} 0 & \frac12 & 0 & \frac12 \\ \frac13 & 0 & \frac13 & \frac13 \\ 0 & \frac12 & 0 & \frac12 \\ \frac13 & \frac13 & \frac13 & 0 \end{pmatrix}, \quad P_H = \begin{pmatrix} 0 & \frac12 & \frac12 \\ \frac12 & 0 & \frac12 \\ \frac12 & \frac12 & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{P}_{G_1}}$, $\Delta_{\mathcal{P}_{G_2}}$ and $\Delta_{\mathcal{P}_H}$ are $\{0,1,1,2\}$, $\{0,1,\frac43,\frac53\}$, and $\{0,\frac32,\frac32\}$. Their $(1,1)$-probabilistic transition matrices are the same and are given as

$$\mathcal{Q}_{G_i \widehat{+} H_{(1,1)}} = \begin{pmatrix} P_{G_i} & O \\ O & P_H \end{pmatrix} \begin{pmatrix} O & \frac13 M_{12} \\ \frac14 M_{21} & O \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac13 & \frac13 & \frac13 \\ 0 & 0 & 0 & 0 & \frac13 & \frac13 & \frac13 \\ 0 & 0 & 0 & 0 & \frac13 & \frac13 & \frac13 \\ 0 & 0 & 0 & 0 & \frac13 & \frac13 & \frac13 \\ \frac14 & \frac14 & \frac14 & \frac14 & 0 & 0 & 0 \\ \frac14 & \frac14 & \frac14 & \frac14 & 0 & 0 & 0 \\ \frac14 & \frac14 & \frac14 & \frac14 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of (1, 1)-probabilistic transition Laplacians of their joined Kähler graphs $G_1\widehat{+}H$, $G_2\widehat{+}H$ are $\{0,1,1,1,1,1,2\}$. We should note that they are not (1, 1)-probabilistic transitionary isospectral because their principal graphs are not transitionary isospectral.

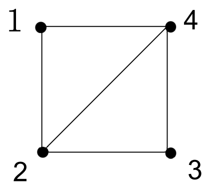

FIG. 24

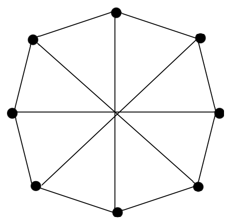

FIG. 25

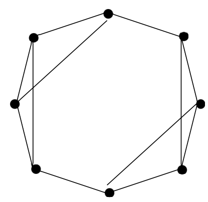

FIG. 26

EXAMPLE 4.44. Let H be a 3-circuit, and G_1, G_2 be graphs of $n_{G_1} = n_{G_2} = 8$ and $d_{G_1} = d_{G_2} = 3$ given in Figs. 25 and 26. Their adjacency operators are

$$A_{G_1}=\begin{pmatrix}0&1&0&0&1&0&0&1\\1&0&1&0&0&1&0&0\\0&1&0&1&0&0&1&0\\0&0&1&0&1&0&0&1\\1&0&0&1&0&1&0&0\\0&1&0&0&1&0&1&0\\0&0&1&0&0&1&0&1\\1&0&0&1&0&0&1&0\end{pmatrix},\ A_{G_2}=\begin{pmatrix}0&1&1&0&0&0&0&1\\1&0&1&1&0&0&0&0\\1&1&0&1&0&0&0&0\\0&1&1&0&1&0&0&0\\0&0&0&1&0&1&1&0\\0&0&0&0&1&0&1&1\\0&0&0&0&1&1&0&1\\1&0&0&0&0&1&1&0\end{pmatrix},\ A_H=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}.$$

The eigenvalues of $\Delta_{\mathcal{A}_{G_1}}, \Delta_{\mathcal{A}_{G_2}}$ and $\Delta_{\mathcal{A}_H}$ are

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{G_1}}\big) = \big\{0,2,2,\ 4-\sqrt{2},\ 4-\sqrt{2},\ 4+\sqrt{2},\ 4+\sqrt{2}\big\},$$
$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{G_2}}\big) = \big\{0,\ 3-\sqrt{5},\ 2,4,4,4,4,\ 3+\sqrt{5}\big\} \qquad ,$$

and $\mathrm{Spec}\big(\Delta_{\mathcal{A}_H}\big) = \big\{0,3,3\big\}$. We take joined Kähler graphs $G_1\widehat{+}H$ and $G_2\widehat{+}H$. Their (1, 1)-adjacency matrices are the same and are given as

$$\mathcal{A}_{G_i\widehat{+}H_{(1,1)}} = \begin{pmatrix} A_{G_i} & O \\ O & A_H \end{pmatrix}\begin{pmatrix} O & \frac{1}{3}M_{12} \\ \frac{1}{8}M_{21} & O \end{pmatrix} = \begin{pmatrix} 0&0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1 \\ \frac14&\frac14&\frac14&\frac14&\frac14&\frac14&\frac14&\frac14&0&0&0 \\ \frac14&\frac14&\frac14&\frac14&\frac14&\frac14&\frac14&\frac14&0&0&0 \\ \frac14&\frac14&\frac14&\frac14&\frac14&\frac14&\frac14&\frac14&0&0&0 \end{pmatrix}.$$

Their $(1,1)$-probabilistic transition matrices are the same and are given as The eigenvalues of $(1,1)$-combinatorial Laplacians of their joined Kähler graphs $G_1\widehat{+}H,\ G_2\widehat{+}H$ are $\{0,2,2,3,3,3,3,3,3,3,5\}$. We should note that these are not $(1,1)$-combinatorially isospectral.

7.3. Isospectrality of Kähler graphs of commutative product type. At the end of this section we study isospectral condition on Kähler graphs of commutative product type.

PROPOSITION 4.18. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular ordinary graphs satisfying $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. Then their Kähler graphs $G_1 \boxplus H_1,\ G_2 \boxplus H_2$ of Cartesian-tensor product type are $(1,1)$-isospectral.*

PROOF. For eigenvalues μ of $\Delta_{\mathcal{P}_G}$ and ν of $\Delta_{\mathcal{P}_H}$, we take functions f, g satisfying $\Delta_{\mathcal{P}_G} f = \mu f$ and $\Delta_{\mathcal{P}_H} g = \nu g$. Then we have $\mathcal{A}_G f = d_G(1-\mu)f$ and $\mathcal{A}_H g = d_H(1-\nu)g$. As the adjacency matrix of the principal graph is given as $A^{(p)}_{G\boxplus H} = \big(a^G_{ij}\delta_{\alpha\beta} + \delta_{ij}a^H_{\alpha\beta}\big)$ by use of the adjacency matrices $A_G = (a^G_{ij}$ and $A_H = (a^H_{\alpha\beta})$, we find

$$\mathcal{A}_{G\boxplus H_{(1,1)}}\varphi_{f,g} = \big\{d_G(1-\mu) + d_H(1-\nu)\big\}\varphi_{f,g},$$

where $\varphi_{f,g}$ is a function defined by $\varphi_{f,g}(v,w) = f(v)g(w)$. Hence the eigenvalues of the combinatorial Laplacian of the principal graph are

$$\operatorname{Spec}\big(\Delta_{\mathcal{A}_{G\boxplus H^{(p)}}}\big) = \big\{d_G(1-\mu) + d_H(1-\nu) \bigm| \mu \in \operatorname{Spec}\big(\Delta_{\mathcal{P}_G}\big),\ \nu \in \operatorname{Spec}\big(\Delta_{\mathcal{P}_H}\big)\big\}.$$

Therefore we get the conclusion directly from Theorem 4.16. □

PROPOSITION 4.19. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular ordinary graphs satisfying $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. If these four graphs are connected, then their Kähler graphs $G_1 \boxdot H_1$, $G_2 \boxdot H_2$ of Cartesian-complement product type are $(1,1)$-isospectral.*

PROOF. Since the principal graph of $G \boxdot H$ is the same as that of $G \boxplus H$, we get the conclusion by Theorem 4.17. □

PROPOSITION 4.20. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular ordinary graphs satisfying $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. If these four graphs are connected, then their Kähler graphs $G_1 \Diamond H_1$, $G_2 \Diamond H_2$ of Cartesian-lexicographic product type are $(1,1)$-isospectral.*

PROOF. Since the principal graph of $G \Diamond H$ is the same as that of $G \boxplus H$, we get the conclusion by Theorem 4.18. □

PROPOSITION 4.21. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular ordinary graphs satisfying $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. If these four graphs are connected, then their Kähler graphs $G_1 \circledast H_1$, $G_2 \circledast H_2$ of strong-complement product type are $(1,1)$-isospectral.*

PROOF. For eigenvalues μ of $\Delta_{\mathcal{P}_G}$ and ν of $\Delta_{\mathcal{P}_H}$, we take functions f, g satisfying $\Delta_{\mathcal{P}_G} f = \mu f$ and $\Delta_{\mathcal{P}_H} g = \nu g$. Then we have $\mathcal{A}_G f = d_G(1-\mu)f$ and $\mathcal{A}_H g = d_H(1-\nu)g$. As the adjacency matrix of the principal graph is given as $A^{(p)}_{G\circledast H} = \big(a^G_{ij}\delta_{\alpha\beta} + \delta_{ij}a^H_{\alpha\beta} + a^G_{ij}a^H_{\alpha\beta}\big)$ by use of the adjacency matrices $A_G = (a^G_{ij}$ and $A_H = (a^H_{\alpha\beta})$, we find

$$\mathcal{A}_{G\circledast H_{(1,1)}}\varphi_{f,g} = \big\{d_G(1-\mu) + d_H(1-\nu) + d_G d_H(1-\mu)(1-\nu)\big\}\varphi_{f,g},$$

where $\varphi_{f,g}$ is a function defined by $\varphi_{f,g}(v,w) = f(v)g(w)$. Hence the eigenvalues of the combinatorial Laplacian of the principal graph are

$$\begin{aligned}&\mathrm{Spec}\big(\Delta_{\mathcal{A}_{G\circledast H^{(p)}}}\big)\\ &\quad = \big\{d_G(1-\mu) + d_H(1-\nu) + d_G d_H(1-\mu)(1-\nu) \bigm| \mu \in \mathrm{Spec}\big(\Delta_{\mathcal{P}_G}\big),\ \nu \in \mathrm{Spec}\big(\Delta_{\mathcal{P}_H}\big)\big\}.\end{aligned}$$

Therefore we get the conclusion directly from Theorem 4.19. □

PROPOSITION 4.22. *Let* G_1, G_2 *and* H_1, H_2 *be two pairs of isospectral regular ordinary graphs satisfying* $d_{G_1} = d_{G_2}$ *and* $d_{H_1} = d_{H_2}$. *If these four graphs are connected, then their Kähler graphs* $G_1 \spadesuit H_1,\ G_2 \spadesuit H_2$ *of complement-tensor product type are* $(1,1)$-*isospectral.*

PROOF. For eigenvalues μ of $\Delta_{\mathcal{P}_G}$ and ν of $\Delta_{\mathcal{P}_H}$, we take functions f, g satisfying $\Delta_{\mathcal{P}_G} f = \mu f$ and $\Delta_{\mathcal{P}_H} g = \nu g$. Then we have $\mathcal{A}_G f = d_G(1-\mu)f$ and $\mathcal{A}_H g = d_H(1-\nu)g$. As the adjacency matrix of the principal graph is given as $A^{(p)}_{G\spadesuit H} = \big(a^G_{ij}(1-a^H_{\alpha\beta}) + (1-a^G_{ij})a^H_{\alpha\beta}\big)$ by use of the adjacency matrices $A_G = (a^G_{ij}$ and $A_H = (a^H_{\alpha\beta})$, we find

$$\mathcal{A}_{G\spadesuit H_{(1,1)}} \varphi_{f,g} = \big\{ d_G(1-\mu)(n_H - d_G + d_G\nu) + (n_G - d_G + d_G\mu)d_H(1-\nu)\big\}\varphi_{f,g},$$

where $\varphi_{f,g}$ is a function defined by $\varphi_{f,g}(v,w) = f(v)g(w)$. Hence the eigenvalues of the combinatorial Laplacian of the principal graph are

$$\begin{aligned}&\mathrm{Spec}\big(\Delta_{\mathcal{A}_{G\spadesuit H^{(p)}}}\big)\\ &= \left\{ d_G(1-\mu)(n_H - d_G + d_G\nu) + d_H(n_G - d_G + d_G\mu)(1-\nu) \;\middle|\; \begin{matrix}\mu \in \mathrm{Spec}\big(\Delta_{\mathcal{P}_G}\big),\\ \nu \in \mathrm{Spec}\big(\Delta_{\mathcal{P}_H}\big)\end{matrix}\right\}.\end{aligned}$$

Therefore we get the conclusion directly from Theorem 4.20. □

PROPOSITION 4.23. *Let* G_1, G_2 *and* H_1, H_2 *be two pairs of isospectral regular ordinary graphs satisfying* $d_{G_1} = d_{G_2}$ *and* $d_{H_1} = d_{H_2}$. *If these four graphs are connected, then their Kähler graphs* $G_1 \clubsuit H_1,\ G_2 \clubsuit H_2$ *of tensor-complement product type are* $(1,1)$-*isospectral.*

PROOF. For eigenvalues μ of $\Delta_{\mathcal{P}_G}$ and ν of $\Delta_{\mathcal{P}_H}$, we take functions f, g satisfying $\Delta_{\mathcal{P}_G} f = \mu f$ and $\Delta_{\mathcal{P}_H} g = \nu g$. Then we have $\mathcal{A}_G f = d_G(1-\mu)f$ and $\mathcal{A}_H g = d_H(1-\nu)g$. As the adjacency matrix of the principal graph is given as $A^{(p)}_{G\clubsuit H} = \big(a^G_{ij}a^H_{\alpha\beta}\big)$ by use of the adjacency matrices $A_G = (a^G_{ij}$ and $A_H = (a^H_{\alpha\beta})$, we find

$$\mathcal{A}_{G\clubsuit H_{(1,1)}} \varphi_{f,g} = \big\{ d_G d_H(1-\mu)(1-\nu)\big\}\varphi_{f,g},$$

where $\varphi_{f,g}$ is a function defined by $\varphi_{f,g}(v,w) = f(v)g(w)$. Hence the eigenvalues of the combinatorial Laplacian of the principal graph are

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{G\clubsuit H(p)}}\big) = \big\{ d_G d_H (1-\mu)(1-\nu) \bigm| \mu \in \mathrm{Spec}\big(\Delta_{\mathcal{P}_G}\big), \nu \in \mathrm{Spec}\big(\Delta_{\mathcal{P}_H}\big)\big\}.$$

Therefore we get the conclusion directly from Theorem 4.21. □

CHAPTER 5

Eigenvalues of (p, q)-Laplacians for Kähler graphs

In this chapter we study eigenvalues of (p, q)-Laplacians mainly for finite regular Kähler graphs.

1. Polynomial representations of eigenvalues of (p, q)-Laplacians

Given a positive integer d, we define a sequence $\{F_n(t; d)\}_{n=0}^{\infty}$ of monic polynomials by the relations

$$\begin{cases} F_{n+1}(t; d) = tF_n(t; d) - (d-1)F_{n-1}(t; d) \quad (n \geq 2), \\ F_0(t; d) = 1, \ F_1(t; d) = t, \ F_2(t; d) = t^2 - d. \end{cases}$$

For example, we have

$$F_3(t; d) = t^3 - (2d-1)t, \quad F_4(t; d) = t^4 - (3d-2)t^2 + d(d-1),$$

$$F_5(t; d) = t^5 - (4d-3)t^3 + (d-1)(3d-1)t,$$

$$F_6(t; d) = t^6 - (5d-4)t^4 + 3(d-1)(2d-1)t^2 - d(d-1),$$

$$F_7(t; d) = t^7 - (6d-5)t^5 + 2(d-1)(5d-3)t^3 - (4d-1)(d-1)^2 t.$$

Lemma 5.1. *The polynomials $F_n(t; d)$ $(n \geq 1)$ satisfy the following properties*:

(1) $F_n(d; d) = d(d-1)^{n-1}$;

(2) $F_{2k-1}(0; d) = 0$ *and* $F_{2k}(0; d) = (-1)^k d(d-1)^{k-1}$;

(3) $F_{2k-1}(t; d)$ *contains only terms of odd degrees, and* $F_{2k}(t; d)$ *contains only terms of even degrees.*

Proof. We show our assertion by induction.

(1) By definition we have $F_1(d; d) = d, \ F_2(d; d) = d^2 - d = d(d-1)$. If we suppose $F_n(d; d) = d(d-1)^{n-1}, \ F_{n+1}(d; d) = d(d-1)^n$, then we have

$$F_{n+2}(d; d) = dF_{n+1}(d; d) - (d-1)F_n(d; d) = d^2(d-1)^n - (d-1)d(d-1)^{n-1} = d(d-1)^{n+1}$$

hence get the first assertion.

(2) By definition we have $F_1(0;d)=0,\ F_2(0;d)=-d$. If we suppose $F_{2k-1}(0;d)=0$ and $F_{2k}(0;d)=(-1)^k d(d-1)^{k-1}$, we have

$$F_{2k+1}(0;d)=0F_{2k}(0;d)-(d-1)F_{2k-1}(0;d)=0,$$

$$F_{2k+2}(0;d)=0F_{2k+1}(0;d)-(d-1)F_{2k}(0;d)=(-1)^{k+1}d(d-1)^k.$$

(3) It is clear that the third assertion holds for $k=1,2$. If we suppose the assertion holds for $2k-1$ and $2k$, we have

$$F_{2k+1}(t;d)=t\ (\text{terms of odd degrees})-(d-1)(\text{terms of even degrees})$$

contains only terms of even degrees, and

$$F_{2k+2}(t;d)=t\ (\text{terms of even degrees})-(d-1)(\text{terms of odd degrees})$$

contains only terms of odd degrees. □

The n-step adjacency and transition operators of regular ordinary graphs are expressed by use of these polynomials. Given an ordinary graph G we denote by $G_{[n]}$ its n-step derived graph (see §3.3).

Proposition 5.1. *Let $G=(V,E)$ be a regular ordinary graph of degree d_G.*

(1) *The adjacency operator $\mathcal{A}_{G_{[n]}}$ by n-step paths on G without backtracking is given as $F_n(\mathcal{A}_G;d_G)$.*
(2) *The transition operator $\mathcal{Q}_{G_{[n]}}$ by n-step paths on G without backtracking is given as $\dfrac{1}{d_G(d_G-1)}F_n(\mathcal{A}_G;d_G)$.*

Here, for a positive k the operator $\mathcal{A}_G^k$ means the kth-composition $\overbrace{\mathcal{A}_G\circ\cdots\circ\mathcal{A}_G}^{k}$ and $\mathcal{A}_G^0=\mathcal{I}$. Thus for a polynomial $F(t)=a_nt^n+a_1t^{n-1}+\cdots+a_{n-1}t+a_n$ the operator $F(\mathcal{A})$ means $a_n\mathcal{A}^n+\cdots+a_{n-1}\mathcal{A}+a_n\mathcal{I}$.

Proof. (1) Since $F_1(\mathcal{A}_G;d_G)=\mathcal{A}_G$ and the 1-step adjacency operator is the ordinary adjacency operator, we have $\mathcal{A}_{G_{[1]}}=F_1(\mathcal{A}_G;d_G)$. We study the case $n=2$. A 2-step path on G with backtracking is of the form (v_0,v_1,v_0) with $v_0\sim v_1$. As G

is regular of degree d_G, we have d_G 2-step paths with backtracking emanating from a given vertex. Since $\mathcal{A}_G \circ \mathcal{A}_G$ shows adjacency by 2-step paths with or without backtracking, we see the adjacency operator by 2-step paths without backtracking is express by the operator $\mathcal{A}_G^2 - d_G\mathcal{I}$.

We now study general case by induction. We suppose the assertion holds in the case $n-1$ and n. We consider the case $n+1$. We take an n-step path $(v_0, v_1, \cdots, v_n)$ without backtracking. When we consider a sequence $(v_0, v_1, \cdots, v_n, v)$ of vertices, we see that this is an $(n+1)$-step path if and only if v adjacent to v_n (i.e. $v_n \sim v$). Thus, we find that the adjacency by $(n+1)$-step paths whose first n-step subpaths do not contain backtracking is expressed by $F_n(\mathcal{A}_G; d_G) \circ \mathcal{A}_G$. When this sequence $(v_0, v_1, \cdots, v_n, v)$ is a path, it does not have backtracking if and only if $v_{n-1} \neq v$. Therefore, we find that there is one $(n+1)$-step path containing backtracking whose first n step coincide with the given n-step path $(v_0, \dots, v_n)$. It is $(v_0, \dots, v_n, v_{n-1})$. We hence get a bijective correspondence of the set of $(n+1)$-step paths whose first n-step subpaths coincide with $(v_0, \dots, v_n)$ and that contain backtracking to $\{(v_0, \dots, v_{n-1})\}$. Since $v_n \neq v_{n-2}$, we see that given an $(n-1)$-step path $(v_0, \dots, v_{n-1})$ we can construct $d_G(v_{n-1}) - 1$ n-step paths $(v_0, \dots, v_n)$ without backtracking. Since G is regular, we hence find that $(n+1)$-step adjacency is expressed by $\mathcal{A}_{G_{[n+1]}} = A_{G_{[n]}}A_G - (d_G - 1)A_{G_{[n-1]}}$. As A_G and $F_n(A_G; d_G)$ are commutative, we get the first assertion.

Since G is regular, we see $\mathcal{Q}_{G_{[n]}} = \dfrac{1}{d_{G(p)}^{(p)}(d_{G(p)}^{(p)} - 1)}\mathcal{A}_{G_{[n]}}$. Hence we get the second assertion. □

Example 5.1. We take a 4-circuit G. Its adjacency matrix is given by

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

We here study adjacency matrices by $2,3$ and 4-step paths on G, and A_G^2, A_G^3, A_G^4. They are expressed as

$$A_{G_{[2]}} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \qquad A_G^2 = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix},$$

$$A_{G_{[3]}} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = A_G, \quad A_G^3 = \begin{pmatrix} 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{pmatrix} = 4A_G,$$

$$A_{G_{[4]}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 2I, \qquad A_G^4 = 4A_G^2.$$

Thus we see $A_{G_{[2]}} = A_G^2 - 2I$, $A_{G_{[3]}} = A_G^3 - 2A_G - A_G$ and $A_{G_{[4]}} = A_G^4 - 3A_G^2 - A_{G_{[2]}}$.

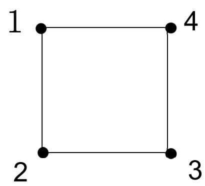

FIG. 1

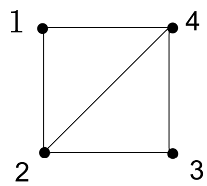

FIG. 2

When the graph is not regular, the situation is not simple.

EXAMPLE 5.2. We take a graph G in Fig. 2. Its adjacency matrix is given by

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

The adjacency matrices by 2 and 3-step paths on G are given as

$$A_{G_{[2]}} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}, \qquad A_{G_{[3]}} = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 4 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 4 \end{pmatrix}.$$

On the other hand, A_G^2 and A_G^3 are given as

$$A_G^2 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}, \qquad A_G^3 = \begin{pmatrix} 2 & 5 & 2 & 5 \\ 5 & 4 & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{pmatrix}.$$

We now study (p, q)-Laplacians for Kähler graphs. By Lemma 4.3, we know that the (p, q)-adjacency operator and the (p, q)-probabilistic transition operator of a Kähler graph is decomposed to operators concerning its principal graph and its auxiliary graph, hence their properties are important.

THEOREM 5.1. *Let $G = (V, E^{(p)} \cup E^{(a)})$ be a finite regular Kähler graph. Suppose that its adjacency operators $\mathcal{A}_{G^{(p)}}, \mathcal{A}_{G^{(a)}}$ of the principal and the auxiliary graphs are commutative ($\mathcal{A}_{G^{(p)}} \circ \mathcal{A}_{G^{(a)}} = \mathcal{A}_{G^{(a)}} \circ \mathcal{A}_{G^{(p)}}$), We denote the eigenvalues of $\mathcal{A}_{G^{(p)}}$ by λ_i ($i = 1, \ldots, n_G$), and denote the eigenvalues of $\mathcal{A}_{G^{(a)}}$ by η_i ($i = 1, \ldots, n_G$), where we attach the indices so that for each i both λ_i and η_i have the same eigenfunctions. Then eigenvalues of the (p, q)-combinatorial Laplacian $\Delta_{\mathcal{A}_{(p,q)}}$ are*

$$d_{G^{(p)}}(d_{G^{(p)}} - 1)^{p-1} - \frac{F_p(\lambda_i; d_{G^{(p)}})F_q(\eta_i; d_{G^{(a)}})}{d_{G^{(a)}}(d_{G^{(a)}} - 1)^{q-1}} \qquad (i = 1, \ldots, n_G), \tag{1.1}$$

and the eigenvalues of the (p, q)-probabilistic transition Laplacian $\Delta_{\mathcal{Q}_{(p,q)}}$ are

$$1 - \frac{F_p(\lambda_i; d_{G^{(p)}})F_q(\eta_i; d_{G^{(a)}})}{d_{G^{(p)}}(d_{G^{(p)}} - 1)^{p-1} d_{G^{(a)}}(d_{G^{(a)}} - 1)^{q-1}} \qquad (i = 1, \ldots, n_G). \tag{1.2}$$

PROOF. We take an eigenfunction f_i satisfying $\mathcal{A}_{G^{(p)}} f_i = \lambda_i f_i$ and $\mathcal{A}_{G^{(a)}} f_i = \eta_i f_i$. We note that by the condition of simultaneously diagonalizable ($\mathcal{A}_{G^{(p)}} \circ \mathcal{A}_{G^{(a)}} = \mathcal{A}^{(a)} \circ \mathcal{A}_{G^{(p)}}$) of symmetric operators, we have such an eigenfunction (see Note 1.3). Thus for positive integer k we have

$$\mathcal{A}_{G^{(p)}}^k f_i = \mathcal{A}_{G^{(p)}}^{k-1} \lambda_i f_i = \lambda_i \mathcal{A}_{G^{(p)}}^{k-1} f_i = \cdots = \lambda_i^k f_i.$$

Similarly we have $\mathcal{A}_{G^{(a)}}^k f_i = \eta_i^k f_i$. Therefore we find that

$$F(\mathcal{A}_{G^{(p)}}; d_{G^{(p)}}) f_i = F(\lambda_i; d_{G^{(p)}}) f_i \quad \text{and} \quad F(\mathcal{A}_{G^{(q)}}; d_{G^{(q)}}) f_i = F(\eta_i; d_{G^{(a)}}) f_i.$$

Since $G^{(a)}$ is regular, we see $\mathcal{Q}_{(0,q)}$ coincides with q-step transition operator $\mathcal{P}_{G^{(a)}_{[q]}}$ (see Lemma 4.2). We note $\mathcal{P}_{G^{(a)}_{[q]}} = \dfrac{1}{d_{G^{(a)}}(d_{G^{(a)}}-1)^{q-1}}\mathcal{A}_{G^{(a)}_{[q]}}$. Thus we have by Lemma 4.3 and Proposition 5.1 that

$$\begin{aligned}\mathcal{A}_{(p,q)}f_i = \mathcal{A}_{(p,0)}\mathcal{Q}_{(0,q)}f_i &= \frac{1}{d_{G^{(a)}}(d_{G^{(a)}}-1)^{q-1}}F_p(\mathcal{A}_{G^{(p)}};d_{G^{(p)}})F_q(\mathcal{A}_{G^{(a)}};d_{G^{(a)}})f_i \\ &= \frac{F_p(\lambda_i;d_{G^{(p)}})F_q(\eta_i;d_{G^{(a)}})}{d_{G^{(a)}}(d_{G^{(a)}}-1)^{q-1}}f_i.\end{aligned}$$

We hence get the conclusion. □

Proposition 5.2. *If the adjacency operators $\mathcal{A}_{G^{(p)}}, \mathcal{A}_{G^{(a)}}$ of a regular Kähler graph G are commutative, then its (p,q)-adjacency operator $\mathcal{A}_{(p,q)}$ and its (p,q)-probabilistic transition operator $\mathcal{Q}_{(p,q)}$ are symmetric.*

Proof. Since $\mathcal{A}_{G^{(p)}}$ and $\mathcal{A}_{G^{(a)}}$ are symmetric, we see $\mathcal{A}^k_{G^{(p)}}$ and $\mathcal{A}^k_{G^{(a)}}$ are symmetric for an arbitrary nonnegative integer k, hence both $F_p\big(\mathcal{A}_{G^{(p)}};d_{G^{(p)}}\big)$ and $F_q\big(\mathcal{A}_{G^{(a)}};d_{G^{(a)}}\big)$ are symmetric. Moreover, as $\mathcal{A}_{G^{(p)}}\circ\mathcal{A}_{G^{(a)}} = \mathcal{A}_{G^{(a)}}\circ\mathcal{A}_{G^{(p)}}$ we have $\mathcal{A}^k_{G^{(p)}}\circ\mathcal{A}^\ell_{G^{(a)}} = \mathcal{A}^\ell_{G^{(a)}}\circ\mathcal{A}^k_{G^{(p)}}$ for arbitrary nonnegative integers k,ℓ, hence have

$$F_p\big(\mathcal{A}_{G^{(p)}};d_{G^{(p)}}\big)\circ F_q\big(\mathcal{A}_{G^{(a)}};d_{G^{(a)}}\big) = F_q\big(\mathcal{A}_{G^{(a)}};d_{G^{(a)}}\big)\circ F_p\big(\mathcal{A}_{G^{(p)}};d_{G^{(p)}}\big).$$

We take arbitrary functions $f,g\in C(V,\mathbb{C})$. We then have

$$\begin{aligned}\langle\mathcal{A}_{(p,q)}f,g\rangle &= \left\langle\frac{1}{d_{G^{(a)}}(d_{G^{(a)}}-1)^{q-1}}F_p\big(\mathcal{A}_{G^{(p)}};d_{G^{(p)}}\big)F_q\big(\mathcal{A}_{G^{(a)}};d_{G^{(a)}}\big)f,g\right\rangle \\ &= \frac{1}{d_{G^{(a)}}(d_{G^{(a)}}-1)^{q-1}}\big\langle F_p\big(\mathcal{A}_{G^{(p)}};d_{G^{(p)}}\big)F_q\big(\mathcal{A}_{G^{(a)}};d_{G^{(a)}}\big)f,g\big\rangle \\ &= \frac{1}{d_{G^{(a)}}(d_{G^{(a)}}-1)^{q-1}}\big\langle F_q\big(\mathcal{A}_{G^{(a)}};d_{G^{(a)}}\big)f,F_p\big(\mathcal{A}_{G^{(p)}};d_{G^{(p)}}\big)g\big\rangle \\ &= \frac{1}{d_{G^{(a)}}(d_{G^{(a)}}-1)^{q-1}}\big\langle f,F_q\big(\mathcal{A}_{G^{(a)}};d_{G^{(a)}}\big)F_p\big(\mathcal{A}_{G^{(p)}};d_{G^{(p)}}\big)g\big\rangle \\ &= \langle f,\mathcal{A}_{(p,q)}g\rangle.\end{aligned}$$

Hence we find that $\mathcal{A}_{(p,q)}$ is symmetric. As we have $\mathcal{Q}_{(p,q)} = \dfrac{1}{d_{G^{(p)}}(d_{G^{(p)}}-1)^{p-1}}\mathcal{A}_{(p,q)}$, it is also symmetric.

Though our proof is completed, we here give a proof by matrix representations in the case that G is finite. We denote by $A_{G^{(p)}},\ A_{G^{(a)}}$ the adjacency matrices of the principal and the auxiliary graphs of G. Then they satisfy ${}^tA_{G^{(p)}} = A_{G^{(p)}}$, ${}^tA_{G^{(a)}} = A_{G^{(a)}}$, and satisfy $A_{G^{(p)}}A_{G^{(a)}} = A_{G^{(a)}}A_{G^{(p)}}$ by the assumption. Generally we have ${}^t(AB) = {}^tB{}^tA$ for arbitrary matrices A, B. Thus for arbitrary positive integer k we have

$${}^t(A^k_{G^{(p)}}) = {}^tA_{G^{(p)}} \cdots {}^tA_{G^{(p)}} = A_{G^{(p)}} \cdots A_{G^{(p)}} = A^k_{G^{(p)}} \quad \text{and} \quad {}^t(A^k_{G^{(a)}}) = A^k_{G^{(a)}},$$

hence have

$${}^t\big\{F_p\big(A_{G^{(p)}}; d_{G^{(p)}}\big)\big\} = F_p\big(A_{G^{(p)}}; d_{G^{(p)}}\big) \quad \text{and} \quad {}^t\big\{F_q\big(A_{G^{(a)}}; d_{G^{(a)}}\big)\big\} = F_q\big(A_{G^{(a)}}; d_{G^{(a)}}\big).$$

Also, by the property $A_{G^{(p)}}A_{G^{(a)}} = A_{G^{(a)}}A_{G^{(p)}}$, we have

$$F_p\big(A_{G^{(p)}}; d_{G^{(p)}}\big)F_q\big(A_{G^{(a)}}; d_{G^{(a)}}\big) = F_q\big(A_{G^{(a)}}; d_{G^{(a)}}\big)F_p\big(A_{G^{(p)}}; d_{G^{(p)}}\big).$$

Since we have

$$\begin{aligned} A_{(p,q)} &= \frac{1}{d_{G^{(a)}}(d_{G^(a)} - 1)^{q-1}}\, A_{G^{(p)}_{[p]}} A_{G^{(a)}_{[q]}} \\ &= \frac{1}{d_{G^{(a)}}(d_{G^(a)} - 1)^{q-1}}\, F_p\big(A_{G^{(p)}}; d_{G^{(p)}}\big)F_q\big(A_{G^{(a)}}; d_{G^{(a)}}\big), \end{aligned}$$

we find

$$\begin{aligned} {}^tA_{(p,q)} &= \frac{1}{d_{G^{(a)}}(d_{G^(a)} - 1)^{q-1}}\, {}^t\big\{F_q\big(A_{G^{(a)}}; d_{G^{(a)}}\big)\big\}{}^t\big\{F_p\big(A_{G^{(p)}}; d_{G^{(p)}}\big)\big\} \\ &= \frac{1}{d_{G^{(a)}}(d_{G^(a)} - 1)^{q-1}}\, F_q\big(A_{G^{(a)}}; d_{G^{(a)}}\big)F_p\big(AG^{(p)}; d_{G^{(p)}}\big) \\ &= \frac{1}{d_{G^{(a)}}(d_{G^(a)} - 1)^{q-1}}\, F_p\big(A_{G^{(p)}}; d_{G^{(p)}}\big)F_q\big(A_{G^{(a)}}; d_{G^{(a)}}\big) = A_{(p,q)}. \end{aligned}$$

As we have $Q_{(p,q)} = \dfrac{1}{d_{G^{(p)}}(d_{G^(p)} - 1)^{p-1}}\, A_{(p,q)}$, we get the conclusion. □

Corollary 5.1. *If the adjacency operators $\mathcal{A}_{G^{(p)}}, \mathcal{A}_{G^{(a)}}$ of a regular Kähler graph G are commutative, then all eigenvalues of the (p,q)-adjacency Laplacian $\Delta_{\mathcal{A}_{(p,q)}}$ and those of the (p,q)-probabilistic transition Laplacian $\Delta_{\mathcal{Q}_{(p,q)}}$ are real for an arbitrary pair (p,q) of relatively prime positive integers.*

The condition that the adjacency operators of the principal and the auxiliary graphs are commutative is a strong condition, but we have many Kähler graphs satisfying this condition, complement-filled Kähler graphs, Kähler graphs of Cartesian product type, of strong product type, and so on, for example. For more study on commutativity of the adjacency operators, see [**3**].

2. (p, q)-Laplacians of complement-filled Kähler graphs

We now apply Theorem 5.1 to some typical examples of Kähler graphs.

2.1. Eigenvalues of (p, q)-Laplacians of complement-filled Kähler graphs.

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First we study complement-filled Kähler graphs.

THEOREM 5.2. *Let $G = (V, E)$ be a connected regular finite graph of degree $2 \leq d_{G^{(p)}} \leq n_G - 3$. We denote the eigenvalues of $\Delta_{\mathcal{A}_G}$ by $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{n_G}$. Then the eigenvalues of $(p.q)$-combinatorial Laplacian $\Delta_{\mathcal{A}_{p,q}}$ of the complement-filled Kähler graph G^K are*

$$0 \quad \textit{and} \quad d_G^{(p)}(d_G^{(p)} - 1)^{p-1} - \frac{F_p(d_G^{(p)} - \lambda_i; d_G^{(p)})F_q(\lambda_i - d_G^{(p)} - 1; n_G - d_G^{(p)} - 1)}{(n_G - d_G^{(p)} - 1)(n_G - d_G^{(p)} - 2)^{q-1}}$$

for $i = 2, \cdots, n_G$.

PROOF. We use the same notations as in §4.2.1. Since G is regular, we have

$$(\mathcal{A}_G \circ \mathcal{M})f(v) = \mathcal{A}_G \sum_{w \in V} f(w) = \Big(\sum_{w \in V} f(w)\Big)\mathcal{A}_G 1 = d_G \sum_{w \in V} f(w),$$
$$(\mathcal{M} \circ \mathcal{A}_G)f(v) = \sum_{v \in V} \sum_{w : w \sim v} f(w) = \sum_{w \in V} d_G f(w) = d_G \sum_{w \in V} f(w),$$

hence $\mathcal{A}_G \circ \mathcal{M} = \mathcal{M} \circ \mathcal{A}_G$. As $\mathcal{A}_{G^c} = \mathcal{M} - \mathcal{I} - \mathcal{A}_G$, we find that $\mathcal{A}_G$ and $\mathcal{A}_{G^c}$ are commutative.

We recall the argument in the proof of Theorem 4.1. We take an eigenfunction $f_i; V \to \mathbb{R}$ corresponding to λ_i. We have $\mathcal{A}_G f_i = (d_G - \lambda_i) f_i$. When $i = 1$, we see $\lambda_1 = 0$ and the eigenfunction f_1 is a non-zero constant function. Hence we have $\mathcal{A}_G f_1 = d_G f_1$ and

$$\mathcal{A}_{G^c} f_1 = (\mathcal{M} f_1 - f_1 - \mathcal{A}_G f_1) = (n_G - d_G - 1) f_1.$$

Therefore by Theorem 5.1 and Lemma 5.1 we obtain

$$\Delta_{\mathcal{A}_{p,q}} f_1 = d_G(d_G - 1)^{p-1} f_1 - \frac{F_p(d_G; d_G)F_q(n_G - d_G - 1; n_G - d_G - 1)}{(n_G - d_G - 1)(n_G - d_G - 2)^{q-1}} f_1$$
$$= d_G(d_G - 1)^{p-1} f_1 - d_{G^{(p)}}(d_{G^{(p)}} - 1)^{p-1} f_1 = 0.$$

When $i = 2, \cdots, n_G$, as the graph G is connected, we find that f_i is orthogonal to f_1 (Note 1.1) hence satisfies $\sum_{v\in V} f_i = 0$. Therefore we have

$$\mathcal{A}_{G^c} f_i = (\mathcal{M} f_i - f_i - \mathcal{A}_G f_i) = (-1 - d_G + \lambda_i) f_i.$$

Hence we obtain

$$\Delta_{\mathcal{A}_{p,q}} f_i = \{d_{G^{(p)}}(d_{G^{(p)}} - 1)^{p-1} f_i - \frac{F_p(d_G - \lambda_i; d_G) F_q(\lambda_i - d_{G^{(p)}} - 1; n_G - d_G - 1)}{(n_G - d_G - 1)(n_G - d_G - 2)^{q-1}} f_i,$$

and get the conclusion. □

Corollary 5.2. *Let $G = (V, E)$ be a connected regular finite graph of degree $2 \le d_{G^{(p)}} \le n_G - 3$. We denote the eigenvalues of $\Delta_{\mathcal{P}_G}$ by $0 = \mu_1 < \mu_2 \le \cdots \le \mu_{n_G}$. Then the eigenvalues of $(p.q)$-combinatorial Laplacian $\Delta_{\mathcal{Q}_{p,q}}$ of the complement-filled Kähler graph G^K are*

$$0 \quad and \quad 1 - \frac{F_p(d_G^{(p)}(1-\mu_i); d_G^{(p)}) F_q(d_G \mu_i - d_G^{(p)} - 1; n_G - d_G^{(p)} - 1)}{d_G^{(p)}(d_G^{(p)} - 1)^{p-1}(n_G - d_G^{(p)} - 1)(n_G - d_G^{(p)} - 2)^{q-1}}$$

for $i = 2, \cdots, n_G$.

Proof. We take an eigenfunction $f_i; V \to \mathbb{R}$ corresponding to μ_i. We have $\mathcal{A}_G f_i = d_G(1 - \mu_i) f_i$. We hence have

$$\mathcal{A}_{G^c} f_i = \begin{cases} (n_G - d_G - 1) f_i, & \text{when } i = 1, \\ (-1 - d_G - d_G \mu_i) f_i, & \text{when } i \ne 1. \end{cases}$$

Thus we get the assertion by Theorem 5.1. □

2.2. (p, q)-isospectral Kähler graphs. Given a pair of Kähler graphs we say that they are (p, q)-*combinatorially isospectral* (resp. (p, q)-*probabilistic transitionally isospectral*) if they satisfy the following conditions:

i) Their combinatorial (p, q)-Laplacians (resp. probabilistic transitional (p, q)-Laplacians) have the same eigenvalues by taking account of their multiplicities;
ii) Their principal graphs are combinatorially (resp. transitionary) isospectral.

Clearly, two Kähler graphs are (p, q)-combinatorially isospectral if and only if they are (p, q)-probabilistic transitionally isospectral when their principal graphs are regular. In this case we just say that they are (p, q)-isospectral.

As a direct consequence of Theorem 5.1 we have the following.

THEOREM 5.3. *Let $G_1=(V_1,E_1^{(p)}\cup E_1^{(a)})$ and $G_2=(V_2,E_2^{(p)}\cup E_2^{(a)})$ be two regular Kähler graphs satisfying $d_{G_1}^{(p)}=d_{G_2}^{(p)}$ and $d_{G_1}^{(a)}=d_{G_2}^{(a)}$. We suppose that their adjacency operators of their principal and auxiliary graphs are simultaneously diagonalizable, that is, $\mathcal{A}_{G_i^{(p)}}\circ\mathcal{A}_{G_i^{(a)}}=\mathcal{A}_{G_i^{(a)}}\circ\mathcal{A}_{G_i^{(p)}}$ for $i=1,2$. If their principal graphs $(V_1,E_1^{(p)}),(V_2,E_2^{(p)})$ are isospectral and if their auxiliary graphs $(V_1,E_1^{(a)}),(V_2,E_2^{(a)})$ are isospectral, then they are (p,q)-isospectral for an arbitrary pair (p,q) of relatively prime positive integers.*

Applying this result to complement-filled Kähler graphs we obtain the following.

