# A Survey of Extremal Holomorphic Curves for the Truncated Defect Relation

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Holomorphic curves extremal for the truncated defect relation are considered in this article and several results on the defect are given. These results correspond mainly to those obtained for holomorphic curves extremal for the non-truncated defect relation in the references [10] and [11].

## 1. Introduction

Let $f=[f_1,\cdots,f_{n+1}]$ be a holomorphic curve from $\boldsymbol{C}$ into the $n$-dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1,\cdots,f_{n+1}):\boldsymbol{C}\rightarrow\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\},$$

where $n$ is a positive integer.

We use the following notations:

$$\|f(z)\|=(|f_1(z)|^2+\cdots+|f_{n+1}(z)|^2)^{1/2}$$

and for a vector $\boldsymbol{a}=(a_1,\cdots,a_{n+1})\in\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$

$$\|\boldsymbol{a}\|=(|a_1|^2+\cdots+|a_{n+1}|^2)^{1/2},\quad (\boldsymbol{a},f)=a_1f_1+\cdots+a_{n+1}f_{n+1},\quad (\boldsymbol{a},f(z))=a_1f_1(z)+\cdots+a_{n+1}f_{n+1}(z).$$

The characteristic function of $f$ is defined as follows (see [12]):

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta-\log\|f(0)\|$$

We suppose throughout the paper that $f$ is transcendental; that is to say,

$$\lim_{r\to\infty}\frac{T(r,f)}{\log r}=\infty$$

and $f$ is linearly non-degenerate over $\boldsymbol{C}$; namely, $f_1,\cdots,f_{n+1}$ are linearly independent over $\boldsymbol{C}$.

It is well-known that $f$ is linearly non-degenerate over $\boldsymbol{C}$ if and only if the Wronskian $W=W(f_1,\cdots,f_{n+1})$ of $f_1,\cdots,f_{n+1}$ is not identically equal to zero.

For meromorphic functions in the complex plane we use the standard notation of the Nevanlinna theory of meromorphic functions ([4], [5]).

For $\boldsymbol{a}\in\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$ we write

$$m(r,\boldsymbol{a},f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|\boldsymbol{a}\|\|f(re^{i\theta})\|}{|(\boldsymbol{a},f(re^{i\theta}))|}d\theta,\quad N(r,\boldsymbol{a},f)=N(r,\frac{1}{(\boldsymbol{a},f)}).$$

We then have the First Fundamental Theorem ([12], p.76):

$$T(r,f)=m(r,\boldsymbol{a},f)+N(r,\boldsymbol{a},f)+O(1).$$

We call the quantity

$$\delta(\boldsymbol{a},f)=1-\limsup_{r\to\infty}N(r,\boldsymbol{a},f)/T(r,f)=\liminf_{r\to\infty}m(r,\boldsymbol{a},f)/T(r,f)$$

the deficiency (or defect ) of $\boldsymbol{a}$ with respect to $f$. We have $0 \le \delta(\boldsymbol{a}, f) \le 1$.

Further, let $\nu(c)$ be the order of zero of $(\boldsymbol{a}, f(z))$ at $z=c$ and for a positive integer $k$ let

$$n_k(r, \boldsymbol{a}, f) = \sum_{|c| \le r} \min\{\nu(c), k\}.$$

Then, we put for $r>0$

$$N_k(r, \boldsymbol{a}, f) = \int_0^r \frac{n_k(t, \boldsymbol{a}, f) - n_k(0, \boldsymbol{a}, f)}{t} dt + n_k(0, \boldsymbol{a}, f) \log r$$

and put $\delta_k(\boldsymbol{a}, f) = 1 - \limsup_{r \to \infty} N_k(r, \boldsymbol{a}, f)/T(r, f)$. Then,

$$0 \le \delta(\boldsymbol{a}, f) \le \delta_k(\boldsymbol{a}, f) \le 1 \tag{1}$$

Let $X$ be a subset of $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ in $N$-subgeneral position; that is to say, $\#X \ge N+1$ and any $N+1$ elements of $X$ generate $\boldsymbol{C}^{n+1}$, where $N$ is an integer satisfying $N \ge n$.

Cartan ([1], $N=n$) and Nochka ([6], $N>n$ ) gave the following

**Theorem A (Truncated Defect Relation).** For any $q$ elements $\boldsymbol{a}_j$ ($j = 1, \cdots, q$) of $X$,

$$\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) \le 2N - n + 1$$

where $2N-n+1<q \le \infty$ (see also [2] or [3]).

We are interested in a holomorphic curve $f$ for which the defect relation is extremal:

$$\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) = 2N-n+1. \tag{2}$$

Put $X(0) = \{\boldsymbol{a} = (a_1, \cdots, a_{n+1}) \in X \mid a_{n+1} = 0\}$. Then, it is easy to see that $0 \le \#X(0) \le N$ since $X$ is in $N$-subgeneral position. Further we put (see Definition 1 in [7])

$$u(z) = \max_{1 \le j \le n} |f_j(z)|, \quad t(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \{\log u(re^{i\theta}) - \log u(e^{i\theta})\} d\theta$$

and

$$\Omega = \limsup_{r \to \infty} t(r, f)/T(r, f).$$

**Proposition A** (see [7]). (a) $t(r, f)$ is independent of the choice of reduced representation of $f$.
(b) $t(r, f) \le T(r, f) + O(1)$. (c) $N(r, 1/f_j) \le T(r, f) + O(1)$ ($j=1, \cdots, n$). (d) $0 \le \Omega \le 1$.

