

Periodic harmonic functions on lattices and Chebyshev polynomials

Masakazu Yamagishi

Department of Mathematics,

Nagoya Institute of Technology,

Gokiso-cho, Showa-ku, Nagoya, Aichi 466-8555, Japan

e-mail: yamagishi.masakazu@nitech.ac.jp

Abstract

We shall give an explicit expression of the dimension of the space of harmonic functions on the Cartesian product of path (resp. cycle) graphs in terms of Chebyshev polynomials of the second (resp. first) kind. As an application, we obtain several identities among the dimensions, some are new and some are known but obtained previously by other methods. Our motivation for this study is the “Lights Out” puzzle.

Keywords: graph Laplacian; Cartesian product; Lights Out puzzle; Chebyshev polynomial.

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1 Introduction

Let $G = (V, E)$ be a finite undirected simple graph and K an arbitrary field. A function on V with values in K is called a configuration. Let $C_{G,K}$ denote the set of all configurations. It is regarded as a vector space over K . For $a \in K$, we define the endomorphism $\Delta_{G,K,a}$ of $C_{G,K}$, which we call the a -Laplacian, by

$$\Delta_{G,K,a}(f)(v) := af(v) + \sum_{(u,v) \in E} f(u).$$

In the case where G is r -regular, the ordinary Laplacian is $-\Delta_{G,K,-r}$. We are interested in the dimension of the space of “ a -harmonic functions”

$$d(G, K, a) := \dim_K \ker \Delta_{G,K,a}.$$

Let \mathbf{P}_n denote the path graph with n vertices ($n \geq 2$) and \mathbf{C}_n the cycle graph with n vertices ($n \geq 3$). Let $G \times H$ denote the Cartesian product of graphs G and H . The number $d(\mathbf{P}_n \times \mathbf{P}_n, \mathbb{F}_2, 1)$ (resp. $d(\mathbf{C}_n \times \mathbf{C}_n, \mathbb{F}_2, 1)$) has attracted special attention in

connection with the ‘‘Lights Out’’ puzzle (resp. the torus version of this puzzle); see the references [1],[2], [3], [4], [8], [9], [10], [11], [12], [13]. The behavior of these numbers is rather mysterious; see [3, Table 1] for the values of $d(\mathbf{C}_n \times \mathbf{C}_n, \mathbb{F}_2, 1)$, $n \leq 300$.

In this paper we shall give an explicit expression of $d(\mathbf{P}_m \times \mathbf{P}_n, K, a)$ and $d(\mathbf{C}_m \times \mathbf{C}_n, K, a)$ in terms of Chebyshev polynomials of the second and first kind, respectively. A configuration for $\mathbf{C}_m \times \mathbf{C}_n$ is naturally identified with a function on \mathbb{Z}^2 which is (m, n) -periodic, hence the title of this paper. The normalized Chebyshev polynomials of the first and the second kind are defined by

$$\begin{aligned} C_0(x) &= 2, & C_1(x) &= x, & C_n(x) &= xC_{n-1}(x) - C_{n-2}(x) \quad (n \geq 2), \\ S_0(x) &= 1, & S_1(x) &= x, & S_n(x) &= xS_{n-1}(x) - S_{n-2}(x) \quad (n \geq 2), \end{aligned}$$

respectively. We put $\tilde{C}_n(x) := C_n(x) - 2$. Let $\text{ord}_p(n)$ denote the p -adic additive valuation of n . The main result is the following.

Theorem 1.1. (i) $d(\mathbf{P}_m \times \mathbf{P}_n, K, a) = \deg \gcd_K(S_m(x), S_n(-x - a))$.

(ii) $d(\mathbf{C}_m \times \mathbf{C}_n, K, a) = 2 \deg \gcd_K(\tilde{C}_m(x), \tilde{C}_n(-x - a)) - \varepsilon$, where

- $\varepsilon = 2$ if $\text{char } K = p \geq 3$, $a = 0$, $\text{ord}_p(m) = \text{ord}_p(n)$, and both m, n are even,
- $\varepsilon = 1$ if either
 - $\text{char } K = 2$, $a = 0$, and both m, n are odd,
 - $\text{char } K = p \geq 3$, $a = -4$, $\text{ord}_p(m) = \text{ord}_p(n)$,
 - $\text{char } K = p \geq 3$, $a = 4$, $\text{ord}_p(m) = \text{ord}_p(n)$, and both m, n are even,
or
 - $\text{char } K = p \geq 3$, $a = 0$, $\text{ord}_p(m) = \text{ord}_p(n)$, and either m or n is even,
- $\varepsilon = 0$ otherwise.

Theorem 1.1 (i) was essentially known in the case $\text{char } K = 2$, $a = 0$ (cf. Remark 4.2). In the case $a = 1$, (i) was proved in [10] (see also [4]). The equality (ii) for $a = 1$ was conjectured in [3] in the case $\text{char } K = 2$ and in [11] in the case $\text{char } K = p > 0$.