COROLLARY 5.3. *If two finite regular graphs G_1, G_2 are isospectral and have the same degrees, then their compliment-filled Kähler graphs are (p,q)-isospectral for an arbitrary pair (p,q) of relatively prime positive integers.*

We here study the example given in §4.2.

EXAMPLE 5.3. Let G_1,G_2 be the pair of isospectral regular graphs of $n_{G_1}=n_{G_2}=10$ given in Example 4.9. Their complement-filled Kähler graphs G_1^K, G_2^K are (p,q)-isospectral. We here list eigenvalues of some (p,q)-combinatorial Laplacians:

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^K_{(2,1)}}})=\Big\{0,\frac{54}{5},\ \frac{56}{5},\ \frac{56}{5},\ \frac{1}{5}\big(61-\sqrt{5}\big),\ 12,12,12,12,\ \frac{1}{5}\big(61+\sqrt{5}\big)\Big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^K_{(3,1)}}})=\left\{\begin{matrix}0,\ \frac{2}{5}\big(85-\sqrt{17}\big),\ \frac{2}{5}\big(85-\sqrt{5}\big),\ \frac{168}{5},\ \frac{2}{5}\big(85+\sqrt{5}\big),\\ \frac{2}{5}\big(85-\sqrt{17}\big),\ 36,36,36,36\end{matrix}\right\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^K_{(1,2)}}})=\left\{\begin{matrix}0,\frac{1}{20}\big(70-\sqrt{5}\big),\ \frac{1}{20}\big(70+\sqrt{5}\big),\ \frac{1}{40}\big(151-\sqrt{17}\big),\\ \frac{15}{4},\ \frac{15}{4},\ \frac{15}{4},\ \frac{15}{4},\ \frac{1}{40}\big(151+\sqrt{17}\big),\ \frac{81}{20}\end{matrix}\right\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^K_{(1,3)}}})=\left\{\begin{matrix}0,\frac{1}{40}\big(151-\sqrt{17}\big),\ \frac{7}{80}\big(45-\sqrt{5}\big),\ \frac{31}{8},\\ \frac{1}{40}\big(151+\sqrt{17}\big),\ 4,4,4,4,\ \frac{7}{80}\big(45+\sqrt{5}\big),\ \frac{81}{20}\end{matrix}\right\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^K_{(3,2)}}}) = \left\{ \begin{array}{l} 0, \frac{357}{10},\ \frac{1}{20}\bigl(727-\sqrt{17}\bigr),\ \frac{1}{20}\bigl(717+\sqrt{17}\bigr),\ \frac{1}{10}\bigl(370-\sqrt{5}\bigr), \\ \frac{1}{10}\bigl(370+\sqrt{5}\bigr),\ \frac{75}{2},\ \frac{75}{2},\ \frac{75}{2},\ \frac{75}{2} \end{array} \right\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^K_{(2,3)}}}) = \left\{ \begin{array}{l} 0, 12, 12, 12, 12,\ \frac{1}{80}\bigl(967-\sqrt{5}\bigr),\ \frac{1}{80}\bigl(967+\sqrt{5}\bigr), \\ \frac{1}{40}\bigl(489-\sqrt{17}\bigr),\ \frac{1}{40}\bigl(489+\sqrt{17}\bigr),\ \frac{99}{8} \end{array} \right\}.$$

As we have

$$A_{G_{1[2]}}= \begin{pmatrix} 0&1&2&1&1&1&3&1&1&1\\ 1&0&1&1&1&2&1&3&1&1\\ 2&1&0&1&2&0&2&1&1&2\\ 1&1&1&0&1&2&1&1&3&1\\ 1&1&2&1&0&1&1&1&1&3\\ 1&2&0&2&1&0&1&2&2&1\\ 3&1&2&1&1&1&0&1&1&1\\ 1&3&1&1&1&2&1&0&1&1\\ 1&1&1&3&1&2&1&1&0&1\\ 1&1&2&1&3&1&1&1&1&0 \end{pmatrix},\ A_{G_{2[2]}}= \begin{pmatrix} 0&1&2&1&1&1&2&2&0&2\\ 1&0&1&1&1&2&2&2&2&0\\ 2&1&0&1&2&0&2&1&1&2\\ 1&1&1&0&1&2&0&2&2&2\\ 1&1&2&1&0&1&2&0&2&2\\ 1&2&0&2&1&0&1&2&2&1\\ 2&2&2&0&2&1&0&1&1&1\\ 2&2&1&2&0&2&1&0&1&1\\ 0&2&1&2&2&2&1&1&0&1\\ 2&0&2&2&2&1&1&1&1&0 \end{pmatrix},$$

$$A_{G_{1[3]}}= \begin{pmatrix} 6&3&4&5&4&2&0&3&5&4\\ 3&4&2&6&5&4&3&4&0&5\\ 4&2&4&2&4&8&4&2&2&4\\ 5&6&2&4&3&4&5&0&4&3\\ 4&5&4&3&6&2&4&5&3&0\\ 2&4&8&4&2&4&2&4&4&2\\ 0&3&4&5&4&2&6&3&5&4\\ 3&4&2&0&5&4&3&4&6&5\\ 5&0&2&4&0&2&4&5&3&6 \end{pmatrix},\ A_{G_{2[3]}}= \begin{pmatrix} 4&3&4&5&6&2&1&6&2&3\\ 3&6&2&4&5&4&2&1&3&6\\ 4&2&4&2&4&8&4&2&2&4\\ 5&4&2&6&3&4&6&3&1&2\\ 6&5&4&3&4&2&3&2&6&1\\ 2&4&8&4&2&4&2&4&4&2\\ 1&2&4&6&3&2&6&3&5&4\\ 6&1&2&3&2&4&3&4&6&5\\ 2&3&2&1&6&4&5&6&4&3\\ 3&6&4&2&1&2&4&5&3&6 \end{pmatrix},$$

$$A_{G^c_{1[2]}}= \begin{pmatrix} 0&3&2&1&1&3&5&3&1&1\\ 3&0&3&1&1&2&3&3&3&1\\ 2&3&0&3&2&0&2&3&3&2\\ 1&1&3&0&3&2&1&3&3&3\\ 1&1&2&3&0&3&1&1&3&5\\ 3&2&0&2&3&0&3&2&2&3\\ 5&3&2&1&1&3&0&3&1&1\\ 3&3&3&3&1&2&3&0&1&1\\ 1&3&3&3&3&2&1&1&0&3\\ 1&1&2&3&5&3&1&1&3&0 \end{pmatrix},\ A_{G^c_{2[2]}}= \begin{pmatrix} 0&3&2&1&1&3&4&2&2&2\\ 3&0&3&1&1&2&4&4&2&0\\ 2&3&0&3&2&0&2&3&3&2\\ 1&1&3&0&3&2&0&2&4&4\\ 1&1&2&3&0&3&2&2&2&4\\ 3&2&0&2&3&0&3&2&2&3\\ 4&4&2&0&2&3&0&3&1&1\\ 2&4&3&2&2&2&3&0&1&1\\ 2&2&3&4&2&2&1&1&0&3\\ 2&0&2&4&4&3&1&1&3&0 \end{pmatrix},$$

$$A_{G^c_{1[3]}}=\begin{pmatrix} 6 & 9 & 6 & 8 & 9 & 10 & 6 & 9 & 8 & 9 \\ 9 & 8 & 10 & 7 & 8 & 6 & 9 & 3 & 12 & 8 \\ 6 & 10 & 8 & 10 & 6 & 8 & 6 & 10 & 10 & 6 \\ 8 & 7 & 10 & 8 & 9 & 6 & 8 & 12 & 3 & 9 \\ 9 & 8 & 6 & 9 & 6 & 10 & 9 & 8 & 9 & 6 \\ 10 & 6 & 8 & 6 & 10 & 8 & 10 & 6 & 6 & 10 \\ 6 & 9 & 6 & 8 & 9 & 10 & 6 & 9 & 8 & 9 \\ 9 & 3 & 10 & 12 & 8 & 6 & 9 & 8 & 7 & 8 \\ 8 & 12 & 10 & 3 & 9 & 6 & 8 & 7 & 8 & 9 \\ 9 & 8 & 6 & 9 & 6 & 10 & 9 & 8 & 9 & 6 \end{pmatrix}, \; A_{G^c_{2[3]}}=\begin{pmatrix} 8 & 9 & 6 & 8 & 7 & 10 & 8 & 4 & 13 & 7 \\ 9 & 6 & 10 & 9 & 8 & 6 & 7 & 8 & 7 & 10 \\ 6 & 10 & 8 & 10 & 6 & 8 & 6 & 10 & 10 & 6 \\ 8 & 9 & 10 & 6 & 9 & 6 & 10 & 7 & 8 & 7 \\ 7 & 8 & 6 & 9 & 8 & 10 & 7 & 13 & 4 & 8 \\ 10 & 6 & 8 & 6 & 10 & 8 & 10 & 6 & 6 & 10 \\ 8 & 7 & 6 & 10 & 7 & 10 & 6 & 9 & 8 & 9 \\ 4 & 8 & 10 & 7 & 13 & 6 & 9 & 8 & 7 & 8 \\ 13 & 7 & 10 & 8 & 4 & 6 & 8 & 7 & 8 & 9 \\ 7 & 10 & 6 & 7 & 8 & 10 & 9 & 8 & 9 & 6 \end{pmatrix},$$

we have

$$A_{G^K_{1(2,1)}}=\frac{1}{5}\begin{pmatrix} 6 & 5 & 6 & 6 & 7 & 6 & 6 & 5 & 6 & 7 \\ 5 & 8 & 6 & 5 & 6 & 6 & 5 & 5 & 8 & 6 \\ 6 & 6 & 8 & 6 & 6 & 4 & 6 & 6 & 6 & 6 \\ 6 & 5 & 6 & 8 & 5 & 6 & 6 & 8 & 5 & 5 \\ 7 & 6 & 6 & 5 & 6 & 6 & 7 & 6 & 5 & 6 \\ 6 & 6 & 4 & 6 & 6 & 8 & 6 & 6 & 6 & 6 \\ 6 & 5 & 6 & 6 & 7 & 6 & 6 & 5 & 6 & 7 \\ 5 & 5 & 6 & 8 & 6 & 6 & 5 & 8 & 5 & 6 \\ 6 & 8 & 6 & 5 & 5 & 6 & 6 & 5 & 8 & 5 \\ 7 & 6 & 6 & 5 & 6 & 6 & 7 & 6 & 5 & 6 \end{pmatrix}, \; A_{G^K_{2(2,1)}}=\frac{1}{5}\begin{pmatrix} 8 & 5 & 6 & 6 & 5 & 6 & 6 & 4 & 7 & 7 \\ 5 & 6 & 6 & 7 & 6 & 6 & 5 & 6 & 7 & 6 \\ 6 & 6 & 8 & 6 & 6 & 4 & 6 & 6 & 6 & 6 \\ 6 & 7 & 6 & 6 & 5 & 6 & 6 & 7 & 6 & 5 \\ 5 & 6 & 6 & 5 & 8 & 6 & 7 & 7 & 4 & 6 \\ 6 & 6 & 4 & 6 & 6 & 8 & 6 & 6 & 6 & 6 \\ 6 & 5 & 6 & 6 & 7 & 6 & 6 & 5 & 6 & 7 \\ 4 & 6 & 6 & 7 & 7 & 6 & 5 & 8 & 5 & 6 \\ 7 & 7 & 6 & 6 & 4 & 6 & 6 & 5 & 8 & 5 \\ 7 & 6 & 6 & 5 & 6 & 6 & 7 & 6 & 5 & 6 \end{pmatrix},$$

$$A_{G^K_{1(3,1)}}= \frac{1}{5}\begin{pmatrix} 22 & 18 & 16 & 16 & 16 & 20 & 22 & 18 & 16 & 16 \\ 18 & 24 & 20 & 14 & 16 & 16 & 18 & 18 & 20 & 16 \\ 16 & 20 & 24 & 20 & 16 & 12 & 16 & 20 & 20 & 16 \\ 16 & 14 & 20 & 24 & 18 & 16 & 16 & 20 & 18 & 18 \\ 16 & 16 & 16 & 18 & 22 & 20 & 16 & 16 & 18 & 22 \\ 20 & 16 & 12 & 16 & 20 & 24 & 20 & 16 & 16 & 20 \\ 22 & 18 & 16 & 16 & 16 & 20 & 22 & 18 & 16 & 16 \\ 18 & 18 & 20 & 20 & 16 & 16 & 18 & 24 & 14 & 16 \\ 16 & 20 & 20 & 18 & 18 & 16 & 16 & 14 & 24 & 18 \\ 16 & 16 & 16 & 18 & 22 & 20 & 16 & 16 & 18 & 22 \end{pmatrix},$$

$$A_{G^K_{2(3,1)}}= \frac{1}{5}\begin{pmatrix} 24 & 18 & 16 & 16 & 14 & 20 & 20 & 16 & 18 & 18 \\ 18 & 22 & 20 & 16 & 16 & 16 & 20 & 20 & 18 & 14 \\ 16 & 20 & 24 & 20 & 16 & 12 & 16 & 20 & 20 & 16 \\ 16 & 16 & 20 & 22 & 18 & 16 & 14 & 18 & 20 & 20 \\ 14 & 16 & 16 & 18 & 24 & 20 & 18 & 18 & 16 & 20 \\ 20 & 16 & 12 & 16 & 20 & 24 & 20 & 16 & 16 & 20 \\ 20 & 20 & 16 & 14 & 18 & 20 & 22 & 18 & 16 & 16 \\ 16 & 20 & 20 & 18 & 18 & 16 & 18 & 24 & 14 & 16 \\ 18 & 18 & 20 & 20 & 16 & 16 & 16 & 14 & 24 & 18 \\ 18 & 14 & 16 & 20 & 20 & 20 & 16 & 16 & 18 & 22 \end{pmatrix},$$

$$A_{G^K_{1(1,2)}}=\frac{1}{20}\begin{pmatrix}
14 & 8 & 8 & 7 & 6 & 7 & 9 & 8 & 7 & 6\\
8 & 12 & 7 & 8 & 7 & 8 & 8 & 10 & 5 & 7\\
8 & 7 & 12 & 7 & 8 & 8 & 8 & 7 & 7 & 8\\
7 & 8 & 7 & 12 & 8 & 8 & 7 & 5 & 10 & 8\\
6 & 7 & 8 & 8 & 14 & 7 & 6 & 7 & 8 & 9\\
7 & 8 & 8 & 8 & 7 & 12 & 7 & 8 & 8 & 7\\
9 & 8 & 8 & 7 & 6 & 7 & 14 & 8 & 7 & 6\\
8 & 10 & 7 & 5 & 7 & 8 & 8 & 12 & 8 & 7\\
7 & 5 & 7 & 10 & 8 & 8 & 7 & 8 & 12 & 8\\
6 & 7 & 8 & 8 & 9 & 7 & 6 & 7 & 8 & 14
\end{pmatrix},$$

$$A_{G^K_{2(1,2)}}=\frac{1}{20}\begin{pmatrix}
12 & 8 & 8 & 7 & 8 & 7 & 8 & 10 & 5 & 7\\
8 & 14 & 7 & 6 & 7 & 8 & 9 & 8 & 7 & 6\\
8 & 7 & 12 & 7 & 8 & 8 & 8 & 7 & 7 & 8\\
7 & 6 & 7 & 14 & 8 & 8 & 6 & 7 & 8 & 9\\
8 & 7 & 8 & 8 & 12 & 7 & 7 & 5 & 10 & 8\\
7 & 8 & 8 & 8 & 7 & 12 & 7 & 8 & 8 & 7\\
8 & 9 & 8 & 6 & 7 & 7 & 14 & 8 & 7 & 6\\
10 & 8 & 7 & 7 & 5 & 8 & 8 & 12 & 8 & 7\\
5 & 7 & 7 & 8 & 10 & 8 & 7 & 8 & 12 & 8\\
7 & 6 & 8 & 9 & 8 & 7 & 6 & 7 & 8 & 14
\end{pmatrix},$$

$$A_{G^K_{1(1,3)}}=\frac{1}{80}\begin{pmatrix}
34 & 26 & 34 & 33 & 35 & 30 & 34 & 26 & 33 & 35\\
26 & 40 & 30 & 29 & 33 & 34 & 26 & 35 & 34 & 33\\
34 & 30 & 40 & 30 & 34 & 24 & 34 & 30 & 30 & 34\\
33 & 29 & 30 & 40 & 26 & 34 & 33 & 34 & 35 & 26\\
35 & 33 & 34 & 26 & 34 & 30 & 35 & 33 & 26 & 34\\
30 & 34 & 24 & 34 & 30 & 40 & 30 & 34 & 34 & 30\\
34 & 26 & 34 & 33 & 35 & 30 & 34 & 26 & 33 & 35\\
26 & 35 & 30 & 34 & 33 & 34 & 26 & 40 & 29 & 33\\
33 & 34 & 30 & 35 & 26 & 34 & 33 & 29 & 40 & 26\\
35 & 33 & 34 & 26 & 34 & 30 & 35 & 33 & 26 & 34
\end{pmatrix},$$

$$A_{G^K_{2(1,3)}}=\frac{1}{80}\begin{pmatrix}
40 & 26 & 34 & 33 & 29 & 30 & 31 & 30 & 29 & 38\\
26 & 34 & 30 & 35 & 33 & 34 & 29 & 31 & 38 & 30\\
34 & 30 & 40 & 30 & 34 & 24 & 34 & 30 & 30 & 34\\
33 & 35 & 30 & 34 & 26 & 34 & 30 & 38 & 31 & 29\\
29 & 33 & 34 & 26 & 40 & 30 & 38 & 29 & 30 & 31\\
30 & 34 & 24 & 34 & 30 & 40 & 30 & 34 & 34 & 30\\
31 & 29 & 34 & 30 & 38 & 30 & 34 & 26 & 33 & 35\\
30 & 31 & 30 & 38 & 29 & 34 & 26 & 40 & 29 & 33\\
29 & 38 & 30 & 31 & 30 & 34 & 33 & 29 & 40 & 26\\
38 & 30 & 34 & 29 & 31 & 30 & 35 & 33 & 26 & 34
\end{pmatrix},$$

$$A_{G^K_{1(2,3)}} = \frac{1}{80}\begin{pmatrix} 92 & 99 & 94 & 98 & 95 & 98 & 92 & 99 & 98 & 95 \\ 99 & 84 & 98 & 103 & 98 & 94 & 99 & 99 & 88 & 98 \\ 94 & 98 & 88 & 98 & 94 & 104 & 94 & 98 & 98 & 94 \\ 98 & 103 & 98 & 84 & 99 & 94 & 98 & 88 & 99 & 99 \\ 95 & 98 & 94 & 99 & 92 & 98 & 95 & 98 & 99 & 92 \\ 98 & 94 & 104 & 94 & 98 & 88 & 98 & 94 & 94 & 98 \\ 92 & 99 & 94 & 98 & 95 & 98 & 92 & 99 & 98 & 95 \\ 99 & 99 & 98 & 88 & 98 & 94 & 99 & 84 & 103 & 98 \\ 98 & 88 & 98 & 99 & 99 & 94 & 98 & 103 & 84 & 99 \\ 95 & 98 & 94 & 99 & 92 & 98 & 95 & 98 & 99 & 92 \end{pmatrix},$$

$$A_{G^K_{2(2,3)}} = \frac{1}{80}\begin{pmatrix} 84 & 99 & 94 & 98 & 103 & 98 & 94 & 104 & 93 & 93 \\ 99 & 92 & 98 & 95 & 98 & 94 & 97 & 94 & 93 & 100 \\ 94 & 98 & 88 & 98 & 94 & 104 & 94 & 98 & 98 & 94 \\ 98 & 95 & 98 & 92 & 99 & 94 & 100 & 93 & 94 & 97 \\ 103 & 98 & 94 & 99 & 84 & 98 & 93 & 93 & 104 & 94 \\ 98 & 94 & 104 & 94 & 98 & 88 & 98 & 94 & 94 & 98 \\ 94 & 97 & 94 & 100 & 93 & 94 & 99 & 84 & 103 & 98 \\ 93 & 93 & 98 & 94 & 104 & 94 & 98 & 103 & 84 & 99 \\ 93 & 100 & 94 & 97 & 94 & 98 & 95 & 98 & 99 & 92 \end{pmatrix},$$

$$A_{G^K_{1(3,2)}} = \frac{1}{20}\begin{pmatrix} 50 & 71 & 76 & 73 & 76 & 74 & 80 & 71 & 73 & 76 \\ 71 & 60 & 74 & 66 & 73 & 76 & 71 & 72 & 84 & 73 \\ 76 & 74 & 56 & 74 & 76 & 64 & 76 & 74 & 74 & 76 \\ 73 & 66 & 74 & 60 & 71 & 76 & 73 & 84 & 72 & 71 \\ 76 & 73 & 76 & 71 & 50 & 74 & 76 & 73 & 71 & 80 \\ 74 & 76 & 64 & 76 & 74 & 56 & 74 & 76 & 76 & 74 \\ 80 & 71 & 76 & 73 & 76 & 74 & 50 & 71 & 73 & 76 \\ 71 & 72 & 74 & 84 & 73 & 76 & 71 & 60 & 66 & 73 \\ 73 & 84 & 74 & 72 & 71 & 76 & 73 & 66 & 60 & 71 \\ 76 & 73 & 76 & 71 & 80 & 74 & 76 & 73 & 71 & 50 \end{pmatrix},$$

$$A_{G^K_{2(3,2)}} = \frac{1}{20}\begin{pmatrix} 60 & 71 & 76 & 73 & 66 & 74 & 77 & 66 & 78 & 79 \\ 71 & 50 & 74 & 76 & 73 & 76 & 74 & 77 & 79 & 70 \\ 76 & 74 & 56 & 74 & 76 & 64 & 76 & 74 & 74 & 76 \\ 73 & 76 & 74 & 50 & 71 & 76 & 70 & 79 & 77 & 74 \\ 66 & 73 & 76 & 71 & 60 & 74 & 79 & 78 & 66 & 77 \\ 74 & 76 & 64 & 76 & 74 & 56 & 74 & 76 & 76 & 74 \\ 77 & 74 & 76 & 70 & 79 & 74 & 50 & 71 & 73 & 76 \\ 66 & 77 & 74 & 79 & 78 & 76 & 71 & 60 & 66 & 73 \\ 78 & 79 & 74 & 77 & 66 & 76 & 73 & 66 & 60 & 71 \\ 79 & 70 & 76 & 74 & 77 & 74 & 76 & 73 & 71 & 50 \end{pmatrix}.$$

3. (p,q)-Laplacians of Kähler graphs of product type whose principal graphs are unions of original graphs

In this section, we study (p,q)-step combinatorial and transitional Laplacians for those four Kähler graphs of product types $G\widehat{\Box}H, G\widehat{\boxtimes}H, G\widehat{\otimes}H, G \rhd H$.

3.1. (p,q)-Laplacians of Kähler graphs of Cartesian product type.

THEOREM 5.4. *Let $G=(V,E), H=(W,F)$ be regular finite graphs. We denote by $0=\mu_1 \le \cdots \le \mu_{n_G}$ the eigenvalues of the transitional Laplacian $\Delta_{\mathcal{P}_G}$ of G, and by $0=\nu_1 \le \cdots \le \nu_{n_H}$ the eigenvalues of the transitional Laplacian $\Delta_{\mathcal{P}_H}$ of H. Then the eigenvalues of the (p,q)-probabilistic transition Laplacian $\Delta_{\mathcal{Q}_{(p,q)}}$ of $G\widehat{\Box}H$ are*

$$1-\frac{F_p\big(d_G(1-\mu_i);d_G\big)F_q\big(d_H(1-\nu_\alpha);d_H\big)}{d_Gd_H(d_G-1)^{p-1}(d_H-1)^{q-1}}$$

for $i=1,\dots,n_G$ and $\alpha=1,\dots,n_H$. The eigenvalues of the (p,q)-combinatorial Laplacian $\Delta_{\mathcal{A}_{(p,q)}}$ of $G\widehat{\Box}H$ are

$$d_G(d_G-1)^{p-1}-\frac{F_p\big(d_G(1-\mu_i);d_G\big)F_q\big(d_H(1-\nu_\alpha);d_H\big)}{d_H(d_H-1)^{q-1}}$$

for $i=1,\dots,n_G$ and $\alpha=1,\dots,n_H$.

PROOF. As we see in §2.2.2, we have

$$d^{(p)}_{G\widehat{\Box}H}=d_G,\quad d^{(a)}_{G\widehat{\Box}H}=d_H.$$

By using the same notations as in Theorem 4.4, the adjacency matrices of the principal and the auxiliary graphs of $G\widehat{\Box}H$ are expressed as

$$A^{(p)}_{G\widehat{\Box}H}=(a^{(p)}_{(i,\alpha),(j,\beta)})=\Big(a^G_{ij}\delta_{\alpha,\beta}\Big),\qquad A^{(a)}_{G\widehat{\Box}H}=(a^{(a)}_{(i,\alpha),(j,\beta)})=\Big(\delta_{ij}a^H_{\alpha,\beta}\Big)$$

by use of the adjacency matrices $A_G=(a^G_{ij})$ and $A_H=(a^H_{\alpha\beta})$ of G and H (c.f. §4.3.1). By these expressions we find that they are commutative.

We take an eigenfunction $f:V\to\mathbb{R}$ corresponding to an eigenvalue μ of $\Delta_{\mathcal{P}_G}$ and an eigenfunction $g:W\to\mathbb{R}$ corresponding to an eigenvalue ν of $\Delta_{\mathcal{P}_H}$. Then we have $\mathcal{A}_Gf=d_G(1-\mu)f$ and $\mathcal{A}_Hg=d_H(1-\nu)g$. We take a function $\varphi_{f,g}:V\times W\to\mathbb{R}$

defined by $\varphi_{f,g}(v,w) = f(v)g(w)$. By using canonical basis, we can correspond to these functions f, g and $\varphi_{f,g}$ to vectors as

$$f \leftrightarrow \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{n_G} \end{pmatrix}, \qquad g \leftrightarrow \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{n_G} \end{pmatrix}, \qquad \varphi_{f,g} \leftrightarrow \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_1\eta_{n_H} \\ \vdots \\ \zeta_{n_G}\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_G} \end{pmatrix}.$$

As we have

$$A_G \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{n_G} \end{pmatrix} = d_G(1-\mu) \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{n_G} \end{pmatrix} \quad \text{and} \quad A_H \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{n_G} \end{pmatrix} = d_H(1-\nu) \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{n_G} \end{pmatrix},$$

we find

$$A^{(p)}_{G\widehat{\square}H} \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_1\eta_{n_H} \\ \vdots \\ \zeta_{n_G}\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_G} \end{pmatrix} = d_G(1-\mu) \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_1\eta_{n_H} \\ \vdots \\ \zeta_{n_G}\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_G} \end{pmatrix},$$

$$A^{(a)}_{G\widehat{\square}H} \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_1\eta_{n_H} \\ \vdots \\ \zeta_{n_G}\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_G} \end{pmatrix} = d_H(1-\nu) \begin{pmatrix} \zeta_1\eta_1 \\ \vdots \\ \zeta_1\eta_{n_H} \\ \vdots \\ \zeta_{n_G}\eta_1 \\ \vdots \\ \zeta_{n_G}\eta_{n_G} \end{pmatrix}.$$

This means that

$$\mathcal{A}^{(p)}_{G\widehat{\square}H}\varphi_{f,g} = d_G(1-\mu)\varphi_{f,g} \quad \text{and} \quad \mathcal{A}^{(a)}_{G\widehat{\square}H}\varphi_{f,g} = d_H(1-\nu)\varphi_{f,g}.$$

Since $G\widehat{\square}H$ is regular, we get the conclusion by Theorem 5.1. □

3.2. (p,q)-Laplacians of Kähler graphs of strong product type.

THEOREM 5.5. *Let $G=(V,E), H=(W,F)$ be finite regular graphs. We denote by $0=\mu_1\le\cdots\le\mu_{n_G}$ the eigenvalues of the transitional Laplacian $\Delta_{\mathcal{P}_G}$ of G, and by $0=\nu_1\le\cdots\le\nu_{n_H}$ the eigenvalues of the transitional Laplacian $\Delta_{\mathcal{P}_H}$ of H. Then the eigenvalues of the transitional (p,q) Laplacian$\Delta_{\mathcal{P}_{(p,q)}}$ of $G\widehat{\boxtimes}H$ are*

$$1-\frac{F_p\big(d_G(1-\mu_i);d_G\big)F_q\big(d_H(d_G-d_G\mu_i+1)(1-\nu_\alpha);d_H(d_G+1)\big)}{d_Gd_H(d_G+1)(d_G-1)^{p-1}(d_Gd_H+d_H-1)^{q-1}}.$$

The eigenvalues of the (p,q)-combinatorial Laplacian $\Delta_{\mathcal{A}_{(p,q)}}$ of $G\widehat{\boxtimes}H$ are

$$d_G(d_G-1)^{p-1}-\frac{F_p\big(d_G(1-\mu_i);d_G\big)F_q\big(d_H(d_G-d_G\mu_j+1)(1-\nu_\alpha);d_H(d_G+1)\big)}{d_H(d_G+1)(d_Gd_H+d_H-1)^{q-1}}.$$

PROOF. As we see in §2.2.2, we have

$$d^{(p)}_{G\widehat{\boxtimes}H}=d_G,\quad d^{(a)}_{G\widehat{\boxtimes}H}=d_H(d_G+1).$$

By using the same notations as in Theorem 4.4, the adjacency matrices of the principal and the auxiliary graphs of $G\widehat{\boxtimes}H$ are expressed as

$$A^{(p)}_{G\widehat{\boxtimes}H}=(a^{(p)}_{(i,\alpha),(j,\beta)})=\Big(a^G_{ij}\delta_{\alpha,\beta}\Big),\qquad A^{(a)}_{G\widehat{\boxtimes}H}=(a^{(a)}_{(i,\alpha),(j,\beta)})=\Big((a^G_{ij}+\delta_{ij})a^H_{\alpha,\beta}\Big)$$

by use of the adjacency matrices $A_G=(a^G_{ij})$ and $A_H=(a^H_{\alpha\beta})$ of G and H (c.f. §4.3.2). By these expressions we find that they are commutative.

We take an eigenfunction $f:V\to\mathbb{R}$ corresponding to an eigenvalue μ of $\Delta_{\mathcal{P}_G}$ and an eigenfunction $g:W\to\mathbb{R}$ corresponding to an eigenvalue ν of $\Delta_{\mathcal{P}_H}$. Then we have $\mathcal{A}_Gf=d_G(1-\mu)f$ and $\mathcal{A}_Hg=d_H(1-\nu)g$. We take a function $\varphi_{f,g}:V\times W\to\mathbb{R}$ defined by $\varphi_{f,g}(v,w)=f(v)g(w)$. By a similar computation as in the proof of Theorem 5.4, we have

$$\mathcal{A}^{(p)}_{G\widehat{\boxtimes}H}\varphi_{f,g}=d_G(1-\mu)\varphi_{f,g}\quad\text{and}\quad\mathcal{A}^{(a)}_{G\widehat{\boxtimes}H}\varphi_{f,g}=d_H(1-\nu)\{d_G(1-\mu)+1\}\varphi_{f,g}.$$

Since $G\widehat{\boxtimes}H$ is regular, we get the conclusion by Theorem 5.1. □

3.3. (p,q)-Laplacians of Kähler graphs of semi-tensor product type.

THEOREM 5.6. *Let $G=(V,E), H=(W,F)$ be finite regular graphs. We denote by $0=\mu_1\le\cdots\le\mu_{n_G}$ the eigenvalues of the transition Laplacian $\Delta_{\mathcal{P}_G}$ of G, and by $0=\nu_1\le\cdots\le\nu_{n_H}$ the eigenvalues of the transitional Laplacian $\Delta_{\mathcal{P}_H}$ of H. Then the eigenvalues of the (p,q)-probabilistic transitional Laplacian $\Delta_{\mathcal{P}_{(p,q)}}$ of $G\widehat{\otimes}H$ are*

$$1-\frac{F_p\big(d_G(1-\mu_i);d_G\big)F_q\big(d_Gd_H(1-\mu_i)(1-\nu_\alpha);d_Hd_G\big)}{d_G^2d_H(d_G-1)^{p-1}(d_Gd_H-1)^{q-1}}.$$

The eigenvalues of the combinatorial (p,q) Laplacian$\Delta_{\mathcal{A}_{(p,q)}}$ of $G\widehat{\otimes}H$ are

$$d_G(d_G-1)^{p-1}-\frac{F_p\big(d_G(1-\mu_i);d_G\big)F_q\big(d_Gd_H(1-\mu_i)(1-\nu_\alpha);d_Hd_G\big)}{d_Gd_H(d_Gd_H-1)^{q-1}}.$$

PROOF. As we see in §2.2.2, we have

$$d^{(p)}_{G\widehat{\otimes}H}=d_G,\quad d^{(a)}_{G\widehat{\otimes}H}=d_Gd_H.$$

By using the same notations as in Theorem 4.4, the adjacency matrices of the principal and the auxiliary graphs of $G\widehat{\otimes}H$ are expressed as

$$A^{(p)}_{G\widehat{\otimes}H}=(a^{(p)}_{(i,\alpha),(j,\beta)})=\Big(a^G_{ij}\delta_{\alpha,\beta}\Big),\qquad A^{(a)}_{G\widehat{\otimes}H}=(a^{(a)}_{(i,\alpha),(j,\beta)})=\Big(a^G_{ij}a^H_{\alpha,\beta}\Big)$$

by use of the adjacency matrices $A_G=(a^G_{ij})$ and $A_H=(a^H_{\alpha\beta})$ of G and H (c.f. §4.3.3). By these expressions we find that they are commutative.

We take an eigenfunction $f:V\to\mathbb{R}$ corresponding to an eigenvalue μ of $\Delta_{\mathcal{P}_G}$ and an eigenfunction $g:W\to\mathbb{R}$ corresponding to an eigenvalue ν of $\Delta_{\mathcal{P}_H}$. Then we have $\mathcal{A}_Gf=d_G(1-\mu)f$ and $\mathcal{A}_Hg=d_H(1-\nu)g$. We take a function $\varphi_{f,g}:V\times W\to\mathbb{R}$ defined by $\varphi_{f,g}(v,w)=f(v)g(w)$. By a similar computation as in the proof of Theorem 5.4, we have

$$\mathcal{A}^{(p)}_{G\widehat{\otimes}H}\varphi_{f,g}=d_G(1-\mu)\varphi_{f,g}\quad\text{and}\quad\mathcal{A}^{(a)}_{G\widehat{\otimes}H}\varphi_{f,g}=d_Gd_H(1-\mu)(1-\nu)\varphi_{f,g}.$$

Since $G\widehat{\otimes}H$ is regular, we get the conclusion by Theorem 5.1. □

3.4. (p,q)-Laplacians of Kähler graphs of lexicographical product type.

PROPOSITION 5.3. *Let $G=(V,E), H=(W,F)$ be finite regular graphs. Suppose G is connected. We denote by $0=\mu_1<\cdots\le\mu_{n_G}$ the eigenvalues of the transition Laplacian $\Delta_{\mathcal{P}_G}$ of G, by $0=\nu_1\le\cdots\le\nu_{n_H}$ the eigenvalues of the transitional Laplacian $\Delta_{\mathcal{P}_H}$ of H. Also we denote by k_H the numbers of connected components of H. Then the eigenvalues of the (p,q)-probabilistic transitional Laplacian $\Delta_{\mathcal{P}_{(p,q)}}$ of their Kähler graph $G\rhd H$ of lexicographical product type are as follows*:

(1) *When q is odd, they are $0,1$ and*
$$1-\frac{F_q\big(n_Gd_H(1-\nu_\alpha);n_Gd_H\big)}{n_Gd_H(n_Gd_H-1)^{q-1}},\qquad(\alpha=k_H+1,\dots,n_H)$$
where the first 0 appears k_H times, the second 1 appears $(n_G-1)n_H$ times.

(2) *When q is even, they are 0 and*
$$1-\frac{F_q\big(n_Gd_H(1-\nu_\alpha);n_Gd_H\big)}{n_Gd_H(n_Gd_H-1)^{q-1}},\qquad(\alpha=k_H+1,\dots,n_H),$$
$$1-\frac{(-1)^{q/2}F_p\big(d_H(1-\mu_i);d_H\big)}{d_G(d_G-1)^{p-1}(n_Gd_H-1)^{q/2}},\qquad(i=2,\dots,n_G),$$
where the first 0 appears k_H times, and each of the last form appears n_H times.

PROOF. As we see in §2.2.2, we have
$$d^{(p)}_{G\rhd H}=d_G,\quad d^{(a)}_{G\rhd H}=n_Gd_H.$$
By using the same notations as in Theorem 4.4, the adjacency matrices of the principal and the auxiliary graphs of $G\rhd H$ are expressed as
$$A^{(p)}_{G\rhd H}=(a^{(p)}_{(i,\alpha),(j,\beta)})=\Big(a^G_{ij}\delta_{\alpha\beta}\Big),\qquad A^{(a)}_{G\rhd H}=(a^{(a)}_{(i,\alpha),(j,\beta)})=\Big(a^H_{\alpha\beta}\Big)$$
by use of the adjacency matrices $A_G=(a^G_{ij})$ and $A_H=(a^H_{\alpha\beta})$ of G and H (c.f. §4.3.4). Since G is regular, we have
$$A^{(p)}_{G\rhd H}A^{(a)}_{G\rhd H}=\Big(\sum_{k=1}^{n_G}\sum_{\gamma=1}^{n_H}a^G_{ik}\delta_{\alpha\gamma}a^H_{\gamma\beta}\Big)=\big(d_Ga^H_{\alpha\beta}\big),$$
$$A^{(a)}_{G\rhd H}A^{(p)}_{G\rhd H}=\Big(\sum_{k=1}^{n_G}\sum_{\gamma=1}^{n_H}a^H_{\alpha\gamma}a^G_{kj}\delta_{\gamma\beta}\Big)=\big(d_Ga^H_{\alpha\beta}\big),$$

hence find that they are commutative.

We take an eigenfunction $f : V \to \mathbb{R}$ corresponding to an eigenvalue μ of $\Delta_{\mathcal{P}_G}$ and an eigenfunction $g : W \to \mathbb{R}$ corresponding to an eigenvalue ν of $\Delta_{\mathcal{P}_H}$. Then we have $\mathcal{A}_G f = d_G(1-\mu)f$ and $\mathcal{A}_H g = d_H(1-\nu)g$. We take a function $\varphi_{f,g} : V \times W \to \mathbb{R}$ defined by $\varphi_{f,g}(v,w) = f(v)g(w)$. We then have $\mathcal{A}^{(p)}_{G\rhd H}\varphi_{f,g} = d_G(1-\mu)\varphi_{f,g}$ and

$$\mathcal{A}^{(a)}_{G\rhd H}\varphi_{f,g} = \begin{cases} n_G d_H(1-\nu)\varphi_{f,g}, & \text{when } \mu = 0,\\ 0, & \text{when } \mu \neq 0,\end{cases}$$

because f is constant when $\mu = 0$ and $\sum_{v\in V} f(v) = 0$ when $\mu \neq 0$ by the property that G is connected. By Theorem 5.1, we find that the eigenvalues of (p,q)-probabilistic transition Laplacian $\Delta_{\mathcal{Q}_{(p,q)}}$ are

$$1 - \frac{F_p(d_G; d_G)F_q\big(n_G d_H(1-\nu_\alpha); n_G d_H\big)}{d_G(d_G-1)^{p-1} n_G d_H (n_G d_H - 1)^{q-1}}, \qquad (\alpha = 1, \dots, n_H),$$

$$1 - \frac{F_p(d_G(1-\mu_i); d_G)F_q\big(0; n_G d_H\big)}{d_G(d_G-1)^{p-1} n_G d_H (n_G d_H - 1)^{q-1}}, \qquad (i = 2, \dots, n_G),$$

where each of the former form appears k_G times and each of the latter form appears n_H time. Here, we have $F_p(d_G; d_G) = d_G(d_G-1)^{p-1}$, $F_q(n_G d_H; n_G d_H) = n_G d_H(n_G d_H - 1)^{q-1}$ and $F_{2\ell-1}(0; n_G d_H) = 0$, $F_{2\ell}(0; n_G d_H) = (-1)^{\ell} n_G d_H (n_G d_H - 1)^{\ell-1}$. Since $\mu_2 > 0$ and $\nu_1 = \cdots = \nu_{k_H} = 0$, we get the conclusion. □

We can extend the above result to the case that the former component is not regular. When q is odd, we can show that the same assertions as in Proposition 5.3 hold.

Theorem 5.7. *Let $G = (V,E), H = (W,F)$ be finite graphs. We suppose H is regular. We denote by $0 = \nu_1 \leq \cdots \leq \nu_{n_H}$ the eigenvalues of the transitional Laplacian $\Delta_{\mathcal{P}_H}$ of H, and by k_H the number of connected components of H. If q is odd, the eigenvalues of the (p,q)-probabilistic transitional Laplacian $\Delta_{\mathcal{P}_{(p,q)}}$ of their Kähler graph $G \rhd H$ of lexicographical product type are $0, 1$ and*

$$1 - \frac{F_q\big(n_G d_H(1-\nu_\alpha); n_G d_H\big)}{n_G d_H (n_G d_H - 1)^{q-1}}, \qquad (\alpha = k_H + 1, \dots, n_H)$$

where the first 0 appears k_H times, the second 1 appears $(n_G - 1)n_H$ times.