We proved the following

**Theorem B.** Suppose that (i) $N>n \ge 2$ and (ii) there are vectors $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ in $X$ satisfying

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N-n+1,$$

where $2N-n+1<q \le \infty$.

(I) If $\Omega < 1$, then (a) $\#X(0) = N$ ; (b) There exists a subset $P$ of $\{1, 2, \cdots, q\}$ satisfying

$$\#P = N-n+1, \quad \delta(\boldsymbol{a}_j, f) = 1 \ (j \in P) \quad \text{and} \quad X(0) \cap \{\boldsymbol{a}_j \mid j \in P\} = \phi;$$

(c) Any $n$ elements of $X - \{\boldsymbol{a}_j \mid j \in P\}$ are linearly independent (see [11]).

(II) If $n=2m$ ($m \in N$), then there are at least $[(2N-n+1)/(n+1)]+1$ vectors $\boldsymbol{a} \in \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q\}$ satisfying $\delta(\boldsymbol{a}, f) = 1$ ([10]).

Our main purpose of this article is to change the assumption (ii) in Theorem B for a weaker condition (2) and to obtain a similar result to Theorem B.

## 2 Preliminaries and lemma

We shall give some lemmas for later use. Let $f=[f_1,\cdots,f_{n+1}]$, $X$, $X(0)$ etc. be as in Section 1, $q$ any integer satisfying $2N-n+1<q<\infty$ and we put $Q=\{1,2,\cdots,q\}$. Let $\{\boldsymbol{a}_j \mid j\in Q\}$ be a family of vectors in $X$. For a non-empty subset $P$ of $Q$, we denote

$$V(P)= \text{the vector space spanned by } \{\boldsymbol{a}_j \mid j\in P\} \quad \text{and} \quad d(P)=\dim V(P)$$

and we put $\mathcal{O}=\{P\subset Q \mid 0<\#P\le N+1\}$.

**Lemma 2-1** ((2.4.3) in [3], p.68). For $P\in\mathcal{O}$, $\#P-d(P)\le N-n$.

For $\{\boldsymbol{a}_j \mid j\in Q\}$, let $\omega : Q \to (0,1]$ be the Nochka weight function given in [3, p.72] and $\theta$ the reciprocal number of the Nochka constant given in [3, p.72]. Then they have the following properties:

**Lemma 2-2** (see[3], Theorem 2.11.4).

(a) $0<\omega(j)\theta\le 1$ for all $j\in Q$; (b) $q-2N+n-1=\theta\,(\sum_{j=1}^{q}\omega(j)-n-1)$;

(c) $(N+1)/(n+1)\le\theta\le(2N-n+1)/(n+1)$; (d) If $P\in\mathcal{O}$, then $\sum_{j\in P}\omega(j)\le d(P)$.

**Note 2-1.** (c) of Lemma 2-2 can be refined as floolws:

$$N/n\le\theta\le(2N-n+1)/(n+1).$$

Proof. When $\theta=(2N-n+1)/(n+1)$, there is nothing to prove as $N/n\le(2N-n+1)/(n+1)$.

When $\theta<(2N-n+1)/(n+1)$, then $N>n$ and there is an element $P\in\mathcal{O}$ satisfying

$$\theta=(2N-n+1-\#P)/(n+1-d(P)) \quad (1\le d(P)\le n)$$

by the definition of $\theta$. By Lemma 2-1 we have

$$\theta=(2N-n+1-\#P)/(n+1-d(P))\ge(N+1-d(P))/(n+1-d(P))\ge N/n$$

since $d(P)\ge 1$ in this case.

**Lemma 2-3** ([8], Theorem 2). For any $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q\in X$, we have the inequality

$$\sum_{j=1}^{q}\omega(j)m(r,\boldsymbol{a}_j,f)+N(r,1/W)\le(1+d)T(r,f)+(n-d)t(r,f)+S(r,f),$$

where $d=\sum_{\boldsymbol{a}_j\in X(0)}\omega(j)$.

**Corollary 2-1 (Defect relation).** For any $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q\in X$, we have the following inequalities:

(I) $\sum_{j=1}^{q}\omega(j)\delta_n(\boldsymbol{a}_j,f)\le d+1+(n-d)\Omega$;

(II) $\sum_{j=1}^{q}\delta_n(\boldsymbol{a}_j,f)\le 2N-n+1-N(n-d)(n-\Omega)/n$,

where $d$ is that given in Lemma 2-3.

Proof. From the inequality (3.2.14) in [3] we easily obtain the inequality

$$\sum_{j=1}^{q}\omega(j)(N(r,\boldsymbol{a}_j,f)-N_n(r,\boldsymbol{a}_j,f))\le N(r,1/W)+O(\log r).$$

By using this inequality we obtain (I) from Lemma 2-3 and by applying Lemma 2-2 and Note 2-1 to (I) we obtain (II) as uaual.