The organization of this paper is as follows. Basic properties of Chebyshev polynomials are gathered in Section 2. It should be pointed out that Chebyshev polynomials of the (somewhat minor) third and fourth kind will be proved to be useful for our purposes. In Section 3 we prove Theorem 1.1. Using Theorem 1.1, we obtain various identities concerning $d(\mathbf{P}_m \times \mathbf{P}_n, K, a)$ and $d(\mathbf{C}_m \times \mathbf{C}_n, K, a)$ in Section 4. Some of them are already known but proved by different methods. We can give a unified proof using Theorem 1.1 and basic properties of Chebyshev polynomials.

We use the following notation. \mathbb{F}_q denotes the finite field with q elements, I_n the identity matrix of degree n , $\text{char } K$ the characteristic of a field K , and $A \otimes B$ the Kronecker product of matrices A and B .

2 Chebyshev polynomials

The Chebyshev polynomials T_n, U_n, V_n , and W_n of the first, second, third, and fourth kind, respectively, are characterized by

$$\begin{aligned} T_n(\cos \theta) &= \cos n\theta, & U_n(\cos \theta) &= \frac{\sin(n+1)\theta}{\sin \theta}, \\ V_n(\cos \theta) &= \frac{\cos(n+1/2)\theta}{\cos \theta/2}, & W_n(\cos \theta) &= \frac{\sin(n+1/2)\theta}{\sin \theta/2}, \end{aligned}$$

where n is an integer (cf.[5],[6]). Note that $C_n(x) = 2T_n(x/2)$, $S_n(x) = U_n(x/2)$. We adopt Schur's notation $\mathcal{S}_n(x) := S_{n-1}(x)$. For odd n we define

$$\mathcal{V}_n(x) := V_{(n-1)/2}(x/2), \quad \mathcal{W}_n(x) := W_{(n-1)/2}(x/2),$$

which we propose to call the normalized Chebyshev polynomials of the third and fourth kind, respectively. We always assume odd indices for \mathcal{V} and \mathcal{W} . The polynomials $C_n(x), \mathcal{S}_n(x), \mathcal{V}_n(x), \mathcal{W}_n(x)$ have integral coefficients and are monic except for $C_0(x) = 2$ and $\mathcal{S}_0(x) = 0$. The following properties of Chebyshev polynomials are well known and easily verified. Some of them will not be used in the following, but are here for the sake of completeness.

Lemma 2.1. (i) *We have*

$$\begin{aligned} C_n(z+z^{-1}) &= z^n + z^{-n}, \\ \mathcal{S}_n(z+z^{-1}) &= \frac{z^n - z^{-n}}{z - z^{-1}} = z^{n-1} + z^{n-3} + \dots + z^{-(n-3)} + z^{-(n-1)}, \\ \mathcal{V}_n(z+z^{-1}) &= \frac{z^{n/2} + z^{-n/2}}{z^{1/2} + z^{-1/2}} = z^{(n-1)/2} - z^{(n-3)/2} + \dots - z^{-(n-3)/2} + z^{-(n-1)/2}, \\ \mathcal{W}_n(z+z^{-1}) &= \frac{z^{n/2} - z^{-n/2}}{z^{1/2} - z^{-1/2}} = z^{(n-1)/2} + z^{(n-3)/2} + \dots + z^{-(n-3)/2} + z^{-(n-1)/2}. \end{aligned}$$

$$(ii) \quad C_n(2) = 2, \mathcal{S}_n(2) = n, \mathcal{V}_n(2) = 1, \mathcal{W}_n(2) = n.$$

$$(iii) \quad C_n(-x) = (-1)^n C_n(x), S_n(-x) = (-1)^n S_n(x), \mathcal{V}_n(-x) = (-1)^{(n-1)/2} \mathcal{V}_n(x).$$

$$(iv) \quad C_{-n}(x) = C_n(x), \mathcal{S}_{-n}(x) = -\mathcal{S}_n(x), \mathcal{V}_{-n}(x) = \mathcal{V}_n(x), \mathcal{W}_{-n}(x) = -\mathcal{W}_n(x).$$

(v) *We have*

$$\begin{aligned} C_n(x)^2 - C_{n+1}(x)C_{n-1}(x) &= 4 - x^2, \\ \mathcal{S}_n(x)^2 - \mathcal{S}_{n+1}(x)\mathcal{S}_{n-1}(x) &= 1, \\ \mathcal{V}_n(x)^2 - \mathcal{V}_{n+2}(x)\mathcal{V}_{n-2}(x) &= 2 - x, \\ \mathcal{W}_n(x)^2 - \mathcal{W}_{n+2}(x)\mathcal{W}_{n-2}(x) &= 2 + x. \end{aligned}$$