Proof. First we study the q-step adjacency operator $\mathcal{A}_{(G\triangleright H)^{(a)}_{[q]}}$ of the auxiliary graph of $G\triangleright H$. The adjacency matrix of the auxiliary graph is given as

$$A^{(a)}_{G\triangleright H}=\begin{pmatrix}A_H & \cdots & A_H\\ \vdots & & \vdots\\ A_H & \cdots & A_H\end{pmatrix},$$

hence we have

$$(A^{(a)}_{G\triangleright H})^k=\begin{pmatrix}n_G^{k-1}A_H^k & \cdots & n_G^{k-1}A_H^k\\ \vdots & & \vdots\\ n_G^{k-1}A_H^k & \cdots & n_G^{k-1}A_H^k\end{pmatrix}.$$

As a matter of fact, this holds when $k=1$. If this holds for k, we have

$$\begin{aligned}(A^{(a)}_{G\triangleright H})^{k+1}&=(A^{(a)}_{G\triangleright H})^{k}A^{(a)}_{G\triangleright H}\\ &=\begin{pmatrix}n_G^{k-1}A_H^k & \cdots & n_G^{k-1}A_H^k\\ \vdots & & \vdots\\ n_G^{k-1}A_H^k & \cdots & n_G^{k-1}A_H^k\end{pmatrix}\begin{pmatrix}A_H & \cdots & A_H\\ \vdots & & \vdots\\ A_H & \cdots & A_H\end{pmatrix}=\begin{pmatrix}n_G^{k}A_H^{k+1} & \cdots & n_G^{k}A_H^{k+1}\\ \vdots & & \vdots\\ n_G^{k}A_H^{k+1} & \cdots & n_G^{k}A_H^{k+1}\end{pmatrix}.\end{aligned}$$

Thus by mathematical induction we find that $(A^{(a)}_{G\triangleright H})^k$ is of the form.

We now study the q-step probabilistic transition matrix

$$Q_{(G\triangleright H)^{(a)}_{[q]}}=\{n_Gd_H(d_Gd_H-1)^{q-1}\}^{-1}F_q(A^{(a)}_{G\triangleright H};n_Gd_H)$$

of the auxiliary graph. We here show

$$\begin{aligned}&F_q(A^{(a)}_{G\triangleright H};n_Gd_H)\\ &=\begin{cases}\begin{pmatrix}N_q & \cdots & N_q\\ \vdots & & \vdots\\ N_q & \cdots & N_q\end{pmatrix}, & \text{when } q \text{ is odd},\\[2ex] \begin{pmatrix}N_q & \cdots & N_q\\ \vdots & & \vdots\\ N_q & \cdots & N_q\end{pmatrix}+(-1)^{q/2}n_Gd_H(n_Gd_H-1)^{(q/2)-1}I, & \text{when } q \text{ is even},\end{cases}\end{aligned}$$

where

$$N_q=\begin{cases}n_G^{-1}F_q(n_GA_H;n_Gd_H), & \text{when } q \text{ is odd},\\ n_G^{-1}F_q(n_GA_H;n_Gd_H)-(-1)^{q/2}d_H(n_Gd_H-1)^{(q/2)-1}I, & \text{when } q \text{ is even},\end{cases}$$

by mathematical induction. Since we have

$$F_1(A^{(a)}_{G\triangleright H};n_Gd_H)=A^{(a)}_{G\triangleright H},$$

$$\begin{aligned}
&F_2(A^{(a)}_{G\rhd H}; n_G d_H) = (A^{(a)}_{G\rhd H})^2 - n_G d_H I \\
&\quad = \begin{pmatrix} n_G^{-1}(n_G A_H)^2 & \cdots & n_G^{-1}(n_G A_H)^2 \\ \vdots & & \vdots \\ n_G^{-1}(n_G A_H)^2 & \cdots & n_G^{-1}(n_G A_H)^2 \end{pmatrix} - n_G d_H I \\
&\quad = \begin{pmatrix} n_G^{-1}\{F_2(n_G A_H; n_G d_H) + n_G d_H I\} & \cdots & n_G^{-1}\{F_2(n_G A_H; n_G d_H) + n_G d_H I\} \\ \vdots & & \vdots \\ n_G^{-1}\{F_2(n_G A_H; n_G d_H) + n_G d_H I\} & \cdots & n_G^{-1}\{F_2(n_G A_H; n_G d_H) + n_G d_H I\} \end{pmatrix} - n_G d_H I,
\end{aligned}$$

the above expressions hold for $q = 1, 2$. We here suppose that the above expression holds for $1 \le q \le 2\ell$ $(\ell \ge 1)$. As we have

$$F_{q+1}(A^{(a)}_{G\rhd H}; n_G d_H) = F_q(A^{(a)}_{G\rhd H}; n_G d_H) A^{(a)}_{G\rhd H} - (n_G d_H - 1) F_{q-1}(A^{(a)}_{G\rhd H}; n_G d_H),$$

we find that

$$\begin{aligned}
&F_{2\ell+1}(A^{(a)}_{G\rhd H}; n_G d_H) \\
&\quad = \begin{pmatrix} N_{2\ell} & \cdots & N_{2\ell} \\ \vdots & & \vdots \\ N_{2\ell} & \cdots & N_{2\ell} \end{pmatrix} \begin{pmatrix} A_H & \cdots & A_H \\ \vdots & & \vdots \\ A_H & \cdots & A_H \end{pmatrix} + (-1)^\ell n_G d_H (n_G d_H - 1)^{\ell-1} A^{(a)}_{G\rhd H} \\
&\qquad - (n_G d_H - 1) \begin{pmatrix} N_{2\ell-1} & \cdots & N_{2\ell-1} \\ \vdots & & \vdots \\ N_{2\ell-1} & \cdots & N_{2\ell-1} \end{pmatrix} \\
&\quad = \begin{pmatrix} n_G N_{2\ell} A_H & \cdots & n_G N_{2\ell} A_H \\ \vdots & & \vdots \\ n_G N_{2\ell} A_H & \cdots & n_G N_{2\ell} A_H \end{pmatrix} + (-1)^\ell n_G d_H (n_G d_H - 1)^{\ell-1} A^{(a)}_{G\rhd H} \\
&\qquad - (n_G d_H - 1) \begin{pmatrix} N_{2\ell-1} & \cdots & N_{2\ell-1} \\ \vdots & & \vdots \\ N_{2\ell-1} & \cdots & N_{2\ell-1} \end{pmatrix}.
\end{aligned}$$

As we have

$$\begin{aligned}
&n_G N_{2\ell} A_H + (-1)^\ell n_G d_H (n_G d_H - 1)^{\ell-1} A_H - (n_G d_H - 1) N_{2\ell-1} \\
&\quad = n_G^{-1}\big\{F_{2\ell}(n_G A_H; n_G d_H) n_G A_H - (n_G d_H - 1) F_{2\ell-1}(n_G A_H; n_G d_H)\big\} \\
&\quad = n_G^{-1} F_{2\ell+1}(n_G A_H; n_G d_H),
\end{aligned}$$

we find that the expression of $F_q(A^{(a)}; n_G d_H)$ holds for $q = 2\ell+1$. We therefore have

$$\begin{aligned}
&F_{2\ell+2}(A^{(a)}; n_g d_H) \\
&= \begin{pmatrix} n_G N_{2\ell+1} A_H & \cdots & n_G N_{2\ell+1} A_H \\ \vdots & & \vdots \\ n_G N_{2\ell+1} A_H & \cdots & n_G N_{2\ell+1} A_H \end{pmatrix} \\
&\qquad - (n_G d_H - 1) \begin{pmatrix} N_{2\ell} & \cdots & N_{2\ell} \\ \vdots & & \vdots \\ N_{2\ell} & \cdots & N_{2\ell} \end{pmatrix} - (-1)^{\ell} n_G d_H (n_G d_H - 1)^{\ell} I_{n_H}.
\end{aligned}$$

As we have

$$\begin{aligned}
&n_G N_{2\ell+1} A_H - (n_G d_H - 1) N_{2\ell} \\
&= n_G^{-1} \big\{ F_{2\ell+1}(n_G A_H; n_G d_H) n_G A_H - (n_G d_H - 1) F_{2\ell}(n_G A_H; n_G d_H) \big\} \\
&\quad - (-1)^{\ell+1} n_G d_H (n_G d_H - 1)^{\ell} I_{n_H} \\
&= n_G^{-1} F_{2\ell+2}(n_G A_H; n_G d_H) - (-1)^{\ell+1} n_G d_H (n_G d_H - 1)^{\ell} I_{n_H},
\end{aligned}$$

we find that the expression of $F_q(A^{(a)}; n_G d_H)$ holds for $q = 2\ell+2$. Thus we get the form of $F_q(A^{(a)}; n_G d_H)$ by induction.

We set $N_q = (b_{\alpha\beta;q})$. Then $Q_{(G\triangleright H)^{(a)}_{[q]}}$ is expressed as

$$\begin{cases} (c_{\alpha\beta;q}), & \text{when } q \text{ is odd}, \\ (c_{\alpha\beta;q}) + (-1)^{q/2}(n_G d_H - 1)^{-q/2}(\delta_{ij}\delta_{\alpha\beta}), & \text{when } q \text{ is even}, \end{cases}$$

where $c_{\alpha\beta;q} = b_{\alpha\beta;q}/\{n_G d_H (n_G d_H - 1)^{q-1}\}$. We denote by $A_{G_{[p]}} = (a^G_{ij;p})$ the p-step adjacency matrix of G. Then denoting by $d_{G:p}(v)$ cardinality of the set of all p-step paths without backtracking emanating from v, we have $d_{G:p}(v_i) = \sum_j a^G_{ij;p}$. Since the principal graph of $G \triangleright H$ is isomorphic to a disjoint union of n_H-copies of G, the p-step adjacency matrix of the principal graph of $G \triangleright H$ is expressed as $A_{(G\triangleright H)^{(p)}_{[p]}} = (a^{(p)}_{(i,\alpha),(j,\beta);p}) = (a^G_{ij;p}\delta_{\alpha\beta})$.

When q is odd, the (p,q)-probabilistic transition matrix $Q_{G\triangleright H_{(p,q)}}$ is given as

$$\begin{aligned}
Q_{G\triangleright H_{(p,q)}} &= Q_{(G\triangleright H)^{(p)}_{[p]}} Q_{(G\triangleright H)^{(a)}_{[q]}} \\
&= \big(d_{G;p}(i)^{-1} a^G_{ij;p}\delta_{\alpha\beta}\big)(c_{\alpha\beta;q}) = \Big(d_{G;p}(i)^{-1}\big(\sum_j a^G_{ij;p}\big) c_{\alpha\beta;q}\Big) = (c_{\alpha\beta;q}),
\end{aligned}$$

where $d_{G;p}(i)$ denotes the cardinality of p-step paths on G without backtracking emanating from v_i. We define functions ϵ_k $(k = 1, \ldots, n_G)$ on V by $\epsilon_1 \equiv 1$, and by

$\epsilon_k = \delta_{v_1} - \delta_{v_i}$ for $2 \le k \le n_G$. For a function g_α on W satisfying $\Delta_{\mathcal{P}_H} g = \nu_\gamma g_\gamma$ we define a function $\varphi_{\epsilon_k, g_\gamma}$ on $V \times W$ by $\varphi_{\epsilon_k, g_\gamma}(v,w) = \epsilon_i(v) g_\gamma(w)$. If we represent $\varphi_{\epsilon_k, g_\gamma}$ by vectors $(\zeta_i^{(k)} \eta_\alpha^{(\gamma)})$, we have

$$\begin{aligned} Q_{G\rhd H_{(p,q)}}(\zeta_j^{(k)}\eta_\beta^{(\gamma)}) &= \Big(\sum_{j=1}^{n_G}\sum_{\beta=1}^{n_H} c_{\alpha\beta;q}\zeta_j^{(k)}\eta_\beta^{(\gamma)}\Big) \\ &= \begin{cases} \Big(n_G \sum_{\beta=1}^{n_H} c_{\alpha\beta;q}\eta_\beta^{(\gamma)}\Big), & \text{when } k=1, \\ 0, & \text{when } k \ne 1. \end{cases} \end{aligned}$$

Hence we have

$$\mathcal{Q}_{p,q}\psi_{1,\alpha} = \frac{F_q(n_G d_H(1-\nu_\gamma); n_G d_H)}{n_G d_H (n_G d_H - 1)^{q-1}} \varphi_{\epsilon_1, g_\gamma}, \qquad \mathcal{Q}_{p,q}\varphi_{\epsilon_k, g_\gamma} = 0 \quad (k = 2, \dots, n_G),$$

and get the conclusion. □

EXAMPLE 5.4. Let G be a 3-circuit and H be a 4-circuit. The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0, \frac{3}{2}, \frac{3}{2}\}$ and $\{0,1,1,2\}$, respectively. We take their Kähler graph $G \rhd H$ of lexicographical product type. The eigenvalues of some (p,q)-probabilistic transition Laplacian are as follows:

$$\begin{aligned} &\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(1,1)}}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(1,3)}}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(2,1)}}}\big) \\ &= \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(2,3)}}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(3,1)}}}\big) \\ &\qquad = \{0,1,1,1,1,1,1,1,1,1,1,2\}, \end{aligned}$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(1,2)}}}\big) = \Big\{0,0,\frac{9}{10},\frac{9}{10},\frac{9}{10},\frac{9}{10},\frac{9}{10},\frac{9}{10},\frac{9}{10},\frac{9}{10},\frac{6}{5},\frac{6}{5}\Big\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(3,2)}}}\big) = \Big\{0,0,\frac{6}{5},\frac{6}{5},\frac{6}{5},\frac{6}{5},\frac{6}{5},\frac{6}{5},\frac{6}{5},\frac{6}{5},\frac{6}{5},\frac{6}{5}\Big\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(3,4)}}}\big) = \Big\{0,0,\frac{24}{25},\frac{24}{25},\frac{24}{25},\frac{24}{25},\frac{24}{25},\frac{24}{25},\frac{24}{25},\frac{24}{25},\frac{24}{25},\frac{24}{25}\Big\}$$

Since G is a 3-circuit and H is a 4-circuit, we have $\mathcal{Q}_{(G\rhd H)^{(p)}_{[3\ell+1]}} = \mathcal{Q}_{(G\rhd H)^{(p)}_{[3\ell+1]}} = \mathcal{Q}_{(G\rhd H)^{(p)}}$ and $\mathcal{Q}_{(G\rhd H)^{(p)}_{[3\ell]}} = \mathcal{Q}_{(G\rhd H)^{(p)}_{[3]}}$. We therefore have

$$\begin{aligned} \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(3\ell+1,q)}}}\big) &= \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(3\ell+2,q)}}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(1,q)}}}\big), \\ \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(3\ell,q)}}}\big) &= \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(3,q)}}}\big). \end{aligned}$$

We note $F_2(t;6)=t^2-6,\ F_3(t;6)=t^3-11t,\ F_4(t;6)=t^4-16t^2+30$.

Example 5.5. Let G be a non-regular graph with $n_G=4$ and H be a 4-circuit. The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0,1,\frac{4}{3},\frac{5}{3}\}$ and $\{0,1,1,2\}$, respectively. We take their Kähler graph $G\triangleright H$ of lexicographical product type. The eigenvalues of some (p,q)-probabilistic transition Laplacian are as follows:

$$\begin{aligned}\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(1,1)}}}\big) &= \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(1,3)}}}\big)=\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(1,5)}}}\big)\\ &=\{0,1,1,1,1,1,1,1,1,1,1,2\},\end{aligned}$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(1,2)}}}\big)=\left\{0,0,\frac{19}{21},\frac{19}{21},\frac{19}{21},\frac{19}{21},\frac{20}{21},\frac{20}{21},\frac{20}{21},\frac{20}{21},1,1,1,1,\frac{8}{7},\frac{8}{7}\right\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(1,4)}}}\big)=\left\{0,0,\frac{48}{49},\frac{48}{49},1,1,1,1,\frac{148}{147},\frac{148}{147},\frac{148}{147},\frac{148}{147},\frac{149}{147},\frac{149}{147},\frac{149}{147},\frac{149}{147}\right\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(1,6)}}}\big)=\left\{0,0,\frac{1027}{1029},\frac{1027}{1029},\frac{1027}{1029},\frac{1027}{1029},\frac{1028}{1029},\frac{1028}{1029},\frac{344}{343},\frac{344}{343}\right\}.$$

Example 5.6. Let G be a union of two 3-circuit and H be a 4-circuit. The eigenvalues of $\Delta_{\mathcal{P}_H}$ are $\{0,1,1,2\}$. We take their Kähler graph $G\triangleright H$ of lexicographical product type (see Example 4.22 in §4.3). The eigenvalues of some (p,q)-probabilistic transition Laplacian are as follows:

$$\begin{aligned}&\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(1,3)}}}\big)=\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(1,5)}}}\big)=\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(2,1)}}}\big)=\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(3,1)}}}\big)\\ &=\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(4,1)}}}\big)=\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(5,1)}}}\big)=\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(2,3)}}}\big)=\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(4,3)}}}\big)\\ &\qquad=\{0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2\},\end{aligned}$$

$$\begin{aligned}\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(1,2)}}}\big)&=\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(5,2)}}}\big)\\ &=\left\{\begin{array}{l}0,0,\frac{21}{22},\frac{21}{22},\frac{21}{22},\frac{21}{22},\frac{21}{22},\frac{21}{22},\frac{21}{22},\frac{21}{22},\frac{21}{22},\frac{21}{22},\frac{21}{22},\frac{21}{22},\\ \frac{21}{22},\frac{21}{22},\frac{21}{22},\frac{21}{22},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11}\end{array}\right\},\end{aligned}$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(3,2)}}}\big)=\left\{\begin{array}{l}0,0,\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\\ \frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11},\frac{12}{11}\end{array}\right\},$$

$$
\begin{aligned}
\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(1,4)}}}\big) &= \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(5,4)}}}\big)\\
&= \left\{\begin{array}{l} 0,0,\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\\ \frac{243}{242},\frac{243}{242},\frac{243}{242},\frac{243}{242},\frac{243}{242},\frac{243}{242},\frac{243}{242},\frac{243}{242},\\ \frac{243}{242},\frac{243}{242},\frac{243}{242},\frac{243}{242},\frac{243}{242},\frac{243}{242},\frac{243}{242},\frac{243}{242}\end{array}\right\},
\end{aligned}
$$

$$
\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(3,4)}}}\big) = \left\{\begin{array}{l} 0,0,\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\\ \frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\frac{120}{121},\end{array}\right\}.
$$

Since G is a union of two 3-circuits, we note that $\mathcal{Q}_{(G\rhd H)^{(p)}_{[3\ell+1]}} = \mathcal{Q}_{(G\rhd H)^{(p)}_{[3\ell+1]}} = \mathcal{Q}_{(G\rhd H)^{(p)}}$ and $\mathcal{Q}_{(G\rhd H)^{(p)}_{[3\ell]}} = \mathcal{Q}_{(G\rhd H)^{(p)}_{[3]}}$. We therefore have

$$
\begin{aligned}
\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(3\ell+1,q)}}}\big) &= \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(3\ell+2,q)}}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(1,q)}}}\big),\\
\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(3\ell,q)}}}\big) &= \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(3,q)}}}\big),
\end{aligned}
$$

for an arbitrary positive integer q.

EXAMPLE 5.7. Let G be a union of a 3-circuit and a 4-circuit, and H be a 4-circuit. The eigenvalues of $\Delta_{\mathcal{P}_G}$ and $\Delta_{\mathcal{P}_H}$ are $\{0,0,1,1,\frac{3}{2},\frac{3}{2},2\}$ and $\{0,1,1,2\}$, respectively. We take their Kähler graph $G\rhd H$ of lexicographical product type (see Example 5.7 in §4.3). The eigenvalues of some (p,q)-probabilistic transition Laplacian are as follows:

$$
\begin{aligned}
\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(1,3)}}}\big) &= \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(1,5)}}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(2,1)}}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(3,1)}}}\big)\\
= \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(4,1)}}}\big) &= \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(5,1)}}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(2,3)}}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(4,3)}}}\big)\\
&= \{0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2\},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(1,2)}}}\big) &= \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\rhd H)_{(5,2)}}}\big)\\
&= \left\{\begin{array}{l} 0,0,\frac{12}{13},\frac{12}{13},\frac{12}{13},\frac{12}{13},\frac{25}{26},\frac{25}{26},\frac{25}{26},\frac{25}{26},\frac{25}{26},\frac{25}{26},\frac{25}{26},\frac{25}{26},\\ 1,1,1,1,1,1,1,1,\frac{14}{13},\frac{14}{13},\frac{14}{13},\frac{14}{13},\frac{14}{13},\frac{14}{13}\end{array}\right\},
\end{aligned}
$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(1,4)}}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(5,4)}}}\big) = \left\{ \begin{array}{l} 0, 0, \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, 1, 1, 1, 1, 1, 1, 1, 1, \\ \frac{338}{339}, \frac{338}{339}, \frac{338}{339}, \frac{338}{339}, \frac{338}{339}, \frac{338}{339}, \frac{338}{339}, \frac{338}{339}, \frac{170}{169}, \frac{170}{169}, \frac{170}{169}, \frac{170}{169} \end{array} \right\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(3,2)}}}\big) = \left\{ \begin{array}{l} 0, 0, \frac{12}{13}, \frac{12}{13}, \frac{12}{13}, \frac{12}{13}, 1, 1, 1, 1, 1, 1, 1, 1, \\ \frac{14}{13}, \frac{14}{13}, \frac{14}{13}, \frac{14}{13}, \frac{14}{13}, \frac{14}{13}, \frac{14}{13}, \frac{14}{13}, \frac{14}{13}, \frac{14}{13}, \frac{14}{13}, \frac{14}{13}, \frac{14}{13}, \frac{14}{13}, \end{array} \right\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{Q}_{(G\triangleright H)_{(3,4)}}}\big) = \left\{ \begin{array}{l} 0, 0, \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, \\ \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, \frac{168}{169}, \\ 1, 1, 1, 1, 1, 1, 1, 1, \frac{170}{169}, \frac{170}{169}, \frac{170}{169}, \frac{170}{169} \end{array} \right\}.$$

3.5. **(p,q)-isospectrality of Kähler graphs of product types whose principal graphs are union of original graphs.** We here summarize conditions for isospectral Kähler graphs of product types discussed in this section.

Corollary 5.4. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular finite graphs. We suppose $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. Then their Kähler graphs $G_1 \widehat{\square} H_1$, $G_2 \widehat{\square} H_2$ of Cartesian product type are (p,q)-isospectral for an arbitrary pair (p,q) of relatively prime positive integers.*

Corollary 5.5. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular finite graphs. We suppose $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. Then their Kähler graphs $G_1 \widehat{\boxtimes} H_1$, $G_2 \widehat{\boxtimes} H_2$ of strong product type are (p,q)-isospectral for an arbitrary pair (p,q) of relatively prime positive integers.*

Corollary 5.6. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular finite graphs. We suppose $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. Then their Kähler graphs $G_1 \widehat{\otimes} H_1$, $G_2 \widehat{\otimes} H_2$ of semi-tensor product type are (p,q)-isospectral for an arbitrary pair (p,q) of relatively prime positive integers.*

Corollary 5.7. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular finite graphs. We suppose that G_1, G_2 are connected and that $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$ hold. Then their Kähler graphs $G_1 \triangleright H_1$, $G_2 \triangleright H_2$ of lexicographical product type are (p,q)-isospectral for an arbitrary pair (p,q) of relatively prime positive integers.*

Proposition 5.4. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular finite graphs. We suppose that G_1, G_2 have the same numbers of connected components (i.e. $k_{G_1} = k_{G_2}$) and that $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$ hold. Then their Kähler graphs $G_1 \triangleright H_1$, $G_2 \triangleright H_2$ of lexicographical product type are (p,q)-isospectral for an arbitrary pair (p,q) of relatively prime positive integers with odd q.*

4. (p,q)-Laplacians of Kähler graphs of product type obtained by commutative operations

In this section we treat Kähler graph of product type made by regular graphs. By making use of Theorem 5.1, we calculated eigenvalues of the following Kähler graphs produce type; $G \boxplus H,\ G \boxdot H,\ G\Diamond H,\ G \divideontimes H, G\spadesuit H$ and $G\clubsuit H$.

4.1. (p,q)-Laplacians of Kähler graphs of Cartesian-tensor product type.

THEOREM 5.8. *Let $G=(V,E), H=(W,F)$ be finite regular ordinary graphs. We denote by $0=\mu_1\le\cdots\le\mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0=\nu_1\le\cdots\le\nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(p,q)}}$ for their Kähler graph $G\boxplus H$ are*

$$1-\frac{F_p\big(d_G(1-\mu_i)+d_H(1-\nu_\alpha);d_G+d_H\big)F_q\big(d_Gd_H(1-\mu_i)(1-\nu_\alpha);d_Gd_H\big)}{d_Gd_H(d_G+d_H)(d_G+d_H-1)^{p-1}(d_Gd_H-1)^{q-1}}$$

for $i=1,\cdots,n_G$ and $\alpha=1,\ldots,n_H$.

PROOF. As we see in §2.2.2, we have

$$d^{(p)}_{G\boxplus H}=d_G+d_H,\quad d^{(a)}_{G\boxplus H}=d_Gd_H.$$

By using the same notations as in Theorem 4.16, the adjacency matrices of the principal and the auxiliary graphs of $G\boxplus H$ are expressed as

$$A^{(p)}_{G\boxplus H}=(a^{(p)}_{(i,\alpha),(j,\beta)})=\Big(a^G_{ij}\delta_{\alpha\beta}+\delta_{ij}a^H_{\alpha\beta}\Big),\qquad A^{(a)}_{G\boxplus H}=(a^{(a)}_{(i,\alpha),(j,\beta)})=\Big(a^G_{ij}a^H_{\alpha\beta}\Big)$$

by use of the adjacency matrices $A_G=(a^G_{ij})$ and $A_H=(a^H_{\alpha\beta})$ of G and H (see §4.6.1). By these expressions we find that they are commutative.

We take an eigenfunction $f:V\to\mathbb{R}$ corresponding to an eigenvalue μ of $\Delta_{\mathcal{P}_G}$ and an eigenfunction $g:W\to\mathbb{R}$ corresponding to an eigenvalue ν of $\Delta_{\mathcal{P}_H}$. Then we have $\mathcal{A}_Gf=d_G(1-\mu)f$ and $\mathcal{A}_Hg=d_H(1-\nu)g$. We take a function $\varphi_{f,g}:V\times W\to\mathbb{R}$ defined by $\varphi_{f,g}(v,w)=f(v)g(w)$. By the same computation as in the proof of Theorem

4.16, we obtain

$$\mathcal{A}^{(p)}_{G\boxplus H}\varphi_{f,g} = \{d_G(1-\mu)+d_H(1-\nu)\}\varphi_{f,g},$$
$$\mathcal{A}^{(a)}_{G\boxplus H}\varphi_{f,g} = d_G(1-\mu)d_H(1-\nu)\varphi_{f,g}.$$

Hence we get the conclusion by Theorem 5.1. □

COROLLARY 5.8. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular finite graphs. We suppose $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. Then their Kähler graphs $G_1 \boxplus H_1$, $G_2 \boxplus H_2$ of Cartesian-tensor product type are (p,q)-isospectral for an arbitrary pair (p,q) of relatively prime positive integers.*

4.2. (p,q)-Laplacians of Kähler graphs of Cartesian-complement product type.

THEOREM 5.9. *Let $G=(V,E), H=(W,F)$ be finite regular ordinary graphs. Suppose G and H are connected. We denote by $0=\mu_1<\mu_2\le\cdots\le\mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0=\nu_1<\nu_2\le\cdots\le\nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(p,q)}}$ for their Kähler graph $G\boxdot H$ are*

$0,$

$$1-\frac{F_p\big(d_G+d_H(1-\nu_\alpha)\,;\,d_G+d_H\big)}{(d_G+d_H)(d_G+d_H-1)^{p-1}} \times\frac{F_q\big(d_H(1-\nu_\alpha)(n_G-2d_G-1)-d_G\,;\,d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\big)}{\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\}\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)-1\}^{q-1}},$$

$$1-\frac{F_p\big(d_G(1-\mu_i)+d_H\,;\,d_G+d_H\big)}{(d_G+d_H)(d_G+d_H-1)^{p-1}} \times\frac{F_q\big(d_G(1-\mu_i)(n_H-2d_H-1)-d_H\,;\,d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\big)}{\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\}\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)-1\}^{q-1}},$$

$$1-\frac{F_p\big(d_G(1-\mu_i)+d_H(1-\nu_\alpha)\,;\,d_G+d_H\big)}{(d_G+d_H)(d_G+d_H-1)^{p-1}} \times\frac{F_q\big(-2d_Gd_H(1-\mu_i)(1-\nu_\alpha)-d_G(1-\mu_i)-d_H(1-\nu_\alpha)\,;\,d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\big)}{\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\}\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)-1\}^{q-1}}.$$

for $i=2,\cdots,n_G$ and $\alpha=2,\ldots,n_H$.

PROOF. As we see in §2.2.2, we have

$$d^{(p)}_{G\boxdot H} = d_G + d_H, \quad d^{(a)}_{G\boxdot H} = d_G(n_H - d_H - 1) + d_H(n_G - d_G - 1).$$

By using the same notations as in Theorem 4.17, the adjacency matrices of the principal and the auxiliary graphs of $G \boxdot H$ are expressed as

$$A^{(p)}_{G\boxdot H} = (a^{(p)}_{(i,\alpha),(j,\beta)}) = \Big(a^G_{ij}\delta_{\alpha\beta} + \delta_{ij}a^H_{\alpha\beta}\Big), \qquad A^{(a)}_{G\boxdot H} = (a^{(a)}_{(i,\alpha),(j,\beta)}) = \Big(a^G_{ij}a^{H^c}_{\alpha\beta} + a^{G^c}_{ij}a^H_{\alpha\beta}\Big)$$

by use of the adjacency matrices $A_G = (a^G_{ij})$, $A_H = (a^H_{\alpha\beta})$, $A_{G^c} = (a^{G^c}_{ij})$, $A_{H^c} = (a^{H^c}_{\alpha\beta})$ of G, H and their complement graphs G^c, H^c (see §4.6.2). Since $a^{G^c}_{ij} = 1 - \delta_{ij} - a^G_{ij}$ and $a^{H^c}_{\alpha\beta} = 1 - \delta_{\alpha\beta} - a^H_{\alpha\beta}$, by these expressions we find that they are commutative.

We take an eigenfunction $f : V \to \mathbb{R}$ corresponding to an eigenvalue μ of $\Delta_{\mathcal{P}_G}$ and an eigenfunction $g : W \to \mathbb{R}$ corresponding to an eigenvalue ν of $\Delta_{\mathcal{P}_H}$. Then we have $\mathcal{A}_G f = d_G(1-\mu)f$, $\mathcal{A}_H g = d_H(1-\nu)g$ and

$$\mathcal{A}_{G^c} f = \begin{cases} (n_G - d_G - 1)f, & \text{when } \mu = 0, \\ \{-1 - d_G(1-\mu)\}f, & \text{when } \mu \neq 0, \end{cases}$$

$$\mathcal{A}_{H^c} g = \begin{cases} (n_H - d_H - 1)g, & \text{when } \nu = 0, \\ \{-1 - d_H(1-\nu)\}g, & \text{when } \nu \neq 0. \end{cases}$$

We consider the function $\varphi_{f,g} : V \times W \to \mathbb{R}$ defined by $\varphi_{f,g}(v,w) = f(v)g(w)$. By the expressions of adjacency matrices of principal and the auxiliary graphs, doing the same computation as in the proof of Theorem 4.17, we have

$$\mathcal{A}^{(p)}_{G\boxdot H}\varphi_{f,g} = \{d_G(1-\mu) + d_H(1-\nu)\}\varphi_{f,g},$$

$$\mathcal{A}^{(a)}_{G\boxdot H}\varphi_{f,g} = \begin{cases} \{d_G(n_H - d_H - 1) + d_H(n_G - d_G - 1)\}\varphi_{f,g}, & \mu = \nu = 0, \\ \{d_G(d_H\nu - d_H - 1) + d_H(1-\nu)(n_G - d_G - 1)\}\varphi_{f,g}, & \mu = 0, \nu \neq 0, \\ \{d_G(1-\mu)(n_H - d_H - 1) + d_H(d_G\mu - d_G - 1)\}\varphi_{f,g}, & \mu \neq 0, \nu = 0, \\ \{d_G(1-\mu)(n_H - d_H - 1) + d_H(1-\nu)(n_G - d_G - 1)\}\varphi_{f,g}, & \mu \neq 0, \nu \neq 0. \end{cases}$$

Hence we get the conclusion by Theorem 5.1. □

COROLLARY 5.9. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular finite connected graphs. We suppose $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. Then their Kähler graphs $G_1 \boxdot H_1$, $G_2 \boxdot H_2$ of Cartesian-complement product type are (p,q)-isospectral for an arbitrary pair (p,q) of relatively prime positive integers.*

4.3. (p,q)**-Laplacians of Kähler graphs of Cartesian-lexicographical product type.**

THEOREM 5.10. *Let $G=(V,E), H=(W,F)$ be finite regular ordinary graphs. Suppose G and H are connected. We denote by $0=\mu_1<\mu_2\le\cdots\le\mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0=\nu_1<\nu_2\le\cdots\le\nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues of $\Delta_{\mathcal{Q}_{(p,q)}}$ for their Kähler graph $G\lozenge H$ are*

$$\begin{aligned}
&0,\\
&1-\frac{F_p\bigl(d_G+d_H(1-\nu_\alpha)\,;\,d_G+d_H\bigr)}{(d_G+d_H)(d_G+d_H-1)^{p-1}}\\
&\qquad\times\frac{F_q\bigl(d_H(1-\nu_\alpha)(n_G-1)-d_G\,;\,d_G(n_H-1)+d_H(n_G-1)\bigr)}{\{d_G(n_H-1)+d_H(n_G-1)\}\{d_G(n_H-1)+d_H(n_G-1)-1\}^{q-1}},\\
&1-\frac{F_p\bigl(d_G(1-\mu_i)+d_H\,;\,d_G+d_H\bigr)}{(d_G+d_H)(d_G+d_H-1)^{p-1}}\\
&\qquad\times\frac{F_q\bigl(d_G(1-\mu_i)(n_H-1)-d_H\,;\,d_G(n_H-1)+d_H(n_G-1)\bigr)}{\{d_G(n_H-1)+d_H(n_G-1)\}\{d_G(n_H-1)+d_H(n_G-1)-1\}^{q-1}},\\
&1-\frac{F_p\bigl(d_G(1-\mu_i)+d_H(1-\nu_\alpha)\,;\,d_G+d_H\bigr)}{(d_G+d_H)(d_G+d_H-1)^{p-1}}\\
&\qquad\times\frac{F_q\bigl(-d_G(1-\mu_i)-d_H(1-\nu_\alpha)\,;\,d_G(n_H-1)+d_H(n_G-1)\bigr)}{\{d_G(n_H-1)+d_H(n_G-1)\}\{d_G(n_H-1)+d_H(n_G-1)-1\}^{q-1}}.
\end{aligned}$$

for $i=2,\cdots,n_G$ and $\alpha=2,\ldots,n_H$.

PROOF. As we see in §2.2.2, we have

$$d^{(p)}_{G\lozenge H}=d_G+d_H,\quad d^{(a)}_{G\lozenge H}=d_H(n_G-1)+d_G(n_H-1).$$

By using the same notations as in Theorem 4.18, the adjacency matrices of the principal and the auxiliary graphs of $G\lozenge H$ are expressed as

$$\begin{aligned}
A^{(p)}_{G\lozenge H}&=(a^{(p)}_{(i,\alpha),(j,\beta)})=\Bigl(a^G_{ij}\delta_{\alpha\beta}+\delta_{ij}a^H_{\alpha\beta}\Bigr),\\
A^{(a)}_{G\lozenge H}&=(a^{(a)}_{(i,\alpha),(j,\beta)})=\Bigl(a^G_{ij}(1-\delta_{\alpha\beta})+a^H_{\alpha\beta}(1-\delta_{ij})\Bigr)
\end{aligned}$$

by use of the adjacency matrices $A_G=(a^G_{ij}), A_H=(a^H_{\alpha\beta})$ of G and H (see §4.6.3). We take an eigenfunction $f:V\to\mathbb{R}$ corresponding to an eigenvalue μ of $\Delta_{\mathcal{P}_G}$ and an eigenfunction $g:W\to\mathbb{R}$ corresponding to an eigenvalue ν of $\Delta_{\mathcal{P}_H}$, and consider the function $\varphi_{f,g} : V \times W \to \mathbb{R}$ defined by $\varphi_{f,g}(v,w) = f(v)g(w)$. Then we have $\mathcal{A}_G f = d_G(1-\mu)f$ and $\mathcal{A}_H g = d_H(1-\nu)g$. Since we have

$$\sum_{v\in V} f(v) = \begin{cases} n_G f(*), & \text{when } \mu = 0,\\ 0, & \text{when } \mu \neq 0, \end{cases} \qquad \sum_{w\in W} g(w) = \begin{cases} n_H g(*), & \text{when } \nu = 0,\\ 0, & \text{when } \nu \neq 0, \end{cases}$$

by the expressions of adjacency matrices of principal and the auxiliary graphs, doing the same computation as in the proof of Theorem 4.18, we have

$$\mathcal{A}^{(p)}_{G\Diamond H}\varphi_{f,g} = \{d_G(1-\mu) + d_H(1-\nu)\}\varphi_{f,g},$$

$$\mathcal{A}^{(a)}_{G\Diamond H}\varphi_{f,g} = \begin{cases} \{d_G(n_H - 1) + d_H(n_G - 1)\}\varphi_{f,g}, & \mu = \nu = 0,\\ \{-d_G + d_H(1-\nu)(n_G-1)\}\varphi_{f,g}, & \mu = 0, \nu \neq 0,\\ \{d_G(1-\mu)(n_H-1) - d_H\}\varphi_{f,g}, & \mu \neq 0, \nu = 0,\\ \{-d_G(1-\mu) - d_H(1-\nu)\}\varphi_{f,g}, & \mu \neq 0, \nu \neq 0. \end{cases}$$

Hence we get the conclusion by Theorem 5.1. □

Corollary 5.10. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular finite connected graphs. We suppose $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. Then their Kähler graphs $G_1\Diamond H_1$, $G_2\Diamond H_2$ of Cartesian-lexicographical product type are (p,q)-isospectral for an arbitrary pair (p,q) of relatively prime positive integers.*

4.4. (p,q)-Laplacians of Kähler graphs of strong-complement product type.

Theorem 5.11. *Let $G = (V,E), H = (W,F)$ be finite regular ordinary graphs. Suppose G and H are connected. We denote by $0 = \mu_1 < \mu_2 \le \cdots \le \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 < \nu_2 \le \cdots \le \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues*

of $\Delta_{\mathcal{Q}_{(p,q)}}$ *for their Kähler graph* $G \circledast H$ *of strong-complement product type are*

$$
\begin{aligned}
&0,\\
&1-\frac{F_p\bigl(d_G+d_H(1-\nu_\alpha)\,;\,d_G+d_H+d_Gd_H\bigr)}{(d_G+d_H+d_Gd_H)(d_G+d_H+d_Gd_H-1)^{p-1}}\\
&\qquad\times\frac{F_q\bigl(d_H(1-\nu_\alpha)(n_G-2d_G-1)-d_G\,;\,d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\bigr)}{\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\}\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)-1\}^{q-1}},\\
&1-\frac{F_p\bigl(d_G(1-\mu_i)+d_H\,;\,d_G+d_H+d_Gd_H\bigr)}{(d_G+d_H+d_Gd_H)(d_G+d_H+d_Gd_H-1)^{p-1}}\\
&\qquad\times\frac{F_q\bigl(d_G(1-\mu_i)(n_H-2d_H-1)-d_H\,;\,d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\bigr)}{\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\}\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)-1\}^{q-1}},\\
&1-\frac{F_p\bigl(d_G(1-\mu_i)+d_H(1-\nu_\alpha)\,;\,d_G+d_H+d_Gd_H\bigr)}{(d_G+d_H+d_Gd_H)(d_G+d_H+d_Gd_H-1)^{p-1}}\\
&\qquad\times\frac{F_q\bigl(-2d_Gd_H(1-\mu_i)(1-\nu_\alpha)-d_G(1-\mu_i)-d_H(1-\nu_\alpha)\,;\,d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\bigr)}{\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)\}\{d_G(n_H-d_H-1)+d_H(n_G-d_G-1)-1\}^{q-1}}.
\end{aligned}
$$

for $i=2,\cdots,n_G$ *and* $\alpha=2,\dots,n_H$.

Proof. As we see in §2.2.2, we have

$$d^{(p)}_{G\circledast H}=d_G+d_H+d_Gd_H,\quad d^{(a)}_{G\circledast H}=d_G(n_G-d_H-1)+d_H(n_G-d_G-1).$$

We use the same notations as in the proof of Theorem 5.9 (or in the proof of Theorem 4.19). The adjacency matrices of the principal and the auxiliary graphs of $G \circledast H$ are expressed as

$$
\begin{aligned}
A^{(p)}_{G\circledast H}&=(a^{(p)}_{(i,\alpha),(j,\beta)})=\Bigl(a^G_{ij}\delta_{\alpha\beta}+\delta_{ij}a^H_{\alpha\beta}+a^G_{ij}a^H_{\alpha\beta}\Bigr),\\
A^{(a)}_{G\circledast H}&=(a^{(a)}_{(i,\alpha),(j,\beta)})=\Bigl(a^G_{ij}a^{H^c}_{\alpha\beta}+a^{G^c}_{ij}a^H_{\alpha\beta}\Bigr)
\end{aligned}
$$

(see §4.6.4). By these expressions we find that they are commutative.

We take an eigenfunction $f:V\to\mathbb{R}$ corresponding to an eigenvalue μ of $\Delta_{\mathcal{P}_G}$ and an eigenfunction $g:W\to\mathbb{R}$ corresponding to an eigenvalue ν of $\Delta_{\mathcal{P}_H}$, and consider the function $\varphi_{f,g}:V\times W\to\mathbb{R}$ defined by $\varphi_{f,g}(v,w)=f(v)g(w)$. Since we have

$\mathcal{A}_G f = d_G(1-\mu)f,\ \mathcal{A}_H g = d_H(1-\nu)g$ and

$$\mathcal{A}_{G^c} f = \begin{cases} (n_G - d_G - 1)f, & \text{when } \mu = 0, \\ \{-1 - d_G(1-\mu)\}f, & \text{when } \mu \neq 0, \end{cases}$$
$$\mathcal{A}_{H^c} g = \begin{cases} (n_H - d_H - 1)g, & \text{when } \nu = 0, \\ \{-1 - d_H(1-\nu)\}g, & \text{when } \nu \neq 0, \end{cases}$$

by the expressions of adjacency matrices of principal and the auxiliary graphs, by the expressions of adjacency matrices of principal and the auxiliary graphs, doing the same computation as in the proof of Theorem 4.19, we have

$$\mathcal{A}^{(p)}_{G \circledast H}\varphi_{f,g} = \{d_G(1-\mu) + d_H(1-\nu) + d_G d_H(1-\mu)(1-\nu)\}\varphi_{f,g},$$
$$\mathcal{A}^{(a)}_{G \circledast H}\varphi_{f,g} = \begin{cases} \{d_G(n_H - d_H - 1) + d_H(n_G - d_G - 1)\}\varphi_{f,g}, & \mu = \nu = 0, \\ \{d_G(d_H\nu - d_H - 1) + d_H(1-\nu)(n_G - d_G - 1)\}\varphi_{f,g}, & \mu = 0, \nu \neq 0, \\ \{d_G(1-\mu)(n_H - d_H - 1) + d_H(d_G\mu - d_G - 1)\}\varphi_{f,g}, & \mu \neq 0, \nu = 0, \\ \{d_G(1-\mu)(d_H\nu - d_H - 1) + d_H(1-\nu)(d_G\mu - d_G - 1)\}\varphi_{f,g}, & \mu \neq 0, \nu \neq 0. \end{cases}$$

Hence we get the conclusion by Theorem 5.1. □

Corollary 5.11. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular finite connected graphs. We suppose $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. Then their Kähler graphs $G_1 \circledast H_1$, $G_2 \circledast H_2$ of strong-complement product type are (p,q)-isospectral for an arbitrary pair (p,q) of relatively prime positive integers.*

4.5. (p,q)-Laplacians of Kähler graphs of complement-tensor product type.