**Remark 2-1.** This is an amelioration of Theorem A since $d\le n$, $\Omega\le 1$ and so

$$d+1+(n-d)\Omega\le n+1 \quad \text{and} \quad 2N-n+1-N(n-d)(1-\Omega)/n\le 2N-n+1.$$

**Lemma 2-4** (see [9], Lemma 3). Suppose that $N>n$. For $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q\in X$, the maximal deficiency sum

$$\sum_{j=1}^{q}\delta_n(\boldsymbol{a}_j,f)=2N-n+1$$

holds if and only if the following two relations hold:

1) $(1-\theta\omega(j))(1-\delta_n(\boldsymbol{a}_j,f))=0 \quad (j=1,\cdots,q)$; 2) $\sum_{j=1}^{q}\omega(j)\delta_n(\boldsymbol{a}_j,f)=n+1$.

Proof. The proof is similar to that of Lemma 3 in [9]. We easily have that

$$\theta\sum_{j=1}^{q}\omega(j)\delta_n(\boldsymbol{a}_j,f)+q-\theta\sum_{j=1}^{q}\omega(j)=\sum_{j=1}^{q}\{\delta_n(\boldsymbol{a}_j,f)+(1-\theta\omega(j))(1-\delta_n(\boldsymbol{a}_j,f))\},$$

which reduces to

$$2N-n+1-\sum_{j=1}^{q}\delta_n(\boldsymbol{a}_j,f)=\theta(n+1-\sum_{j=1}^{q}\omega(j)\delta_n(\boldsymbol{a}_j,f))+\sum_{j=1}^{q}(1-\theta\omega(j))(1-\delta_n(\boldsymbol{a}_j,f)) \qquad (3)$$

by Lemma 2-2(b). By Corollary 2-1 and Remark 2-1, we easily obtain this lemma since

$$(1-\theta\omega(j))(1-\delta_n(\boldsymbol{a}_j,f))\geq 0 \quad (j=1,\cdots,q)$$

from (1) and Lemma 2-2(a).

**Corollary 2-2.** Suppose that $N>n$ and that for $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q\in X$, the equality

$$\sum_{j=1}^{q}\delta_n(\boldsymbol{a}_j,f)=2N-n+1$$

holds. If $\theta\omega(j)<1$ for some $j\in Q$, then $\delta_n(\boldsymbol{a}_j,f)=1$.

**Definition 2-1** ([9], Definition 1). We put

$$\lambda=\min_{P\in\mathcal{O}} d(P)/\#P \quad \text{and} \quad \tau(j)=\lambda \quad (j\in Q)$$

Then, $\lambda$ and $\tau$ have the following properties.

**Lemma 2-5** ([9], Proposition 2). (a) $1/(N-n+1)\leq\lambda\leq(n+1)/(N+1)$;

(b) For any $P\in\mathcal{O}$, $\sum_{j\in P}\tau(j)\leq d(P)$.

**Remark 2-2.** (a) If $\lambda<(n+1)/(2N-n+1)$, then $\lambda=\min_{1\leq j\leq q}\omega(j)$, $\omega(j)=\lambda$ and $\theta\omega(j)<1$ $(j\in P_o)$ for an element $P_o\in\mathcal{O}$ satisfying $\lambda=d(P_o)/\#P_o$.

(b) If $\lambda\geq(n+1)/(2N-n+1)$, then $\omega(j)=1/\theta=(n+1)/(2N-n+1)$ $(j=1,\cdots,q)$.

In fact, the first assertion of (a) is given in the proof of Proposition 2.4.4 ([3],p.68) and by the definition of $\omega(j)$ ([3],p.72). For the second assertion of (a), as $(n+1)/(2N-n+1)\leq 1/\theta$ and $\omega(j)=\lambda$ $(j\in P_o)$, we have the conclusion.

(b) See the definitions of $\omega(j)$ and $\theta$ ([3], p.72).

**Lemma 2-6.** Suppose that there exists a function $\tau: Q\to(0,1]$ which satisfies the following condition $(*)$:

$(*)$ For any $P\in\mathcal{O}$, $\sum_{j\in P}\tau(j)\leq d(P)$.

Then, for any $P\in\mathcal{O}$ satisfying $\#P=N+1$ and for real numbers $E_1,\cdots,E_q$ satisfying $E_j\geq 1$ $(j\in Q)$, there exists a subset $B$ of $P$ satisfying the followings:

(a) $\#B=n+1$; (b) $\{\boldsymbol{a}_j\,|\,j\in B\}$ is a basis of $\boldsymbol{C}^{n+1}$; (c) $\prod_{j\in P}E_j^{\tau(j)}\leq\prod_{j\in B}E_j$.

Proof. By $(*)$, we can prove this proposition as in the case of Proposition 2.4.15 in [3], p.75. To make sure of it we shall give a proof of this proposition. We suppose without loss of generality that $E_1\geq E_2\geq\cdots\geq E_q$. We choose $j_1,\cdots,j_{n+1}$ by induction as follows:

1) Let $j_1$ be the minimum number in $P$. We put

$$P_1=\{j_1\} \quad \text{and} \quad R_1=\{j\in P\mid \boldsymbol{a}_j\in V(P_1)\}.$$

2) Suppose that $J_1,\cdots,j_k$ are chosen. We put for $k\geq 1$

$$P_k=\{j_1,\cdots,j_k\} \quad \text{and} \quad R_k=\{j\in P-R_{k-1}\mid \boldsymbol{a}_j\in V(P_k)\},$$

where $R_0=\phi$. We choose $j_{k+1}$ $(1\leq k\leq n)$ as follows.