(vi) We have

$$\begin{aligned}
C_m(x)C_n(x) &= C_{m+n}(x) + C_{m-n}(x), \\
(x^2 - 4)\mathcal{S}_m(x)\mathcal{S}_n(x) &= C_{m+n}(x) - C_{m-n}(x), \\
(x + 2)\mathcal{V}_m(x)\mathcal{V}_n(x) &= C_{(m+n)/2}(x) + C_{(m-n)/2}(x), \\
(x - 2)\mathcal{W}_m(x)\mathcal{W}_n(x) &= C_{(m+n)/2}(x) - C_{(m-n)/2}(x), \\
C_m(x)\mathcal{S}_n(x) &= \mathcal{S}_{m+n}(x) - \mathcal{S}_{m-n}(x), \\
C_m(x)\mathcal{V}_n(x) &= \mathcal{V}_{2m+n}(x) + \mathcal{V}_{2m-n}(x), \\
C_m(x)\mathcal{W}_n(x) &= \mathcal{W}_{2m+n}(x) - \mathcal{W}_{2m-n}(x), \\
(x + 2)\mathcal{S}_m(x)\mathcal{V}_n(x) &= \mathcal{W}_{2m+n}(x) + \mathcal{W}_{2m-n}(x), \\
(x - 2)\mathcal{S}_m(x)\mathcal{W}_n(x) &= \mathcal{V}_{2m+n}(x) - \mathcal{V}_{2m-n}(x), \\
\mathcal{V}_m(x)\mathcal{W}_n(x) &= \mathcal{S}_{(m+n)/2}(x) - \mathcal{S}_{(m-n)/2}(x).
\end{aligned}$$

(vii) We have

$$C_n(x) + 2 = \begin{cases} C_{n/2}(x)^2 & (n : \text{even}), \\ (x + 2)\mathcal{V}_n(x)^2 & (n : \text{odd}), \end{cases}$$

$$C_n(x) - 2 = \begin{cases} (x^2 - 4)\mathcal{S}_{n/2}(x)^2 & (n : \text{even}), \\ (x - 2)\mathcal{W}_n(x)^2 & (n : \text{odd}), \end{cases} \quad (1)$$

$$\mathcal{S}_n(x) = \begin{cases} \mathcal{S}_{n/2}(x)C_{n/2}(x) & (n : \text{even}), \\ \mathcal{V}_n(x)\mathcal{W}_n(x) & (n : \text{odd}), \end{cases} \quad (2)$$

$$\mathcal{S}_n(x) + 1 = \begin{cases} \mathcal{S}_{(n+1)/2}(x)C_{(n-1)/2}(x) & (n : \text{odd}), \\ \mathcal{V}_{n-1}(x)\mathcal{W}_{n+1}(x) & (n : \text{even}), \end{cases}$$

$$\mathcal{S}_n(x) - 1 = \begin{cases} \mathcal{S}_{(n-1)/2}(x)C_{(n+1)/2}(x) & (n : \text{odd}), \\ \mathcal{V}_{n+1}(x)\mathcal{W}_{n-1}(x) & (n : \text{even}). \end{cases}$$

(viii) We have

$$\begin{aligned}
C_{mn}(x) &= C_m(C_n(x)), \\
\mathcal{S}_{mn}(x) &= \mathcal{S}_m(C_n(x))\mathcal{S}_n(x), \\
\mathcal{V}_{mn}(x) &= \mathcal{V}_m(C_n(x))\mathcal{V}_n(x), \\
\mathcal{W}_{mn}(x) &= \mathcal{W}_m(C_n(x))\mathcal{W}_n(x).
\end{aligned}$$

(ix) Let p be a prime number and $e \geq 0$. We have the following polynomial congru-

ences modulo p :

$$\begin{aligned} C_{p^e}(x) &\equiv x^{p^e}, \\ \mathcal{S}_{p^e}(x) &\equiv (x^2 - 4)^{(p^e-1)/2}, \\ \mathcal{V}_{p^e}(x) &\equiv (x + 2)^{(p^e-1)/2}, \\ \mathcal{W}_{p^e}(x) &\equiv (x - 2)^{(p^e-1)/2}. \end{aligned}$$

Note that the second congruence makes sense even if $p = 2$, and we assume that p is odd in the third and the fourth congruences.

$$(x) \quad C_n(x) \equiv x\mathcal{S}_n(x) \pmod{2}.$$

(xi) The solution of the linear recurrence equation $x_n = ax_{n-1} - x_{n-2}$ is given by $x_n = \mathcal{S}_n(a)x_1 - \mathcal{S}_{n-1}(a)x_0$.

Since the normalized Chebyshev polynomials have integral coefficients, they can be considered over any field. Recall that we put $\tilde{C}_n(x) = C_n(x) - 2$.