Theorem 5.12. *Let $G = (V,E), H = (W,F)$ be finite regular ordinary graphs. Suppose G and H are connected. We denote by $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 < \nu_2 \leq \cdots \leq \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues* *of* $\Delta_{\mathcal{Q}_{(p,q)}}$ *for their Kähler graph* $G♠H$ *of complement-tensor product type are*

$$\begin{aligned}
&0,\\
&1-\frac{F_p\big(d_H(1-\nu_\alpha)(n_G-2d_G)\,;\,d_G(n_G-d_H)+d_H(n_G-d_G)\big)}{\{d_G(n_G-d_H)+d_H(n_G-d_G)\}\{d_G(n_G-d_H)+d_H(n_G-d_G)-1\}^{p-1}}\\
&\qquad\times\frac{F_q\big(d_Gd_H(1-\nu_\alpha)\,;\,d_Gd_H\big)}{d_Gd_H(d_Gd_H-1)^{q-1}},\\
&1-\frac{F_p\big(d_G(1-\mu_i)(n_H-2d_H)\,;\,d_G(n_G-d_H)+d_H(n_G-d_G)\big)}{\{d_G(n_G-d_H)+d_H(n_G-d_G)\}\{d_G(n_G-d_H)+d_H(n_G-d_G)-1\}^{p-1}}\\
&\qquad\times\frac{F_q\big(d_Gd_H(1-\mu_i)\,;\,d_Gd_H\big)}{d_Gd_H(d_Gd_H-1)^{q-1}},\\
&1-\frac{F_p\big(-(d_G+d_H)(1-\mu_i)(1-\nu_\alpha)\,;\,d_G(n_G-d_H)+d_H(n_G-d_G)\big)}{\{d_G(n_G-d_H)+d_H(n_G-d_G)\}\{d_G(n_G-d_H)+d_H(n_G-d_G)-1\}^{p-1}}\\
&\qquad\times\frac{F_q\big(d_Gd_H(1-\mu_i)(1-\nu_\alpha)\,;\,d_Gd_H\big)}{d_Gd_H(d_Gd_H-1)^{q-1}}.
\end{aligned}$$

for $i=2,\cdots,n_G$ *and* $\alpha=2,\ldots,n_H$.

Proof. As we see in §2.2.2, we have

$$d^{(p)}_{G♠H}=d_G(n_G-d_H)+d_H(n_G-d_G)\quad d^{(a)}_{G♠H}=d_Gd_H.$$

We use the same notations as in the proof of Theorem 5.9 (or in the proof of Theorem 4.20). The adjacency matrices of the principal and the auxiliary graphs of $G♠H$ are expressed as

$$\begin{aligned}
A^{(p)}_{G♠H}=(a^{(p)}_{(i,\alpha),(j,\beta)})&=\Big(a^G_{ij}(a^{H^c}_{\alpha\beta}+\delta_{\alpha\beta})+(a^{G^c}_{ij}+\delta_{ij})a^H_{\alpha\beta}\Big)\\
&=\Big(a^G_{ij}(1-a^H_{\alpha\beta})+(1-a^{G^c}_{ij})a^H_{\alpha\beta}\Big),\\
A^{(a)}_{G♠H}=(a^{(a)}_{(i,\alpha),(j,\beta)})&=\Big(a^G_{ij}a^H_{\alpha\beta}\Big).
\end{aligned}$$

(see §4.6.5). By these expressions we find that they are commutative.

We take an eigenfunction $f:V\to\mathbb{R}$ corresponding to an eigenvalue μ of $\Delta_{\mathcal{P}_G}$ and an eigenfunction $g:W\to\mathbb{R}$ corresponding to an eigenvalue ν of $\Delta_{\mathcal{P}_H}$, and consider the function $\varphi_{f,g}:V\times W\to\mathbb{R}$ defined by $\varphi_{f,g}(v,w)=f(v)g(w)$. We have

$\mathcal{A}_G f = d_G(1-\mu)f,\ \mathcal{A}_H g = d_H(1-\nu)g$ and

$$\sum_{v\in V} f(v) = \begin{cases} n_G f(*), & \text{when } \mu = 0,\\ 0, & \text{when } \mu \neq 0,\end{cases} \qquad \sum_{w\in W} g(w) = \begin{cases} n_H g(*), & \text{when } \nu = 0,\\ 0, & \text{when } \nu \neq 0.\end{cases}$$

By the expressions of adjacency matrices of principal and the auxiliary graphs, doing the same computation as in the proof of Theorem 4.20, we have

$$\mathcal{A}^{(p)}_{G\spadesuit H}\varphi_{f,g} = \begin{cases} \{d_G(n_H - d_H) + d_H(n_G - d_G)\}\varphi_{f,g}, & \mu = \nu = 0,\\ \{d_G d_H(\nu-1) + d_H(1-\nu)(n_G - d_G)\}\varphi_{f,g}, & \mu = 0, \nu \neq 0,\\ \{d_G(1-\mu)(n_H - d_H) + d_G d_H(\mu-1)\}\varphi_{f,g}, & \mu \neq 0, \nu = 0,\\ \{d_G(1-\mu)(\nu-1) + d_H(1-\nu)(\mu-1)\}\varphi_{f,g}, & \mu \neq 0, \nu \neq 0.\end{cases}$$
$$\mathcal{A}^{(a)}_{G\spadesuit H}\varphi_{f,g} = d_G d_H(1-\mu)(1-\nu)\varphi_{f,g},$$

Hence we get the conclusion by Theorem 5.1. □

COROLLARY 5.12. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular finite connected graphs. We suppose $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. Then their Kähler graphs $G_1\spadesuit H_1$, $G_2\spadesuit H_2$ of complement-tensor product type are (p,q)-isospectral for an arbitrary pair (p,q) of relatively prime positive integers.*

4.6. (p,q)-Laplacians of Kähler graphs of tensor-complement product type.

THEOREM 5.13. *Let $G = (V,E), H = (W,F)$ be finite regular ordinary graphs. Suppose G and H are connected. We denote by $0 = \mu_1 < \mu_2 \le \cdots \le \mu_{n_G}$ the eigenvalues of $\Delta_{\mathcal{P}_G}$, and by $0 = \nu_1 < \nu_2 \le \cdots \le \nu_{n_H}$ that of $\Delta_{\mathcal{P}_H}$. Then the eigenvalues*

of $\Delta_{\mathcal{Q}_{(p,q)}}$ for their Kähler graph $G\clubsuit H$ of tensor-complement product type are

$$
\begin{aligned}
&0,\\
&1-\frac{F_p\big(d_Gd_H(1-\nu_\alpha)\,;\,d_Gd_H\big)}{d_Gd_H(d_Gd_H-1)^{p-1}}\\
&\qquad\times\frac{F_q\big(d_H(1-\nu_\alpha)(n_G-2d_G-1)-d_G\,;\,d_G(n_G-d_H-1)+d_H(n_G-d_G-1)\big)}{\{d_G(n_G-d_H-1)+d_H(n_G-d_G-1)\}\{d_G(n_G-d_H-1)+d_H(n_G-d_G-1)-1\}^{q-1}},\\
&1-\frac{F_p\big(d_Gd_H(1-\mu_i)\,;\,d_Gd_H\big)}{d_Gd_H(d_Gd_H-1)^{p-1}}\\
&\qquad\times\frac{F_q\big(d_G(1-\mu_i)(n_H-2d_H-1)-d_H\,;\,d_G(n_G-d_H-1)+d_H(n_G-d_G-1)\big)}{\{d_G(n_G-d_H-1)+d_H(n_G-d_G-1)\}\{d_G(n_G-d_H-1)+d_H(n_G-d_G-1)-1\}^{q-1}},\\
&1-\frac{F_p\big(d_Gd_H(1-\mu_i)(1-\nu_\alpha)\,;\,d_Gd_H\big)}{d_Gd_H(d_Gd_H-1)^{p-1}}\\
&\qquad\times\frac{F_q\left(\begin{matrix}-2d_Gd_H(1-\mu_i)(1-\nu_\alpha)\\ +d_G(1-\mu_i)+d_H(1-\nu_\alpha)\end{matrix}\,;\,d_G(n_G-d_H-1)+d_H(n_G-d_G-1)\right)}{\{d_G(n_G-d_H-1)+d_H(n_G-d_G-1)\}\{d_G(n_G-d_H-1)+d_H(n_G-d_G-1)-1\}^{q-1}}.
\end{aligned}
$$

for $i=2,\cdots,n_G$ and $\alpha=2,\dots,n_H$.

Proof. As we see in §2.2.2, we have

$$d^{(p)}_{G\clubsuit H}=d_Gd_H,\quad d^{(a)}_{G\clubsuit H}=d_G(n_G-d_H-1)+d_H(n_G-d_G-1).$$

We use the same notations as in the proof of Theorem 5.9 (or in the proof of Theorem 4.21). The adjacency matrices of the principal and the auxiliary graphs of $G\clubsuit H$ are expressed as

$$A^{(p)}_{G\clubsuit H}=\big(a^{(p)}_{(i,\alpha),(j,\beta)}\big)=\Big(a^G_{ij}a^H_{\alpha\beta}\Big),\qquad A^{(a)}_{G\clubsuit H}=\big(a^{(a)}_{(i,\alpha),(j,\beta)}\big)=\Big(a^G_{ij}a^{H^c}_{\alpha\beta}+a^{G^c}_{ij}a^H_{\alpha\beta}\Big).$$

(see §4.6.6). Since $a^{G^c}_{ij}=1-\delta_{ij}-a^G_{ij}$ and $a^{H^c}_{\alpha\beta}=1-\delta_{\alpha\beta}-a^H_{\alpha\beta}$, by these expressions we find that they are commutative.

We take an eigenfunction $f:V\to\mathbb{R}$ corresponding to an eigenvalue μ of $\Delta_{\mathcal{P}_G}$ and an eigenfunction $g:W\to\mathbb{R}$ corresponding to an eigenvalue ν of $\Delta_{\mathcal{P}_H}$, and consider the function $\varphi_{f,g}:V\times W\to\mathbb{R}$ defined by $\varphi_{f,g}(v,w)=f(v)g(w)$. We have

$\mathcal{A}_G f = d_G(1-\mu)f,\ \mathcal{A}_H g = d_H(1-\nu)g$ and

$$\mathcal{A}_{G^c} f = \begin{cases} (n_G - d_G - 1)f, & \text{when } \mu = 0, \\ \{-1 - d_G(1-\mu)\}f, & \text{when } \mu \neq 0, \end{cases}$$

$$\mathcal{A}_{H^c} g = \begin{cases} (n_H - d_H - 1)g, & \text{when } \nu = 0, \\ \{-1 - d_H(1-\nu)\}g, & \text{when } \nu \neq 0. \end{cases}$$

By the expressions of adjacency matrices of principal and the auxiliary graphs, doing the same computation as in the proof of Theorem 4.21, we have

$\mathcal{A}^{(p)}_{G\clubsuit H}\varphi_{f,g} = d_G d_H(1-\mu)(1-\nu)\varphi_{f,g}$,

$$\mathcal{A}^{(a)}_{G\clubsuit H}\varphi_{f,g} = \begin{cases} \{d_G(n_H - d_H - 1) + d_H(n_G - d_G - 1)\}\varphi_{f,g}, & \mu = \nu = 0, \\ \{-d_G\big(d_H(1-\nu)+1\big) + d_H(1-\nu)(n_G - d_G - 1)\}\varphi_{f,g}, & \mu = 0, \nu \neq 0, \\ \{d_G(1-\mu)(n_H - d_H - 1) - d_H\big(d_G(1-\mu)+1\big)\}\varphi_{f,g}, & \mu \neq 0, \nu = 0, \\ \{-d_G(1-\mu)\big(d_H(1-\nu)+1\big) - d_H(1-\nu)\big(d_G(1-\mu)+1\big)\}\varphi_{f,g}, & \mu \neq 0, \nu \neq 0. \end{cases}$$

Hence we get the conclusion by Theorem 5.1. □

Corollary 5.13. *Let G_1, G_2 and H_1, H_2 be two pairs of isospectral regular finite connected graphs. We suppose $d_{G_1} = d_{G_2}$ and $d_{H_1} = d_{H_2}$. Then their Kähler graphs $G_1\clubsuit H_1$, $G_2\clubsuit H_2$ of complement-tensor product type are (p, q)-isospectral for an arbitrary pair (p, q) of relatively prime positive integers.*

5. Eigenvalues of other typical Kähler graphs

In this section, we study some other typical examples of Kähler graphs.

5.1. Kähler 3-cubes.

First we study Kähler k-cubes.

EXAMPLE 5.8. We take a Kähler 3-cube $G = (Q_3, E^{(p)} \cup E^{(a)})$ with $Q_3 = \{(a_1, a_2, a_3)\} \mid a_i \in \{0,1\}\}$ (see Example 2.29 in §2.3). This is a regular Kähler graph of $d_G^{(p)} = d_G^{(a)} = 3$. The adjacency matrices of the principal and the auxiliary graphs are given as

$$A_{G^{(p)}} = \begin{pmatrix} 0&1&0&1&0&0&0&1\\ 1&0&1&0&0&0&1&0\\ 0&1&0&1&0&1&0&0\\ 1&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&0&1\\ 0&0&1&0&1&0&1&0\\ 0&1&0&0&0&1&0&1\\ 1&0&0&0&1&0&1&0 \end{pmatrix}, \qquad A_{G^{(a)}} = \begin{pmatrix} 0&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&0&1\\ 1&0&0&0&1&0&1&0\\ 0&1&0&0&0&1&0&1\\ 1&0&1&0&0&0&1&0\\ 0&1&0&1&0&0&0&1\\ 1&0&1&0&1&0&0&0\\ 0&1&0&1&0&1&0&0 \end{pmatrix}.$$

They are commutative.

$$A_{G^{(p)}} A_{G^{(a)}} = \begin{pmatrix} 0&2&0&2&0&3&0&2\\ 2&0&2&0&3&0&2&0\\ 0&2&0&2&0&2&0&3\\ 2&0&2&0&2&0&3&0\\ 0&3&0&2&0&2&0&2\\ 3&0&2&0&2&0&2&0\\ 0&2&0&3&0&2&0&2\\ 2&0&3&0&2&0&2&0 \end{pmatrix} = A_{G^{(a)}} A_{G^{(p)}}.$$

Thus we can apply Theorem 5.1. The eigenvalues of adjacency operators of the principal and the auxiliary graphs are

$$\mathrm{Spec}(\mathcal{A}_{G^{(p)}}) = \{-3, -1, -1, -1, 1, 1, 1, 3\},$$
$$\mathrm{Spec}(\mathcal{A}_{G^{(a)}}) = \{-1, -1, -1, -1, -1, -1, 3, 3\}.$$

The eigenvalues of combinatorial Laplacians of the principal and the auxiliary graphs are

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^{(p)}}}) = \{0, 2, 2, 2, 4, 4, 4, 6\}, \qquad \mathrm{Spec}(\Delta_{\mathcal{A}_{G^{(a)}}}) = \{0, 0, 4, 4, 4, 4, 4, 4\}.$$

We note that the auxiliary graph of G is not connected. If we compute directly eigenvalues of some combinatorial Laplacians, as we have

$$\begin{aligned}
A_{G_{(1,1)}} &= \frac{1}{3}A_{G^{(p)}}A_{G^{(a)}}, \\
A_{G_{(2,1)}} &= \frac{1}{3}\big(A^2_{G^{(p)}}-3I\big)A_{G^{(a)}}, & A_{G_{(1,2)}} &= \frac{1}{6}A_{G^{(p)}}\big(A^2_{G^{(a)}}-3I\big), \\
A_{G_{(3,1)}} &= \frac{1}{3}\big(A^2_{G^{(p)}}-5I\big)A_{G^{(p)}}A_{G^{(a)}}, & A_{G_{(3,2)}} &= \frac{1}{6}\big(A^2_{G^{(p)}}-5I\big)A_{G^{(p)}}\big(A^2_{G^{(a)}}-3I\big), \\
A_{G_{(1,3)}} &= \frac{1}{12}A_{G^{(p)}}A_{G^{(a)}}\big(A^2_{G^{(a)}}-5I\big), & A_{G_{(2,3)}} &= \frac{1}{12}\big(A^2_{G^{(p)}}-3I\big)A_{G^{(a)}}\big(A^2_{G^{(a)}}-5I\big),
\end{aligned}$$

that is,

$$A_{G_{(2,1)}} = \frac{1}{3}\begin{pmatrix} 6&0&4&0&4&0&4&0\\ 0&6&0&4&0&4&0&4\\ 4&0&6&0&4&0&4&0\\ 0&4&0&6&0&4&0&4\\ 4&0&4&0&6&0&4&0\\ 0&4&0&4&0&6&0&4\\ 4&0&4&0&4&0&6&0\\ 0&4&0&4&0&4&0&6 \end{pmatrix}, \quad A_{G_{(1,2)}} = \frac{1}{6}\begin{pmatrix} 0&4&0&4&0&6&0&4\\ 4&0&4&0&6&0&4&0\\ 0&4&0&4&0&4&0&6\\ 4&0&4&0&4&0&6&0\\ 0&6&0&4&0&4&0&4\\ 6&0&4&0&4&0&4&0\\ 0&4&0&6&0&4&0&4\\ 4&0&6&0&4&0&4&0 \end{pmatrix},$$

$$A_{G_{(3,1)}} = \frac{1}{3}\begin{pmatrix} 0&10&0&10&0&6&0&10\\ 10&0&10&0&6&0&10&0\\ 0&10&0&10&0&10&0&6\\ 10&0&10&0&10&0&6&0\\ 0&6&0&10&0&10&0&10\\ 6&0&10&0&10&0&10&0\\ 0&10&0&6&0&10&0&10\\ 10&0&6&0&10&0&10&0 \end{pmatrix},$$

$$A_{G_{(3,2)}} = \frac{1}{6}\begin{pmatrix} 0&20&0&20&0&12&0&20\\ 20&0&20&0&12&0&20&0\\ 0&20&0&20&0&20&0&12\\ 20&0&20&0&20&0&12&0\\ 0&12&0&20&0&20&0&20\\ 12&0&20&0&20&0&20&0\\ 0&20&0&12&0&20&0&20\\ 20&0&12&0&20&0&20&0 \end{pmatrix},$$

$$A_{G_{(1,3)}} = \frac{1}{12}\begin{pmatrix} 0 & 10 & 0 & 10 & 0 & 6 & 0 & 10 \\ 10 & 0 & 10 & 0 & 6 & 0 & 10 & 0 \\ 0 & 10 & 0 & 10 & 0 & 10 & 0 & 6 \\ 10 & 0 & 10 & 0 & 10 & 0 & 6 & 0 \\ 0 & 6 & 0 & 10 & 0 & 10 & 0 & 10 \\ 6 & 0 & 10 & 0 & 10 & 0 & 10 & 0 \\ 0 & 10 & 0 & 6 & 0 & 10 & 0 & 10 \\ 10 & 0 & 6 & 0 & 10 & 0 & 10 & 0 \end{pmatrix},$$

$$A_{G_{(2,3)}} = \frac{1}{12}\begin{pmatrix} 12 & 0 & 20 & 0 & 20 & 0 & 20 & 0 \\ 0 & 12 & 0 & 20 & 0 & 20 & 0 & 20 \\ 20 & 0 & 12 & 0 & 20 & 0 & 20 & 0 \\ 0 & 20 & 0 & 12 & 0 & 20 & 0 & 20 \\ 20 & 0 & 20 & 0 & 12 & 0 & 20 & 0 \\ 0 & 20 & 0 & 20 & 0 & 12 & 0 & 20 \\ 20 & 0 & 20 & 0 & 20 & 0 & 12 & 0 \\ 0 & 20 & 0 & 20 & 0 & 20 & 0 & 12 \end{pmatrix},$$

we have

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(1,1)}}}) = \Big\{0, \frac{8}{3}, \frac{8}{3}, \frac{8}{3}, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, 6\Big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(2,1)}}}) = \Big\{0, 0, \frac{16}{3}, \frac{16}{3}, \frac{16}{3}, \frac{16}{3}, \frac{16}{3}, \frac{16}{3}, \frac{16}{3}\Big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(1,2)}}}) = \Big\{0, \frac{8}{3}, \frac{8}{3}, \frac{8}{3}, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, 6\Big\},$$

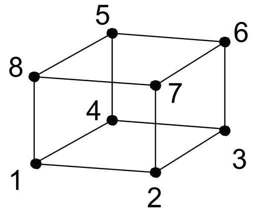
$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(3,1)}}}) = \Big\{0, \frac{32}{3}, \frac{32}{3}, \frac{32}{3}, \frac{40}{3}, \frac{40}{3}, \frac{40}{3}, 24\Big\},$$

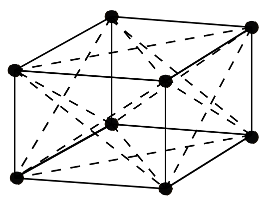
$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(3,2)}}}) = \Big\{0, \frac{32}{3}, \frac{32}{3}, \frac{32}{3}, \frac{40}{3}, \frac{40}{3}, \frac{40}{3}, 24\Big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(1,3)}}}) = \Big\{0, \frac{8}{3}, \frac{8}{3}, \frac{8}{3}, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, 6\Big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(2,3)}}}) = \Big\{0, 0, \frac{20}{3}, \frac{20}{3}, \frac{20}{3}, \frac{20}{3}, \frac{20}{3}, \frac{20}{3}\Big\}.$$

We can hence see that $\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(1,1)}}}) = \mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(1,2)}}}) = \mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(1,3)}}})$, and that $\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(2,1)}}})$ and $\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(2,3)}}})$ contain two null eigenvalues and others are same.





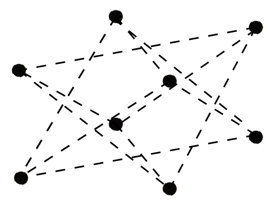


FIG. 3. principal graph FIG. 4. 3-Kähler cube FIG. 5. auxiliary graph

We here study general Kähler cubes.

PROPOSITION 5.5. *The adjacency operators for the principal and the auxiliary graphs of a Kähler k-cube are commutative.*

PROOF. We take a vertex $v=(0,\dots,0)\in Q_k$. It is principally adjacent to

$$(0,\dots,0,\overset{i}{1},0,\dots,0) \qquad (i=1,\dots,k),$$

hence is $(1,1)$-adjacent to

$$\begin{aligned}&(0,\dots,0,\overset{j}{1},0,\dots,0,\overset{i}{1},0,\dots,0,\overset{\ell}{1},0,\dots,0) && (i,j,\ell=1,\dots,k,\ \ i\neq j,\ell,\ \ j<\ell),\\ &(0,\dots,0,\overset{j}{1},0,\dots,0) && (j=1,\dots,k),\end{aligned}$$

where each of the second type appears $k-1$ times. This is because the second type occurs when either $i=j$ or $i=\ell$ and when we choose one other coordinate for ℓ or for j. On the other hand, as v is auxiliary adjacent to

$$(0,\dots,0,\overset{j}{1},0,\dots,0,\overset{\ell}{1},0,\dots,0) \qquad (j,\ell=1,\dots,k,\ \ j<\ell),$$

we find adjacent to the above vertices by 2-step paths formed by auxiliary edges followed by principal edges. Since Kähler cubes are vertex transitive, this shows the assertion. □

5.2. The Cayley Kähler graph of D_4. Next we study The Cayley Kähler graph obtained by a dihedral group D_4

EXAMPLE 5.9. A dihedral group D_4 is generated by two elements in two ways:

$$D_4=\left\langle a,b \;\middle|\; a^4=b^2=1,\ ab=ba^3\right\rangle=\left\langle b,c \;\middle|\; b^2=c^2=1,\ bcbc=cbcb\right\rangle$$

where $c=ab$. Putting $\mathcal{S}^{(p)}=\{b,c\}$ and $\mathcal{S}^{(a)}=\{a,a^3\}$, we get a regular Kähler graph G with $d_G^{(p)}=d_G^{(a)}=2$ given in Example 2.5 in §2.1 which is like Fig. 6. The adjacency matrices of its principal and auxiliary graphs are given as

$$A_{G^{(p)}} = \begin{pmatrix} 0&1&0&0&0&0&0&1\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&0&1&0\\ 0&0&0&0&0&1&0&1\\ 1&0&0&0&0&0&1&0 \end{pmatrix}, \quad A_{G^{(a)}} = \begin{pmatrix} 0&0&1&0&0&0&1&0\\ 0&0&0&1&0&0&0&1\\ 1&0&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0\\ 0&0&1&0&0&0&1&0\\ 0&0&0&1&0&0&0&1\\ 1&0&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0 \end{pmatrix}.$$

We then see the adjacency operators of the principal and the auxiliary graphs are commutative

$$A_{G^{(p)}}A_{G^{(a)}} = \begin{pmatrix} 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0 \end{pmatrix} = A_{G^{(a)}}A_{G^{(p)}}.$$

Thus we can apply Theorem 5.1. The eigenvalues of adjacency operators of the principal and the auxiliary graphs are

$$\mathrm{Spec}(\mathcal{A}_{G^{(p)}}) = \bigl\{-2, -\sqrt{2}, -\sqrt{2}, 0, 0, \sqrt{2}, \sqrt{2}, 2\bigr\},$$
$$\mathrm{Spec}(\mathcal{A}_{G^{(a)}}) = \bigl\{-2, -2, 0, 0, 0, 0, 2, 2\bigr\}.$$

The eigenvalues of combinatorial Laplacians of its principal and auxiliary graphs and those of some combinatorial Laplacians are as follows:

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^{(p)}}}) = \bigl\{0, 2, 2, 2+\sqrt{2},\ 2+\sqrt{2},\ 2+\sqrt{2},\ 2+\sqrt{2}, 4\bigr\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^{(a)}}}) = \bigl\{0, 0, 2, 2, 2, 2, 4, 4\bigr\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(1,1)}}}) = \bigl\{0, 2, 2, 2, 2, 2, 2, 4\bigr\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(2,1)}}}) = \bigl\{0, 0, 0, 0, 2, 2, 2, 2\bigr\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(1,2)}}}) = \bigl\{0, 2-\sqrt{2},\ 2-\sqrt{2},\ 2, 2, 2+\sqrt{2},\ 2+\sqrt{2},\ 4\bigr\}.$$

We note that $A_{G_{(1,1)}} = \frac{1}{2} A_{G^{(p)}} A_{G^{(a)}}$,

$$A_{G_{(2,1)}} = \frac{1}{2}\big(A^2_{G^{(p)}} - 2I\big)A_{G^{(a)}} = \frac{1}{2}\begin{pmatrix} 2&0&0&0&2&0&0&0\\ 0&2&0&0&0&2&0&0\\ 0&0&2&0&0&0&2&0\\ 0&0&0&2&0&0&0&2\\ 2&0&0&0&2&0&0&0\\ 0&2&0&0&0&2&0&0\\ 0&0&2&0&0&0&2&0\\ 0&0&0&2&0&0&0&2 \end{pmatrix},$$

$$A_{G_{(1,2)}} = \frac{1}{2}A_{G^{(p)}}\big(A^2_{G^{(a)}} - 2I\big) = \frac{1}{2}\begin{pmatrix} 0&0&0&2&0&2&0&0\\ 0&0&0&0&2&0&2&0\\ 0&0&0&0&0&2&0&2\\ 2&0&0&0&0&0&2&0\\ 0&2&0&0&0&0&0&2\\ 2&0&2&0&0&0&0&0\\ 0&2&0&2&0&0&0&0\\ 0&0&2&0&2&0&0&0 \end{pmatrix}.$$

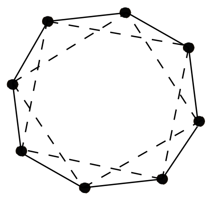


Fig. 6

Here, for the sake of readers' convenience we briefly explain dihedral groups. We take a regular k-polygon ($k \geq 3$). A dihedral group D_k is the group formed by motions of $\mathbb{R}^2$ which preserve this k-polygon. This group is formed by k kinds of rotations and k kinds of reflections. Rotations are the identity, the $2\pi/k$-rotation, the $4\pi/k$-rotation, $\dots$, the $2(k-1)\pi/k$-rotation. Reflections are the following. When k is odd, we take lines which join a vertex and the mid point of its antipodal edge. We have k such lines. Reflection with respect to these lines preserves the regular k-polygon. When k is even, we take lines which join a vertex and its antipodal vertex. We have $k/2$ such lines. Also we take lines which joins the mid point of an edge and the mid point of its antipodal edge. Also we have $k/2$ such lines. Reflection with respect to these lines preserves the regular k-polygon. Thus a dihedral group D_k have $2k$ elements.

5.3. Eigenvalues of Kähler Petersen graphs and the Petersen Kähler graph. In the third we study Kähler graphs obtained from a Petersen graph.

EXAMPLE 5.10. We take a Kähler Petersen graph G given in Example 2.24 in §2.3 which is like Fig. 7. It is a regular Kähler graph of $d_G^{(p)} = 3$ and $d_G^{(a)} = 2$. The adjacency matrices of its principal graph and its auxiliary graph are given as

$$A_{G^{(p)}} = \begin{pmatrix} A & I \\ I & B \end{pmatrix}, \qquad A_{G^{(a)}} = \begin{pmatrix} B & O \\ O & A \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} = A^2 - 2I.$$

As $AB = BA = A + B$, we have

$$A_{G^{(p)}} A_{G^{(a)}} = \begin{pmatrix} AB & A \\ B & AB \end{pmatrix} = {}^t\{A_{G^{(a)}} A_{G^{(p)}}\}.$$

But this shows that the adjacency operators of its principal and auxiliary graphs are not commutative. As we have $A_{G_{(1,1)}} = \frac{1}{2} A_{G^{(p)}} A_{G^{(a)}}$, the eigenvalues of combinatorial Laplacians of its principal graph and those of $(1,1)$-combinatorial Laplacians are as follows:

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^{(p)}}}) = \{0, 2, 2, 2, 2, 2, 5, 5, 5, 5\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(1,1)}}}) = \left\{ \begin{matrix} 0, 2, \dfrac{7+\sqrt{-1}}{2}, \dfrac{7+\sqrt{-1}}{2}, \dfrac{7+\sqrt{-1}}{2}, \dfrac{7+\sqrt{-1}}{2}, \\ \dfrac{7-\sqrt{-1}}{2}, \dfrac{7-\sqrt{-1}}{2}, \dfrac{7-\sqrt{-1}}{2}, \dfrac{7-\sqrt{-1}}{2} \end{matrix} \right\}.$$

As we have $A = B^2 - 2I$, we see

$$A_{G_{(2,1)}} = \frac{1}{2} \begin{pmatrix} A^2-2I & A+B \\ A+B & B^2-2I \end{pmatrix} \begin{pmatrix} B & O \\ O & A \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A+2I & A+2B+2I \\ 2A+B+2I & B+2I \end{pmatrix}.$$

Hence we find that

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(2,1)}}}) = \Big\{0, 5, 5, 5, 5, \frac{11}{2}, \frac{11}{2}, \frac{11}{2}, \frac{11}{2}, 8\Big\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{Q}_{G_{(2,1)}}}) = \Big\{0, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{11}{12}, \frac{11}{12}, \frac{11}{12}, \frac{11}{12}, \frac{4}{3}\Big\}.$$

Similarly, we have

$$A_{G_{(3,1)}} = \begin{pmatrix} B & A+B \\ A+B & A \end{pmatrix}\begin{pmatrix} B & O \\ O & A \end{pmatrix} = \begin{pmatrix} A+2I & A+2B+2I \\ 2A+B+2I & B+2I \end{pmatrix},$$

and find

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(3,1)}}}) = \Big\{0, 10, 10, 10, 10, 11, 11, 11, 11, 16\Big\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{Q}_{G_{(3,1)}}}) = \Big\{0, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{11}{12}, \frac{11}{12}, \frac{11}{12}, \frac{11}{12}, \frac{4}{3}\Big\}.$$

We therefore have

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(3,1)}}}) = 2 \times \mathrm{Spec}(\Delta_{\mathcal{A}_{G(2,1)}}), \qquad \mathrm{Spec}(\Delta_{\mathcal{Q}_{G_{(3,1)}}}) = \mathrm{Spec}(\Delta_{\mathcal{Q}_{G_{(2,1)}}}),$$

where $2 \times S$ means that we multiple 2 on each element of S.

Since we have

$$\begin{aligned} A_{G^{(p)}_{[4]}} &= \begin{pmatrix} 4A+9B+15I & 9A+9B+4I \\ 9A+9B+4I & 9A+4B+15I \end{pmatrix} - 7\begin{pmatrix} B+3I & A+B \\ A+B & A+3I \end{pmatrix} + 6\begin{pmatrix} I & O \\ O & I \end{pmatrix} \\ &= \begin{pmatrix} 4A+2B & 2A+2B+4I \\ 2A+2B+4I & 2A+4B \end{pmatrix}, \end{aligned}$$

we find

$$A_{G_{(4,1)}} = \begin{pmatrix} 2A+B & A+B+2I \\ A+B+2I & A+2B \end{pmatrix}\begin{pmatrix} B & O \\ O & A \end{pmatrix} = \begin{pmatrix} 3A+2B+2I & 3A+2B+2I \\ 2A+3B+2I & 2A+3B+2I \end{pmatrix}.$$

Therefore we have

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(4,1)}}}) = \Big\{0, 24, 24, 24, 24, 24, 25, 25, 25, 25\Big\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{Q}_{G_{(4,1)}}}) = \Big\{0, 1, 1, 1, 1, 1, \frac{25}{12}, \frac{25}{12}, \frac{25}{12}, \frac{25}{12}\Big\}.$$

Similarly, we have

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(5,1)}}}) = \left\{ \begin{array}{l} 0, 40, 2\big(25-\sqrt{5}\big),\ 2\big(25-\sqrt{5}\big),\ 2\big(25-\sqrt{5}\big),\ 2\big(25-\sqrt{5}\big), \\ \ 2\big(25+\sqrt{5}\big),\ 2\big(25+\sqrt{5}\big),\ 2\big(25+\sqrt{5}\big),\ 2\big(25+\sqrt{5}\big) \end{array} \right\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{Q}_{G_{(5,1)}}}) = \left\{ \begin{array}{l} 0, \frac{5}{6}, \frac{25-\sqrt{5}}{24},\ \frac{25-\sqrt{5}}{24},\ \frac{25-\sqrt{5}}{24},\ \frac{25-\sqrt{5}}{24}, \\ \ \frac{25+\sqrt{5}}{24},\ \frac{25+\sqrt{5}}{24},\ \frac{25+\sqrt{5}}{24},\ \frac{25+\sqrt{5}}{24} \end{array} \right\}.$$

We note that $P_{G^{(p)}_{[2]}} = P_{G^{(p)}_{[3]}}$, therefore we find that

$$\mathrm{Spec}(\Delta_{\mathcal{Q}_{G_{(2,q)}}}) = \mathrm{Spec}(\Delta_{\mathcal{Q}_{G_{(3,q)}}})$$

for an arbitrary positive integer q.

As we have

$$\begin{aligned}
A_{G_{(1,2)}} &= \frac{1}{2}\begin{pmatrix} A & I \\ I & B \end{pmatrix}\begin{pmatrix} A & O \\ O & B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} B+2I & B \\ A & A+2I \end{pmatrix},\\
A_{G_{(1,3)}} &= \frac{1}{2}\begin{pmatrix} A & I \\ I & B \end{pmatrix}\begin{pmatrix} A & O \\ O & B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} B+2I & B \\ A & A+2I \end{pmatrix},\\
A_{G_{(1,4)}} &= \frac{1}{2}\begin{pmatrix} A & I \\ I & B \end{pmatrix}\begin{pmatrix} B & O \\ O & A \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A+B & A \\ B & A+B \end{pmatrix} = A_{G_{(1,1)}},\\
A_{G_{(1,5)}} &= \frac{1}{2}\begin{pmatrix} A & I \\ I & B \end{pmatrix}\begin{pmatrix} 2I & O \\ O & 2I \end{pmatrix} = A_{G^{(p)}},
\end{aligned}$$

we find

$$\begin{aligned}
&\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,2)}}) = \mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,3)}}) = \left\{0,2,2,2,2,2,\frac{5}{2},\frac{5}{2},\frac{5}{2},\frac{5}{2}\right\},\\
&\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,4)}}) = \left\{\begin{matrix} 0,2,\dfrac{7+\sqrt{-1}}{2},\ \dfrac{7+\sqrt{-1}}{2},\ \dfrac{7+\sqrt{-1}}{2},\ \dfrac{7+\sqrt{-1}}{2},\\ \dfrac{7-\sqrt{-1}}{2},\ \dfrac{7-\sqrt{-1}}{2},\ \dfrac{7-\sqrt{-1}}{2},\ \dfrac{7-\sqrt{-1}}{2} \end{matrix}\right\},\\
&\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,5)}}) = \left\{0,2,2,2,2,2,5,5,5,5\right\}.
\end{aligned}$$

Since the auxiliary graph is a disjoint union of two 5-circuits, we have

$$P_{G^{(a)}_{[5\ell+1]}} = P_{G^{(a)}_{[5\ell+4]}} = P_{G^{(a)}},\quad P_{G^{(a)}_{[5\ell+2]}} = P_{G^{(a)}_{[5\ell+3]}} = P_{G^{(a)}_{[2]}},\quad P_{G^{(a)}_{[5\ell]}} = I$$

for an arbitrary nonnegative integer ℓ. Hence we find

$$\begin{aligned}
\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(p,5\ell+1)}}}) &= \mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(p,5\ell+4)}}}) = \mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(p,1)}}}),\\
\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(p,5\ell+2)}}}) &= \mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(p,5\ell+3)}}}) = \mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(p,2)}}}),\\
\mathrm{Spec}(\Delta_{\mathcal{A}_{G_{(p,5\ell)}}}) &= \mathrm{Spec}(\Delta_{\mathcal{A}_{G^{(p)}_{[p]}}})
\end{aligned}$$

for an arbitrary positive integer p.