$$j_{k+1}=\min\{\, j\in P \mid \boldsymbol{a}_j\notin V(P_k)\}$$

and put

$$P_{k+1}=\{\, j_1,\cdots,j_{k+1}\} \quad \text{and} \quad R_{k+1}=\{\, j\in P-P_k \mid \boldsymbol{a}_j\in V(P_{k+1})\}.$$

Then, it is easy to see that

(i) $R_1,\cdots,R_{k+1}$ are mutualy disjoint and $\boldsymbol{a}_{j_1},\cdots,\boldsymbol{a}_{j_k+1}$ are linearly independent;

(ii) $P=\cup_{k=1}^{n+1}R_k$ ; (iii) $E_{j_k}\geq E_j$ for $j\in R_k$; (iv) $E_{j_1}\geq E_{j_2}\geq\cdots\geq E_{j_{n+1}}\geq 1$ .

We put for $k=1,\cdots,n+1$

$$T_k=R_1\cup\cdots\cup R_k \quad \text{and} \quad \sum_{j\in R_k}\tau(j)=d_k.$$

Then,

$$\sum_{k=1}^{m}d_k-m\leq 0 \ \ (m=1,2,\cdots,n+1) \tag{4}$$

since

$$\sum_{k=1}^{m}d_k=\sum_{k=1}^{m}\sum_{j\in R_k}\tau(j)\leq d(T_m)=m$$

by $(*)$. Put $B=\{\, j_1,\cdots,j_{n+1}\}$. Then, $B$ satisfies (a), (b) and (c). It is easy to see that (a) and (b) hold. We have only to prove (c). Now, by (4), (iii) and (iv) we have the inequality (c):

$$\begin{aligned}\prod_{j\in P}E_j^{\tau(j)} &=\prod_{k=1}^{n+1}\prod_{j\in R_k}E_j^{\tau(j)}\leq\prod_{k=1}^{n+1}\prod_{j\in R_k}E_{j_k}^{\tau(j)}=\prod_{k=1}^{n+1}E_{j_k}^{d_k}=E_{j_1}E_{j_1}^{-1+d_1}\prod_{k=2}^{n+1}E_{j_k}^{d_k}\leq E_{j_1}E_{j_2}^{-1+d_1}\prod_{k=2}^{n+1}E_{j_k}^{d_k}\\ &=E_{j_1}E_{j_2}^{-1+d_1+d_2}\prod_{k=3}^{n+1}E_{j_k}^{d_k}=E_{j_1}E_{j_2}E_{j_2}^{-2+d_1+d_2}\prod_{k=3}^{n+1}E_{j_k}^{d_k}\leq E_{j_1}E_{j_2}E_{j_3}^{-2+d_1+d_2}\prod_{k=3}^{n+1}E_{j_k}^{d_k}\\ &=E_{j_1}E_{j_2}E_{j_3}^{-2+d_1+d_2+d_3}\prod_{k=4}^{n+1}E_{j_k}^{d_k}=E_{j_1}E_{j_2}E_{j_3}E_{j_3}^{-3+d_1+d_2+d_3}\prod_{k=4}^{n+1}E_{j_k}^{d_k}\\ &\cdots\cdots\\ &\leq E_{j_1}E_{j_2}\cdots E_{j_{n+1}}E_{j_{n+1}}^{-n-1+d_1+\cdots+d_{n+1}}\leq E_{j_1}E_{j_2}\cdots E_{j_{n+1}}=\prod_{j\in B}E_j.\end{aligned}$$

**Lemma 2-7.** Suppose that a function $\tau : Q\rightarrow(0,1]$ satisfies $(*)$ in Lemma 2-6. Then, the following inequality holds.

$$\sum_{j=1}^{q}\tau(j)m(r,\boldsymbol{a}_j,f)\leq(n+1)T(r,f)-N(r,1/W)+S(r,f).$$

Proof. We put $(\boldsymbol{a}_j,f)=F_j\ (j=1,\cdots,q)$. For any $z(\neq 0)$ arbitrarily fixed in $|z|<\infty$ for which $F_j(z)\neq 0\ (j=1,\cdots,q)$, let

$$|F_{j_1}(z)|\leq|F_{j_2}(z)|\leq\cdots\leq|F_{j_q}(z)|,$$

where $j_1,\cdots,j_q$ are distinct and $1\leq j_1,\cdots,j_q\leq q$. Then, there is a positive constant $K$ such that

$$\|f(z)\|\leq K|F_{j_\nu}(z)| \quad (\nu=N+1,\cdots,q). \tag{5}$$

(From now on we denote by $K$ a positive constant, which may be different from each other when it appears.)