Lemma 2.2. *Let m, n be positive integers and $g = \gcd(m, n)$. Over an arbitrary field K , we have the following. (We assume odd indices for \mathcal{V} and \mathcal{W} .)*

$$\gcd(\tilde{C}_m(x), \tilde{C}_n(x)) = \tilde{C}_g(x), \quad (3)$$

$$\gcd(\mathcal{S}_m(x), \mathcal{S}_n(x)) = \mathcal{S}_g(x), \quad (4)$$

$$\gcd(\mathcal{V}_m(x), \mathcal{V}_n(x)) = \mathcal{V}_g(x), \quad (5)$$

$$\gcd(\mathcal{W}_m(x), \mathcal{W}_n(x)) = \mathcal{W}_g(x). \quad (6)$$

Proof. In this proof we abbreviate $\tilde{C}_n(x)$ as \tilde{C}_n etc.

First we prove (4), following [6, Section 5.3] (see also [7, Section 7]). It suffices to show that $(\mathcal{S}_m, \mathcal{S}_n) = (\mathcal{S}_g)$ as ideals of $K[x]$. By Lemma 2.1 (viii) we have $(\mathcal{S}_m, \mathcal{S}_n) \subset (\mathcal{S}_g)$. Let a, b be integers such that $am + bn = g$. By Lemma 2.1 (vi) we have

$$\mathcal{S}_g = \begin{vmatrix} \mathcal{S}_{am} & \mathcal{S}_{-bn} \\ \mathcal{S}_{am+1} & \mathcal{S}_{-bn+1} \end{vmatrix} \in (\mathcal{S}_{am}, \mathcal{S}_{-bn}) \subset (\mathcal{S}_m, \mathcal{S}_n),$$

as desired.

Next we prove (5). We have $(\mathcal{V}_m, \mathcal{V}_n) \subset (\mathcal{V}_g)$ similarly. Since n is odd, there exist odd integers a and b such that $2am + bn = g$. Then by Lemma 2.1 (vi) we have

$$\mathcal{V}_g = C_{(am+bn)/2}\mathcal{V}_{am} - \mathcal{V}_{bn} \in (\mathcal{V}_{am}, \mathcal{V}_{bn}) \subset (\mathcal{V}_m, \mathcal{V}_n),$$

as desired.

Changing the sign of x in (5) and using Lemma 2.1 (iii), we obtain (6).

Before proving (3), we show that $(\mathcal{S}_m, \mathcal{W}_n) = (\mathcal{W}_g)$. Since $(\mathcal{S}_g) \subset (\mathcal{W}_g)$ by (2), we have $(\mathcal{S}_m, \mathcal{W}_n) \subset (\mathcal{W}_g)$. Since n is odd, there exist integers a and b such that b is odd and $4am + bn = g$. Then by Lemma 2.1 (vi) we have

$$\mathcal{W}_g = (x + 2)\mathcal{S}_{am}\mathcal{V}_{(bn+g)/2} - \mathcal{W}_{-bn} \in (\mathcal{S}_{am}, \mathcal{W}_{-bn}) \subset (\mathcal{S}_m, \mathcal{W}_n),$$

Then ${}^t(x_0, x_1, \dots, x_{n-1})$ is an eigenvector for an eigenvalue λ if and only if ${}^t(x_0, x_1, \dots, x_{n-1}) = {}^t(S_0(\lambda), S_1(\lambda), \dots, S_{n-1}(\lambda))x_0$ and $S_n(\lambda)x_0 = 0$. This proves the assertions.

(ii) Similarly, we have

$$A(\mathbf{C}_n) = \begin{pmatrix} & 1 & & 1 \\ 1 & & 1 & \\ & 1 & & \ddots \\ & & \ddots & 1 \\ 1 & & & 1 \end{pmatrix}.$$

Then ${}^t(x_0, x_1, \dots, x_{n-1})$ is an eigenvector for an eigenvalue λ if and only if, introducing two more variables,

$$x_j = \lambda x_{j-1} - x_{j-2} \quad (2 \leq j \leq n+1), \quad x_n = x_0, \quad x_{n+1} = x_1.$$

By Lemma 2.1 (xi), this is equivalent to

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} \mathcal{S}_0(\lambda) \\ \mathcal{S}_1(\lambda) \\ \vdots \\ \mathcal{S}_{n-1}(\lambda) \end{pmatrix} x_1 - \begin{pmatrix} \mathcal{S}_{-1}(\lambda) \\ \mathcal{S}_0(\lambda) \\ \vdots \\ \mathcal{S}_{n-2}(\lambda) \end{pmatrix} x_0$$

and

$$B(\lambda) \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where we put

$$B(\lambda) := \begin{pmatrix} \mathcal{S}_{n-1}(\lambda) + 1 & -\mathcal{S}_n(\lambda) \\ -\mathcal{S}_n(\lambda) & \mathcal{S}_{n+1}(\lambda) - 1 \end{pmatrix}.$$