Example 5.11. We take a Petersen Kähler graph of first kind given in Example 2.24 in §2.3 which is like Fig. 8. It is a regular Kähler graph of $d^{(p)} = d^{(a)} = 3$. The

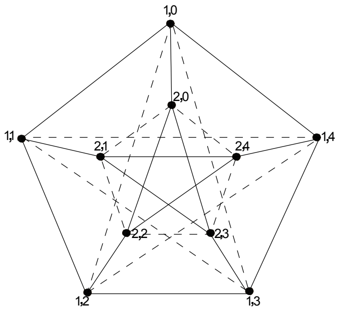


FIG. 7

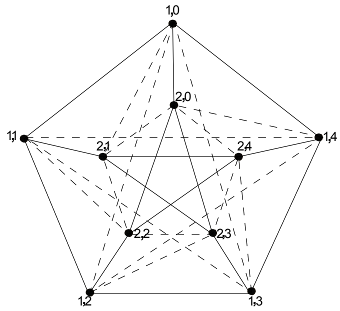


FIG. 8

adjacency matrices of its principal graph and its auxiliary graphs are given as

$$A_{G^{(p)}} = \begin{pmatrix} 0&1&0&0&1&1&0&0&0&0\\ 1&0&1&0&0&0&1&0&0&0\\ 0&1&0&1&0&0&0&1&0&0\\ 0&0&1&0&1&0&0&0&1&0\\ 1&0&0&1&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&1&1&0\\ 0&1&0&0&0&0&0&0&1&1\\ 0&0&1&0&0&1&0&0&0&1\\ 0&0&0&1&0&1&1&0&0&0\\ 0&0&0&0&1&0&1&1&0&0 \end{pmatrix}, \quad A_{G^{(a)}} = \begin{pmatrix} 0&0&1&1&0&0&1&0&0&0\\ 0&0&0&1&1&0&0&1&0&0\\ 1&0&0&0&1&0&0&0&1&0\\ 1&1&0&0&0&0&0&0&0&1\\ 0&1&1&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&1\\ 1&0&0&0&0&1&0&1&0&0\\ 0&1&0&0&0&0&1&0&1&0\\ 0&0&1&0&0&0&0&1&0&1\\ 0&0&0&1&0&1&0&0&1&0 \end{pmatrix},$$

and they are not commutative:

$$A_{G^{(p)}}A_{G^{(a)}} = \begin{pmatrix} 0&1&1&1&2&1&1&1&0&1\\ 2&0&1&1&1&1&1&1&1&0\\ 1&2&0&1&1&0&1&1&1&1\\ 1&1&2&0&1&1&0&1&1&1\\ 1&1&1&2&0&1&1&0&1&1\\ 0&1&2&1&0&0&2&1&1&1\\ 0&0&1&2&1&1&0&2&1&1\\ 1&0&0&1&2&1&1&0&2&1\\ 2&1&0&0&1&1&1&1&0&2\\ 1&2&1&0&0&2&1&1&1&0 \end{pmatrix}, \quad A_{G^{(a)}}A_{G^{(p)}} = \begin{pmatrix} 0&2&1&1&1&0&0&1&2&1\\ 1&0&2&1&1&1&0&0&1&2\\ 1&1&0&2&1&2&1&0&0&1\\ 1&1&1&0&2&1&2&1&0&0\\ 2&1&1&1&0&0&1&2&1&0\\ 1&1&0&1&1&0&1&0&1&2\\ 1&1&1&0&1&2&0&1&1&1\\ 1&1&1&1&0&1&2&0&1&1\\ 0&1&1&1&1&1&1&2&0&1\\ 1&0&1&1&1&1&1&1&2&0 \end{pmatrix},$$

The eigenvalues of combinatorial Laplacians of its principal graph and those of $(1,1)$-combinatorial and $(1,1)$-probabilistic transition Laplacians are as follows:

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^{(p)}}}) = \big\{0,2,2,2,2,2,5,5,5,5\big\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,1)}}) = \Big\{0,\frac{8}{3},\epsilon,\epsilon,\rho,\rho,\varrho,\varrho,\varsigma,\varsigma\Big\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{Q}_{G(1,1)}}) = \Big\{0,\frac{8}{9},\frac{\epsilon}{3},\frac{\epsilon}{3},\frac{\rho}{3},\frac{\rho}{3},\frac{\varrho}{3},\frac{\varrho}{3},\frac{\varsigma}{3},\frac{\varsigma}{3}\Big\},$$

where

$$\begin{cases} \epsilon = \dfrac{41+\sqrt{5}}{12} - \dfrac{\sqrt{-1}}{12}\sqrt{34-10\sqrt{5}}, \\ \rho = \dfrac{41+\sqrt{5}}{12} + \dfrac{\sqrt{-1}}{12}\sqrt{34-10\sqrt{5}} \end{cases} \qquad \begin{cases} \varrho = \dfrac{41-\sqrt{5}}{12} - \dfrac{\sqrt{-1}}{12}\sqrt{34+10\sqrt{5}}, \\ \varsigma = \dfrac{41-\sqrt{5}}{12} + \dfrac{\sqrt{-1}}{12}\sqrt{34+10\sqrt{5}}. \end{cases}$$

As $A_{G^{(p)}_{[2]}} = A^2_{G^{(p)}} - 3I$ we have

$$A_{G^{(p)}_{[2]}} = \begin{pmatrix} 0&0&1&1&0&0&1&1&1&1 \\ 0&0&0&1&1&1&0&1&1&1 \\ 1&0&0&0&1&1&1&0&1&1 \\ 1&1&0&0&0&1&1&1&0&1 \\ 0&1&1&0&0&1&1&1&1&0 \\ 0&1&1&1&1&0&1&0&0&1 \\ 1&0&1&1&1&1&0&1&0&0 \\ 1&1&0&1&1&0&1&0&1&0 \\ 1&1&1&0&1&0&0&1&0&1 \\ 1&1&1&1&0&1&0&0&1&0 \end{pmatrix}, \quad A_{G(2,1)} = \frac{1}{3}\begin{pmatrix} 3&2&1&1&1&2&1&2&3&2 \\ 1&3&2&1&1&2&2&1&2&3 \\ 1&1&3&2&1&3&2&2&1&2 \\ 1&1&1&3&2&2&3&2&2&1 \\ 2&1&1&1&3&1&2&3&2&2 \\ 3&2&1&2&2&3&0&2&2&1 \\ 2&3&2&1&2&1&3&0&2&2 \\ 2&2&3&2&1&2&1&3&0&2 \\ 1&2&2&3&2&2&2&1&3&0 \\ 2&1&2&2&3&0&2&2&1&3 \end{pmatrix},$$

hence we find

$$\mathrm{Spec}(\mathit{\Delta}_{\mathcal{A}_{G(2,1)}}) = \Big\{0, \frac{20}{3}, \epsilon', \epsilon', \rho', \rho', \varrho', \varrho', \varsigma', \varsigma'\Big\},$$

$$\mathrm{Spec}(\mathit{\Delta}_{\mathcal{Q}_{G(2,1)}}) = \Big\{0, \frac{10}{9}, \frac{\epsilon'}{6}, \frac{\epsilon'}{6}, \frac{\rho'}{6}, \frac{\rho'}{6}, \frac{\varrho'}{6}, \frac{\varrho'}{6}, \frac{\varsigma'}{6}, \frac{\varsigma'}{6}\Big\},$$

where

$$\begin{cases} \epsilon' = \dfrac{65-\sqrt{5}}{12} + \dfrac{1}{12}\sqrt{-10+14\sqrt{5}}, \\ \rho' = \dfrac{65-\sqrt{5}}{12} - \dfrac{1}{12}\sqrt{-10+14\sqrt{5}}, \end{cases} \qquad \begin{cases} \varrho' = \dfrac{65+\sqrt{5}}{12} + \dfrac{\sqrt{-1}}{12}\sqrt{10+14\sqrt{5}}, \\ \varsigma' = \dfrac{65+\sqrt{5}}{12} - \dfrac{\sqrt{-1}}{12}\sqrt{10+14\sqrt{5}}, \end{cases}$$

Similarly, as $A_{G^{(p)}_{[3]}} = A^3_{G^{(p)}} - 5A_{G^{(p)}}$ we have

$$A_{G^{(p)}_{[3]}} = \begin{pmatrix} 0&0&2&2&0&0&2&2&2&2 \\ 0&0&0&2&2&2&0&2&2&2 \\ 2&0&0&0&2&2&2&0&2&2 \\ 2&2&0&0&0&2&2&2&0&2 \\ 0&2&2&0&0&2&2&2&2&0 \\ 0&2&2&2&2&0&2&0&0&2 \\ 2&0&2&2&2&2&0&2&0&0 \\ 2&2&0&2&2&0&2&0&2&0 \\ 2&2&2&0&2&0&0&2&0&2 \\ 2&2&2&2&0&2&0&0&2&0 \end{pmatrix}, \quad A_{G(3,1)} = \frac{1}{3}\begin{pmatrix} 6&4&2&2&2&4&2&4&6&4 \\ 2&6&4&2&2&4&4&2&4&4 \\ 2&2&6&4&2&6&4&4&2&4 \\ 2&2&2&6&4&4&6&4&4&2 \\ 4&2&2&2&6&2&4&6&4&4 \\ 6&4&2&4&4&6&0&4&4&2 \\ 4&6&4&2&4&2&6&0&4&4 \\ 4&4&6&4&2&4&2&6&0&4 \\ 2&4&4&6&4&4&4&2&6&0 \\ 4&2&4&4&6&0&4&4&2&6 \end{pmatrix},$$

hence we get

$$\mathrm{Spec}(\varDelta_{\mathcal{A}_{G(3,1)}}) = \Big\{0, \frac{40}{3}, 2\epsilon', 2\epsilon', 2\rho', 2\rho', 2\varrho', 2\varrho', 2\varsigma', 2\varsigma'\Big\},$$

$$\mathrm{Spec}(\varDelta_{\mathcal{Q}_{G(3,1)}}) = \Big\{0, \frac{20}{18}, \frac{\epsilon'}{6}, \frac{\epsilon'}{6}, \frac{\rho'}{6}, \frac{\rho'}{6}, \frac{\varrho'}{6}, \frac{\varrho'}{6}, \frac{\varsigma'}{6}, \frac{\varsigma'}{6}\Big\},$$

where $\epsilon', \rho', \varrho', \varsigma'$ are the same as above. We therefore have

$$\mathrm{Spec}(\varDelta_{\mathcal{A}_{G(3,1)}}) = 2 \times \mathrm{Spec}(\varDelta_{\mathcal{A}_{G(2,1)}}), \qquad \mathrm{Spec}(\varDelta_{\mathcal{Q}_{G(3,1)}}) = \mathrm{Spec}(\varDelta_{\mathcal{Q}_{G(2,1)}}).$$

As $A_{G^{(p)}_{[4]}} = A^4_{G^{(p)}} - 7A^2_{G^{(p)}} + 6I$ we have

$$A_{G^{(p)}_{[4]}} = \begin{pmatrix} 0 & 4 & 2 & 2 & 4 & 4 & 2 & 2 & 2 & 2 \\ 4 & 0 & 4 & 2 & 2 & 2 & 4 & 2 & 2 & 2 \\ 2 & 4 & 0 & 4 & 2 & 2 & 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 0 & 4 & 2 & 2 & 2 & 4 & 2 \\ 4 & 2 & 2 & 4 & 0 & 2 & 2 & 2 & 2 & 4 \\ 4 & 2 & 2 & 2 & 2 & 0 & 2 & 4 & 4 & 2 \\ 2 & 4 & 2 & 2 & 2 & 2 & 0 & 2 & 4 & 4 \\ 2 & 2 & 4 & 2 & 2 & 4 & 2 & 0 & 2 & 4 \\ 2 & 2 & 2 & 4 & 2 & 4 & 4 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 4 & 2 & 4 & 4 & 2 & 0 \end{pmatrix}, \quad A_{G(4,1)} = \frac{1}{3} \begin{pmatrix} 6 & 8 & 6 & 6 & 10 & 8 & 6 & 8 & 6 & 8 \\ 10 & 6 & 8 & 6 & 6 & 8 & 8 & 6 & 8 & 6 \\ 6 & 10 & 6 & 8 & 6 & 6 & 8 & 8 & 6 & 8 \\ 6 & 6 & 10 & 6 & 8 & 8 & 6 & 8 & 8 & 6 \\ 8 & 6 & 6 & 10 & 6 & 6 & 8 & 6 & 8 & 8 \\ 6 & 8 & 10 & 8 & 4 & 6 & 8 & 8 & 8 & 6 \\ 4 & 6 & 8 & 10 & 8 & 6 & 6 & 8 & 8 & 8 \\ 8 & 4 & 6 & 8 & 10 & 8 & 6 & 6 & 8 & 8 \\ 10 & 8 & 4 & 6 & 8 & 8 & 8 & 6 & 6 & 8 \\ 8 & 10 & 8 & 4 & 6 & 8 & 8 & 8 & 6 & 6 \end{pmatrix},$$

hence we find

$$\mathrm{Spec}(\varDelta_{\mathcal{A}_{G(4,1)}}) = \Big\{0, 24, 24, 24, 24, 24, \frac{75-\sqrt{5}}{3}, \frac{75-\sqrt{5}}{3}, \frac{75+\sqrt{5}}{3}, \frac{75-\sqrt{5}}{3}\Big\},$$

$$\mathrm{Spec}(\varDelta_{\mathcal{Q}_{G(4,1)}}) = \Big\{0, 1, 1, 1, 1, 1, \frac{75-\sqrt{5}}{72}, \frac{75-\sqrt{5}}{72}, \frac{75+\sqrt{5}}{72}, \frac{75-\sqrt{5}}{72}\Big\}.$$

Since $A_{G^{(p)}_{[5]}} = A^5_{G^{(p)}} - 9A^3_{G^{(p)}} + 16A_{G^{(p)}}$ we have

$$A_{G^{(p)}_{[5]}} = \begin{pmatrix} 12 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 12 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 12 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 12 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 12 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 12 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 12 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 12 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 12 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 12 \end{pmatrix}, \quad A_{G(5,1)} = \frac{4}{3} \begin{pmatrix} 3 & 3 & 5 & 5 & 3 & 3 & 5 & 3 & 3 & 3 \\ 3 & 3 & 3 & 5 & 5 & 3 & 3 & 5 & 3 & 3 \\ 5 & 3 & 3 & 3 & 5 & 3 & 3 & 3 & 5 & 3 \\ 5 & 5 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5 \\ 3 & 5 & 5 & 3 & 3 & 5 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 5 & 3 & 5 & 3 & 3 & 5 \\ 5 & 3 & 3 & 3 & 3 & 5 & 3 & 5 & 3 & 3 \\ 3 & 5 & 3 & 3 & 3 & 3 & 5 & 3 & 5 & 3 \\ 3 & 3 & 5 & 3 & 3 & 3 & 3 & 5 & 3 & 5 \\ 3 & 3 & 3 & 5 & 3 & 5 & 3 & 3 & 5 & 3 \end{pmatrix},$$

we have

$$\mathrm{Spec}(\varDelta_{\mathcal{A}_{G(5,1)}}) = \Big\{0, \frac{136}{3}, \frac{136}{3}, \frac{136}{3}, \frac{136}{3}, \frac{136}{3}, \frac{160}{3}, \frac{160}{3}, \frac{160}{3}, \frac{160}{3}\Big\},$$

$$\mathrm{Spec}(\varDelta_{\mathcal{Q}_{G(5,1)}}) = \Big\{0, \frac{17}{18}, \frac{17}{18}, \frac{17}{18}, \frac{17}{18}, \frac{17}{18}, \frac{10}{9}, \frac{10}{9}, \frac{10}{9}, \frac{10}{9}\Big\}.$$

By the same reason as in Example 5.10 that $P_{G^{(p)}_{[2]}} = P_{G^{(p)}_{[3]}}$, we find that

$$\mathrm{Spec}(\Delta_{\mathcal{Q}_{G_{(2,q)}}}) = \mathrm{Spec}(\Delta_{\mathcal{Q}_{G_{(3,q)}}})$$

for an arbitrary positive integer q.

As

$$P_{G^{(a)}_{[2]}} = \frac{1}{6}(A^2_{G^{(a)}} - 3I), \qquad P_{G^{(a)}_{[3]}} = \frac{1}{12}(A^3_{G^{(a)}} - 5A_{G^{(a)}}),$$
$$P_{G^{(a)}_{[4]}} = \frac{1}{24}(A^4_{G^{(a)}} - 7A^2_{G^{(a)}} + 6I), \quad P_{G^{(a)}_{[5]}} = \frac{1}{48}(A^5_{G^{(a)}} - 9A^3_{G^{(a)}} + 16A_{G^{(a)}}),$$

we have

$$P_{G^{(a)}_{[2]}} = \frac{1}{6}\begin{pmatrix} 0&1&0&0&1&1&0&1&1&1\\ 1&0&1&0&0&1&1&0&1&1\\ 0&1&0&1&0&1&1&1&0&1\\ 0&0&1&0&1&1&1&1&1&0\\ 1&0&0&1&0&0&1&1&1&1\\ 1&1&1&1&0&0&0&1&1&0\\ 0&1&1&1&1&0&0&0&1&1\\ 1&0&1&1&1&1&0&0&0&1\\ 1&1&0&1&1&1&1&0&0&0\\ 1&1&1&0&1&0&1&1&0&0 \end{pmatrix}, \ P_{G^{(a)}_{[3]}} = \frac{1}{12}\begin{pmatrix} 0&2&0&0&2&2&0&2&2&2\\ 2&0&2&0&0&2&2&0&2&2\\ 0&2&0&2&0&2&2&2&0&2\\ 0&0&2&0&2&2&2&2&2&0\\ 2&0&0&2&0&0&2&2&2&2\\ 2&2&2&2&0&0&0&2&2&0\\ 0&2&2&2&2&0&0&0&2&2\\ 2&0&2&2&2&2&0&0&0&2\\ 2&2&0&2&2&2&2&0&0&0\\ 2&2&2&0&2&0&2&2&0&0 \end{pmatrix},$$

$$P_{G^{(a)}_{[4]}} = \frac{1}{24}\begin{pmatrix} 0&2&4&4&2&2&4&2&2&2\\ 2&0&2&4&4&2&2&4&2&2\\ 4&2&0&2&4&2&2&2&4&2\\ 4&4&2&0&2&2&2&2&2&4\\ 2&4&4&2&0&4&2&2&2&2\\ 2&2&2&2&4&0&4&2&2&4\\ 4&2&2&2&2&4&0&4&2&2\\ 2&4&2&2&2&2&4&0&4&2\\ 2&2&4&2&2&2&2&4&0&4\\ 2&2&2&4&2&4&2&2&4&0 \end{pmatrix}, \ P_{G^{(a)}_{[5]}} = \frac{1}{48}\begin{pmatrix} 12&4&4&4&4&4&4&4&4&4\\ 4&12&4&4&4&4&4&4&4&4\\ 4&4&12&4&4&4&4&4&4&4\\ 4&4&4&12&4&4&4&4&4&4\\ 4&4&4&4&12&4&4&4&4&4\\ 4&4&4&4&4&12&4&4&4&4\\ 4&4&4&4&4&4&12&4&4&4\\ 4&4&4&4&4&4&4&12&4&4\\ 4&4&4&4&4&4&4&4&12&4\\ 4&4&4&4&4&4&4&4&4&12 \end{pmatrix},$$

in particular, we have $P_{G^{(a)}_{[3]}} = P_{G^{(a)}_{[2]}}$ and $P_{G^{(a)}_{[5]}} = \frac{1}{12}(M + 2I)$. Thus we see

$$A_{G_{(1,2)}} = A_{G_{(1,3)}} = \frac{1}{6}\begin{pmatrix} 3&1&2&2&0&1&2&2&3&2\\ 0&3&1&2&2&2&1&2&2&3\\ 2&0&3&1&2&3&2&1&2&2\\ 2&2&0&3&1&2&3&2&1&2\\ 1&2&2&0&3&2&2&3&2&1\\ 2&2&1&2&3&3&1&1&1&2\\ 3&2&2&1&2&2&3&1&1&1\\ 2&3&2&2&1&1&2&3&1&1\\ 1&2&3&2&2&1&1&2&3&1\\ 2&1&2&3&2&1&1&1&2&3 \end{pmatrix},$$

$$A_{G(1,4)} = \frac{1}{24}\begin{pmatrix} 6 & 6 & 8 & 8 & 8 & 6 & 8 & 8 & 6 & 8 \\ 8 & 6 & 6 & 8 & 8 & 8 & 6 & 8 & 8 & 6 \\ 8 & 8 & 6 & 6 & 8 & 6 & 8 & 6 & 8 & 8 \\ 8 & 8 & 8 & 6 & 6 & 8 & 6 & 8 & 6 & 8 \\ 6 & 8 & 8 & 8 & 6 & 8 & 8 & 6 & 8 & 6 \\ 4 & 8 & 10 & 8 & 6 & 6 & 10 & 6 & 6 & 8 \\ 6 & 4 & 8 & 10 & 8 & 8 & 6 & 10 & 6 & 6 \\ 8 & 6 & 4 & 8 & 10 & 6 & 8 & 6 & 10 & 6 \\ 10 & 8 & 6 & 4 & 8 & 6 & 6 & 8 & 6 & 10 \\ 8 & 10 & 8 & 6 & 4 & 10 & 6 & 6 & 8 & 6 \end{pmatrix}, \quad A_{G(1,4)} = \frac{1}{12}\bigl(3M + 2A_{G^{(p)}}\bigr).$$

Hence we obtain

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,2)}}) = \mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,3)}}) = \Bigl\{0, \frac{10}{3}, \frac{\epsilon'}{2}, \frac{\epsilon'}{2}, \frac{\rho'}{2}, \frac{\rho'}{2}, \frac{\varrho'}{2}, \frac{\varrho'}{2}, \frac{\varsigma'}{2}, \frac{\varsigma'}{2}\Bigr\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,4)}}) = \Bigl\{0, 3, 3, 3, 3, 3, \frac{75-\sqrt{5}}{24}, \frac{75-\sqrt{5}}{24}, \frac{75+\sqrt{5}}{24}, \frac{75-\sqrt{5}}{24}\Bigr\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,5)}}) = \Bigl\{0, \frac{17}{6}, \frac{17}{6}, \frac{17}{6}, \frac{17}{6}, \frac{17}{6}, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}\Bigr\},$$

and find that this Kähler graph has a quite interesting property.

$$\mathrm{Spec}(\Delta_{\mathcal{Q}_{G(1,2)}}) = \mathrm{Spec}(\Delta_{\mathcal{Q}_{G(2,1)}}) = \mathrm{Spec}(\Delta_{\mathcal{Q}_{G(1,3)}}) = \mathrm{Spec}(\Delta_{\mathcal{Q}_{G(3,1)}}),$$
$$\mathrm{Spec}(\Delta_{\mathcal{Q}_{G(1,4)}}) = \mathrm{Spec}(\Delta_{\mathcal{Q}_{G(4,1)}}), \quad \mathrm{Spec}(\Delta_{\mathcal{Q}_{G(1,5)}}) = \mathrm{Spec}(\Delta_{\mathcal{Q}_{G(5,1)}}).$$

If we study more, as we have

$$A_{G(2,3)} = \frac{1}{12}\begin{pmatrix} 6 & 8 & 8 & 8 & 10 & 8 & 8 & 6 & 4 & 6 \\ 10 & 6 & 8 & 8 & 8 & 6 & 8 & 8 & 6 & 4 \\ 8 & 10 & 6 & 8 & 8 & 4 & 6 & 8 & 8 & 6 \\ 8 & 8 & 10 & 6 & 8 & 6 & 4 & 6 & 8 & 8 \\ 8 & 8 & 8 & 10 & 6 & 8 & 6 & 4 & 6 & 8 \\ 6 & 6 & 8 & 6 & 6 & 6 & 10 & 8 & 8 & 8 \\ 6 & 6 & 6 & 8 & 6 & 8 & 6 & 10 & 8 & 8 \\ 6 & 6 & 6 & 6 & 8 & 8 & 8 & 6 & 10 & 8 \\ 8 & 6 & 6 & 6 & 6 & 8 & 8 & 8 & 6 & 10 \\ 6 & 8 & 6 & 6 & 6 & 10 & 8 & 8 & 8 & 6 \end{pmatrix}, \; A_{G(2,3)} = \frac{1}{6}\begin{pmatrix} 6 & 8 & 8 & 8 & 10 & 8 & 8 & 6 & 4 & 6 \\ 10 & 6 & 8 & 8 & 8 & 6 & 8 & 8 & 6 & 4 \\ 8 & 10 & 6 & 8 & 8 & 4 & 6 & 8 & 8 & 6 \\ 8 & 8 & 10 & 6 & 8 & 6 & 4 & 6 & 8 & 8 \\ 8 & 8 & 8 & 10 & 6 & 8 & 6 & 4 & 6 & 8 \\ 6 & 6 & 8 & 6 & 6 & 6 & 10 & 8 & 8 & 8 \\ 6 & 6 & 6 & 8 & 6 & 8 & 6 & 10 & 8 & 8 \\ 6 & 6 & 6 & 6 & 8 & 8 & 8 & 6 & 10 & 8 \\ 8 & 6 & 6 & 6 & 6 & 8 & 8 & 8 & 6 & 10 \\ 6 & 8 & 6 & 6 & 6 & 10 & 8 & 8 & 8 & 6 \end{pmatrix},$$

$$A_{G(4,3)} = \frac{1}{3}\begin{pmatrix} 9 & 6 & 8 & 8 & 5 & 6 & 8 & 7 & 8 & 7 \\ 5 & 9 & 6 & 8 & 8 & 7 & 6 & 8 & 7 & 8 \\ 8 & 5 & 9 & 6 & 8 & 8 & 7 & 6 & 8 & 7 \\ 8 & 8 & 5 & 9 & 6 & 7 & 8 & 7 & 6 & 8 \\ 6 & 8 & 8 & 5 & 9 & 8 & 7 & 8 & 7 & 6 \\ 7 & 7 & 6 & 7 & 9 & 9 & 7 & 6 & 6 & 8 \\ 9 & 7 & 7 & 6 & 7 & 8 & 9 & 7 & 6 & 6 \\ 7 & 9 & 7 & 7 & 6 & 6 & 8 & 9 & 7 & 6 \\ 6 & 7 & 9 & 7 & 7 & 6 & 6 & 8 & 9 & 7 \\ 7 & 6 & 7 & 9 & 7 & 7 & 6 & 6 & 8 & 9 \end{pmatrix}, \qquad \begin{aligned} &A_{G(2,5)} = \frac{1}{6}\bigl(3M + A_{G^{(p)}_{[2]}}\bigr), \\ &A_{G(4,5)} = \frac{1}{6}\bigl(12M + A_{G^{(p)}_{[4]}}\bigr), \\ &A_{G(5,2)} = \frac{2}{3}\bigl(3M + A_{G^{(a)}_{[2]}}\bigr) = 2A_{G(5,3)}, \\ &A_{G(5,4)} = \frac{1}{12}\bigl(12M + A_{G^{(a)}_{[4]}}\bigr) \end{aligned}$$

we find

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(2,3)}}) = \Big\{0, \frac{16}{3}, \epsilon'', \epsilon'', \rho'', \rho'', \varrho'', \varrho'', \varsigma'', \varsigma''\Big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(2,5)}}) = \Big\{0, \frac{35}{6}, \frac{35}{6}, \frac{35}{6}, \frac{35}{6}, \frac{19}{6}, \frac{19}{6}, \frac{19}{6}, \frac{19}{6}, \frac{19}{6}\Big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(3,2)}}) = \Big\{0, \frac{32}{3}, 2\epsilon'', 2\epsilon'', 2\rho'', 2\rho'', 2\varrho'', 2\varrho'', 2\varsigma'', 2\varsigma''\Big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(3,4)}}) = \Big\{0, \frac{135-\sqrt{5}}{12}, \frac{135-\sqrt{5}}{12}, \frac{135+\sqrt{5}}{12}, \frac{135+\sqrt{5}}{12}, 12, 12, 12, 12, 12\Big\}$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(3,5)}}) = \Big\{0, \frac{35}{3}, \frac{35}{3}, \frac{35}{3}, \frac{35}{3}, \frac{38}{3}, \frac{38}{3}, \frac{38}{3}, \frac{38}{3}, \frac{38}{3}\Big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(4,3)}}) = \Big\{0, \frac{135-\sqrt{5}}{6}, \frac{135-\sqrt{5}}{6}, \frac{135+\sqrt{5}}{6}, \frac{135+\sqrt{5}}{6}, 24, 24, 24, 24, 24\Big\}$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(4,5)}}) = \Big\{0, 24, 24, 24, 24, 24, 25, 25, 25, 25\Big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(5,2)}}) = \mathrm{Spec}(\Delta_{\mathcal{A}_{G(5,3)}}) = \Big\{0, \frac{140}{3}, \frac{140}{3}, \frac{140}{3}, \frac{140}{3}, \frac{152}{3}, \frac{152}{3}, \frac{152}{3}, \frac{152}{3}, \frac{152}{3}\Big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(5,4)}}) = \Big\{0, 48, 48, 48, 48, 48, 50, 50, 50, 50\Big\},$$

where

$$\epsilon'' = \frac{1}{24}\Big(149 - \sqrt{5} - \sqrt{-1}\sqrt{34 - 10\sqrt{5}}\Big), \quad \rho'' = \frac{1}{24}\Big(149 - \sqrt{5} + \sqrt{-1}\sqrt{34 - 10\sqrt{5}}\Big),$$

$$\varrho'' = \frac{1}{24}\Big(149 + \sqrt{5} - \sqrt{-1}\sqrt{34 - 10\sqrt{5}}\Big), \quad \varsigma'' = \frac{1}{24}\Big(149 + \sqrt{5} + \sqrt{-1}\sqrt{34 - 10\sqrt{5}}\Big).$$

Hence find

$$\mathrm{Spec}(\Delta_{\mathcal{Q}_{G(2,3)}}) = \mathrm{Spec}(\Delta_{\mathcal{Q}_{G(3,2)}}), \quad \mathrm{Spec}(\Delta_{\mathcal{Q}_{G(2,5)}}) = \mathrm{Spec}(\Delta_{\mathcal{Q}_{G(5,2)}})$$

$$\mathrm{Spec}(\Delta_{\mathcal{Q}_{G(3,5)}}) = \mathrm{Spec}(\Delta_{\mathcal{Q}_{G(5,3)}}), \quad \mathrm{Spec}(\Delta_{\mathcal{Q}_{G(4,5)}}) = \mathrm{Spec}(\Delta_{\mathcal{Q}_{G(5,4)}}).$$

Example 5.12. We take a Petersen Kähler graph of second kind given in Example 2.24 in §2.3 which is like Fig. 9. It is a regular Kähler graph of $d^{(p)} = 3$, $d^{(a)} = 4$. The adjacency matrix of its auxiliary graph is given as The adjacency matrix of auxiliary graph is given as

$$A_{G^{(a)}} = \begin{pmatrix} 0&0&1&1&0&0&1&1&0&0\\ 0&0&0&1&1&0&0&1&1&0\\ 1&0&0&0&1&0&0&0&1&1\\ 1&1&0&0&0&1&0&0&0&1\\ 0&1&1&0&0&1&1&0&0&0\\ 0&0&0&1&1&0&1&0&0&1\\ 1&0&0&0&1&1&0&1&0&0\\ 1&1&0&0&0&0&1&0&1&0\\ 0&1&1&0&0&0&0&1&0&1\\ 0&0&1&1&0&1&0&0&1&0 \end{pmatrix}.$$

As the principal graph is a Petersen graph, we find that the adjacency matrices of the principal and the auxiliary graphs are not commutative:

$$A_{G^{(p)}}A_{G^{(a)}} = \begin{pmatrix} 0&1&1&2&2&1&2&1&1&1\\ 2&0&1&1&2&1&1&2&1&1\\ 2&2&0&1&1&1&1&1&2&1\\ 1&2&2&0&1&1&1&1&1&2\\ 1&1&2&2&0&2&1&1&1&1\\ 1&2&2&1&0&0&2&2&1&1\\ 0&1&2&2&1&1&0&2&2&1\\ 1&0&1&2&2&1&1&0&2&2\\ 2&1&0&1&2&2&1&1&0&2\\ 2&2&1&0&1&2&2&1&1&0 \end{pmatrix},\quad A_{G^{(a)}}A_{G^{(p)}} = \begin{pmatrix} 0&2&2&1&1&1&0&1&2&2\\ 1&0&2&2&1&2&1&0&1&2\\ 1&1&0&2&2&2&2&1&0&1\\ 2&1&1&0&2&1&2&2&1&0\\ 2&2&1&1&0&0&1&2&2&1\\ 1&1&1&1&2&0&1&1&2&2\\ 2&1&1&1&1&2&0&1&1&2\\ 1&2&1&1&1&2&2&0&1&1\\ 1&1&2&1&1&1&2&2&0&1\\ 1&1&1&2&1&1&1&2&2&0 \end{pmatrix}.$$

We here compute k-step adjacency. As we have

$$P_{G^{(a)}_{[2]}} = \frac{1}{12}(A^2_{G^{(a)}} - 4I), \qquad P_{G^{(a)}_{[3]}} = \frac{1}{36}(A^3_{G^{(a)}} - 9A_{G^{(a)}}),$$
$$P_{G^{(a)}_{[4]}} = \frac{1}{108}(A^4_{G^{(a)}} - 10A^2_{G^{(a)}} + 12I), \quad P_{G^{(a)}_{[5]}} = \frac{1}{324}\big(A^5_{G^{(a)}} - 13(A^3_{G^{(a)}} + 33(A_{G^{(a)}}\big),$$

we see

$$P_{G^{(a)}_{[2]}} = \frac{1}{12}\begin{pmatrix} 0&2&0&0&2&2&1&1&2&2\\ 2&0&2&0&0&2&2&1&1&2\\ 0&2&0&2&0&2&2&2&1&1\\ 0&0&2&0&2&1&2&2&2&1\\ 2&0&0&2&0&1&1&2&2&2\\ 2&2&2&1&1&0&1&1&1&1\\ 1&2&2&2&1&1&0&1&1&1\\ 1&1&2&2&2&1&1&0&1&1\\ 2&1&1&2&2&1&1&1&0&1\\ 2&2&1&1&2&1&1&1&1&0 \end{pmatrix},\quad P_{G^{(a)}_{[3]}} = \frac{1}{36}\begin{pmatrix} 2&5&3&3&5&5&2&2&5&4\\ 5&2&5&3&3&4&5&2&2&5\\ 3&5&2&5&3&5&4&5&2&2\\ 3&3&5&2&5&2&5&4&5&2\\ 5&3&3&5&2&2&2&5&4&5\\ 5&4&5&2&2&4&1&6&6&1\\ 2&5&4&5&2&1&4&1&6&6\\ 2&2&5&4&5&6&1&4&1&6\\ 5&2&2&5&4&6&6&1&4&1\\ 4&5&2&2&5&1&6&6&1&4 \end{pmatrix},$$

$$P_{G^{(a)}_{[4]}} = \frac{1}{108}\begin{pmatrix} 10 & 9 & 16 & 16 & 9 & 8 & 11 & 11 & 8 & 10 \\ 9 & 10 & 9 & 16 & 16 & 10 & 8 & 11 & 11 & 8 \\ 16 & 9 & 10 & 9 & 16 & 8 & 10 & 8 & 11 & 11 \\ 16 & 16 & 9 & 10 & 9 & 11 & 8 & 10 & 8 & 11 \\ 9 & 16 & 16 & 9 & 10 & 11 & 11 & 8 & 10 & 8 \\ 8 & 10 & 8 & 11 & 11 & 6 & 14 & 13 & 13 & 14 \\ 11 & 8 & 10 & 8 & 11 & 14 & 6 & 14 & 13 & 13 \\ 11 & 11 & 8 & 10 & 8 & 13 & 14 & 6 & 14 & 13 \\ 8 & 11 & 11 & 8 & 10 & 13 & 13 & 14 & 6 & 14 \\ 10 & 8 & 11 & 11 & 8 & 14 & 13 & 13 & 14 & 6 \end{pmatrix},$$

$$P_{G^{(a)}_{[5]}} = \frac{1}{324}\begin{pmatrix} 48 & 29 & 28 & 28 & 29 & 31 & 32 & 32 & 31 & 36 \\ 29 & 48 & 29 & 28 & 28 & 36 & 31 & 32 & 32 & 31 \\ 28 & 29 & 48 & 29 & 28 & 31 & 36 & 31 & 32 & 32 \\ 28 & 28 & 29 & 48 & 29 & 32 & 31 & 36 & 31 & 32 \\ 29 & 28 & 28 & 29 & 48 & 32 & 32 & 31 & 36 & 31 \\ 31 & 36 & 31 & 32 & 32 & 38 & 35 & 27 & 27 & 35 \\ 32 & 31 & 36 & 31 & 32 & 35 & 38 & 35 & 27 & 27 \\ 32 & 32 & 31 & 36 & 31 & 27 & 35 & 38 & 35 & 27 \\ 31 & 32 & 32 & 31 & 36 & 27 & 27 & 35 & 38 & 35 \\ 36 & 31 & 32 & 32 & 31 & 35 & 27 & 27 & 35 & 38 \end{pmatrix}.$$

Thus we have

$$A_{G_{(2,1)}} = \frac{1}{4}\begin{pmatrix} 4 & 3 & 2 & 1 & 2 & 3 & 1 & 2 & 3 & 3 \\ 2 & 4 & 3 & 2 & 1 & 3 & 3 & 1 & 2 & 3 \\ 1 & 2 & 4 & 3 & 2 & 3 & 3 & 3 & 1 & 2 \\ 2 & 1 & 2 & 4 & 3 & 2 & 3 & 3 & 3 & 1 \\ 3 & 2 & 1 & 2 & 4 & 1 & 2 & 3 & 3 & 3 \\ 3 & 2 & 2 & 2 & 3 & 4 & 1 & 2 & 3 & 2 \\ 3 & 3 & 2 & 2 & 2 & 2 & 4 & 1 & 2 & 3 \\ 2 & 3 & 3 & 2 & 2 & 3 & 2 & 4 & 1 & 2 \\ 2 & 2 & 3 & 3 & 2 & 2 & 3 & 2 & 4 & 1 \\ 2 & 2 & 2 & 3 & 3 & 1 & 2 & 3 & 2 & 4 \end{pmatrix}, \quad A_{G_{(4,1)}} = \frac{1}{2}\begin{pmatrix} 4 & 5 & 4 & 5 & 6 & 5 & 5 & 4 & 5 & 5 \\ 6 & 4 & 5 & 4 & 5 & 5 & 5 & 5 & 4 & 5 \\ 5 & 6 & 4 & 5 & 4 & 5 & 5 & 5 & 5 & 4 \\ 4 & 5 & 6 & 4 & 5 & 4 & 5 & 5 & 5 & 5 \\ 5 & 4 & 5 & 6 & 4 & 5 & 4 & 5 & 5 & 5 \\ 5 & 6 & 6 & 4 & 3 & 4 & 5 & 6 & 5 & 4 \\ 3 & 5 & 6 & 6 & 4 & 4 & 4 & 5 & 6 & 5 \\ 4 & 3 & 5 & 6 & 6 & 5 & 4 & 4 & 5 & 6 \\ 6 & 4 & 3 & 5 & 6 & 6 & 5 & 4 & 4 & 5 \\ 6 & 6 & 4 & 3 & 5 & 5 & 6 & 5 & 4 & 4 \end{pmatrix},$$

$$A_{G_{(5,1)}} = 4M + 2A_{G^{(a)}},$$

$$A_{G_{(1,2)}} = \frac{1}{12}\begin{pmatrix} 6 & 2 & 4 & 3 & 1 & 3 & 4 & 4 & 4 & 5 \\ 1 & 6 & 2 & 4 & 3 & 5 & 3 & 4 & 4 & 4 \\ 3 & 1 & 6 & 2 & 4 & 4 & 5 & 3 & 4 & 4 \\ 4 & 3 & 1 & 6 & 2 & 4 & 4 & 5 & 3 & 4 \\ 2 & 4 & 3 & 1 & 6 & 4 & 4 & 4 & 5 & 3 \\ 3 & 4 & 3 & 4 & 6 & 4 & 3 & 2 & 3 & 4 \\ 6 & 3 & 4 & 3 & 4 & 4 & 4 & 3 & 2 & 3 \\ 4 & 6 & 3 & 4 & 3 & 3 & 4 & 4 & 3 & 2 \\ 3 & 4 & 6 & 3 & 4 & 2 & 3 & 4 & 4 & 3 \\ 4 & 3 & 4 & 6 & 3 & 3 & 2 & 3 & 4 & 4 \end{pmatrix}, \quad A_{G_{(1,3)}} = \frac{1}{36}\begin{pmatrix} 15 & 9 & 13 & 10 & 7 & 10 & 8 & 13 & 12 & 11 \\ 7 & 15 & 9 & 13 & 10 & 11 & 10 & 8 & 13 & 12 \\ 10 & 7 & 15 & 9 & 13 & 12 & 11 & 10 & 8 & 13 \\ 13 & 10 & 7 & 15 & 9 & 13 & 12 & 11 & 10 & 8 \\ 9 & 13 & 10 & 7 & 15 & 8 & 13 & 12 & 11 & 10 \\ 9 & 9 & 10 & 12 & 14 & 17 & 9 & 7 & 10 & 11 \\ 14 & 9 & 9 & 10 & 12 & 11 & 17 & 9 & 7 & 10 \\ 12 & 14 & 9 & 9 & 10 & 10 & 11 & 17 & 9 & 7 \\ 10 & 12 & 14 & 9 & 9 & 7 & 10 & 11 & 17 & 9 \\ 9 & 10 & 12 & 14 & 9 & 9 & 7 & 10 & 11 & 17 \end{pmatrix},$$

$$A_{G_{(1,4)}}=\frac{1}{108}\begin{pmatrix} 26 & 36 & 33 & 36 & 37 & 27 & 33 & 32 & 34 & 30 \\ 37 & 26 & 36 & 33 & 36 & 30 & 27 & 33 & 32 & 34 \\ 36 & 37 & 26 & 36 & 33 & 34 & 30 & 27 & 33 & 32 \\ 33 & 36 & 37 & 26 & 36 & 32 & 34 & 30 & 27 & 33 \\ 36 & 33 & 36 & 37 & 26 & 33 & 32 & 34 & 30 & 27 \\ 29 & 31 & 35 & 34 & 27 & 34 & 38 & 31 & 28 & 37 \\ 27 & 29 & 31 & 35 & 34 & 37 & 34 & 38 & 31 & 28 \\ 34 & 27 & 29 & 31 & 35 & 28 & 37 & 34 & 38 & 31 \\ 35 & 34 & 27 & 29 & 31 & 31 & 28 & 37 & 34 & 38 \\ 31 & 35 & 34 & 27 & 29 & 38 & 31 & 28 & 37 & 34 \end{pmatrix},$$

$$A_{G_{(1,5)}}=\frac{1}{324}\begin{pmatrix} 89 & 112 & 88 & 89 & 108 & 106 & 98 & 90 & 95 & 97 \\ 108 & 89 & 112 & 88 & 89 & 97 & 106 & 98 & 90 & 95 \\ 89 & 108 & 89 & 112 & 88 & 95 & 97 & 106 & 98 & 90 \\ 88 & 89 & 108 & 89 & 112 & 90 & 95 & 97 & 106 & 98 \\ 112 & 88 & 89 & 108 & 89 & 98 & 90 & 95 & 97 & 106 \\ 111 & 93 & 91 & 95 & 96 & 85 & 94 & 105 & 104 & 98 \\ 96 & 111 & 93 & 91 & 95 & 98 & 85 & 94 & 105 & 104 \\ 95 & 96 & 111 & 93 & 91 & 104 & 98 & 85 & 94 & 105 \\ 91 & 95 & 96 & 111 & 93 & 105 & 104 & 98 & 85 & 94 \\ 93 & 91 & 95 & 96 & 111 & 94 & 105 & 104 & 98 & 85 \end{pmatrix},$$

Computing the eigenvalues of $\Delta_{\mathcal{A}_{G(p,q)}}$ we have

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,1)}}) = \big\{0, 3, \epsilon_1, \epsilon_1, \rho_1, \rho_1, \varrho_1, \varrho_1, \varsigma_1, \varsigma_1\big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(2,1)}}) = \big\{0, 6, \epsilon_2, \epsilon_2, \rho_2, \rho_2, \varrho_2, \varrho_2, \varsigma_2, \varsigma_2\big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(4,1)}}) = \big\{0, 24, 24, 24, 24, 24, 25, 25, 25, 25\big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(5,1)}}) = \big\{0, 48, \epsilon_3, \epsilon_3, \rho_3, \rho_3, \varrho_3, \varrho_3, \varsigma_3, \varsigma_3\big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,2)}}) = \big\{0, 10/3, \epsilon_4, \epsilon_4, \rho_4, \rho_4, \varrho_4, \varrho_4, \varsigma_4, \varsigma_4\big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,3)}}) = \big\{0, 3, \epsilon_5, \epsilon_5, \rho_5, \rho_5, \varrho_5, \varrho_5, \varsigma_5, \varsigma_5\big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,4)}}) = \big\{0, 26/9, \epsilon_6, \epsilon_6, \rho_6, \rho_6, \varrho_6, \varrho_6, \varsigma_6, \varsigma_6\big\},$$