We have by (5) and Lemma 2-6

$$\prod_{j=1}^{q}\left(\frac{\|a_j\|\|f(z)\|}{|(a_j,f(z))|}\right)^{\tau(j)} \leq K\prod_{j_\nu=1}^{N+1}\left(\frac{\|a_{j_\nu}\|\|f(z)\|}{|F_{j_\nu}(z)|}\right)^{\tau(j_\nu)} = \prod_{j_\nu\in B}\frac{\|a_{j_\nu}\|\|f(z)\|}{|F_{j_\nu}(z)|}$$

$$=K\frac{\|f(z)\|^{n+1}}{|W(z)|}\cdot\frac{|W_B(z)|}{\Pi_{j_\nu\in B}|F_{j_\nu}(z)|}, \tag{6}$$

where $W_B(z)$ is the Wronskian of $F_{j_\nu}$ ($j_\nu\in B$). Note that $W_B(z)=cW(z)$ ($c\neq 0$, constant). From (6) we obtain

$$\sum_{j=1}^{q}\tau(j)\log\frac{\|a_j\|\|f(z)\|}{|(a_j,f(z))|}\leq(n+1)\log\|f(z)\|-\log|W(z)|+\sum_{B\subset Q}\log^+\frac{|W_B(z)|}{\Pi_{j_\nu\in B}|F_{j_\nu}(z)|}+\log K,$$

where summation $\Sigma_{B\subset Q}$ is taken over all $B\subset Q$ satisfying that $\{a_j \mid j\in B\}$ is a basis of $C^{n+1}$. From this inequality we obtain the inequality

$$\sum_{j=1}^{q}\tau(j)m(r,a_j,f)\leq(n+1)T(r,f)-N(r,1/W)+S(r,f).$$

as usual.

For any entire function $h$ and a point $a\in C$, let $\nu_h(a)$ be the order of zero of $h(z)$ at $z=a$.

**Lemma 2-8.** Suppose that a function $\tau: Q\rightarrow(0,1]$ satisfies $(*)$ in Lemma 2-6. Then, for any vectors $a_1,\cdots,a_q\in X$, we have the inequality:

$$\sum_{j=1}^{q}\tau(j)\{N(r,a_j,f)-N_n(r,a_j,f)\}\leq N(r,1/W)+O(\log r).$$

Proof. As in the case of (3.2.14) in [3], we obtain the inequality

$$\sum_{j=1}^{q}\tau(j)(\nu_{F_j}(a)-n)^+\leq\nu_W(a), \tag{7}$$

where $F_j=(a_j,f)$ and $x^+=\max(x,0)$ for a real number $x$.

In fact, as is seen from the proof of the inequality (3.2.14), only (d) of four properties (a), (b), (c) and (d) in Lemma 2-2 is necessary to prove it. Therefore, the proof is effective if we change $\omega$ for our weight function $\tau$ which has the same property $(*)$ as Lemma 2-2 (d) and we have the inequality (7).

From (7), we obtain the inequality

$$\sum_{j=1}^{q}\tau(j)(n(r,a_j,f)-n_n(r,a_j,f))\leq n(r,1/W),$$

from which we have our lemma immediately.

**Lemma 2-9.** Suppose that a function $\tau: Q\rightarrow(0,1]$ satisfies $(*)$ in Lemma 2-6. Then, for vectors $a_1,\cdots,a_q\in X$, we have the inequality:

$$\sum_{j=1}^{q}\tau(j)\delta_n(a_j,f)\leq n+1.$$

Proof. From the First Fundamental Theorem, Lemma 2-7 and Lemma 2-8 we obtain the inequality

$$\sum_{j=1}^{q}\tau(j)(T(r,a_j,f)-N_n(r,a_j,f))\leq(n+1)T(r,f)+S(r,f),$$

from which we obtain our lemma as usual.

**Corollary 2-3.** For vectors $a_1,\cdots,a_q\in X$, we have the inequality

$$\sum_{j=1}^{q}\delta_n(a_j,f)\leq\frac{n+1}{\lambda}.$$

Proof. As $\tau(j)=\lambda$ ($j\in Q$) satisfies the property $(*)$ in Lemma 2-6 (Lemma 2-5(b)), we have this corollary from Lemma 2-

9 immediately.

## 3 Extremal case I : $\Omega < 1$ and $q < \infty$

Let $f$, $X$, $X(0)$ and $\omega$ etc. be as in the previous sections and $q$ an integer satisfying $2N-n+1<q<\infty$. Throughout this section we suppose that

(i) $N > n \geq 2$;

(ii) there are vectors $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q \in X$ satisfying

$$\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) = 2N-n+1;$$

(iii) $\Omega < 1$.

**Proposition 3-1.** For any $\boldsymbol{a} \in X(0)$, $\delta_n(\boldsymbol{a}, f) \geq \delta(\boldsymbol{a}, f) \geq 1-\Omega > 0$.

Proof. From the definitions of $t(r,f)$ and $N(r, \boldsymbol{a}, f)$, for $\boldsymbol{a} \in X(0)$ we easily have the inequality $N(r, \boldsymbol{a}, f) \leq t(r,f) + O(1)$, so that we obtain the inequality

$$\delta_n(\boldsymbol{a}, f) \geq \delta(\boldsymbol{a}, f) \geq 1-\Omega > 0.$$

**Proposition 3-2.** $X(0) \subset \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q\}$.

Proof. If there exists a vector $\boldsymbol{a} \in X(0)$ satisfying $\boldsymbol{a} \notin \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q\}$ then by Proposition 3-1 and Theorem A

$$\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) \leq 2N-n+1-\delta_n(\boldsymbol{a}, f) < 2N-n+1,$$

which is a contradiction to our assumption (ii).

Put $P(0) = \{ j \in Q \mid \boldsymbol{a}_j \in X(0)\}$ and $\sum_{j \in P(0)} \omega(j) = d$.

**Proposition 3-3.** $d = n$.

In fact, according to Corollary 2-1 (II), we easily have that $d = n$ by the assumption (ii) and (iii) as $d \leq n$.