Thus we see that $\chi_{A(\mathbf{C}_n)}(x) = \det B(x)$ and the dimension of the eigenspace of $A(\mathbf{C}_n)$ for λ is equal to $2 - \text{rank}_K B(\lambda)$. By Lemma 2.1 (vii) we have

$$B(\lambda) = \begin{cases} \mathcal{S}_{n/2}(\lambda) \begin{pmatrix} C_{n/2-1}(\lambda) & -C_{n/2}(\lambda) \\ -C_{n/2}(\lambda) & C_{n/2+1}(\lambda) \end{pmatrix} & (n : \text{even}), \\ \mathcal{V}_n(\lambda) \begin{pmatrix} \mathcal{V}_{n-2}(\lambda) & -\mathcal{V}_n(\lambda) \\ -\mathcal{V}_n(\lambda) & \mathcal{V}_{n+2}(\lambda) \end{pmatrix} & (n : \text{odd}), \end{cases}$$

so that $\chi_{A(\mathbf{C}_n)}(x) = \tilde{C}_n(x)$ by Lemma 2.1 (v) and (1). Moreover, we can give a complete description of the value of $\text{rank}_K B(\lambda)$ by using Lemma 2.1 (ii), (iii), and (1).

For matrices A and B , let

$$A \oplus B := \begin{pmatrix} A & O \\ O & B \end{pmatrix}$$

denote the block sum.

Lemma 3.2. (i) *The Jordan canonical form of $A(\mathbf{P}_n)$ is of the form $\bigoplus_{j=1}^r J_{m_j}(\lambda_j)$ where λ_j 's are distinct from each other.*

(ii) *The Jordan canonical form of $A(\mathbf{C}_n)$ is of the form*

$$\begin{cases} J_1(2) \oplus \bigoplus_{j=1}^r J_{m_j/2}(\lambda_j)^{\oplus 2} & (n : \text{odd}, \text{char } K \nmid n), \\ J_1(2) \oplus J_1(-2) \oplus \bigoplus_{j=1}^r J_{m_j/2}(\lambda_j)^{\oplus 2} & (n : \text{even}, \text{char } K \nmid n), \\ J_k(2) \oplus J_{k+1}(2) \oplus \bigoplus_{j=1}^r J_{m_j/2}(\lambda_j)^{\oplus 2} & (n : \text{odd}, \text{char } K \mid n), \\ J_k(2) \oplus J_{k+1}(2) \oplus J_k(-2) \oplus J_{k+1}(-2) \\ \quad \oplus \bigoplus_{j=1}^r J_{m_j/2}(\lambda_j)^{\oplus 2} & (n : \text{even}, 2 \neq \text{char } K \mid n), \\ J_k(0)^{\oplus 2} \oplus \bigoplus_{j=1}^r J_{m_j/2}(\lambda_j)^{\oplus 2} & (2 = \text{char } K \mid n) \end{cases}$$

where $k = \lfloor p^e/2 \rfloor$, $p = \text{char } K$, $e = \text{ord}_p(n)$, $r \geq 0$, $\lambda_j \notin \{\pm 2\}$, and λ_j 's are distinct from each other.

Proof. The claim (i) and (ii) in the case $\text{char } K \nmid n$ is immediate from Lemma 3.1. Suppose n is odd, $p = \text{char } K \mid n$ and put $e = \text{ord}_p(n)$, $m = n/p^e$. By Lemma 2.1 (viii) and (ix) we have

$$\mathscr{W}_n(x) = (x-2)^{(p^e-1)/2} \mathscr{W}_m(x),$$

$$\tilde{C}_n(x) = (x-2)^{p^e} \mathscr{W}_m(x)^2$$

in $K[x]$. Since $\mathscr{W}_m(2) \neq 0$ in K , the multiplicity of the eigenvalue 2 in $\chi_{A(\mathbf{C}_n)}(x) = \tilde{C}_n(x)$ is p^e and that in the minimal polynomial $(x-2)\mathscr{W}_m(x)$ is $(p^e+1)/2$. This proves the claim in this case. The remaining cases can be treated similarly. \square

Lemma 3.3. *The eigenspace of $J_m(\alpha) \otimes I_n + I_m \otimes J_n(\beta)$ for the eigenvalue $\alpha + \beta$ has dimension $\min\{m, n\}$.*