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,5)}}) = \big\{0, 3, \epsilon_7, \epsilon_7, \rho_7, \rho_7, \varrho_7, \varrho_7, \varsigma_7, \varsigma_7\big\},$$

where

$$\begin{cases} \epsilon_1 = \dfrac{1}{8}\Big(27 + \sqrt{-1}\sqrt{11 - 4\sqrt{5}}\Big), \\ \rho_1 = \dfrac{1}{8}\Big(27 - \sqrt{-1}\sqrt{11 - 4\sqrt{5}}\Big), \end{cases} \qquad \begin{cases} \varrho_1 = \dfrac{1}{8}\Big(27 + \sqrt{-1}\sqrt{11 + 4\sqrt{5}}\Big), \\ \varsigma_1 = \dfrac{1}{8}\Big(27 - \sqrt{-1}\sqrt{11 + 4\sqrt{5}}\Big), \end{cases}$$

$$\begin{cases} \epsilon_2 = \dfrac{1}{4}\Big(22 - \sqrt{\sqrt{5}-1}\Big), \\ \rho_2 = \dfrac{1}{4}\Big(22 + \sqrt{\sqrt{5}-1}\Big), \end{cases} \qquad \begin{cases} \varrho_2 = \dfrac{1}{4}\Big(22 + \sqrt{-1}\sqrt{1+\sqrt{5}}\Big), \\ \varsigma_2 = \dfrac{1}{4}\Big(22 - \sqrt{-1}\sqrt{1+\sqrt{5}}\Big), \end{cases}$$

$$\epsilon_3 = 49 - \sqrt{11+2\sqrt{5}}, \quad \rho_3 = 49 - \sqrt{11-2\sqrt{5}}, \quad \varrho_3 = 49 + \sqrt{11-2\sqrt{5}}, \quad \varsigma_3 = 49 + \sqrt{11+2\sqrt{5}},$$

$$\begin{cases} \epsilon_4 = \dfrac{1}{48}\Big(135 - \sqrt{5} + \sqrt{70+66\sqrt{5}}\Big), \\ \rho_4 = \dfrac{1}{48}\Big(135 - \sqrt{5} - \sqrt{70+66\sqrt{5}}\Big), \end{cases} \qquad \begin{cases} \varrho_4 = \dfrac{1}{48}\Big(135 + \sqrt{5} + \sqrt{-1}\sqrt{66\sqrt{5}-70}\Big), \\ \varsigma_4 = \dfrac{1}{48}\Big(135 + \sqrt{5} - \sqrt{-1}\sqrt{66\sqrt{5}-70}\Big), \end{cases}$$

$$\begin{cases} \epsilon_5 = \dfrac{1}{72}\Big(203 + \sqrt{5} + \sqrt{-1}\sqrt{26-6\sqrt{5}}\Big), \\ \rho_5 = \dfrac{1}{72}\Big(203 + \sqrt{5} - \sqrt{-1}\sqrt{26-6\sqrt{5}}\Big), \end{cases} \qquad \begin{cases} \varrho_5 = \dfrac{1}{72}\Big(203 - \sqrt{5} + \sqrt{-1}\sqrt{26+6\sqrt{5}}\Big), \\ \varsigma_5 = \dfrac{1}{72}\Big(203 - \sqrt{5} - \sqrt{-1}\sqrt{26+6\sqrt{5}}\Big), \end{cases}$$

$$\begin{cases} \epsilon_6 = \dfrac{1}{216}\Big(657 - 5\sqrt{5} + \sqrt{194+122\sqrt{5}}\Big), \\ \rho_6 = \dfrac{1}{216}\Big(657 - 5\sqrt{5} - \sqrt{194+122\sqrt{5}}\Big), \end{cases} \qquad \begin{cases} \varrho_6 = \dfrac{1}{216}\Big(657 + 5\sqrt{5} + \sqrt{-1}\sqrt{122\sqrt{5}-194}\Big), \\ \varsigma_6 = \dfrac{1}{216}\Big(657 + 5\sqrt{5} - \sqrt{-1}\sqrt{122\sqrt{5}-194}\Big), \end{cases}$$

$$\begin{cases} \epsilon_7 = \dfrac{1}{1296}\Big(3939 + 13\sqrt{5} + \sqrt{7446-1426\sqrt{5}}\Big), \\ \rho_7 = \dfrac{1}{1296}\Big(3939 + 13\sqrt{5} - \sqrt{7446-1426\sqrt{5}}\Big), \end{cases}$$

$$\begin{cases} \varrho_7 = \dfrac{1}{1296}\Big(3939 - 13\sqrt{5} + \sqrt{7446+1426\sqrt{5}}\Big), \\ \varsigma_7 = \dfrac{1}{1296}\Big(3939 - 13\sqrt{5} + \sqrt{7446+1426\sqrt{5}}\Big). \end{cases}$$

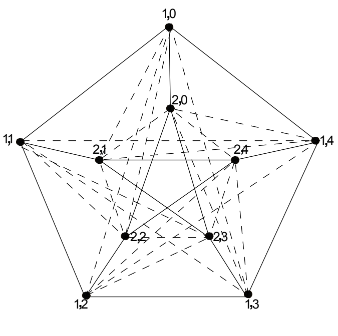


Fig. 9

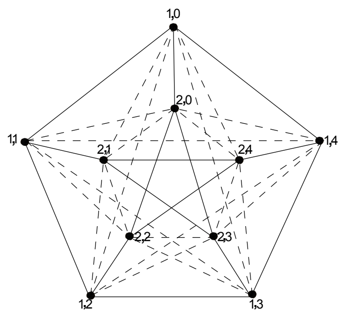


Fig. 10

EXAMPLE 5.13. We take a Kähler graph G obtained from Petersen Kähler graph of second kind given in Example 2.24 in §2.3 which is like Fig. 10. It is a regular Kähler graph of $d_G^{(p)} = 3$ and $d_G^{(a)} = 4$. The adjacency matrix of its auxiliary graph is given as

$$A_{G^{(a)}} = \begin{pmatrix} 0&0&1&1&0&0&1&0&0&1\\ 0&0&0&1&1&1&0&1&0&0\\ 1&0&0&0&1&0&1&0&1&0\\ 1&1&0&0&0&0&0&1&0&1\\ 0&1&1&0&0&1&0&0&1&0\\ 0&1&0&0&1&0&1&0&0&1\\ 1&0&1&0&0&1&0&1&0&0\\ 0&1&0&1&0&0&1&0&1&0\\ 0&0&1&0&1&0&0&1&0&1\\ 1&0&0&1&0&1&0&0&1&0 \end{pmatrix}.$$

We note that they are not commutative:

$$A_{G^{(p)}}A_{G^{(a)}} = \begin{pmatrix} 0&2&1&1&2&2&1&1&1&1\\ 2&0&2&1&1&1&2&1&1&1\\ 1&2&0&2&1&1&1&2&1&1\\ 1&1&2&0&2&1&1&1&2&1\\ 2&1&1&2&0&1&1&1&1&2\\ 0&1&2&2&1&0&2&1&1&2\\ 1&0&1&2&2&2&0&2&1&1\\ 2&1&0&1&2&1&2&0&2&1\\ 2&2&1&0&1&1&1&2&0&2\\ 1&2&2&1&0&2&1&1&2&0 \end{pmatrix}, \quad A_{G^{(a)}}A_{G^{(p)}} = \begin{pmatrix} 0&2&1&1&2&0&1&2&2&1\\ 2&0&2&1&1&1&0&1&2&2\\ 1&2&0&2&1&2&1&0&1&2\\ 1&1&2&0&2&2&2&1&0&1\\ 2&1&1&2&0&1&2&2&1&0\\ 2&1&1&1&1&0&2&1&1&2\\ 1&2&1&1&1&2&0&2&1&1\\ 1&1&2&1&1&1&2&0&2&1\\ 1&1&1&2&1&1&1&2&0&2\\ 1&1&1&1&2&2&1&1&2&0 \end{pmatrix}.$$

The eigenvalues of combinatorial Laplacians of its principal graph and those of $(1,1)$-combinatorial Laplacians are as follows:

$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G^{(p)}}}) = \{0,2,2,2,2,2,5,5,5,5\},$$
$$\mathrm{Spec}(\Delta_{\mathcal{A}_{G(1,1)}}) = \{0,3,\epsilon,\epsilon,\rho,\rho,\varrho,\varrho,\varsigma,\varsigma\},$$

where

$$\begin{cases} \epsilon = \dfrac{27+\sqrt{5}}{8} + \dfrac{\sqrt{-1}}{8}(\sqrt{5}-1), \\ \rho = \dfrac{27+\sqrt{5}}{8} - \dfrac{\sqrt{-1}}{8}(\sqrt{5}-1) \end{cases} \quad \begin{cases} \varrho = \dfrac{27-\sqrt{5}}{8} + \dfrac{\sqrt{-1}}{8}(\sqrt{5}+1), \\ \varsigma = \dfrac{27-\sqrt{5}}{8} - \dfrac{\sqrt{-1}}{8}(\sqrt{5}+1). \end{cases}$$

Like the Petersen Kähler graph of second kind in Example 5.12, it seem that this Kähler graph does not have good properties.

5.4. Eigenvalues of Kähler graphs obtained from a Heawood graph. Next we study Kähler graphs obtained from a Heawood graph.

EXAMPLE 5.14. We take a regular Kähler graph G of $d_G^{(p)} = 3$ and $d_G^{(a)} = 2$ like in Fig. 11. The adjacency matrices of its principal and auxiliary graphs are given as

$$A_G^{(p)} = \begin{pmatrix} 0&1&0&0&0&1&0&0&0&0&0&0&0&1\\ 1&0&1&0&0&0&0&0&0&0&1&0&0&0\\ 0&1&0&1&0&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&1&0&0&0&0&0&0&0&1&0\\ 0&0&0&1&0&1&0&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&1&0&0&0&1&0&0\\ 0&0&1&0&0&0&1&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&1&0&0&0&1\\ 0&0&0&0&1&0&0&0&1&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&1&0&0&0&1&0&1&0\\ 0&0&0&1&0&0&0&0&0&0&0&1&0&1\\ 1&0&0&0&0&0&0&0&1&0&0&0&1&0 \end{pmatrix},$$

$$A_G^{(a)} = \begin{pmatrix} 0&0&0&0&1&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&1&0&0&0&0 \end{pmatrix}.$$

These matrices are commutative:

$$A_G^{(p)}A_G^{(a)} = A_G^{(a)}A_G^{(p)} = \begin{pmatrix} 0&1&0&1&0&1&0&0&0&2&0&1&0&0\\ 1&0&0&0&1&0&2&0&0&0&1&0&1&0\\ 0&0&0&1&0&1&0&1&0&0&0&2&0&1\\ 1&0&1&0&0&0&1&0&2&0&0&0&1&0\\ 0&1&0&0&0&1&0&1&0&1&0&0&0&2\\ 1&0&1&0&1&0&0&0&1&0&2&0&0&0\\ 0&2&0&1&0&0&0&1&0&1&0&1&0&0\\ 0&0&1&0&1&0&1&0&0&0&1&0&2&0\\ 0&0&0&2&0&1&0&0&0&1&0&1&0&1\\ 2&0&0&0&1&0&1&0&1&0&0&0&1&0\\ 0&1&0&0&0&2&0&1&0&0&0&1&0&1\\ 1&0&2&0&0&0&1&0&1&0&1&0&0&0\\ 0&1&0&1&0&0&0&2&0&1&0&0&0&1\\ 0&0&1&0&2&0&0&0&1&0&1&0&1&0 \end{pmatrix}.$$

Thus Theorem 5.1 is applicable. The eigenvalues for these adjacency matrices are

$$\begin{aligned} &\mathrm{Spec}(\mathcal{A}_G^{(p)}) = \Big\{-3, -\sqrt{2}, -\sqrt{2}, -\sqrt{2}, -\sqrt{2}, -\sqrt{2}, -\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, 3\Big\},\\ &\mathrm{Spec}(\mathcal{A}_G^{(a)}) = \{2, 2, \eta_1, \eta_1, \eta_1, \eta_1, \eta_2, \eta_2, \eta_2, \eta_2, \eta_3, \eta_3, \eta_3, \eta_3\}, \end{aligned}$$

where

$$\begin{aligned} &\eta_1 = \frac{1}{3}\big(-7 + 2\sqrt{7}\cos\theta\big), \qquad \eta_2 = -\frac{1}{3}\big(7 + \sqrt{7}\cos\theta + \sqrt{21}\sin\theta\big),\\ &\eta_3 = -\frac{1}{3}\big(7 + \sqrt{7}\cos\theta - \sqrt{21}\sin\theta\big), \end{aligned}$$

with $\cos 3\theta = 1/(2\sqrt{7})$, $\sin 3\theta = (3\sqrt{3})/(2\sqrt{7})$. Here, ± 3 correspond to 2 and $\pm\sqrt{2}$ correspond doubly to η_i.

The eigenvalues of combinatorial Laplacians are

$$\begin{aligned} \mathrm{Spec}\big(\Delta_{\mathcal{A}_{G^{(a)}}}\big) &= \Big\{0, 3\pm\sqrt{2},\ 3\pm\sqrt{2},\ 3\pm\sqrt{2},\ 3\pm\sqrt{2},\ 3\pm\sqrt{2},\ 3\pm\sqrt{2}\Big\},\\ \mathrm{Spec}\big(\Delta_{\mathcal{A}_{G_{(1,1)}}}\big) &= \{0, 6, 3\pm\epsilon_1,\ 3\pm\epsilon_1,\ 3\pm\epsilon_2,\ 3\pm\epsilon_2,\ 3\pm\epsilon_3,\ 3\pm\epsilon_3\}\\ &= \mathrm{Spec}\big(\Delta_{\mathcal{A}_{G_{(1,2)}}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{A}_{G_{(1,3)}}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{A}_{G_{(1,4)}}}\big),\\ \mathrm{Spec}\big(\Delta_{\mathcal{A}_{G_{(2,1)}}}\big) &= \{\epsilon_1', \epsilon_1', \epsilon_1', \epsilon_1', \epsilon_2', \epsilon_2', \epsilon_2', \epsilon_2', \epsilon_3', \epsilon_3', \epsilon_3', \epsilon_3'\}\\ &= \mathrm{Spec}\big(\Delta_{\mathcal{A}_{G_{(2,3)}}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{A}_{G_{(2,5)}}}\big), \end{aligned}$$

where

$$\epsilon_1 = \frac{1}{6}\sqrt{30+12\sqrt{7}\cos\theta}, \quad \epsilon_2 = \frac{1}{6}\sqrt{30-6\sqrt{7}\cos\theta-6\sqrt{21}\sin\theta},$$
$$\epsilon_3 = \frac{1}{6}\sqrt{30-6\sqrt{7}\cos\theta+6\sqrt{21}\sin\theta},$$
$$\epsilon_1' = \frac{35+2\sqrt{7}\cos\theta}{6}, \quad \epsilon_2' = \frac{35-\sqrt{7}\cos\theta-\sqrt{21}\sin\theta}{6}, \quad \epsilon_3' = \frac{35-\sqrt{7}\cos\theta+\sqrt{21}\sin\theta}{6},$$

Since the auxiliary graph is a union of two 7-circuits and 7 is prime, we find

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{G_{(p,q)}}}\big) = \begin{cases} \mathrm{Spec}\big(\Delta_{\mathcal{A}_{G^{(a)}_{[p]}}}\big), & \text{when } q \equiv 0 \pmod 7, \\ \mathrm{Spec}\big(\Delta_{\mathcal{A}_{G_{(p,1)}}}\big), & \text{otherwise.} \end{cases}$$

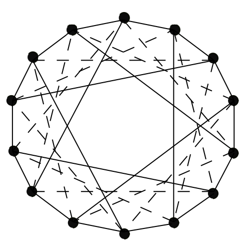


Fig. 11

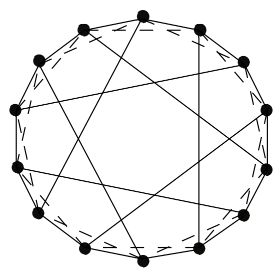


Fig. 12

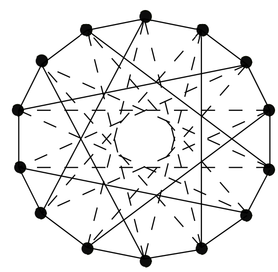


Fig. 13

Example 5.15. We take a regular Kähler graph G of $d_G^{(p)} = 3$ and $d_G^{(a)} = 2$ like in Fig. 12. The adjacency matrix of its auxiliary graph is given as

$$A_G^{(a)} = \begin{pmatrix} 0&0&1&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&1 \\ 1&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&1 \\ 1&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&1&0&0 \end{pmatrix}.$$

The adjacency matrices of the principal ant the auxiliary graphs are commutative:

$$A_G^{(p)} A_G^{(a)} = A_G^{(a)} A_G^{(p)} = \begin{pmatrix} 0&1&0&2&0&0&0&1&0&0&0&1&0&1\\ 1&0&1&0&1&0&0&0&1&0&0&0&2&0\\ 0&1&0&1&0&2&0&0&0&1&0&0&0&1\\ 2&0&1&0&1&0&1&0&0&0&1&0&0&0\\ 0&1&0&1&0&1&0&2&0&0&0&1&0&0\\ 0&0&2&0&1&0&1&0&1&0&0&0&1&0\\ 0&0&0&1&0&1&0&1&0&2&0&0&0&1\\ 1&0&0&0&2&0&1&0&1&0&1&0&0&0\\ 0&1&0&0&0&1&0&1&0&1&0&2&0&0\\ 0&0&1&0&0&0&2&0&1&0&1&0&1&0\\ 0&0&0&1&0&0&0&1&0&1&0&1&0&2\\ 1&0&0&0&1&0&0&0&2&0&1&0&1&0\\ 0&2&0&0&0&1&0&0&0&1&0&1&0&1\\ 1&0&1&0&0&0&1&0&0&0&2&0&1&0 \end{pmatrix}.$$

This Kähler graph has the same eigenvalues of (p, q)-combinatorial Laplacian as those for the Kähler graph in Example 5.14 for arbitrary pair of (p, q). We note that the auxiliary graphs of the graph and the graph in Example 5.14 are isomorphic and that the adjacency matrices of their auxiliary graphs are commutative. As a mater of fact, the adjacency matrix of the auxiliary graph of the Kähler graph in Example 5.14 is given as $(A_G^{(a)})^2 - 2I$ by the adjacency matrix of the auxiliary graph in this example.

Example 5.16. We take a regular Kähler graph G of $d_G^{(p)} = 3$ and $d_G^{(a)} = 2$ like in Fig. 11. The adjacency matrix of its auxiliary graph is given as

$$A_G^{(a)} = \begin{pmatrix} 0&0&0&0&0&0&1&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&1\\ 1&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&1\\ 1&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&1&0&0&0&0&0&0 \end{pmatrix}.$$

The adjacency matrices of the principal ant the auxiliary graphs are commutative:

$$A_G^{(p)}A_G^{(a)} = A_G^{(a)}A_G^{(p)} = \begin{pmatrix} 0&0&0&2&0&1&0&2&0&1&0&1&0&1\\ 0&0&1&0&1&0&1&0&2&0&1&0&0&0\\ 0&1&0&0&0&2&0&1&0&2&0&1&0&1\\ 0&0&0&0&1&0&1&0&1&0&2&0&1&0\\ 0&1&0&1&0&0&0&0&0&1&0&2&0&1\\ 1&0&0&0&0&0&1&0&1&0&1&0&2&0\\ 0&1&0&1&0&1&0&0&0&0&0&1&0&2\\ 2&0&1&0&0&0&0&0&1&0&1&0&1&0\\ 0&2&0&1&0&1&0&1&0&0&0&0&0&1\\ 1&0&2&0&1&0&0&0&0&0&1&0&1&0\\ 0&1&0&2&0&1&0&1&0&1&0&0&0&0\\ 1&0&1&0&2&0&1&0&0&0&0&0&1&0\\ 0&0&0&1&0&2&0&1&0&1&0&1&0&0\\ 1&0&1&0&1&0&2&0&1&0&0&0&0&0 \end{pmatrix}.$$

This Kähler graph has the same eigenvalues of (p, q)-combinatorial Laplacian as those for the Kähler graph in Example 5.14 for an arbitrary pair (p, q). We note that the auxiliary graphs of the graph and the graph in Example 5.14 are isomorphic and that the adjacency matrices of their auxiliary graphs are commutative. As a mater of fact, the adjacency matrix of the auxiliary graph in this example is given as $(A_G^{(a)})^3 - 3A_G^{(a)}$ by the adjacency matrix of the auxiliary graph of the Kähler graph in Example 5.15 .

5.5. Eigenvalues of Kähler flower snark. Let n (≥ 3) be an odd integer. A *flower snark* $J_n = (V, E)$ is a graph given with n copies of star graphs of 4 vertices as

$$\begin{aligned} V &= \{v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4} \mid i = 0, 1, \dots, n-1\}, \\ E &= \big\{\{v_{i,1}, v_{i,2}\},\ \{v_{i,2}, v_{i,3}\},\ \{v_{i,2}, v_{i,4}\},\ \{v_{i,1}, v_{i+1,1}\} \bigm| i = 0, \dots, n-1\big\} \\ &\qquad \bigcup \big\{\{v_{i,3}, v_{i+1,3}\},\ \{v_{i,4}, v_{i+1,4}\} \bigm| i \neq (n-1)/2\big\} \\ &\qquad \bigcup \big\{\{v_{(n-1)/2,3}, v_{(n+1)/2,4}\},\ \{v_{(n-1)/2,4}, v_{(n+1)/2,3}\}\big\}. \end{aligned}$$

where the former index is considered by modulo n (see Fig. 15). It is a regular graph of $d_{J_n} = 3$. We can express J_n in another way as

$$V' = \big\{v'_{1,j},\ v'_{2,k} \bigm| j = 0, \dots, 3n-1,\ k = 0, \dots, n-1\big\},$$

$$E' = \left\{ \begin{array}{l} \{v'_{1,j}, v'_{1,j+1}\},\ \{v'_{1,3k+1}, v'_{1,3k-4}\} \\ \{v'_{1,3k-1}, v'_{1,3k+4}\},\ \{v'_{1,3k}, v'_{2,k}\},\ \{v'_{2,k}, v'_{2,k+1}\} \end{array} \middle|\ \begin{array}{l} j = 0, \dots, 3n-1, \\ k = 0, \dots, n-1 \end{array} \right\},$$

where the latter index for vertices whose former index is 1 is considered by modulo $3n$, and that for vertices whose former index is 2 is considered by modulo n (see Fig. 16). An isomorphism of (V, E) to (V', E') is given as

$$v_{i,1} \mapsto v'_{2,i},\ v_{2j,2} \mapsto v'_{1,6j},\ v_{2j,3} \mapsto v'_{1,6j+1},\ v_{2j,4} \mapsto v'_{1,6j-1},$$
$$v_{2\ell+1,2} \mapsto v'_{1,6\ell+3},\ v_{2\ell+1,3} \mapsto v'_{1,6\ell+2},\ v_{2\ell+1,4} \mapsto v'_{1,6\ell+4},$$

where $1-m \le j \le m-1,\ -m \le \ell \le m-1$ when $n = 4m-1$ and $-m \le j \le m,\ -m \le \ell \le m-1$ when $n = 4m+1$.

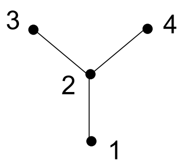


FIG. 14. 4-star

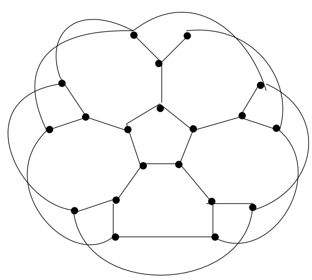


FIG. 15. J_5 original

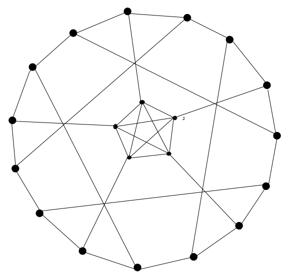


FIG. 16. J_5

By the former representation the adjacency matrix of J_n is given as

$$A_{J_n} = \begin{pmatrix} A & B & O & \cdots & \cdots & \cdots & \cdots & O & B \\ B & A & \ddots & \ddots & & & & & O \\ O & \ddots & \ddots & B & \ddots & & & & \vdots \\ \vdots & \ddots & B & A & C & O & & & \vdots \\ O & & O & C & A & B & O & & O \\ \vdots & & & O & B & A & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & O \\ O & & & & & \ddots & \ddots & \ddots & B \\ B & O & \cdots & \cdots & \cdots & \cdots & O & B & A \end{pmatrix} \begin{matrix} \\ \\ \\ \langle \frac{n-1}{2} \\ \\ \\ \\ \\ \\ \end{matrix}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where vertices are putted by lexicographical order. We set

$$E_1^{(a)} = \{\{v_{i,1}, v_{i,3}\},\ \{v_{i,1}, v_{i,4}\},\ \{v_{i,3}, v_{i+1,2}\},\ \{v_{i,4}, v_{i-1,2}\} \mid i = 0, \dots, n-1\},$$

$$E_2^{(a)} = \{\{v_{i,1}, v_{i+1,4}\},\ \{v_{i,2}, v_{i+1,4}\},\ \{v_{i,3}, v_{i+1,1}\},\ \{v_{i,3}, v_{i+1,2}\} \mid i = 0, \dots, n-1\},$$

$$\begin{aligned} E_3^{(a)} = &\big\{\{v_{i,3}, v_{i,4}\} \bigm| i = 0, \dots, n-1\big\} \\ &\bigcup \Big\{\{v_{i,3}, v_{i+1,4}\},\ \{v_{i,1}, v_{i+1,2}\},\ \{v_{i,2}, v_{i+1,1}\} \Bigm| i \neq \frac{n-1}{2}\Big\} \\ &\bigcup \Big\{\{v_{(n-1)/2,1}, v_{(n+1)/2,4}\},\ \{v_{(n-1)/2,2}, v_{(n+1)/2,2}\},\ \{v_{(n-1)/2,3}, v_{(n+1)/2,1}\}\Big\}, \end{aligned}$$

and consider three Kähler graphs $KJ_n^1 = (V, E \cup E_1^{(a)})$, $KJ_n^2 = (V, E \cup E_2^{(a)})$, $KJ_n^3 = (V, E \cup E_3^{(a)})$ and $KJ_n^4 = \big(V, E \cup (E_1^{(a)} \cup E_3^{(a)})\big)$. We shall call KJ_n^1, KJ_n^2, KJ_n^3 *Kähler flower snarks of first kind* and call KJ_n^4 *Kähler flower snark of second kind.* They are regular Kähler graph with $d^{(a)}_{KJ_n^1} = d^{(a)}_{KJ_n^2} = d^{(a)}_{KJ_n^3} = 2$ and $d^{(a)}_{KJ_n^4} = 4$. By definition the adjacency matrices of auxiliary graphs of KJ_n^1, KJ_n^2, KJ_n^3 are given as

$$A^{(a)}_{KJ_n^j} = \begin{pmatrix} K_j & L_j & O & \cdots & O & {}^tL_j \\ {}^tL_j & K_j & L_j & \ddots & & O \\ O & {}^tL_j & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & L \\ O & & \ddots & {}^tL_j & K_j & L_j \\ L_j & O & \cdots & O & {}^tL_j & K_j \end{pmatrix}$$

with

$$K_1 = \begin{pmatrix} 0&0&1&1\\0&0&0&0\\1&0&0&0\\1&0&0&0 \end{pmatrix}, \quad K_3 = O, \quad L_1 = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&0&0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0&0&0&1\\0&0&0&1\\1&1&0&0\\0&0&0&0 \end{pmatrix},$$

and

$$A^{(a)}_{KJ_n^3} = \begin{pmatrix} C & D & O & \cdots & \cdots & \cdots & \cdots & O & {}^tD \\ {}^tD & C & \ddots & \ddots & & & & & O \\ O & \ddots & \ddots & D & \ddots & & & & \vdots \\ \vdots & \ddots & {}^tD & C & F & O & & & \vdots \\ O & & O & {}^tF & C & D & O & & O \\ \vdots & & & O & {}^tD & C & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & O \\ O & & & & & \ddots & \ddots & \ddots & D \\ D & O & \cdots & \cdots & \cdots & \cdots & O & {}^tD & C \end{pmatrix} \begin{matrix} \\ \\ \\ \langle \frac{n-1}{2} \\ \\ \\ \\ \\ \\ \end{matrix}$$

with

$$C=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\quad D=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix},\quad F=\begin{pmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}.$$

Since $B^tL \neq LB$ and $B^tD \neq DB$, we find that $\mathcal{A}^{(p)}_{KJ_n^1}$, $\mathcal{A}^{(a)}_{KJ_n^1}$ are not commutative nor $\mathcal{A}^{(p)}_{KJ_n^2}$, $\mathcal{A}^{(a)}_{KJ_n^2}$ are. As $A^{(a)}_{KJ_n^4} = A^{(a)}_{KJ_n^1} + A^{(a)}_{KJ_n^3}$ and $B({}^tL + {}^tD) \neq (L+D)B$, we find that $\mathcal{A}^{(p)}_{KJ_n^4}$, $\mathcal{A}^{(p)}_{KJ_n^4}$ are not commutative.

We here show figures of Kähler flower snarks.

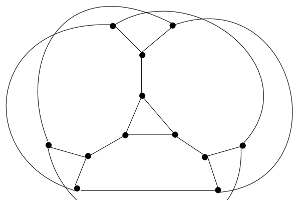


FIG. 17. J_3

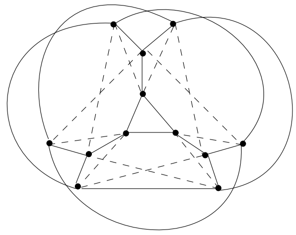


FIG. 18. KJ_3^1

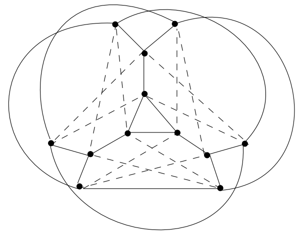


FIG. 19. KJ_3^2

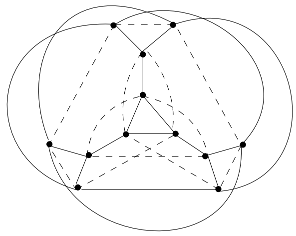


FIG. 20. KJ_3^3

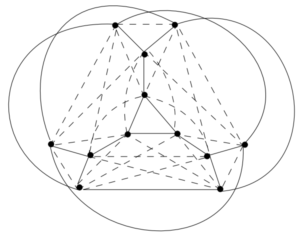


FIG. 21. KJ_3^4

EXAMPLE 5.17. We take a Kähler flower snark KJ_3^1 of first kind. We have

$$A^{(p)}_{KJ_3^1}=\begin{pmatrix}
0&1&0&0&1&0&0&0&1&0&0&0\\
1&0&1&1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&1&0\\
0&1&0&0&0&0&0&1&0&0&0&1\\
1&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&1&0&1&1&0&0&0&0\\
0&0&1&0&0&1&0&0&0&0&1&0\\
0&0&0&1&0&1&0&0&0&0&0&1\\
1&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&1&1\\
0&0&1&0&0&0&1&0&0&1&0&0\\
0&0&0&1&0&0&0&1&0&1&0&0
\end{pmatrix},\ A^{(a)}_{KJ_3^1}=\begin{pmatrix}
0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&1&0\\
1&0&0&0&0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&0&1&0&0\\
0&1&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&1&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&1&0&0&0
\end{pmatrix},$$

hence have

$$A^{(p)}_{KJ^1_3}P^{(a)}_{KJ^1_3} = \frac{1}{2}\begin{pmatrix} 0&0&0&0&0&0&1&2&0&0&2&1\\ 2&0&1&1&0&1&0&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&1&1&1&0\\ 0&1&0&0&1&1&0&1&1&0&1&0\\ 0&0&2&1&0&0&0&0&0&0&1&2\\ 0&1&0&0&2&0&1&1&0&1&0&0\\ 1&0&1&0&0&2&0&0&1&0&0&1\\ 1&1&1&0&0&0&0&0&1&1&0&1\\ 0&0&1&2&0&0&2&1&0&0&0&0\\ 0&1&0&0&0&1&0&0&2&0&1&1\\ 1&1&0&1&1&1&1&0&0&0&0&0\\ 1&0&0&1&1&0&1&0&0&2&0&0 \end{pmatrix}$$

and

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{J_3}}\big) = \Big\{0, 1, \frac{7-\sqrt{13}}{2}, \frac{7-\sqrt{13}}{2}, 2, 2, 4, 4, 4, 5, \frac{7+\sqrt{13}}{2}, \frac{7+\sqrt{13}}{2}\Big\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KJ^1_3)_{(1,1)}}}\big) = \left\{\begin{array}{l} 0, 3, 3, 4, \\ \text{solutions of } 2^4t^4 - 2^3\cdot 26t^3 + 2^2\cdot 249t^2 - 2\cdot 1054t + 1665 = 0 \\ \text{solutions of } 2^4t^4 - 2^3\cdot 26t^3 + 2^2\cdot 249t^2 - 2\cdot 1038t + 1593 = 0 \end{array}\right\}.$$

Example 5.18. We take a Kähler flower snark KJ^2_3 of first kind. We have

$$A^{(a)}_{KJ^2_3} = \begin{pmatrix} 0&0&0&0&0&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&0&1&0&0&1&0\\ 0&0&0&0&1&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&1&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1&1&0&0\\ 1&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&1&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0&0&0 \end{pmatrix},$$

hence get

$$A^{(p)}_{KJ_3^2}P^{(a)}_{KJ_3^2}=\frac{1}{2}\begin{pmatrix}0&0&1&1&0&0&1&1&0&0&1&1\\0&0&0&0&1&1&0&1&1&1&1&0\\1&1&0&0&0&0&0&1&1&1&1&0\\1&1&0&0&1&1&0&1&0&0&1&0\\0&0&1&1&0&0&1&1&0&0&1&1\\1&1&1&0&0&0&0&0&1&1&0&1\\0&0&1&0&2&2&0&0&0&0&0&1\\1&1&1&0&0&0&0&0&1&1&0&1\\0&0&1&1&0&0&1&1&0&0&1&1\\1&1&0&1&1&1&1&0&0&0&0&0\\1&1&0&1&1&1&1&0&0&0&0&0\\0&0&0&1&0&0&1&0&2&2&0&0\end{pmatrix}.$$

The eigenvalues of Laplacians are

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KJ_3^2)(1,1)}}\big)=\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KJ_3^2)(1,3)}}\big)=\Big\{0,3,3,3,3,3,3,3,4,4,\frac{7+\sqrt{-3}}{2},\frac{7-\sqrt{-3}}{2}\Big\},$$

$$\begin{aligned}\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KJ_3^2)(2,1)}}\big)&=\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KJ_3^2)(2,3)}}\big)\\&=\Big\{0,\frac{9}{2},\ 6,6,6,6,6,6,6,\frac{15}{2},\frac{12+\sqrt{-3}}{2},\frac{12-\sqrt{-3}}{2}\Big\}\end{aligned}$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KJ_3^2)(3,1)}}\big)=\Big\{0,\frac{7-\sqrt{2}}{2},\ 12,12,12,12,12,12,12,\frac{7-\sqrt{2}}{2},\frac{21+\sqrt{-6}}{2},\frac{21-\sqrt{-6}}{2}\Big\}$$

$$\begin{aligned}\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KJ_3^2)(4,1)}}\big)&=\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KJ_3^2)(4,3)}}\big)\\&=\Big\{0,20,\frac{41}{2},\frac{45}{2},24,24,24,24,24,24,24,\frac{49}{2},\frac{53}{2}\Big\}\end{aligned}$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KJ_3^2)(1,2)}}\big)=\left\{\begin{array}{l}0,\frac{5+\sqrt{-7}}{2},\frac{5-\sqrt{-7}}{2},\\ \text{solutions of } t^5-14t^4+76t^3-198t^2+242t-99=0\\ \text{solutions of } t^4-9t^3+31t^2-47t+27=0\end{array}\right\}$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KJ_3^2)(1,4)}}\big)=\Big\{0,1,\frac{7-\sqrt{13}}{2},\frac{7-\sqrt{13}}{2},2,2,4,4,4,5,\frac{7+\sqrt{13}}{2},\frac{7+\sqrt{13}}{2}\Big\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KJ_3^2)(3,2)}}\big)=\left\{\begin{array}{l}0,13+\sqrt{-7},\ 13-\sqrt{-7},\\ \text{solutions of } t^5-58t^4+351t^3-15754t^2+91808t-213696=0\\ \text{solutions of } t^4-52t^3+1023t^2-9040t+30240=0\end{array}\right\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KJ_3^2)_{(3,4)}}}\big) = \left\{0, 8, 8, 8, 10, \frac{29-\sqrt{13}}{2}, \frac{29-\sqrt{13}}{2}, 14, 16, 16, \frac{29+\sqrt{13}}{2}, \frac{29+\sqrt{13}}{2}\right\}.$$

We here list the adjacency matrices:

$$A_{{KJ_3^2}_{(2,1)}} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 2 & 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 0 & 0 & 2 & 2 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$A_{{KJ_3^2}_{(3,1)}} = \frac{1}{2}\begin{pmatrix} 4 & 4 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 4 & 4 & 1 & 1 & 4 & 0 & 1 & 1 & 0 & 4 \\ 2 & 2 & 3 & 3 & 2 & 2 & 3 & 0 & 2 & 2 & 0 & 3 \\ 2 & 2 & 3 & 3 & 2 & 2 & 3 & 0 & 2 & 2 & 0 & 3 \\ 2 & 2 & 2 & 1 & 4 & 4 & 1 & 1 & 2 & 2 & 1 & 2 \\ 1 & 1 & 0 & 4 & 2 & 2 & 4 & 4 & 1 & 1 & 4 & 0 \\ 4 & 4 & 0 & 3 & 0 & 0 & 3 & 3 & 2 & 2 & 3 & 0 \\ 0 & 0 & 0 & 3 & 4 & 4 & 3 & 3 & 2 & 2 & 3 & 0 \\ 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 4 & 4 & 1 & 1 \\ 1 & 1 & 4 & 0 & 1 & 1 & 0 & 4 & 2 & 2 & 4 & 4 \\ 0 & 0 & 3 & 0 & 2 & 2 & 0 & 3 & 4 & 4 & 3 & 3 \\ 4 & 4 & 3 & 0 & 2 & 2 & 0 & 3 & 0 & 0 & 3 & 3 \end{pmatrix},$$

$$A_{{KJ_3^2}_{(4,1)}} = \frac{1}{2}\begin{pmatrix} 6 & 6 & 3 & 3 & 5 & 5 & 3 & 2 & 5 & 5 & 2 & 3 \\ 4 & 4 & 5 & 5 & 4 & 4 & 5 & 2 & 4 & 4 & 2 & 5 \\ 4 & 4 & 5 & 5 & 1 & 1 & 5 & 4 & 5 & 5 & 4 & 5 \\ 4 & 4 & 5 & 5 & 5 & 5 & 5 & 4 & 1 & 1 & 4 & 5 \\ 5 & 5 & 2 & 3 & 6 & 6 & 3 & 3 & 5 & 5 & 3 & 2 \\ 4 & 4 & 2 & 5 & 4 & 4 & 5 & 5 & 4 & 4 & 5 & 2 \\ 3 & 3 & 4 & 5 & 6 & 6 & 5 & 5 & 1 & 1 & 5 & 4 \\ 3 & 3 & 4 & 5 & 2 & 2 & 5 & 5 & 5 & 5 & 5 & 4 \\ 5 & 5 & 3 & 2 & 5 & 5 & 2 & 3 & 6 & 6 & 3 & 3 \\ 4 & 4 & 5 & 2 & 4 & 4 & 2 & 5 & 4 & 4 & 5 & 5 \\ 3 & 3 & 5 & 4 & 5 & 5 & 4 & 5 & 2 & 2 & 5 & 5 \\ 3 & 3 & 5 & 4 & 1 & 1 & 4 & 5 & 6 & 6 & 5 & 5 \end{pmatrix},$$

$$A_{KJ^2_3(1,2)} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{KJ^2_3(1,3)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 2 & 0 & 0 \end{pmatrix},$$

$$A_{KJ^2_3(1,4)} = \frac{1}{2}\begin{pmatrix} 6 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 6 & 0 & 0 & 2 & 0 & 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 0 & 6 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 6 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 \\ 2 & 2 & 0 & 0 & 6 & 0 & 2 & 2 & 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 2 & 0 & 6 & 0 & 0 & 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 2 & 0 & 6 & 2 & 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 & 2 & 6 & 0 & 2 & 0 & 2 \\ 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 6 & 0 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 & 0 & 2 & 2 & 0 & 6 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 6 & 2 \\ 0 & 2 & 2 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 2 & 6 \end{pmatrix},$$

$$A_{KJ^2_3(2,3)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 2 & 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 0 & 0 & 2 & 2 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$A_{KJ^2_{3}(3,2)} = \frac{1}{2}\begin{pmatrix} 0 & 4 & 4 & 4 & 2 & 0 & 0 & 4 & 2 & 0 & 4 & 0 \\ 0 & 0 & 2 & 2 & 6 & 2 & 0 & 2 & 6 & 2 & 2 & 0 \\ 0 & 0 & 4 & 0 & 2 & 4 & 4 & 0 & 2 & 4 & 4 & 0 \\ 0 & 0 & 0 & 4 & 2 & 4 & 0 & 4 & 2 & 4 & 0 & 4 \\ 2 & 0 & 4 & 0 & 0 & 4 & 4 & 4 & 2 & 0 & 0 & 4 \\ 6 & 2 & 2 & 0 & 0 & 0 & 2 & 2 & 6 & 2 & 0 & 2 \\ 2 & 4 & 0 & 0 & 0 & 0 & 4 & 4 & 2 & 4 & 4 & 0 \\ 2 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 4 \\ 2 & 0 & 0 & 4 & 2 & 0 & 4 & 0 & 0 & 4 & 4 & 4 \\ 6 & 2 & 0 & 2 & 6 & 2 & 2 & 0 & 0 & 0 & 2 & 2 \\ 2 & 4 & 4 & 4 & 2 & 4 & 4 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 2 & 4 & 0 & 4 & 0 & 0 & 4 & 4 \end{pmatrix},$$

$$A_{KJ^2_{3}(3,4)} = \frac{1}{2}\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 2 & 4 & 4 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 2 & 6 & 2 & 2 & 2 & 6 & 2 & 2 \\ 0 & 0 & 0 & 4 & 4 & 2 & 0 & 4 & 4 & 2 & 0 & 4 \\ 0 & 0 & 4 & 0 & 4 & 2 & 4 & 0 & 4 & 2 & 4 & 0 \\ 0 & 2 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 2 & 4 & 4 \\ 2 & 6 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 6 & 2 & 2 \\ 4 & 2 & 0 & 4 & 0 & 0 & 0 & 4 & 4 & 2 & 4 & 0 \\ 4 & 2 & 4 & 0 & 0 & 0 & 4 & 0 & 4 & 2 & 0 & 4 \\ 0 & 2 & 4 & 4 & 0 & 2 & 4 & 4 & 4 & 0 & 0 & 0 \\ 2 & 6 & 2 & 2 & 2 & 6 & 2 & 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 4 & 4 & 2 & 4 & 0 & 0 & 0 & 0 & 4 \\ 4 & 2 & 4 & 0 & 4 & 2 & 0 & 4 & 0 & 0 & 4 & 0 \end{pmatrix},$$

$$A_{KJ^2_{3}(4,3)} = \frac{1}{2}\begin{pmatrix} 6 & 6 & 3 & 3 & 5 & 5 & 3 & 2 & 5 & 5 & 2 & 3 \\ 4 & 4 & 5 & 5 & 4 & 4 & 5 & 2 & 4 & 4 & 2 & 5 \\ 4 & 4 & 5 & 5 & 1 & 1 & 5 & 4 & 5 & 5 & 4 & 5 \\ 4 & 4 & 5 & 5 & 5 & 5 & 5 & 4 & 1 & 1 & 4 & 5 \\ 5 & 5 & 2 & 3 & 6 & 6 & 3 & 3 & 5 & 5 & 3 & 2 \\ 4 & 4 & 2 & 5 & 4 & 4 & 5 & 5 & 4 & 4 & 5 & 2 \\ 3 & 3 & 4 & 5 & 6 & 6 & 5 & 5 & 1 & 1 & 5 & 4 \\ 3 & 3 & 4 & 5 & 2 & 2 & 5 & 5 & 5 & 5 & 5 & 4 \\ 5 & 5 & 3 & 2 & 5 & 5 & 2 & 3 & 6 & 6 & 3 & 3 \\ 4 & 4 & 5 & 2 & 4 & 4 & 2 & 5 & 4 & 4 & 5 & 5 \\ 3 & 3 & 5 & 4 & 5 & 5 & 4 & 5 & 2 & 2 & 5 & 5 \\ 3 & 3 & 5 & 4 & 1 & 1 & 4 & 5 & 6 & 6 & 5 & 5 \end{pmatrix}.$$