**Proposition 3-4.** $\theta = N/n$, $\#X(0) = N$ and $\theta\omega(j) = 1$ $(j \in P(0))$.

Proof. As $X$ is in $N$-subgeneral position, we have $\#X(0) \leq N$, so that from Proposition 3-3 and Lemma 2-2(a)

$$(*) \quad \theta n = \sum_{j \in P(0)} \theta\omega(j) \leq \sum_{j \in P(0)} 1 = \#P(0) = \#X(0) \leq N,$$

so that we have $\theta \leq N/n$. By Note 2-1 we obtain $\theta = N/n$.

Combining this result with the inequality $(*)$, we have

$$\#X(0) = N \quad \text{and} \quad \theta\omega(j) = 1 \quad (j \in P(0)).$$

**Corollary 3-1.** $\lambda < (n+1)/(2N-n+1)$.

Proof. By Corollary 2-3 and the assumption (ii) we have that $\lambda \leq (n+1)/(2N-n+1)$.

If $\lambda = (n+1)/(2N-n+1)$, then by Remark 2-2 we have

$$\theta = (2N-n+1)/(n+1) > N/n,$$

which is a contradiction to Proposition 3-4. This means that our corollary must hold.

Put $P_1 = \{ j \mid \theta\omega(j) < 1,\ 1 \leq j \leq q\}$. Then, $P_1 \cap P(0) = \phi$ by Proposition 3-4. Note that $\delta_n(\boldsymbol{a}_j, f) = 1$ $(j \in P_1)$ by Corollary 2-2. We have the following

**Proposition 3-5.** $N-n+1 \leq \#P_1 < 2N-n+1$.

Proof. (a) From Lemma 2-2 (b) and Proposition 3-4, Lemma 2-5 (a) and Remark 2-2, we have

$$q-(2N-n+1)=\theta(\sum_{j=1}^{q}\omega(j)-n-1)=\sum_{j\notin P_1}\theta\omega(j)+\sum_{j\in P_1}\theta\omega(j)-\theta n-\theta$$

$$=q-N-\#P_1+\frac{N}{n}(\sum_{j\in P_1}\omega(j)-1)\geq q-N-\#P_1+\frac{N}{n}(\frac{\#P_1}{N-n+1}-1),$$

which reduces to the inequality

$$\#P_1(N-n+1-N/n)\geq(N-n+1-N/n)(N-n+1).$$

As $N-n+1-N/n=(N-n)(n-1)/n>0$, we have $\#P_1\geq N-n+1$.

(b) From Propositions 3-2 and 3-4 we have $\#P_1<2N-n+1$ as $P_1\cap P(0)=\phi$.

Let $P_0$ be an element of $\mathcal{O}$ satisfying $d(P_0)/\#P_0=\lambda$, where $\lambda=\min_{P\in\mathcal{O}}d(P)/\#P$. Then, $\omega(j)=\lambda$ $(j\in P_0)$ (see Remark 2-2) and $P_0$ is a non-empty subset of $P_1$ since $\theta\lambda<1$ by Corollary 3-1 and Remark 2-2 (a).

**Proposition 3-6.** $\#P_0=N-n+1$, $d(P_0)=1$ and $\omega(j)=\lambda=1/(N-n+1)$ $(j\in P_0)$.

Proof. As $\theta=N/n$ (Proposition 3-4) and it is smaller than $(2N-n+1)/(n+1)$, by the definition of $\theta$, there exists a subset $P$ of $Q$ satisfying

$$P_0\subset P,\quad 1\leq d(P)\leq n \text{ and } \theta=(2N-n+1-\#P)/(n+1-d(P)$$

in this case. By Proposition 3-4 and Lemma 2-1 we have the inequality

$$(*)\ 0=\theta-\frac{N}{n}=\frac{2N-n+1-\#P}{n+1-d(P)}-\frac{N}{n}=\frac{(N-n)(n-1)+Nd(P)-n\#P}{n(n+1-d(P))}\geq\frac{(N-n)(d(P)-1)}{n(n+1-d(P))}\geq 0,$$

which implies that $d(P)=1$ and $\#P=N-n+1$, so that by Lemma 2-2 (a), Remark 2-2 and Lemma 2-5 (a) we have

$$1=d(P)\geq\sum_{j\in P}\omega(j)\geq(N-n+1)\lambda\geq 1$$

and $\lambda=1/(N-n+1)=\omega(j)$ $(j\in P)$.

By the choice of $P_0$, $1\leq d(P_0)\leq d(P)=1$ and so we have

$$d(P_0)=1 \quad\text{and}\quad \#P_0=N-n+1.$$

**Remark 3-1.** Note that $1/(N-n+1)<(n+1)/(2N-n+1)$.

**Proposition 3-7.** $\#P_1=N-n+1$ and $P_1=P_0$.

Proof. Suppose that $\#P_1>N-n+1$. Then, by Lemma 2(b), Propositions 3-4 and 3-6 we have

$$q-(2N-n+1)=\theta\sum_{j\notin P(0)\cup P_0}\omega(j)=q-(2N-n+1)-\sum_{j\in(P_1-P_0)}(1-\theta\omega(j))<q-(2N-n+1),$$

which is a contradiction. We have our result From proposition 3-6.