Proof. We may suppose $m \leq n$. Let $\{e_i\}$ and $\{f_j\}$ be the standard bases of K^m and K^n , respectively. Then $\sum_{i,j} c_{ij}(e_i \otimes f_j)$ is an eigenvector of $J_m(\alpha) \otimes I_n + I_m \otimes J_n(\beta)$ for the eigenvalue $\alpha + \beta$ if and only if $c_{i,j+1} + c_{i+1,j} = 0$ holds for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$ where we make the convention $c_{m+1,j} = c_{i,n+1} = 0$. Since any choice of $c_{11}, c_{21}, \dots, c_{m1}$ uniquely determines an eigenvector, the claim follows. \square

Lemma 3.4. *Let A and B be square matrices of degree m and n , respectively, and let d be the dimension over K of the eigenspace of $A \otimes I_n + I_m \otimes B$ for $\lambda \in K$.*

(i) *If both A and B have the property that every eigenspace has dimension 1, then*

$$d = \deg \gcd_K(\chi_A(x), \chi_B(\lambda - x)).$$

(ii) If the Jordan canonical forms of A and B are of the forms

$$\bigoplus_{i=1}^r J_1(\alpha_i) \oplus \bigoplus_{j=1}^s \left(J_{m_j}(\mu_j) \oplus J_{m'_j}(\mu_j) \right)$$

and

$$\bigoplus_{k=1}^t J_1(\beta_k) \oplus \bigoplus_{l=1}^u \left(J_{n_l}(\nu_l) \oplus J_{n'_l}(\nu_l) \right),$$

respectively, where $\alpha_1, \dots, \alpha_r, \mu_1, \dots, \mu_s$ (resp. $\beta_1, \dots, \beta_t, \nu_1, \dots, \nu_u$) are distinct from each other and $m_j \leq m'_j \leq m_j + 1$ ($1 \leq j \leq s$), $n_l \leq n'_l \leq n_l + 1$ ($1 \leq l \leq u$), then

$$d = 2 \deg \gcd_K(\chi_A(x), \chi_B(\lambda - x)) - \sum_{\alpha_i + \beta_k = \lambda} 1 - \sum_{\substack{\mu_j + \nu_l = \lambda \\ (j, l) \in S}} 1,$$

where

$$S = \{(j, l) \mid m'_j = m_j + 1, n'_l = n_l + 1, m_j = n_l\}.$$

Proof.

- (i) Let $\chi_A(x) = \prod_{i=1}^r (x - \mu_i)^{m_i}$, $\chi_B(x) = \prod_{j=1}^s (x - \nu_j)^{n_j}$ be the factorization over K where $\mu_1, \mu_2, \dots, \mu_r$ (resp. $\nu_1, \nu_2, \dots, \nu_s$) are distinct roots. Then the Jordan canonical forms are $\bigoplus_{i=1}^r J_{m_i}(\mu_i)$ and $\bigoplus_{j=1}^s J_{n_j}(\nu_j)$, respectively, so by Lemma 3.3 we have

$$d = \sum_{\mu_i + \nu_j = \lambda} \min\{m_i, n_j\},$$

which is easily seen to be equal to $\deg \gcd_K(\chi_A(x), \chi_B(\lambda - x))$.

- (ii) By Lemma 3.3 we have

$$\begin{aligned} d &= \sum_{\alpha_i + \beta_k = \lambda} 1 + \sum_{\alpha_i + \nu_l = \lambda} 2 + \sum_{\mu_j + \beta_k = \lambda} 2 \\ &\quad + \sum_{\mu_j + \nu_l = \lambda} (\min\{m_j, n_l\} + \min\{m_j, n'_l\} + \min\{m'_j, n_l\} + \min\{m'_j, n'_l\}). \end{aligned}$$

On the other hand, $\gcd_K(\chi_A(x), \chi_B(\lambda - x))$ has degree

$$\sum_{\alpha_i + \beta_k = \lambda} 1 + \sum_{\alpha_i + \nu_l = \lambda} 1 + \sum_{\mu_j + \beta_k = \lambda} 1 + \sum_{\mu_j + \nu_l = \lambda} \min\{m_j + m'_j, n_l + n'_l\}.$$

The claim follows since

$$\begin{aligned} &\min\{m_j, n_l\} + \min\{m_j, n'_l\} + \min\{m'_j, n_l\} + \min\{m'_j, n'_l\} \\ &\quad - 2 \min\{m_j + m'_j, n_l + n'_l\} \end{aligned}$$

is -1 if $m'_j = m_j + 1, n'_l = n_l + 1, m_j = n_l$, and 0 otherwise.