Example 5.19. We take a Kähler flower snark KJ_3^3 of first kind. We have

$$A^{(a)}_{KJ_3^3} = \begin{pmatrix} 0&0&0&0&0&1&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&1&0&0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&1&0&0&0&0&0\\ 1&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&0&1&0 \end{pmatrix},$$

$$A^{(p)}_{KJ_3^3} P^{(a)}_{KJ_3^3} = \frac{1}{2} \begin{pmatrix} 0&2&0&0&1&0&1&0&1&0&0&1\\ 0&0&1&1&0&1&0&1&0&1&1&0\\ 0&0&0&1&1&0&0&1&2&0&0&1\\ 0&0&1&0&2&0&1&0&1&0&1&0\\ 1&1&0&0&0&1&1&0&0&2&0&0\\ 0&1&1&0&0&0&1&1&1&0&0&1\\ 1&0&0&1&1&0&0&1&0&1&1&0\\ 1&0&1&1&0&0&0&0&0&1&1&1\\ 1&1&0&0&0&2&0&0&0&1&0&1\\ 0&1&0&1&1&0&1&0&0&0&1&1\\ 1&0&1&1&0&1&1&1&0&0&0&0\\ 1&0&1&0&0&1&0&1&1&0&1&0 \end{pmatrix}.$$

The eigenvalues of $(1,1)$-Laplacian are

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KJ_3^3)(1,1)}}\big) = \left\{ \begin{array}{l} 0, 7/2, \\ \text{solutions of } 2^3t^4 - 2^2\cdot 25t^3 + 22\cdot 232t^2 - 941t + 646 = 0 \\ \text{solutions of } \begin{array}{c} 2^4t^6 - 2^3\cdot 40t^5 + 2^2\cdot 667t^4 - 2\cdot 5945t^3 \\ +29923t^2 - 40382t + 22860 = 0 \end{array} \end{array} \right\}.$$

We here make mention of the case that n is even. When n is even we shall call the graph $F_n = (V,E)$ defined by

$$V = \{v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4} \mid i = 0,\ldots,n-1\},$$
$$E = \left\{ \begin{array}{l} \{v_{i,1}, v_{i,2}\},\ \{v_{i,2}, v_{i,3}\},\ \{v_{i,2}, v_{i,4}\}, \\ \{v_{i,1}, v_{i+1,1}\}, \{v_{i,3}, v_{i+1,3}\},\ \{v_{i,4}, v_{i+1,4}\} \end{array} \middle|\ i = 0,\ldots,n-1 \right\},$$

a *flower*. It is also represented as

$$V' = \big\{ v'_{1,j},\ v'_{2,k} \ \big|\ j = 0, \dots, 3n-1,\ k = 0, \dots, n-1 \big\},$$

$$E' = \left\{ \begin{array}{l} \{v'_{1,j}, v'_{1,j+1}\},\ \{v'_{1,3k+1}, v'_{1,3k-4}\} \\ \{v'_{1,3k}, v'_{2,k}\},\ \{v'_{2,k}, v'_{2,k+1}\} \end{array} \ \middle|\ \begin{array}{l} j = 0, \dots, 3n-1, \\ k = 0, \dots, n-1 \end{array} \right\},$$

An isomorphism of (V, E) to (V', E') is given as

$$v_{i,1} \mapsto v'_{2,i},\ v_{2j,2} \mapsto v'_{1,6j},\ v_{2j,3} \mapsto v'_{1,6j+1},\ v_{2j,4} \mapsto v'_{1,6j-1},$$

$$v_{2j+1,2} \mapsto v'_{1,6j+3},\ v_{2j+1,3} \mapsto v'_{1,6j+2},\ v_{2j+1,4} \mapsto v'_{1,6j+4} \qquad \Big(0 \le j \le \frac{n-2}{2}\Big).$$

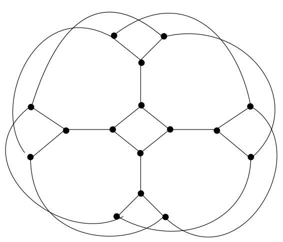


FIG. 22. F_4

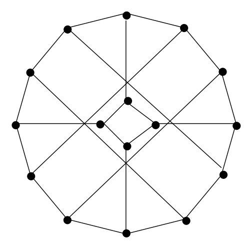


FIG. 23. F_4

The adjacency matrix of F_n is given as

$$A_{F_n} = \begin{pmatrix} A & B & O & \cdots & O & B \\ B & A & B & \ddots & & O \\ O & B & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & O \\ O & & O & B & A & B \\ B & O & \cdots & O & B & A \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

By setting

$$E_1^{(a)} = \Big\{ \{v_{i,1}, v_{i,3}\},\ \{v_{i,1}, v_{i,4}\},\ \{v_{i,3}, v_{i+1,2}\},\ \{v_{i,4}, v_{i-1,2}\} \ \Big|\ i = 0, \dots, n-1 \Big\},$$

$$E_2^{(a)} = \Big\{ \{v_{i,2}, v_{i+1,1}\},\ \{v_{i,2}, v_{i-1,1}\},\ \{v_{i,3}, v_{i,4}\},\ \{v_{i,3}, v_{i+1,4}\} \ \Big|\ i = 0, \dots, n-1 \Big\},$$

$$E_3^{(a)} = \Big\{ \{v_{i,1}, v_{i,3}\},\ \{v_{i,1}, v_{i,4}\},\ \{v_{i,3}, v_{i+1,4}\},\ \{v_{i,2}, v_{i+1,2}\} \ \Big|\ i = 0, \dots, n-1 \Big\},$$

$$E_4^{(a)} = \Big\{ \{v_{i,1}, v_{i+1,4}\},\ \{v_{i,2}, v_{i+1,2}\}\ \{v_{i,3}, v_{i+1,1}\},\ \{v_{i,3}, v_{i+1,4}\}, \ \Big|\ i = 0, \dots, n-1 \Big\},$$

$$E_5^{(a)} = \Big\{ \{v_{i,1}, v_{i+1,4}\},\ \{v_{i,2}, v_{i+1,4}\}\ \{v_{i,3}, v_{i+1,1}\},\ \{v_{i,3}, v_{i+1,2}\}, \ \Big|\ i = 0, \dots, n-1 \Big\},$$

$$E_6^{(a)} = \left\{ \begin{array}{l} \{v_{i,1}, v_{i,3}\},\ \{v_{i,1}, v_{i,4}\},\ \{v_{i,3}, v_{i,4}\}, \\ \{v_{i,1}, v_{i+1,2}\},\ \{v_{i,2}, v_{i+1,2}\},\ \{v_{i,3}, v_{i+1,4}\} \end{array} \ \middle|\ i = 0, \dots, n-1 \right\},$$

we obtain five flower like Kähler graphs $KF_n^j = (V, E \cup E_j^{(a)})$ $(j = 1,2,3,4,5)$ of auxiliary degree 2, a flower like Kähler graph $KF_n^6 = (V, E \cup E_6^{(a)})$ of auxiliary degree 3, and four flower like Kähler graphs

$$KF_n^7 = \big(V, E \cup (E_1^{(a)} \cup E_2^{(a)})\big), \quad KF_n^8 = \big(V, E \cup (E_1^{(a)} \cup E_4^{(a)})\big),$$
$$KF_n^9 = \big(V, E \cup (E_2^{(a)} \cup E_5^{(a)})\big), \quad KF_n^{10} = \big(V, E \cup (E_3^{(a)} \cup E_5^{(a)})\big)$$

of auxiliary degree 4.

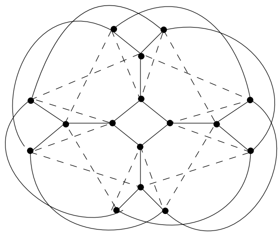


Fig. 24. KF_4^1

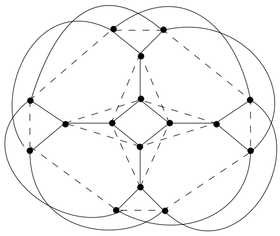


Fig. 25. KF_4^2

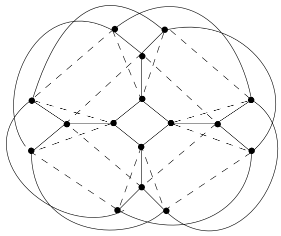


Fig. 26. KF_4^3

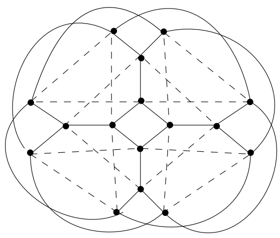


Fig. 27. KF_4^4

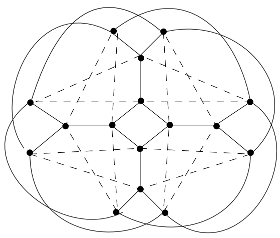


Fig. 28. KF_4^5

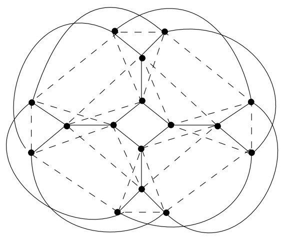


Fig. 29. KF_4^6

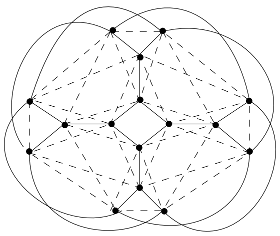


Fig. 30. KF_4^7

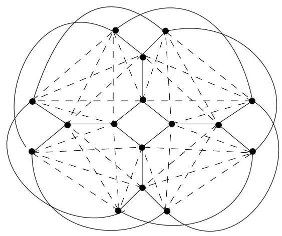


Fig. 31. KF_4^8

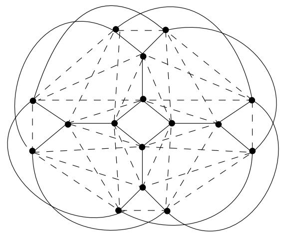


FIG. 32. KF_4^9

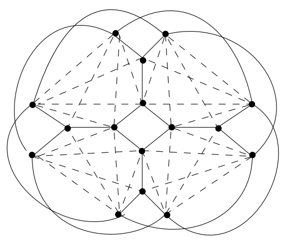


FIG. 33. KF_4^{10}

The adjacency matrices of their auxiliary graphs are given as

$$A^{(a)}_{KF_n^j} = \begin{pmatrix} K_j & L_j & O & \cdots & O & {}^tL_j \\ {}^tL_j & K_j & L_j & \ddots & & O \\ O & {}^tL_j & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & L_j \\ O & & \ddots & {}^tL_j & K_j & L_j \\ L_j & O & \cdots & O & {}^tL_j & K_j \end{pmatrix}$$

with

$$K_1 = K_3 = \begin{pmatrix} 0&0&1&1\\0&0&0&0\\1&0&0&0\\1&0&0&0 \end{pmatrix},\ K_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix},\ K_4 = K_5 = O,\ K_6 = \begin{pmatrix} 0&0&1&1\\0&0&0&0\\1&0&0&1\\1&0&1&0 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&0&0 \end{pmatrix},\quad L_2 = \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix},\quad L_3 = \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix},$$

$$L_4 = \begin{pmatrix} 0&0&0&1\\0&1&0&0\\1&0&0&1\\0&0&0&0 \end{pmatrix},\quad L_5 = \begin{pmatrix} 0&0&0&1\\0&0&0&1\\1&1&0&0\\0&0&0&0 \end{pmatrix},\quad L_6 = \begin{pmatrix} 0&1&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}.$$

As we can see $AK_j + B{}^tL_j + BL_j \neq K_jA + L_jB + {}^tL_jB$, the adjacency operators of the principal and the auxiliary graphs of KF_n^j are not commutative.

Since these Kähler graphs are more "symmetric" than Kähler flower snarks, their eigenvalues are tamer.

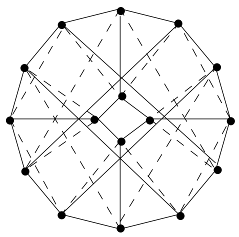


FIG. 34. KF_4^1

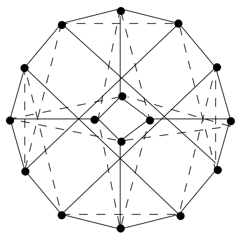


FIG. 35. KF_4^2

EXAMPLE 5.20. We take a Kähler flower KF_4^1. We have

$$A^{(p)}_{KF_4^1} = \begin{pmatrix}
0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&0\\
1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&1\\
1&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&1&0&0&0&0&1&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&1\\
1&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1\\
0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0\\
0&0&0&1&0&0&0&0&0&0&0&1&0&1&0&0
\end{pmatrix},$$

$$A^{(a)}_{KF_4^1} = \begin{pmatrix}
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0\\
1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0\\
0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0
\end{pmatrix},$$

hence get

$$A^{(p)}_{KF^1_4}P^{(a)}_{KF^1_4} = \frac{1}{2}\begin{pmatrix}
0&0&0&0&0&0&1&2&0&0&0&0&0&0&2&1\\
2&0&1&1&0&1&0&0&0&0&0&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0&1&0&0&1&0&1&0\\
0&1&0&0&1&0&0&1&0&1&0&0&1&0&1&0\\
0&0&2&1&0&0&0&0&0&0&1&2&0&0&0&0\\
0&1&0&0&2&0&1&1&0&1&0&0&0&0&0&0\\
1&0&1&0&0&1&0&0&1&0&0&1&0&1&0&0\\
1&0&1&0&0&1&0&0&1&0&0&1&0&1&0&0\\
0&0&0&0&0&0&2&1&0&0&0&0&0&0&1&2\\
0&0&0&0&0&1&0&0&2&0&1&1&0&1&0&0\\
0&1&0&0&1&0&1&0&0&1&0&0&1&0&0&1\\
0&1&0&0&1&0&1&0&0&1&0&0&1&0&0&1\\
0&0&1&2&0&0&0&0&0&0&2&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&1&0&0&2&0&1&1\\
1&0&0&1&0&1&0&0&1&0&1&0&0&1&0&0\\
1&0&0&1&0&1&0&0&1&0&1&0&0&1&0&0
\end{pmatrix}.$$

The eigenvalues of Laplacians are

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{F_4}}\big) = \big\{0,1,1,3-\sqrt{3},\ 3-\sqrt{3},\ 2,3,3,3,3,4,3+\sqrt{3},\ 3+\sqrt{3},5,5,6\big\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^1_4)(1,1)}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^1_4)(1,3)}}\big) = \big\{0,2,3,3,3,3,3,3,3,3,3,3,3,3,4,6\big\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^1_4)(2,1)}}\big) = \big\{0,0,4,4,6,6,6,6,6,6,6,6,6,6,6,6\big\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^1_4)(3,1)}}\big) = \big\{0,12,12,12,12,12,12,12,12,12,12,12,12,12,12,24\big\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^1_4)(1,2)}}\big) = \left\{\begin{array}{l} 0,1,\dfrac{5+\sqrt{-7}}{2},\ \dfrac{5-\sqrt{-7}}{2},\ 3,3,3,3,3,3,3,3, \\ \quad \dfrac{7+\sqrt{-7}}{2},\ \dfrac{5-\sqrt{-7}}{2},\ 5,6 \end{array}\right\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^1_4)(3,2)}}\big) = \left\{\begin{array}{l} 0,10,11+\sqrt{-7},\ 11-\sqrt{-7},\ 12,12,12,12, \\ 12,12,12,12,13+\sqrt{-7},\ 13-\sqrt{-7},\ 14,24 \end{array}\right\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^1_4)(2,3)}}\big) = \big\{0,0,6,6,6,6,6,6,6,6,6,6,6,6,8,8\big\}.$$

We here list adjacency matrices:

$$A_{(KF_4^1)_{(2,1)}} = \frac{1}{2}\begin{pmatrix} 2&0&1&1&0&1&0&0&0&0&3&3&0&1&0&0\\ 0&2&0&0&2&0&1&1&0&2&0&0&2&0&1&1\\ 1&0&2&2&0&0&0&0&2&0&1&1&0&3&0&0\\ 1&0&2&2&0&3&0&0&2&0&1&1&0&0&0&0\\ 0&1&0&0&2&0&1&1&0&1&0&0&0&0&3&3\\ 2&0&1&1&0&2&0&0&2&0&1&1&0&2&0&0\\ 0&3&0&0&1&0&2&2&0&0&0&0&2&0&1&1\\ 0&0&0&0&1&0&2&2&0&3&0&0&2&0&1&1\\ 0&0&3&3&0&1&0&0&2&0&1&1&0&1&0&0\\ 0&2&0&0&2&0&1&1&0&2&0&0&2&0&1&1\\ 2&0&1&1&0&3&0&0&1&0&2&2&0&0&0&0\\ 2&0&1&1&0&0&0&0&1&0&2&2&0&3&0&0\\ 0&1&0&0&0&0&3&3&0&1&0&0&2&0&1&1\\ 2&0&1&1&0&2&0&0&2&0&1&1&0&2&0&0\\ 0&0&0&0&2&0&1&1&0&3&0&0&1&0&2&2\\ 0&3&0&0&2&0&1&1&0&0&0&0&1&0&2&2 \end{pmatrix},$$

$$A_{(KF_4^1)_{(3,1)}} = \frac{1}{2}\begin{pmatrix} 0&4&0&0&4&0&3&1&0&4&0&0&4&0&1&3\\ 0&0&3&3&0&2&0&0&4&0&5&5&0&2&0&0\\ 0&3&0&0&3&0&4&2&0&3&0&0&3&0&2&4\\ 0&3&0&0&3&0&4&2&0&3&0&0&3&0&2&4\\ 4&0&1&3&0&4&0&0&4&0&3&1&0&4&0&0\\ 0&2&0&0&0&0&3&3&0&2&0&0&4&0&5&5\\ 3&0&2&4&0&3&0&0&3&0&4&2&0&3&0&0\\ 3&0&2&4&0&3&0&0&3&0&4&2&0&3&0&0\\ 0&4&0&0&4&0&1&3&0&4&0&0&4&0&3&1\\ 4&0&5&5&0&2&0&0&0&0&3&3&0&2&0&0\\ 0&3&0&0&3&0&2&4&0&3&0&0&3&0&4&2\\ 0&3&0&0&3&0&2&4&0&3&0&0&3&0&4&2\\ 4&0&3&1&0&4&0&0&4&0&1&3&0&4&0&0\\ 0&2&0&0&4&0&5&5&0&2&0&0&0&0&3&3\\ 3&0&4&2&0&3&0&0&3&0&2&4&0&3&0&0\\ 3&0&4&2&0&3&0&0&3&0&2&4&0&3&0&0 \end{pmatrix},$$

$$A_{(KF_4^1)_{(1,2)}} = \frac{1}{2}\begin{pmatrix} 0&2&0&0&1&0&0&0&0&2&0&0&1&0&0&0\\ 0&0&1&1&0&1&0&0&0&0&1&1&0&1&0&0\\ 0&0&0&0&1&0&0&2&0&0&0&0&1&0&0&2\\ 0&0&0&0&1&0&2&0&0&0&0&0&1&0&2&0\\ 1&0&0&0&0&2&0&0&1&0&0&0&0&2&0&0\\ 0&1&0&0&0&0&1&1&0&1&0&0&0&0&1&1\\ 1&0&0&2&0&0&0&0&1&0&0&2&0&0&0&0\\ 1&0&2&0&0&0&0&0&1&0&2&0&0&0&0&0\\ 0&2&0&0&1&0&0&0&0&2&0&0&1&0&0&0\\ 0&0&1&1&0&1&0&0&0&0&1&1&0&1&0&0\\ 0&0&0&0&1&0&0&2&0&0&0&0&1&0&0&2\\ 0&0&0&0&1&0&2&0&0&0&0&0&1&0&2&0\\ 1&0&0&0&0&2&0&0&1&0&0&0&0&2&0&0\\ 0&1&0&0&0&0&1&1&0&1&0&0&0&0&1&1\\ 1&0&0&2&0&0&0&0&1&0&0&2&0&0&0&0\\ 1&0&2&0&0&0&0&0&1&0&2&0&0&0&0&0 \end{pmatrix},$$

$$A_{(KF_4^1)_{(1,3)}} = \frac{1}{2}\begin{pmatrix}
0&0&0&0&0&0&2&1&0&0&0&0&0&0&1&2\\
0&0&0&0&0&1&0&0&2&0&1&1&0&1&0&0\\
0&1&0&0&1&0&1&0&0&1&0&0&1&0&0&1\\
0&1&0&0&1&0&1&0&0&1&0&0&1&0&0&1\\
0&0&1&2&0&0&0&0&0&0&2&1&0&0&0&0\\
0&1&0&0&0&0&0&0&1&0&0&2&0&1&1&0\\
1&0&0&1&0&1&0&0&1&0&1&0&0&1&0&0\\
1&0&0&1&0&1&0&0&1&0&1&0&0&1&0&0\\
0&0&0&0&0&0&1&2&0&0&0&0&0&0&2&1\\
2&0&1&1&0&1&0&0&0&0&0&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0&1&0&0&1&0&1&0\\
0&1&0&0&1&0&0&1&0&1&0&0&1&0&1&0\\
0&0&2&1&0&0&0&0&0&0&1&2&0&0&0&0\\
0&1&0&0&2&0&1&1&0&1&0&0&0&0&0&0\\
1&0&1&0&0&1&0&0&1&0&0&1&0&1&0&0\\
1&0&1&0&0&1&0&0&1&0&0&1&0&1&0&0
\end{pmatrix},$$

$$A_{(KF_4^1)_{(2,3)}} = \frac{1}{2}\begin{pmatrix}
0&0&3&3&0&1&0&0&2&0&1&1&0&1&0&0\\
0&2&0&0&2&0&1&1&0&2&0&0&2&0&1&1\\
2&0&1&1&0&3&0&0&1&0&2&2&0&0&0&0\\
2&0&1&1&0&0&0&0&1&0&2&2&0&3&0&0\\
0&1&0&0&0&3&3&0&1&0&0&2&0&1&1&0\\
2&0&1&1&0&2&0&0&2&0&1&1&0&2&0&0\\
0&0&0&0&2&0&1&1&0&3&0&0&1&0&2&2\\
0&3&0&0&2&0&1&1&0&0&0&0&1&0&2&2\\
2&0&1&1&0&1&0&0&0&3&3&0&1&0&0&0\\
0&2&0&0&2&0&1&1&0&2&0&0&2&0&1&1\\
1&0&2&2&0&0&0&0&2&0&1&1&0&3&0&0\\
1&0&2&2&0&3&0&0&2&0&1&1&0&0&0&0\\
0&1&0&0&2&0&1&1&0&1&0&0&0&0&3&3\\
2&0&1&1&0&2&0&0&2&0&1&1&0&2&0&0\\
0&3&0&0&1&0&2&2&0&0&0&0&2&0&1&1\\
0&0&0&0&1&0&2&2&0&3&0&0&2&0&1&1
\end{pmatrix},$$

$$A_{(KF_4^1)_{(3,2)}} = \frac{1}{2}\begin{pmatrix}
0&2&0&0&2&0&4&4&0&2&0&0&2&0&4&4\\
6&0&2&2&0&2&0&0&6&0&2&2&0&2&0&0\\
0&4&0&0&2&0&4&2&0&4&0&0&2&0&4&2\\
0&4&0&0&2&0&2&4&0&4&0&0&2&0&2&4\\
2&0&4&4&0&2&0&0&2&0&4&4&0&2&0&0\\
0&2&0&0&6&0&2&2&0&2&0&0&6&0&2&2\\
2&0&4&2&0&4&0&0&2&0&4&2&0&4&0&0\\
2&0&2&4&0&4&0&0&2&0&2&4&0&4&0&0\\
0&2&0&0&2&0&4&4&0&2&0&0&2&0&4&4\\
6&0&2&2&0&2&0&0&6&0&2&2&0&2&0&0\\
0&4&0&0&2&0&4&2&0&4&0&0&2&0&4&2\\
0&4&0&0&2&0&2&4&0&4&0&0&2&0&2&4\\
2&0&4&4&0&2&0&0&2&0&4&4&0&2&0&0\\
0&2&0&0&6&0&2&2&0&2&0&0&6&0&2&2\\
2&0&4&2&0&4&0&0&2&0&4&2&0&4&0&0\\
2&0&2&4&0&4&0&0&2&0&2&4&0&4&0&0
\end{pmatrix}.$$

EXAMPLE 5.21. We take a Kähler flower KF_4^2. The adjacency matrix of its auxiliary graph is given as

$$A^{(a)}_{KF_4^2} = \begin{pmatrix} 0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&1&1&0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1\\ 0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0 \end{pmatrix},$$

hence get

$$A^{(p)}_{KF_4^2} P^{(a)}_{KF_4^2} = \frac{1}{2} \begin{pmatrix} 0&2&0&0&1&0&0&0&0&2&0&0&1&0&0&0\\ 0&0&1&1&0&1&0&1&0&0&0&0&0&1&1&0\\ 0&0&0&1&1&0&0&1&0&0&0&1&1&0&0&1\\ 0&0&1&0&1&0&1&0&0&0&1&0&1&0&1&0\\ 1&0&0&0&0&2&0&0&1&0&0&0&0&2&0&0\\ 0&1&1&0&0&0&1&1&0&1&0&1&0&0&0&0\\ 1&0&0&1&0&0&0&1&1&0&0&1&0&0&0&1\\ 1&0&1&0&0&0&1&0&1&0&1&0&0&0&1&0\\ 0&2&0&0&1&0&0&0&0&2&0&0&1&0&0&0\\ 0&0&0&0&0&1&1&0&0&0&1&1&0&1&0&1\\ 0&0&0&1&1&0&0&1&0&0&0&1&1&0&0&1\\ 0&0&1&0&1&0&1&0&0&0&1&0&1&0&1&0\\ 1&0&0&0&0&2&0&0&1&0&0&0&0&2&0&0\\ 0&1&0&1&0&0&0&0&0&1&1&0&0&0&1&1\\ 1&0&0&1&0&0&0&1&1&0&0&1&0&0&0&1\\ 1&0&1&0&0&0&1&0&1&0&1&0&0&0&1&0 \end{pmatrix}.$$

The eigenvalues of Laplacians are

$$\begin{aligned}
\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^2)(1,1)}}\big) &= \mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^2)(1,3)}}\big) \\
&= \Big\{0, \frac{5-\sqrt{-7}}{2}, \frac{5+\sqrt{-7}}{2}, 3,3,3,3,3,3,3,3,3,3,4,4,5\Big\}, \\
\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^2)(2,1)}}\big) &= \left\{ \begin{array}{l} 0,4,4, \frac{11-\sqrt{7}}{2}, \frac{11-\sqrt{7}}{2}, \frac{11-\sqrt{-7}}{2}, \frac{11+\sqrt{-7}}{2}, \\ 6,6,6,6,6,6, \frac{11+\sqrt{7}}{2}, \frac{11+\sqrt{7}}{2}, 7 \end{array} \right\}, \\
\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^2)(3,1)}}\big) &= \left\{ \begin{array}{l} 0,5,2(7-\sqrt{3}),\ 12,12,12,12,12,12,12,12,12,12, \\ 2(7+\sqrt{3}),\ 13-\sqrt{-7},\ 13+\sqrt{-7} \end{array} \right\}, \\
\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^2)(1,2)}}\big) &= \left\{ \begin{array}{l} 0,1,1,1,2,2,3,3,3,3,4,4,4, \\ \text{solutions of } t^3-9t^2+24t-24=0 \end{array} \right\}, \\
\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^2)(2,3)}}\big) &= \left\{ \begin{array}{l} 0,4,4, \frac{13-\sqrt{7}}{2}, \frac{13-\sqrt{7}}{2}, 6,6,6,6,6,6, \\ \frac{13+\sqrt{7}}{2}, \frac{13-\sqrt{7}}{2}, \frac{11-\sqrt{-7}}{2}, \frac{11+\sqrt{-7}}{2} \end{array} \right\}, \\
\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^2)(3,2)}}\big) &= \left\{ \begin{array}{l} 0,4,5,5,12,12,12,12,14,14,14,14,14, \\ \text{solutions of } t^3-38t^2+480t-2112=0 \end{array} \right\}.
\end{aligned}$$

We here list adjacency matrices:

$$A_{(KF_4^2)(2,1)} = \frac{1}{2}\begin{pmatrix}
2&0&1&1&0&2&0&1&2&0&0&0&0&2&1&0\\
0&2&1&1&0&0&1&1&0&2&1&1&0&0&1&1\\
2&0&1&0&0&1&0&0&2&0&0&2&0&1&1&2\\
2&0&0&1&0&1&2&1&2&0&2&0&0&1&0&0\\
0&2&1&0&2&0&1&1&0&2&0&1&2&0&0&0\\
0&0&1&1&0&2&1&1&0&0&1&1&0&2&1&1\\
0&1&1&2&2&0&1&0&0&1&0&0&2&0&0&2\\
0&1&0&0&2&0&0&1&0&1&2&1&2&0&2&0\\
2&0&0&0&0&2&1&0&2&0&1&1&0&2&0&1\\
0&2&1&1&0&0&1&1&0&2&1&1&0&0&1&1\\
2&0&0&2&0&1&1&2&2&0&1&0&0&1&0&0\\
2&0&2&0&0&1&0&0&2&0&0&1&0&1&2&1\\
0&2&0&1&2&0&0&0&0&2&1&0&2&0&1&1\\
0&0&1&1&0&2&1&1&0&0&1&1&0&2&1&1\\
0&1&0&0&2&0&0&2&0&1&1&2&2&0&1&0\\
0&1&2&1&2&0&2&0&0&1&0&0&2&0&0&1
\end{pmatrix},$$

$$A_{(KF_4^2)_{(3,1)}} = \frac{1}{2}\begin{pmatrix}
0&2&2&2&2&0&2&2&0&2&2&2&2&0&2&2\\
6&0&0&0&0&2&2&0&6&0&2&2&0&2&0&2\\
0&4&2&1&2&0&2&1&0&4&2&1&2&0&2&1\\
0&4&1&2&2&0&1&2&0&4&1&2&2&0&1&2\\
2&0&2&2&0&2&2&2&2&0&2&2&0&2&2&2\\
0&2&0&2&6&0&0&0&0&2&2&0&6&0&2&2\\
2&0&2&1&0&4&2&1&2&0&2&1&0&4&2&1\\
2&0&1&2&0&4&1&2&2&0&1&2&0&4&1&2\\
0&2&2&2&2&0&2&2&0&2&2&2&2&0&2&2\\
6&0&2&2&0&2&0&2&6&0&0&0&0&2&2&0\\
0&4&2&1&2&0&2&1&0&4&2&1&2&0&2&1\\
0&4&1&2&2&0&1&2&0&4&1&2&2&0&1&2\\
2&0&2&2&0&2&2&2&2&0&2&2&0&2&2&2\\
0&2&2&0&6&0&2&2&0&2&0&2&6&0&0&0\\
2&0&2&1&0&4&2&1&2&0&2&1&0&4&2&1\\
2&0&1&2&0&4&1&2&2&0&1&2&0&4&1&2
\end{pmatrix},$$

$$A_{(KF_4^2)_{(1,2)}} = \frac{1}{2}\begin{pmatrix}
0&0&0&0&2&0&0&0&0&2&0&0&2&0&0&0\\
0&0&0&0&0&0&1&1&2&0&0&0&0&0&1&1\\
0&0&2&0&0&0&0&0&0&2&2&0&0&0&0&0\\
0&0&0&2&0&0&0&0&0&2&0&2&0&0&0&0\\
2&0&0&0&0&0&0&0&2&0&0&0&0&2&0&0\\
0&0&1&1&0&0&0&0&0&0&1&1&2&0&0&0\\
0&0&0&0&0&0&2&0&0&0&0&0&0&2&2&0\\
0&0&0&0&0&0&0&2&0&0&0&0&0&2&0&2\\
0&2&0&0&2&0&0&0&0&0&0&0&2&0&0&0\\
2&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1\\
0&2&2&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&2&0&2&0&0&0&0&0&0&0&2&0&0&0&0\\
2&0&0&0&0&2&0&0&2&0&0&0&0&0&0&0\\
0&0&1&1&2&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&2&2&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&2&0&2&0&0&0&0&0&0&0&2
\end{pmatrix},$$

$$A_{(KF_4^2)_{(1,3)}} = \frac{1}{2}\begin{pmatrix}
0&2&0&0&1&0&0&0&0&2&0&0&1&0&0&0\\
0&0&0&0&0&1&1&0&0&0&1&1&0&1&0&1\\
0&0&0&1&1&0&0&1&0&0&0&1&1&0&0&1\\
0&0&1&0&1&0&1&0&0&0&1&0&1&0&1&0\\
1&0&0&0&0&2&0&0&1&0&0&0&0&2&0&0\\
0&1&0&1&0&0&0&0&0&1&1&0&0&0&1&1\\
1&0&0&1&0&0&0&1&1&0&0&1&0&0&0&1\\
1&0&1&0&0&0&1&0&1&0&1&0&0&0&1&0\\
0&2&0&0&1&0&0&0&0&2&0&0&1&0&0&0\\
0&0&1&1&0&1&0&1&0&0&0&0&0&1&1&0\\
0&0&0&1&1&0&0&1&0&0&0&1&1&0&0&1\\
0&0&1&0&1&0&1&0&0&0&1&0&1&0&1&0\\
1&0&0&0&0&2&0&0&1&0&0&0&0&2&0&0\\
0&1&1&0&0&0&1&1&0&1&0&1&0&0&0&0\\
1&0&0&1&0&0&0&1&1&0&0&1&0&0&0&1\\
1&0&1&0&0&0&1&0&1&0&1&0&0&0&1&0
\end{pmatrix},$$

$$A_{(KF_4^2)_{(2,3)}} = \frac{1}{2}\begin{pmatrix}
2&0&0&0&0&2&1&0&2&0&1&1&0&2&0&1\\
0&2&1&1&0&0&1&1&0&2&1&1&0&0&1&1\\
2&0&0&2&0&1&1&2&2&0&1&0&0&1&0&0\\
2&0&2&0&0&1&0&0&2&0&0&1&0&1&2&1\\
0&2&0&1&2&0&0&0&0&2&1&0&2&0&1&1\\
0&0&1&1&0&2&1&1&0&0&1&1&0&2&1&1\\
0&1&0&0&2&0&0&2&0&1&1&2&2&0&1&0\\
0&1&2&1&2&0&2&0&0&1&0&0&2&0&0&1\\
2&0&1&1&0&2&0&1&2&0&0&0&0&2&1&0\\
0&2&1&1&0&0&1&1&0&2&1&1&0&0&1&1\\
2&0&1&0&0&1&0&0&2&0&0&2&0&1&1&2\\
2&0&0&1&0&1&2&1&2&0&2&0&0&1&0&0\\
0&2&1&0&2&0&1&1&0&2&0&1&2&0&0&0\\
0&0&1&1&0&2&1&1&0&0&1&1&0&2&1&1\\
0&1&1&2&2&0&1&0&0&1&0&0&2&0&0&2\\
0&1&0&0&2&0&0&1&0&1&2&1&2&0&2&0
\end{pmatrix},$$

$$A_{(KF_4^2)_{(3,2)}} = \frac{1}{2}\begin{pmatrix}
0&4&4&4&2&0&0&0&0&0&4&4&2&0&0&0\\
4&0&0&0&0&6&2&2&0&0&0&0&0&6&2&2\\
0&4&2&4&4&0&0&0&0&0&2&4&4&0&0&0\\
0&4&4&2&4&0&0&0&0&0&4&2&4&0&0&0\\
2&0&0&0&0&4&4&4&2&0&0&0&0&0&4&4\\
0&6&2&2&4&0&0&0&0&6&2&2&0&0&0&0\\
4&0&0&0&0&4&2&4&4&0&0&0&0&0&2&4\\
4&0&0&0&0&4&4&2&4&0&0&0&0&0&4&2\\
0&0&4&4&2&0&0&0&0&4&4&4&2&0&0&0\\
0&0&0&0&0&6&2&2&4&0&0&0&0&6&2&2\\
0&0&2&4&4&0&0&0&0&4&2&4&4&0&0&0\\
0&0&4&2&4&0&0&0&0&4&4&2&4&0&0&0\\
2&0&0&0&0&0&4&4&2&0&0&0&0&4&4&4\\
0&6&2&2&0&0&0&0&0&6&2&2&4&0&0&0\\
4&0&0&0&0&0&2&4&4&0&0&0&0&4&2&4\\
4&0&0&0&0&0&4&2&4&0&0&0&0&4&4&2
\end{pmatrix}.$$

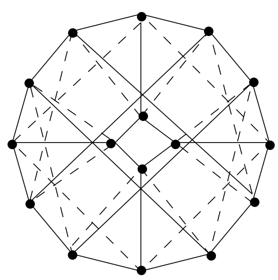


Fig. 36. KF_4^3

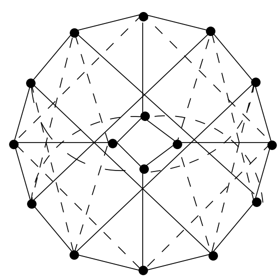


Fig. 37. KF_4^4

EXAMPLE 5.22. We take a Kähler flower KF_4^3. We have

$$A^{(a)}_{KF_4^3} = \begin{pmatrix} 0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\ 0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0 \end{pmatrix},$$

hence get

$$A^{(p)}_{KF_4^3} P^{(a)}_{KF_4^3} = \frac{1}{2} \begin{pmatrix} 0&0&0&0&0&1&1&1&0&0&0&0&0&1&1&1\\ 2&0&1&1&0&0&0&1&0&0&0&0&0&0&1&0\\ 0&0&0&1&1&1&0&0&0&0&0&1&1&1&0&0\\ 0&0&1&0&1&1&0&0&0&0&1&0&1&1&0&0\\ 0&1&1&1&0&0&0&0&0&1&1&1&0&0&0&0\\ 0&0&1&0&2&0&1&1&0&0&0&1&0&0&0&0\\ 1&1&0&0&0&0&0&1&1&1&0&0&0&0&0&1\\ 1&1&0&0&0&0&1&0&1&1&0&0&0&0&1&0\\ 0&0&0&0&0&1&1&1&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&0&2&0&1&1&0&0&0&1\\ 0&0&0&1&1&1&0&0&0&0&0&1&1&1&0&0\\ 0&0&1&0&1&1&0&0&0&0&1&0&1&1&0&0\\ 0&1&1&1&0&0&0&0&0&1&1&1&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&1&0&2&0&1&1\\ 1&1&0&0&0&0&0&1&1&1&0&0&0&0&0&1\\ 1&1&0&0&0&0&1&0&1&1&0&0&0&0&1&0 \end{pmatrix}.$$

The eigenvalues of Laplacians are

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^3)_{(1,1)}}}\big) = \left\{ \begin{matrix} 0,3,3,3,3,3,3,3,3,4,4,4,4, \\ \text{solutions of } t^3-8t^2+20t-12=0 \end{matrix} \right\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^3)_{(2,1)}}}\big) = \left\{ \begin{matrix} 0, \dfrac{10-\sqrt{7}}{2}, \dfrac{10-\sqrt{7}}{2}, \dfrac{11}{2}, 6,6,6,6, \dfrac{10+\sqrt{7}}{2}, \dfrac{10+\sqrt{7}}{2}, 8 \\ \text{solutions of } 2t^3-35t^2+208t-432=0 \end{matrix} \right\},$$

$$\mathrm{Spec}\big(\mathit{\Delta}_{\mathcal{A}_{(KF_4^3)(3,1)}}\big) = \left\{\begin{matrix} 0, 8, 11, 11, 11, 12, 12, 12, 12, 12, 12, 12, 12, \\ \text{solutions of } t^3 - 35t^2 + 392t - 1344 = 0 \end{matrix}\right\},$$

$$\mathrm{Spec}\big(\mathit{\Delta}_{\mathcal{A}_{(KF_4^3)(1,2)}}\big) = \left\{\begin{matrix} 0, 2, 3, 3, 3, 3, 3+\sqrt{-1},\ 3+\sqrt{-1},\ 3-\sqrt{-1},\ 3-\sqrt{-1},\ 4, 4, 4, \\ \text{solutions of } t^3 - 10t^2 + 32t - 36 = 0 \end{matrix}\right\},$$

$$\mathrm{Spec}\big(\mathit{\Delta}_{\mathcal{A}_{(KF_4^3)(1,3)}}\big) = \big\{0, 0, 1, 1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4\big\},$$

$$\mathrm{Spec}\big(\mathit{\Delta}_{\mathcal{A}_{(KF_4^2)(2,3)}}\big) = \big\{0, 4, 5, 5, 6, 6, 6, 6, 6, 6, 6, 6, 7, 7, 8, 12\big\},$$

$$\mathrm{Spec}\big(\mathit{\Delta}_{\mathcal{A}_{(KF_4^3)(3,2)}}\big) = \left\{\begin{matrix} 0, 8, 11, 11, 12, 12, 12, 12, 13, \\ 12+2\sqrt{-1},\ 12+2\sqrt{-1},\ 12-2\sqrt{-1},\ 12-2\sqrt{-1}, \\ \text{solutions of } t^3 - 37t^2 + 440t - 1728 = 0 \end{matrix}\right\},$$

$$\mathrm{Spec}\big(\mathit{\Delta}_{\mathcal{A}_{(KF_4^3)(4,1)}}\big) = \left\{\begin{matrix} 0, 21, 21, 24, 24, 24, 24, 24, 24, 27, \dfrac{43+\sqrt{-71}}{2}, \dfrac{43-\sqrt{-71}}{2}, \\ 28+\sqrt{-5},\ 28+\sqrt{-5},\ 28-\sqrt{-5},\ 28-\sqrt{-5} \end{matrix}\right\},$$

$$\mathrm{Spec}\big(\mathit{\Delta}_{\mathcal{A}_{(KF_4^3)(4,3)}}\big) = \big\{0, 18, 18, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 30, 30, 48\big\},$$

$$\mathrm{Spec}\big(\mathit{\Delta}_{\mathcal{A}_{(KF_4^3)(5,1)}}\big) = \left\{\begin{matrix} 0, 44, 44, 44, 48, 48, 48, 48, 48, 48, 48, 48, 56, 56, \\ 2\big(25+\sqrt{-47}\big),\ 2\big(25-\sqrt{-47}\big) \end{matrix}\right\},$$

$$\begin{aligned} \mathrm{Spec}\big(\mathit{\Delta}_{\mathcal{A}_{(KF_4^3)(5,2)}}\big) &= \mathrm{Spec}\big(\mathit{\Delta}_{\mathcal{A}_{(KF_4^3)(5,4)}}\big) \\ &= \left\{\begin{matrix} 0, 40, 44, 44, 48, 48, 48, 48, 52, 56, 2\big(23+\sqrt{-47}\big),\ 2\big(23-\sqrt{-47}\big), \\ 48+2\sqrt{-1},\ 48+2\sqrt{-1},\ 48-2\sqrt{-1},\ 48-2\sqrt{-1} \end{matrix}\right\}, \end{aligned}$$

$$\mathrm{Spec}\big(\mathit{\Delta}_{\mathcal{A}_{(KF_4^3)(5,3)}}\big) = \big\{0, 0, 48, 48, 48, 48, 48, 48, 48, 48, 56, 56, 56, 56, 56, 56\big\},$$

$$\mathrm{Spec}\big(\mathit{\Delta}_{\mathcal{A}_{(KF_4^3)(7,3)}}\big) = \left\{\begin{matrix} 0, 0, 184, 184, 184, 184, 184, 184, \\ 192, 192, 192, 192, 192, 192, 192, 192 \end{matrix}\right\}.$$