Summarizing these propositions, we obtain the following

**Theorem 3-1.** Suppose that

(i) N>n≥2 and (ii) there are vectors $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q\in X$ $(2N-n+1<q<\infty)$ satisfying

$$\sum_{j=1}^{q}\delta_n(\boldsymbol{a}_j,f)=2N-n+1.$$

If $\Omega<1$, then (a) $\#X(0)=N$ and (b) there is a subset $P\subset Q$ satisfying

$$\#P=N-n+1,\ d(P)=1,\ \delta_n(\boldsymbol{a}_j,f)=1\ (j\in P) \text{ and } X(0)\cap\{\boldsymbol{a}_j\mid j\in P\}=\phi.$$

## 4 Extremal case II : $\Omega<1$ and $q=\infty$

Let $[f_1,\cdots,f_{n+1}]$, $X$, $X(0)$, $\omega$ and $\theta$ etc. be as in Section 1, 2 or 3. From Theorem A, it is easy to see that the set $\{\boldsymbol{a}\in X \mid \delta_n(\boldsymbol{a},f)>0\}$ is at most countable and

$$\sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a},f)\le 2N-n+1.$$

In this section we consider a holomorphic curve $f$ with an infinite number of vectors $\boldsymbol{a}_j\in X$ such that $\delta_n(\boldsymbol{a}_j,f)>0$ $(j=1,2,3,\cdots)$. We put

$$\boldsymbol{N}=\{1,2,3,\cdots\} \text{ (the set of positive integers)};\ Y=\{\boldsymbol{a}_j \mid j\in\boldsymbol{N}\};\ \mathcal{P}=\{P\subset\boldsymbol{N} \mid 0<\#P\le N+1\}$$

and for any non-empty, finite subset $P$ of $\boldsymbol{N}$, we use $V(P)$ and $d(P)$ as in Section 2.

**Definition 4-1** (see [9], p.144). We put

$$\mu=\min_{P\in\mathcal{P}} d(P)/\#P \quad\text{and}\quad \tau(j)=\mu\ (j\in\boldsymbol{N}).$$

Note that $\{d(P)/\#P \mid P\in\mathcal{P}\}$ is a finite set. We have the following ([9],p.144):

(4-a) $1/(N-n+1)\le\mu\le(n+1)/(N+1)$;　　(4-b) For any $P\in\mathcal{P}$, $\sum_{j\in P}\tau(j)\le d(P)$.

We denote by $P_0$ an element of $\mathcal{P}$ satisfying $\mu=d(P_0)/\#P_0$.

**Lemma 4-1.** $\sum_{j=1}^{\infty}\delta_n(\boldsymbol{a}_j,f)\le(n+1)/\mu$.

Proof. Let $q$ be an integer satisfying $Q=\{1,\cdots,q\}\supset P_0$ and $2N-n+1<q<\infty$. Then, by Corollary 2-3 we have the inequality

$$\sum_{j=1}^{q}\delta_n(\boldsymbol{a}_j,f)\le(n+1)/\mu$$

by (4-b). As $q$ can be taken as large as possible, by letting $q\to\infty$ we have our lemma.

From now on throughtout this section we suppose that

(i) $N>n\ge2$; (ii) there exists a subset $Y=\{\boldsymbol{a}_j \mid j\in\boldsymbol{N}\}$ of $X$ satisfying $\delta_n(\boldsymbol{a}_j,f)>0$ and

$$\sum_{j=1}^{\infty}\delta_n(\boldsymbol{a}_j,f)=2N-n+1$$

and (iii) $\Omega<1$.

Note that we obtain the inequality

$$\mu\le(n+1)/(2N-n+1) \tag{8}$$

from (ii) and Lemma 4-1. For any positive number $\varepsilon$ satisfying

$$0<\varepsilon<(N-n)(1-\Omega)/(N-n+1)(n+1) \tag{9}$$

there exists $p\in\boldsymbol{N}$ satisfying $\{1,2,\cdots,p\}\supset P_0$, $2N-n+1<p<\infty$ and

$$2N-n+1-\varepsilon<\sum_{j=1}^{p}\delta_n(\boldsymbol{a}_j,f). \tag{10}$$

For any integer $q$ not less that $p$, we put $Q=\{1,2,\cdots,q\}$. For this $Q$, we use $\theta_q$, $\omega_q$ and $\lambda_q$ instead of $\theta$, $\omega$ and $\lambda$ in Section 2 respectively. Note that

$$\lambda_q=\mu \tag{11}$$

since $Q\supset P_0$. Further we obtain the following inequalities from (3) and (10):

$$n+1-\varepsilon/\theta_q<\sum_{j=1}^{p}\omega_q(j)\delta_n(\boldsymbol{a}_j,f), \tag{12}$$

$$\sum_{j=1}^{p}(1-\theta_q\,\omega_q(j))(1-\delta_n(\boldsymbol{a}_j,f))<\varepsilon. \tag{13}$$

From now on we put $\varepsilon_1=\varepsilon/(1-\Omega)$.

**Proposition 4-1.** For any $\boldsymbol{a}\in X(0)$, $0<1-\Omega\leq\delta(\boldsymbol{a},f)\leq\delta_n(\boldsymbol{a},f)$.

We obtain this proposition as in Proposition 3-1.