□

Proof of Theorem 1.1. Recall that $d(\mathbf{P}_m \times \mathbf{P}_n, K, a)$ is the dimension of the eigenspace of the adjacency matrix $A(\mathbf{P}_m \times \mathbf{P}_n)$ for the value $-a$. In view of the fact

$$A(G \times H) = A(G) \otimes I_n + I_m \otimes A(H)$$

for graphs G, H with m, n vertices, respectively, we can apply Lemma 3.2 (i) and Lemma 3.4 (i) to obtain

$$d(\mathbf{P}_m \times \mathbf{P}_n, K, a) = \deg \gcd_K(\chi_{A(\mathbf{P}_m)}(x), \chi_{A(\mathbf{P}_n)}(-x - a)).$$

Now by Lemma 3.1 (i) we complete the proof of Theorem 1.1 (i). The proof of (ii) is similar, though tedious. □

4 Application

Applying Theorem 1.1, we obtain some corollaries. Let ε be defined as in Theorem 1.1.

Corollary 4.1. (i) $d(\mathbf{P}_{m-1} \times \mathbf{P}_{n-1}, K, 0) = \gcd(m, n) - 1$.

(ii)

$$d(\mathbf{C}_m \times \mathbf{C}_n, K, 0) = \begin{cases} 2 \gcd(m, n) - \varepsilon & (\text{char } K = 2 \text{ or } mn: \text{even}), \\ 0 & (\text{char } K \neq 2 \text{ and } mn: \text{odd}). \end{cases}$$

Proof. (i) By Lemma 2.2.

(ii) By Lemma 2.1 (iii) we have $\tilde{C}_n(-x) = \tilde{C}_n(x)$ if $\text{char } K = 2$ or n is even. In this case the claim follows from Lemma 2.2. Changing m and n , we also cover the case where m is even. Suppose $\text{char } K \neq 2$ and mn is odd. By Lemma 2.1 (vii), noting that $\mathcal{W}_m(-2)\mathcal{V}_n(2) \neq 0$, we have

$$\gcd(C_m(x) - 2, C_n(x) + 2) = \gcd(\mathcal{W}_m(x), \mathcal{V}_n(x))^2.$$

We show that $(\mathcal{W}_m(x), \mathcal{V}_n(x)) = (1)$ as ideals of $K[x]$. Put $l = mn$. As in the proof of Lemma 2.2, we have $(\mathcal{W}_m(x), \mathcal{V}_n(x)) \supset (\mathcal{W}_l(x), \mathcal{V}_l(x))$. By Lemma 2.1 (vi) we have

$$\begin{aligned} \mathcal{V}_l(x) &= \mathcal{S}_{(l+1)/2}(x) - \mathcal{S}_{(l-1)/2}(x), \\ \mathcal{W}_l(x) &= \mathcal{S}_{(l+1)/2}(x) + \mathcal{S}_{(l-1)/2}(x), \end{aligned}$$

so that we have, noting that $\text{char } K \neq 2$,

$$(\mathcal{W}_l(x), \mathcal{V}_l(x)) = (\mathcal{S}_{(l+1)/2}(x), \mathcal{S}_{(l-1)/2}(x)) = (\mathcal{S}_1(x)) = (1),$$

as desired. The claim follows from this. □

□

Remark 4.2. In the case $\text{char } K = 2$, Corollary 4.1 (i) was proved in [1] and [8]. See also [2].

Corollary 4.3. *Suppose $\text{char } K = p > 0$.*

(i)

$$d(\mathbf{P}_{p^m-1} \times \mathbf{P}_{p^n-1}, K, a) = \begin{cases} p^{\min\{m,n\}} - 1 & (a = 0), \\ (p^{\min\{m,n\}} - 1)/2 & (p \geq 3, a = \pm 4), \\ 0 & (\text{otherwise}). \end{cases}$$

(ii)

$$d(\mathbf{C}_{p^m} \times \mathbf{C}_{p^n}, K, a) = \begin{cases} 2p^{\min\{m,n\}} & (p = 2, a = 0), \\ 2p^{\min\{m,n\}} & (p \geq 3, a = -4, m \neq n), \\ 2p^m - 1 & (p \geq 3, a = -4, m = n), \\ 0 & (\text{otherwise}). \end{cases}$$

(iii) $d(\mathbf{C}_{p^m} \times \mathbf{C}_{p^n}, K, a) + \varepsilon = p(d(\mathbf{C}_m \times \mathbf{C}_n, K, a) + \varepsilon)$.

(iv) *For any power q of p with $q \geq 4$ and for any $a \in K \cap \mathbb{F}_q$, we have*

$$d(\mathbf{C}_{q+1} \times \mathbf{C}_{q+1}, K, a) = d(\mathbf{C}_{q-1} \times \mathbf{C}_{q-1}, K, a) + 4.$$

Proof. (i) By Lemma 2.1 (ix).

(ii) By Lemma 2.1 (ix) we have $\tilde{C}_{p^n}(x) \equiv x^{p^n} - 2 \equiv (x - 2)^{p^n} \pmod{p}$.

(iii) By Lemma 2.1 (viii) and (ix) we have $\tilde{C}_{pn}(x) = C_p(C_m(x)) - 2 \equiv C_m(x)^p - 2 \equiv \tilde{C}_n(x)^p \pmod{p}$.