We find KF_4^3 has an interesting property on $(p,3)$-Laplacians. At least for $p = 1, 2, 4, 5, 7$ we find that $(p,3)$-adjacency matrices are symmetric. We here list adjacency matrices:

$$A_{(KF_4^3)_{(2,1)}} = \frac{1}{2}\begin{pmatrix}
2&2&0&0&0&0&0&1&0&2&2&2&0&0&1&0\\
0&0&1&1&2&0&1&1&0&0&1&1&2&0&1&1\\
1&2&1&1&0&0&0&0&2&2&0&0&0&0&1&2\\
1&2&1&1&0&0&2&1&2&2&0&0&0&0&0&0\\
0&0&1&0&2&2&0&0&0&0&0&1&0&2&2&2\\
2&0&1&1&0&0&1&1&2&0&1&1&0&0&1&1\\
0&0&1&2&1&2&1&1&0&0&0&0&2&2&0&0\\
0&0&0&0&1&2&1&1&0&0&2&1&2&2&0&0\\
0&2&2&2&0&0&1&0&2&2&0&0&0&0&0&1\\
0&0&1&1&2&0&1&1&0&0&1&1&2&0&1&1\\
2&2&0&0&0&0&1&2&1&2&1&1&0&0&0&0\\
2&2&0&0&0&0&0&0&1&2&1&1&0&0&2&1\\
0&0&0&1&0&2&2&2&0&0&1&0&2&2&0&0\\
2&0&1&1&0&0&1&1&2&0&1&1&0&0&1&1\\
0&0&0&0&2&2&0&0&0&0&1&2&1&2&1&1\\
0&0&2&1&2&2&0&0&0&0&0&0&1&2&1&1
\end{pmatrix},$$

$$A_{(KF_4^3)_{(3,1)}} = \frac{1}{2}\begin{pmatrix}
0&0&2&2&4&2&1&1&0&0&2&2&4&2&1&1\\
0&6&0&0&0&0&2&0&4&6&2&2&0&0&0&2\\
0&0&2&1&3&2&2&2&0&0&2&1&3&2&2&2\\
0&0&1&2&3&2&2&2&0&0&1&2&3&2&2&2\\
4&2&1&1&0&0&2&2&4&2&1&1&0&0&2&2\\
0&0&0&2&0&6&0&0&0&0&2&0&4&6&2&2\\
3&2&2&2&0&0&2&1&3&2&2&2&0&0&2&1\\
3&2&2&2&0&0&1&2&3&2&2&2&0&0&1&2\\
0&0&2&2&4&2&1&1&0&0&2&2&4&2&1&1\\
4&6&2&2&0&0&0&2&0&6&0&0&0&0&2&0\\
0&0&2&1&3&2&2&2&0&0&2&1&3&2&2&2\\
0&0&1&2&3&2&2&2&0&0&1&2&3&2&2&2\\
4&2&1&1&0&0&2&2&4&2&1&1&0&0&2&2\\
0&0&2&0&4&6&2&2&0&0&0&2&0&6&0&0\\
3&2&2&2&0&0&2&1&3&2&2&2&0&0&2&1\\
3&2&2&2&0&0&1&2&3&2&2&2&0&0&1&2
\end{pmatrix},$$

$$A_{(KF_4^3)_{(1,2)}} = \frac{1}{2}\begin{pmatrix}
0&0&1&1&0&0&0&0&0&2&1&1&0&0&0&0\\
0&0&1&1&1&0&0&1&0&0&0&0&1&0&1&0\\
1&0&0&0&0&0&0&1&1&2&0&0&0&0&0&1\\
1&0&0&0&0&0&0&1&1&2&0&0&0&0&0&1\\
0&0&0&0&0&0&1&1&0&0&0&0&0&2&1&1\\
1&0&1&0&0&0&1&1&1&0&0&1&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&1&1&2&0&0\\
0&0&1&0&1&0&0&0&0&0&1&0&1&2&0&0\\
0&2&1&1&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&1&0&1&0&0&0&1&1&1&0&0&1\\
1&2&0&0&0&0&0&1&1&0&0&0&0&0&0&1\\
1&2&0&0&0&0&1&0&1&0&0&0&0&0&1&0\\
0&0&0&0&0&2&1&1&0&0&0&0&0&0&1&1\\
1&0&0&1&0&0&0&0&1&0&1&0&0&0&1&1\\
0&0&0&1&1&2&0&0&0&0&0&1&1&0&0&0\\
0&0&1&0&1&2&0&0&0&0&1&0&1&0&0&0
\end{pmatrix},$$

$$A_{(KF_4^3)_{(1,3)}} = \frac{1}{2}\begin{pmatrix}
2&0&0&0&0&1&0&0&2&0&0&0&0&1&0&0\\
0&0&0&0&1&0&1&1&0&0&0&0&1&0&1&1\\
0&0&2&0&0&1&0&0&0&0&2&0&0&1&0&0\\
0&0&0&2&0&1&0&0&0&0&0&2&0&1&0&0\\
0&1&0&0&2&0&0&0&0&1&0&0&2&0&0&0\\
1&0&1&1&0&0&0&0&1&0&1&1&0&0&0&0\\
0&1&0&0&0&0&2&0&0&1&0&0&0&0&2&0\\
0&1&0&0&0&0&0&2&0&1&0&0&0&0&0&2\\
2&0&0&0&0&1&0&0&2&0&0&0&0&1&0&0\\
0&0&0&0&1&0&1&1&0&0&0&0&1&0&1&1\\
0&0&2&0&0&1&0&0&0&0&2&0&0&1&0&0\\
0&0&0&2&0&1&0&0&0&0&0&2&0&1&0&0\\
0&1&0&0&2&0&0&0&0&1&0&0&2&0&0&0\\
1&0&1&1&0&0&0&0&1&0&1&1&0&0&0&0\\
0&1&0&0&0&0&2&0&0&1&0&0&0&0&2&0\\
0&1&0&0&0&0&0&2&0&1&0&0&0&0&0&2
\end{pmatrix},$$

$$A_{(KF_4^3)_{(2,3)}} = \frac{1}{2}\begin{pmatrix}
0&2&0&0&2&0&1&1&0&2&0&0&2&0&1&1\\
2&0&2&2&0&0&0&0&2&0&2&2&0&0&0&0\\
0&2&0&0&1&0&2&1&0&2&0&0&1&0&2&1\\
0&2&0&0&1&0&1&2&0&2&0&0&1&0&1&2\\
2&0&1&1&0&2&0&0&2&0&1&1&0&2&0&0\\
0&0&0&0&2&0&2&2&0&0&0&0&2&0&2&2\\
1&0&2&1&0&2&0&0&1&0&2&1&0&2&0&0\\
1&0&1&2&0&2&0&0&1&0&1&2&0&2&0&0\\
0&2&0&0&2&0&1&1&0&2&0&0&2&0&1&1\\
2&0&2&2&0&0&0&0&2&0&2&2&0&0&0&0\\
0&2&0&0&1&0&2&1&0&2&0&0&1&0&2&1\\
0&2&0&0&1&0&1&2&0&2&0&0&1&0&1&2\\
2&0&1&1&0&2&0&0&2&0&1&1&0&2&0&0\\
0&0&0&0&2&0&2&2&0&0&0&0&2&0&2&2\\
1&0&2&1&0&2&0&0&1&0&2&1&0&2&0&0\\
1&0&1&2&0&2&0&0&1&0&1&2&0&2&0&0
\end{pmatrix},$$

$$A_{(KF_4^3)_{(3,2)}} = \frac{1}{2}\begin{pmatrix}
4&4&1&1&0&0&2&2&4&0&1&1&0&0&2&2\\
0&0&0&0&2&6&2&0&0&0&2&2&2&6&0&2\\
3&4&2&2&0&0&2&1&3&0&2&2&0&0&2&1\\
3&4&2&2&0&0&1&2&3&0&2&2&0&0&1&2\\
0&0&2&2&4&4&1&1&0&0&2&2&4&0&1&1\\
2&6&0&2&0&0&0&0&2&6&2&0&0&0&2&2\\
0&0&2&1&3&4&2&2&0&0&2&1&3&0&2&2\\
0&0&1&2&3&4&2&2&0&0&1&2&3&0&2&2\\
4&0&1&1&0&0&2&2&4&4&1&1&0&0&2&2\\
0&0&2&2&2&6&0&2&0&0&0&0&2&6&2&0\\
3&0&2&2&0&0&2&1&3&4&2&2&0&0&2&1\\
3&0&2&2&0&0&1&2&3&4&2&2&0&0&1&2\\
0&0&2&2&4&0&1&1&0&0&2&2&4&4&1&1\\
2&6&2&0&0&0&2&2&2&6&0&2&0&0&0&0\\
0&0&2&1&3&0&2&2&0&0&2&1&3&4&2&2\\
0&0&1&2&3&0&2&2&0&0&1&2&3&4&2&2
\end{pmatrix}$$

$$A_{(KF_4^3)_{(4,3)}} = \frac{1}{2}\begin{pmatrix}
0 & 6 & 0 & 0 & 2 & 0 & 8 & 8 & 0 & 6 & 0 & 0 & 2 & 0 & 8 & 8 \\
6 & 0 & 6 & 6 & 0 & 6 & 0 & 0 & 6 & 0 & 6 & 6 & 0 & 6 & 0 & 0 \\
0 & 6 & 0 & 0 & 8 & 0 & 2 & 8 & 0 & 6 & 0 & 0 & 8 & 0 & 2 & 8 \\
0 & 6 & 0 & 0 & 8 & 0 & 8 & 2 & 0 & 6 & 0 & 0 & 8 & 0 & 8 & 2 \\
2 & 0 & 8 & 8 & 0 & 6 & 0 & 0 & 2 & 0 & 8 & 8 & 0 & 6 & 0 & 0 \\
0 & 6 & 0 & 0 & 6 & 0 & 6 & 6 & 0 & 6 & 0 & 0 & 6 & 0 & 6 & 6 \\
8 & 0 & 2 & 8 & 0 & 6 & 0 & 0 & 8 & 0 & 2 & 8 & 0 & 6 & 0 & 0 \\
8 & 0 & 8 & 2 & 0 & 6 & 0 & 0 & 8 & 0 & 8 & 2 & 0 & 6 & 0 & 0 \\
0 & 6 & 0 & 0 & 2 & 0 & 8 & 8 & 0 & 6 & 0 & 0 & 2 & 0 & 8 & 8 \\
6 & 0 & 6 & 6 & 0 & 6 & 0 & 0 & 6 & 0 & 6 & 6 & 0 & 6 & 0 & 0 \\
0 & 6 & 0 & 0 & 8 & 0 & 2 & 8 & 0 & 6 & 0 & 0 & 8 & 0 & 2 & 8 \\
0 & 6 & 0 & 0 & 8 & 0 & 8 & 2 & 0 & 6 & 0 & 0 & 8 & 0 & 8 & 2 \\
2 & 0 & 8 & 8 & 0 & 6 & 0 & 0 & 2 & 0 & 8 & 8 & 0 & 6 & 0 & 0 \\
0 & 6 & 0 & 0 & 6 & 0 & 6 & 6 & 0 & 6 & 0 & 0 & 6 & 0 & 6 & 6 \\
8 & 0 & 2 & 8 & 0 & 6 & 0 & 0 & 8 & 0 & 2 & 8 & 0 & 6 & 0 & 0 \\
8 & 0 & 8 & 2 & 0 & 6 & 0 & 0 & 8 & 0 & 8 & 2 & 0 & 6 & 0 & 0
\end{pmatrix},$$

$$A_{(KF_4^3)_{(5,3)}} = \begin{pmatrix}
3 & 0 & 7 & 7 & 0 & 7 & 0 & 0 & 3 & 0 & 7 & 7 & 0 & 7 & 0 & 0 \\
0 & 3 & 0 & 0 & 7 & 0 & 7 & 7 & 0 & 3 & 0 & 0 & 7 & 0 & 7 & 7 \\
7 & 0 & 3 & 7 & 0 & 7 & 0 & 0 & 7 & 0 & 3 & 7 & 0 & 7 & 0 & 0 \\
7 & 0 & 7 & 3 & 0 & 7 & 0 & 0 & 7 & 0 & 7 & 3 & 0 & 7 & 0 & 0 \\
0 & 7 & 0 & 0 & 3 & 0 & 7 & 7 & 0 & 7 & 0 & 0 & 3 & 0 & 7 & 7 \\
7 & 0 & 7 & 7 & 0 & 3 & 0 & 0 & 7 & 0 & 7 & 7 & 0 & 3 & 0 & 0 \\
0 & 7 & 0 & 0 & 7 & 0 & 3 & 7 & 0 & 7 & 0 & 0 & 7 & 0 & 3 & 7 \\
0 & 7 & 0 & 0 & 7 & 0 & 7 & 3 & 0 & 7 & 0 & 0 & 7 & 0 & 7 & 3 \\
3 & 0 & 7 & 7 & 0 & 7 & 0 & 0 & 3 & 0 & 7 & 7 & 0 & 7 & 0 & 0 \\
0 & 3 & 0 & 0 & 7 & 0 & 7 & 7 & 0 & 3 & 0 & 0 & 7 & 0 & 7 & 7 \\
7 & 0 & 3 & 7 & 0 & 7 & 0 & 0 & 7 & 0 & 3 & 7 & 0 & 7 & 0 & 0 \\
7 & 0 & 7 & 3 & 0 & 7 & 0 & 0 & 7 & 0 & 7 & 3 & 0 & 7 & 0 & 0 \\
0 & 7 & 0 & 0 & 3 & 0 & 7 & 7 & 0 & 7 & 0 & 0 & 3 & 0 & 7 & 7 \\
7 & 0 & 7 & 7 & 0 & 3 & 0 & 0 & 7 & 0 & 7 & 7 & 0 & 3 & 0 & 0 \\
0 & 7 & 0 & 0 & 7 & 0 & 3 & 7 & 0 & 7 & 0 & 0 & 7 & 0 & 3 & 7 \\
0 & 7 & 0 & 0 & 7 & 0 & 7 & 3 & 0 & 7 & 0 & 0 & 7 & 0 & 7 & 3
\end{pmatrix},$$

$$A_{(KF_4^3)_{(5,3)}} = \begin{pmatrix}
27 & 0 & 23 & 23 & 0 & 23 & 0 & 0 & 27 & 0 & 23 & 23 & 0 & 23 & 0 & 0 \\
0 & 27 & 0 & 0 & 23 & 0 & 23 & 23 & 0 & 27 & 0 & 0 & 23 & 0 & 23 & 23 \\
23 & 0 & 27 & 23 & 0 & 23 & 0 & 0 & 23 & 0 & 27 & 23 & 0 & 23 & 0 & 0 \\
23 & 0 & 23 & 27 & 0 & 23 & 0 & 0 & 23 & 0 & 23 & 27 & 0 & 23 & 0 & 0 \\
0 & 23 & 0 & 0 & 27 & 0 & 23 & 23 & 0 & 23 & 0 & 0 & 27 & 0 & 23 & 23 \\
23 & 0 & 23 & 23 & 0 & 27 & 0 & 0 & 23 & 0 & 23 & 23 & 0 & 27 & 0 & 0 \\
0 & 23 & 0 & 0 & 23 & 0 & 27 & 23 & 0 & 23 & 0 & 0 & 23 & 0 & 27 & 23 \\
0 & 23 & 0 & 0 & 23 & 0 & 23 & 27 & 0 & 23 & 0 & 0 & 23 & 0 & 23 & 27 \\
27 & 0 & 23 & 23 & 0 & 23 & 0 & 0 & 27 & 0 & 23 & 23 & 0 & 23 & 0 & 0 \\
0 & 27 & 0 & 0 & 23 & 0 & 23 & 23 & 0 & 27 & 0 & 0 & 23 & 0 & 23 & 23 \\
23 & 0 & 27 & 23 & 0 & 23 & 0 & 0 & 23 & 0 & 27 & 23 & 0 & 23 & 0 & 0 \\
23 & 0 & 23 & 27 & 0 & 23 & 0 & 0 & 23 & 0 & 23 & 27 & 0 & 23 & 0 & 0 \\
0 & 23 & 0 & 0 & 27 & 0 & 23 & 23 & 0 & 23 & 0 & 0 & 27 & 0 & 23 & 23 \\
23 & 0 & 23 & 23 & 0 & 27 & 0 & 0 & 23 & 0 & 23 & 23 & 0 & 27 & 0 & 0 \\
0 & 23 & 0 & 0 & 23 & 0 & 27 & 23 & 0 & 23 & 0 & 0 & 23 & 0 & 27 & 23 \\
0 & 23 & 0 & 0 & 23 & 0 & 23 & 27 & 0 & 23 & 0 & 0 & 23 & 0 & 23 & 27
\end{pmatrix}.$$

EXAMPLE 5.23. We take a Kähler flower KF_4^4. The adjacency matrix of its auxiliary graph is given as

$$A^{(a)}_{KF_4^4} = \begin{pmatrix}
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0
\end{pmatrix},$$

hence we have

$$A^{(p)}_{KF_4^4} P^{(a)}_{KF_4^4} = \frac{1}{2} \begin{pmatrix}
0&0&1&1&0&1&0&0&0&0&1&1&0&1&0&0\\
0&0&0&0&1&0&0&2&0&0&0&0&1&0&2&0\\
1&0&0&1&0&1&0&0&1&0&0&1&0&1&0&0\\
1&0&1&0&0&1&0&0&1&0&1&0&0&1&0&0\\
0&1&0&0&0&0&1&1&0&1&0&0&0&0&1&1\\
1&0&2&0&0&0&0&0&1&0&0&2&0&0&0&0\\
0&1&0&0&1&0&0&1&0&1&0&0&1&0&0&1\\
0&1&0&0&1&0&1&0&0&1&0&0&1&0&1&0\\
0&0&1&1&0&1&0&0&0&0&1&1&0&1&0&0\\
0&0&0&0&1&0&2&0&0&0&0&0&1&0&0&2\\
1&0&0&1&0&1&0&0&1&0&0&1&0&1&0&0\\
1&0&1&0&0&1&0&0&1&0&1&0&0&1&0&0\\
0&1&0&0&0&0&1&1&0&1&0&0&0&0&1&1\\
1&0&0&2&0&0&0&0&1&0&2&0&0&0&0&0\\
0&1&0&0&1&0&0&1&0&1&0&0&1&0&0&1\\
0&1&0&0&1&0&1&0&0&1&0&0&1&0&1&0
\end{pmatrix}.$$

The eigenvalues of Laplacians are

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^4)_{(1,1)}}}\big) = \big\{0,0,3,3,3,3,3,3,3,3,4,4,4,4,4,4\big\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^4)_{(2,1)}}}\big) = \left\{\begin{matrix} 0,4,\dfrac{12-\sqrt{3}}{2},\dfrac{12-\sqrt{3}}{2},\dfrac{11}{2},\dfrac{11}{2},6,6,6,6, \\ \dfrac{13}{2},\dfrac{13}{2},\dfrac{12+\sqrt{3}}{2},\dfrac{12+\sqrt{3}}{2},8,12 \end{matrix}\right\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^4)_{(3,1)}}}\big) = \big\{0,0,8,8,11,11,11,11,12,12,12,12,12,12,12,12\big\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^4)_{(1,2)}}}\big) = \left\{ \begin{matrix} 0, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 6, \\ 3+\sqrt{-1},\ 3+\sqrt{-1},\ 3-\sqrt{-1},\ 3-\sqrt{-1}, \end{matrix} \right\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^4)_{(1,3)}}}\big) = \big\{0, 0, 1, 1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4\big\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^4)_{(2,3)}}}\big) = \big\{0, 4, 5, 5, 6, 6, 6, 6, 6, 6, 6, 6, 7, 7, 8, 12\big\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF_4^4)_{(3,2)}}}\big) = \left\{ \begin{matrix} 0, 8, 11, 11, 12, 12, 12, 12, 13, 13, 16, 24, \\ 12+2\sqrt{-1},\ 12+2\sqrt{-1},\ 12-2\sqrt{-1},\ 12-2\sqrt{-1}, \end{matrix} \right\}.$$

We here list adjacency matrices:

$$A_{(KF_4^4)_{(2,1)}} = \frac{1}{2} \begin{pmatrix}
0 & 2 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 2 \\
2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 3 & 0 & 2 & 2 \\
0 & 2 & 0 & 0 & 3 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 \\
3 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 2 & 2 & 0 & 2 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & 2 & 1 \\
2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 3 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 2 & 2 \\
1 & 0 & 2 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 3 & 0 & 2 & 2 & 0 & 2 & 0 & 0 \\
3 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0
\end{pmatrix},$$

$$A_{(KF_4^4)_{(3,1)}} = \frac{1}{2} \begin{pmatrix}
4 & 0 & 3 & 3 & 0 & 2 & 0 & 0 & 4 & 0 & 3 & 3 & 0 & 2 & 0 & 0 \\
0 & 6 & 0 & 0 & 2 & 0 & 4 & 0 & 0 & 6 & 0 & 0 & 2 & 0 & 0 & 4 \\
3 & 0 & 4 & 3 & 0 & 2 & 0 & 0 & 3 & 0 & 4 & 3 & 0 & 2 & 0 & 0 \\
3 & 0 & 3 & 4 & 0 & 2 & 0 & 0 & 3 & 0 & 3 & 4 & 0 & 2 & 0 & 0 \\
0 & 2 & 0 & 0 & 4 & 0 & 3 & 3 & 0 & 2 & 0 & 0 & 4 & 0 & 3 & 3 \\
2 & 0 & 0 & 4 & 0 & 6 & 0 & 0 & 2 & 0 & 4 & 0 & 0 & 6 & 0 & 0 \\
0 & 2 & 0 & 0 & 3 & 0 & 4 & 3 & 0 & 2 & 0 & 0 & 3 & 0 & 4 & 3 \\
0 & 2 & 0 & 0 & 3 & 0 & 3 & 4 & 0 & 2 & 0 & 0 & 3 & 0 & 3 & 4 \\
4 & 0 & 3 & 3 & 0 & 2 & 0 & 0 & 4 & 0 & 3 & 3 & 0 & 2 & 0 & 0 \\
0 & 6 & 0 & 0 & 2 & 0 & 0 & 4 & 0 & 6 & 0 & 0 & 2 & 0 & 4 & 0 \\
3 & 0 & 4 & 3 & 0 & 2 & 0 & 0 & 3 & 0 & 4 & 3 & 0 & 2 & 0 & 0 \\
3 & 0 & 3 & 4 & 0 & 2 & 0 & 0 & 3 & 0 & 3 & 4 & 0 & 2 & 0 & 0 \\
0 & 2 & 0 & 0 & 4 & 0 & 3 & 3 & 0 & 2 & 0 & 0 & 4 & 0 & 3 & 3 \\
2 & 0 & 4 & 0 & 0 & 6 & 0 & 0 & 2 & 0 & 0 & 4 & 0 & 6 & 0 & 0 \\
0 & 2 & 0 & 0 & 3 & 0 & 4 & 3 & 0 & 2 & 0 & 0 & 3 & 0 & 4 & 3 \\
0 & 2 & 0 & 0 & 3 & 0 & 3 & 4 & 0 & 2 & 0 & 0 & 3 & 0 & 3 & 4
\end{pmatrix},$$

$$A_{(KF_4^4)_{(1,2)}} = \frac{1}{2}\begin{pmatrix} 0&0&0&0&0&0&1&1&0&2&0&0&0&0&1&1\\ 2&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&1&0&0&1&0&2&0&0&1&0&0&1\\ 0&0&0&0&1&0&1&0&0&2&0&0&1&0&1&0\\ 0&0&1&1&0&0&0&0&0&0&1&1&0&2&0&0\\ 0&0&0&0&2&0&1&1&0&0&0&0&0&0&1&1\\ 1&0&0&1&0&0&0&0&1&0&0&1&0&2&0&0\\ 1&0&1&0&0&0&0&0&1&0&1&0&0&2&0&0\\ 0&2&0&0&0&0&1&1&0&0&0&0&0&0&1&1\\ 0&0&1&1&0&0&0&0&2&0&1&1&0&0&0&0\\ 0&2&0&0&1&0&0&1&0&0&0&0&1&0&0&1\\ 0&2&0&0&1&0&1&0&0&0&0&0&1&0&1&0\\ 0&0&1&1&0&2&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&1&1&0&0&0&0&2&0&1&1\\ 1&0&0&1&0&2&0&0&1&0&0&1&0&0&0&0\\ 1&0&1&0&0&2&0&0&1&0&1&0&0&0&0&0 \end{pmatrix},$$

$$A_{(KF_4^4)_{(1,3)}} = \frac{1}{2}\begin{pmatrix} 2&0&0&0&0&1&0&0&2&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&1&1&0&0&0&0&1&0&1&1\\ 0&0&2&0&0&1&0&0&0&0&2&0&0&1&0&0\\ 0&0&0&2&0&1&0&0&0&0&0&2&0&1&0&0\\ 0&1&0&0&2&0&0&0&0&1&0&0&2&0&0&0\\ 1&0&1&1&0&0&0&0&1&0&1&1&0&0&0&0\\ 0&1&0&0&0&0&2&0&0&1&0&0&0&0&2&0\\ 0&1&0&0&0&0&0&2&0&1&0&0&0&0&0&2\\ 2&0&0&0&0&1&0&0&2&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&1&1&0&0&0&0&1&0&1&1\\ 0&0&2&0&0&1&0&0&0&0&2&0&0&1&0&0\\ 0&0&0&2&0&1&0&0&0&0&0&2&0&1&0&0\\ 0&1&0&0&2&0&0&0&0&1&0&0&2&0&0&0\\ 1&0&1&1&0&0&0&0&1&0&1&1&0&0&0&0\\ 0&1&0&0&0&0&2&0&0&1&0&0&0&0&2&0\\ 0&1&0&0&0&0&0&2&0&1&0&0&0&0&0&2 \end{pmatrix},$$

$$A_{(KF_4^4)_{(2,3)}} = \frac{1}{2}\begin{pmatrix} 0&2&0&0&2&0&1&1&0&2&0&0&2&0&1&1\\ 2&0&2&2&0&0&0&0&2&0&2&2&0&0&0&0\\ 0&2&0&0&1&0&2&1&0&2&0&0&1&0&2&1\\ 0&2&0&0&1&0&1&2&0&2&0&0&1&0&1&2\\ 2&0&1&1&0&2&0&0&2&0&1&1&0&2&0&0\\ 0&0&0&0&2&0&2&2&0&0&0&0&2&0&2&2\\ 1&0&2&1&0&2&0&0&1&0&2&1&0&2&0&0\\ 1&0&1&2&0&2&0&0&1&0&1&2&0&2&0&0\\ 0&2&0&0&2&0&1&1&0&2&0&0&2&0&1&1\\ 2&0&2&2&0&0&0&0&2&0&2&2&0&0&0&0\\ 0&2&0&0&1&0&2&1&0&2&0&0&1&0&2&1\\ 0&2&0&0&1&0&1&2&0&2&0&0&1&0&1&2\\ 2&0&1&1&0&2&0&0&2&0&1&1&0&2&0&0\\ 0&0&0&0&2&0&2&2&0&0&0&0&2&0&2&2\\ 1&0&2&1&0&2&0&0&1&0&2&1&0&2&0&0\\ 1&0&1&2&0&2&0&0&1&0&1&2&0&2&0&0 \end{pmatrix},$$

$$A_{(KF_4^4)_{(3,2)}} = \frac{1}{2}\begin{pmatrix} 0 & 4 & 0 & 0 & 4 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 4 & 0 & 3 & 3 \\ 0 & 0 & 2 & 2 & 0 & 6 & 0 & 0 & 4 & 0 & 2 & 2 & 0 & 6 & 0 & 0 \\ 0 & 4 & 0 & 0 & 3 & 0 & 4 & 3 & 0 & 0 & 0 & 0 & 3 & 0 & 4 & 3 \\ 0 & 4 & 0 & 0 & 3 & 0 & 3 & 4 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 4 \\ 4 & 0 & 3 & 3 & 0 & 4 & 0 & 0 & 4 & 0 & 3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 6 & 0 & 0 & 4 & 0 & 2 & 2 \\ 3 & 0 & 4 & 3 & 0 & 4 & 0 & 0 & 3 & 0 & 4 & 3 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 4 & 0 & 4 & 0 & 0 & 3 & 0 & 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 3 & 3 & 0 & 4 & 0 & 0 & 4 & 0 & 3 & 3 \\ 4 & 0 & 2 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 4 & 3 & 0 & 4 & 0 & 0 & 3 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 3 & 0 & 3 & 4 & 0 & 4 & 0 & 0 & 3 & 0 & 3 & 4 \\ 4 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 4 & 0 & 3 & 3 & 0 & 4 & 0 & 0 \\ 0 & 6 & 0 & 0 & 4 & 0 & 2 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 2 & 2 \\ 3 & 0 & 4 & 3 & 0 & 0 & 0 & 0 & 3 & 0 & 4 & 3 & 0 & 4 & 0 & 0 \\ 3 & 0 & 3 & 4 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 4 & 0 & 4 & 0 & 0 \end{pmatrix}.$$

EXAMPLE 5.24. We take a Kähler flower KF_4^5. The adjacency matrix of its auxiliary graph is given as

$$A^{(a)}_{KF_4^5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

hence we have

$$A^{(p)}_{KF^5_4} P^{(a)}_{KF^5_4} = \frac{1}{2}\begin{pmatrix} 0&0&1&1&0&0&0&1&0&0&1&1&0&0&1&0\\ 0&0&0&0&1&1&0&1&0&0&0&0&1&1&1&0\\ 1&1&0&0&0&0&0&1&1&1&0&0&0&0&1&0\\ 1&1&0&0&0&0&0&1&1&1&0&0&0&0&1&0\\ 0&0&1&0&0&0&1&1&0&0&0&1&0&0&1&1\\ 1&1&1&0&0&0&0&0&1&1&0&1&0&0&0&0\\ 0&0&1&0&1&1&0&0&0&0&0&1&1&1&0&0\\ 0&0&1&0&1&1&0&0&0&0&0&1&1&1&0&0\\ 0&0&1&1&0&0&1&0&0&0&1&1&0&0&0&1\\ 0&0&0&0&1&1&1&0&0&0&0&0&1&1&0&1\\ 1&1&0&0&0&0&1&0&1&1&0&0&0&0&0&1\\ 1&1&0&0&0&0&1&0&1&1&0&0&0&0&0&1\\ 0&0&0&1&0&0&1&1&0&0&1&0&0&0&1&1\\ 1&1&0&1&0&0&0&0&1&1&1&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0&0&1&0&1&1&0&0\\ 0&0&0&1&1&1&0&0&0&0&1&0&1&1&0&0 \end{pmatrix}.$$

Since the auxiliary graph is a union of 4-circuits, the eigenvalues of Laplacians are

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^5_4)(1,4\ell+1)}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^5_4)(1,4\ell+3)}}\big) = \big\{0,3,3,3,3,3,3,3,3,3,3,3,3,4,4,4\big\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^5_4)(1,4\ell+2)}}\big) = \left\{\begin{matrix} 0,1,2,2,2,2,3,3,3,5, \dfrac{5-\sqrt{-7}}{2}, \dfrac{5+\sqrt{-7}}{2}, \\ \text{solutions of } t^3-9t^2+24t-24=0 \end{matrix}\right\},$$

$$\begin{aligned}\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^5_4)(1,4\ell)}}\big) &= \mathrm{Spec}\big(\Delta_{\mathcal{A}_{F_4}}\big)\\ &= \big\{0,1,1,3-\sqrt{3},\ 3-\sqrt{3},\ 2,3,3,3,3,4,3+\sqrt{3},\ 3+\sqrt{3},5,5,6\big\},\end{aligned}$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^5_4)(2,4\ell+1)}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^5_4)(2,4\ell+3)}}\big) = \left\{\begin{matrix} 0,6-\sqrt{3},\ 6-\sqrt{3},\ 6,6,6,6,6,6,6,6,6, \\ 6+\sqrt{3},\ 6+\sqrt{3},\ 5-\sqrt{-3},\ 5+\sqrt{-3}, \end{matrix}\right\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^5_4)(3,4\ell+1)}}\big) = \mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^5_4)(3,4\ell+3)}}\big) = \left\{\begin{matrix} 0,2\big(5-\sqrt{3}\big),\ 12,12,12,12,12,12, \\ 12,12,12,12,12,12,12,12,2\big(5+\sqrt{3}\big) \end{matrix}\right\},$$

$$\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^5_4)(3,4\ell+2)}}\big) = \left\{\begin{matrix} 0,10,12,12,12,12,14,14,14,14,14,13-\sqrt{-7},\ 13+\sqrt{-7}, \\ \text{solutions of } t^3-38t^2+480t-2112=0 \end{matrix}\right\},$$

$$\begin{aligned}\mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^5_4)(3,4\ell)}}\big) &= \mathrm{Spec}\big(\Delta_{\mathcal{A}_{(KF^5_4)[3]}}\big)\\ &= \left\{\begin{matrix} 0,8,2\big(6-\sqrt{3}\big),\ 2\big(6-\sqrt{3}\big),\ 10,10,12,12,12,12, \\ 14,14,2\big(6+\sqrt{3}\big),\ 2\big(6+\sqrt{3}\big),\ 16,24 \end{matrix}\right\}.\end{aligned}$$

We here list adjacency matrices:

$$A_{(KF_4^5)_{(1,2)}} = \frac{1}{2}\begin{pmatrix}
2&0&0&0&0&2&0&0&0&0&0&0&0&2&0&0\\
0&2&0&0&0&0&0&0&0&0&2&2&0&0&0&0\\
2&0&0&0&0&0&0&2&0&0&0&0&0&0&0&2\\
2&0&0&0&0&0&2&0&0&0&0&0&0&0&2&0\\
0&2&0&0&2&0&0&0&0&2&0&0&0&0&0&0\\
0&0&0&0&0&2&0&0&0&0&0&0&0&0&2&2\\
0&0&0&2&2&0&0&0&0&0&0&2&0&0&0&0\\
0&0&2&0&2&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&2&0&0&2&0&0&0&0&2&0&0\\
0&0&2&2&0&0&0&0&0&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&2&0&0&0&0&0&0&2\\
0&0&0&0&0&0&2&0&2&0&0&0&0&0&2&0\\
0&2&0&0&0&0&0&0&0&2&0&0&2&0&0&0\\
0&0&0&0&0&0&2&2&0&0&0&0&0&2&0&0\\
0&0&0&2&0&0&0&0&0&0&0&2&2&0&0&0\\
0&0&2&0&0&0&0&0&0&0&2&0&2&0&0&0
\end{pmatrix},$$

$$A_{(KF_4^5)_{(2,1)}} = \frac{1}{2}\begin{pmatrix}
0&0&1&1&1&1&2&0&0&0&1&1&1&1&0&2\\
2&2&1&1&0&0&0&0&2&2&1&1&0&0&0&0\\
0&0&1&1&0&0&0&1&0&0&1&1&3&3&1&0\\
0&0&1&1&3&3&0&1&0&0&1&1&0&0&1&0\\
1&1&0&2&0&0&1&1&1&1&2&0&0&0&1&1\\
0&0&0&0&2&2&1&1&0&0&0&0&2&2&1&1\\
3&3&1&0&0&0&1&1&0&0&0&1&0&0&1&1\\
0&0&1&0&0&0&1&1&3&3&0&1&0&0&1&1\\
0&0&1&1&1&1&0&2&0&0&1&1&1&1&2&0\\
2&2&1&1&0&0&0&0&2&2&1&1&0&0&0&0\\
0&0&1&1&3&3&1&0&0&0&1&1&0&0&0&1\\
0&0&1&1&0&0&1&0&0&0&1&1&3&3&0&1\\
1&1&2&0&0&0&1&1&1&1&0&2&0&0&1&1\\
0&0&0&0&2&2&1&1&0&0&0&0&2&2&1&1\\
0&0&0&1&0&0&1&1&3&3&1&0&0&0&1&1\\
3&3&0&1&0&0&1&1&0&0&1&0&0&0&1&1
\end{pmatrix},$$

$$A_{(KF_4^5)_{(3,1)}} = \frac{1}{2}\begin{pmatrix}
4&4&1&1&0&0&2&0&4&4&1&1&0&0&0&2\\
0&0&3&3&2&2&2&0&0&0&3&3&2&2&0&2\\
3&3&2&2&0&0&2&0&3&3&2&2&0&0&0&2\\
3&3&2&2&0&0&2&0&3&3&2&2&0&0&0&2\\
0&0&0&2&4&4&1&1&0&0&2&0&4&4&1&1\\
2&2&0&2&0&0&3&3&2&2&2&0&0&0&3&3\\
0&0&0&2&3&3&2&2&0&0&2&0&3&3&2&2\\
0&0&0&2&3&3&2&2&0&0&2&0&3&3&2&2\\
4&4&1&1&0&0&0&2&4&4&1&1&0&0&2&0\\
0&0&3&3&2&2&0&2&0&0&3&3&2&2&2&0\\
3&3&2&2&0&0&0&2&3&3&2&2&0&0&2&0\\
3&3&2&2&0&0&0&2&3&3&2&2&0&0&2&0\\
0&0&2&0&4&4&1&1&0&0&0&2&4&4&1&1\\
2&2&2&0&0&0&3&3&2&2&0&2&0&0&3&3\\
0&0&2&0&3&3&2&2&0&0&0&2&3&3&2&2\\
0&0&2&0&3&3&2&2&0&0&0&2&3&3&2&2
\end{pmatrix},$$

$$A_{(KF_4^5)_{(3,2)}} = \frac{1}{2}\begin{pmatrix}
0&0&0&0&0&2&4&4&4&0&0&0&0&2&4&4\\
0&0&4&4&6&0&0&0&0&4&0&0&6&0&0&0\\
0&0&0&0&0&4&4&2&4&0&0&0&0&4&4&2\\
0&0&0&0&0&4&2&4&4&0&0&0&0&4&2&4\\
0&2&4&4&0&0&0&0&0&2&4&4&4&0&0&0\\
6&0&0&0&0&0&4&4&6&0&0&0&0&4&0&0\\
0&4&4&2&0&0&0&0&0&4&4&2&4&0&0&0\\
0&4&2&4&0&0&0&0&0&4&2&4&4&0&0&0\\
4&0&0&0&0&2&4&4&0&0&0&0&0&2&4&4\\
0&4&0&0&6&0&0&0&0&0&4&4&6&0&0&0\\
4&0&0&0&0&4&4&2&0&0&0&0&0&4&4&2\\
4&0&0&0&0&4&2&4&0&0&0&0&0&4&2&4\\
0&2&4&4&4&0&0&0&0&2&4&4&0&0&0&0\\
6&0&0&0&0&4&0&0&6&0&0&0&0&0&4&4\\
0&4&4&2&4&0&0&0&0&4&4&2&0&0&0&0\\
0&4&2&4&4&0&0&0&0&4&2&4&0&0&0&0
\end{pmatrix},$$

$$A_{(KF_4^5)_{[3]}} = \begin{pmatrix}
0&0&0&0&1&0&2&2&0&2&0&0&1&0&2&2\\
0&0&0&0&0&3&0&0&2&0&2&2&0&3&0&0\\
0&0&0&0&2&0&1&2&0&2&0&0&2&0&1&2\\
0&0&0&0&2&0&2&1&0&2&0&0&2&0&2&1\\
1&0&2&2&0&0&0&0&1&0&2&2&0&2&0&0\\
0&3&0&0&0&0&0&0&0&3&0&0&2&0&2&2\\
2&0&1&2&0&0&0&0&2&0&1&2&0&2&0&0\\
2&0&2&1&0&0&0&0&2&0&2&1&0&2&0&0\\
0&2&0&0&1&0&2&2&0&0&0&0&1&0&2&2\\
2&0&2&2&0&3&0&0&0&0&0&0&0&3&0&0\\
0&2&0&0&2&0&1&2&0&0&0&0&2&0&1&2\\
0&2&0&0&2&0&2&1&0&0&0&0&2&0&2&1\\
1&0&2&2&0&2&0&0&1&0&2&2&0&0&0&0\\
0&3&0&0&2&0&2&2&0&3&0&0&0&0&0&0\\
2&0&1&2&0&2&0&0&2&0&1&2&0&0&0&0\\
2&0&2&1&0&2&0&0&2&0&2&1&0&0&0&0
\end{pmatrix}.$$

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