(iv) By Lemma 2.1 (vi), (viii), and (ix) we obtain $\tilde{C}_{q+1}(x)\tilde{C}_{q-1}(x) = (C_q(x) - x)^2 \equiv (x^q - x)^2 \pmod{p}$. Since $(x^q - x)^2$ is stable under $x \mapsto -x - a$, we have

$$\begin{aligned} \tilde{C}_{q+1}(x)\tilde{C}_{q-1}(x)\tilde{C}_{q-1}(-x-a) &\equiv (x^q - x)^2\tilde{C}_{q-1}(-x-a), \\ \tilde{C}_{q+1}(-x-a)\tilde{C}_{q-1}(-x-a)\tilde{C}_{q-1}(x) &\equiv (x^q - x)^2\tilde{C}_{q-1}(x). \end{aligned}$$

By taking “2 deg gcd” of both sides, we obtain the desired formula. \square

Remark 4.4. In the case $p = 2, a = 1$ (this implies $\varepsilon = 0$), Corollary 4.3 (iii) was first observed in [3] and was proved there by an elementary method. In the same paper, (iv) was also observed for the first time and was proved by using an elliptic curve. Subsequently, in [11], the author applied the same method to prove (iv) in the case where p is general and $a = 1$.

Corollary 4.5. Let $\mu(n, a)$ denote the multiplicity of $x - a$ in the factorization of $\mathcal{S}_n(x)$ over K , and introduce the notation

$$\delta_{m,n}(a) = \begin{cases} 2 & (\text{if } \mu(m, a) > \mu(n, 2)), \\ 0 & (\text{otherwise}). \end{cases}$$

(i) If $\text{char } K = 2$, then

$$d(\mathbf{C}_m \times \mathbf{C}_n, K, a) = 2d(\mathbf{P}_{m-1} \times \mathbf{P}_{n-1}, K, a) + \delta,$$

where

$$\delta = \begin{cases} 2 - \varepsilon & (a = 0), \\ \delta_{m,n}(a) + \delta_{n,m}(a) & (a \neq 0). \end{cases}$$

(ii) If $\text{char } K \neq 2$, then

$$d(\mathbf{C}_{2m} \times \mathbf{C}_{2n}, K, a) = 4d(\mathbf{P}_{m-1} \times \mathbf{P}_{n-1}, K, a) + \delta,$$

where

$$\delta = \begin{cases} 4 - \varepsilon & (a = 0), \\ 2 - \varepsilon + \delta_{m,n}(6) + \delta_{n,m}(6) & (a = \pm 4), \\ \delta_{m,n}(a+2) + \delta_{m,n}(a-2) + \delta_{n,m}(a+2) + \delta_{n,m}(a-2) & (a \neq 0, \pm 4). \end{cases}$$

Proof. (i) By Lemma 2.1 (x) we have

$$d(\mathbf{C}_m \times \mathbf{C}_n, K, a) = 2 \deg \gcd_K(x\mathcal{S}_m(x), (-x-a)\mathcal{S}_n(-x-a)) - \varepsilon,$$

and the claim follows as follows. It is clear for $a = 0$. If $a \neq 0$, then we have

$$\begin{aligned} v_0(\mathcal{S}_n(-x-a)) > v_0(\mathcal{S}_m(x)) &\iff \mu(n, -a) > \mu(m, 0), \\ v_{-a}(\mathcal{S}_m(x)) > v_{-a}(\mathcal{S}_n(-x-a)) &\iff \mu(m, -a) > \mu(n, 0), \end{aligned}$$

where $v_b(f)$ denotes the order of a polynomial $f \in K[x]$ at $x = b \in K$. Also note that $\varepsilon = 0$ if $a \neq 0$.

(ii) Use $\tilde{\mathbf{C}}_{2n}(x) = (x^2 - 4)\mathcal{S}_n(x)^2$ from Lemma 2.1 (vii) (1) instead. □

Remark 4.6. (i) Consider the case $a = 1$. Corollary 4.5 (i) was conjectured in [3], and under the restriction that mn is prime to $\text{char } K$, both (i) and (ii) were obtained essentially in [11, Proposition 2.2].

(ii) This result says that there is a relation between usual Lights Out puzzle and the torus version of the puzzle. It would be interesting to find out a combinatorial interpretation of this result.

(iii) Let $n \geq 1$ and put $e = \text{ord}_p(n)$ in the case $\text{char } K = p > 0$. Using Lemma 2.1, we can easily verify

$$\mu(n, 2) = \begin{cases} 0 & (\text{char } K = 0), \\ 2^e - 1 & (\text{char } K = 2), \\ (p^e - 1)/2 & (\text{char } K = p \geq 3). \end{cases}$$

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