**Proposition 4-2.** $X(0)\subset\{\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q\}$.

Proof. If there exists a vector $\boldsymbol{a}\in X(0)$ satisfying $\boldsymbol{a}\notin\{\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q\}$, then by Proposition 4-1, Theorem A and (10)

$$2N-n+1-\varepsilon<\sum_{j=1}^{q}\delta_n(\boldsymbol{a}_j,f)\leq 2N-n+1-\delta_n(\boldsymbol{a},f)\leq 2N-n+1-(1-\Omega)<2N-n+1-\varepsilon$$

as $p\leq q$ and $\varepsilon<1-\Omega$ from (9). This is a contradiction. We have this proposition.

We put $P(0)=\{j\in Q\mid \boldsymbol{a}_j\in X(0)\}$ and $d_q=\Sigma_{j=P(0)}\,\omega_q(j)$. Note that

$$\#P(0)\leq N \quad\text{and}\quad d_q\leq d(P(0))\leq n. \tag{14}$$

**Proposition 4-3.** $n-\varepsilon_1/\theta_q<d_q$.

Proof. From (12) and Corollary 2-1 (I) we have the inequality

$$n+1-\varepsilon/\theta_q<\sum_{j=1}^{q}\omega_q(j)\delta_n(\boldsymbol{a}_j,f)\leq d_q+1+(n-d_q)\Omega$$

from which we obtain $n-\varepsilon_1/\theta_q<d_q$ as $\Omega<1$.

**Proposition 4-4.** (a) $\#P(0)=N$ and (b) $\theta_q\leq(N+\varepsilon_1)/n$.

Proof. From (14), Proposition 4-3, the definition of $d_q$, Lemma 2-2 (a) and Note 1 we have

$$N-\varepsilon_1\leq\theta_q(n-\varepsilon_1/\theta_q)<\theta_q d_q=\theta_q\sum_{j\in P(0)}\omega_q(j)\leq\#P(0)\leq N$$

from which we obtain that $N-\varepsilon_1\leq\#P(0)\leq N$ and $\theta_q(n-\varepsilon_1/\theta_q)<N$. As $0<\varepsilon_1<1$ from (9), we have (a) of this proposition. Next, we obtain (b) from the second inequality.

**Corollary 4-1.** $\theta_q\lambda_q<1$.

Proof. From (8), (11) and Proposition 4-4 we have by (9)

$$\theta_q\lambda_q<\{(N+\varepsilon_1/n\}\{(n+1)/(2N-n+1)\}<1.$$

Put $P_1=\{j\in Q\mid \theta_q\lambda_q(j)<1; j\notin P(0)\}$.

**Proposition 4-5.** $N-n+1\leq\#P_1$.

Proof. From Lemma 2-2 (b) and Proposition 4-3, we have the inequality

$$\begin{aligned} q-(2N-n+1) &=\theta_q\{\sum_{j=1}^{q}\omega_q(j)-n-1\}\\ &=\theta_q\{\sum_{j\in P(0)}\omega_q(j)+\sum_{j\in P_1}\omega_q(j)+\sum_{j\notin P(0)\cup P_1}\omega_q(j)-n-1\}\\ &>\theta_q\{\sum_{j\in P_1}\omega_q(j)+\sum_{j\notin P(0)\cup P_1}\omega_q(j)-(1+\varepsilon_1/\theta_q)\}\end{aligned}$$

and by (4-a), (11) and Remark 2-2

$$\geq\theta_q\#P_1/(N-n+1)+q-\#P(0)-\#P_1-\theta-\varepsilon_1,$$

from which and Proposition 4-4 we obtain the inequality

$$\#P_1(1-\frac{\theta_q}{N-n+1})>N-n+1-\theta_q-\varepsilon_1,$$

which reduces to the inequality

$$\#P_1(N-n+1-\theta_q)>(N-n+1)(N-n+1-\theta_q-\varepsilon_1).$$

Since

$$N-n+1-\theta_q>N-n+1-(N+\varepsilon_1)/n>0$$

by Proposition 4-4 and (9), we have

$$\begin{aligned}\#P_1 &> (N-n+1)(1-\varepsilon_1/(N-n+1-\theta_q))\\ &> (N-n+1)(1-\varepsilon_1/(N-n+1-(2N-n+1)/(n+1)))\\ &= (N-n+1)(1-(n+1)\varepsilon_1/(N-n)(n-1)) > N-n\end{aligned}$$

by Lemma 2-2 (c) and (9). This means that $\#P_1 \geq N-n+1$.

**Proposition 4-6.** $\#P_0=N-n+1$, $d(P_0)=1$ and $\theta_q=N/n$.

Proof. In this case, by the definition of $\theta_q$ and the choice of $P_0$, there exists a set $P$ satisfying

$$P_0 \subset P,\quad 1 \leq d(P) \leq n \quad \text{and} \quad \theta_q=(2N-n+1-\#P)/(n+1-d(P)). \tag{15}$$

By Proposition 4-4(b), Lemma 2-1 and (15)

$$\begin{aligned}(*)\qquad 0 &> \theta_q-(N+\varepsilon_1)/n=\theta_q-N/n-\varepsilon_1/n\\ &=\{(N-n)(n-1)+Nd(P)-n\#P\}/\{n(n+1-d(P))\}-\varepsilon_1/n\\ &\geq (N-n)(d(P)-1)/\{n(n+1-d(P))\}-\varepsilon_1/n.\end{aligned}$$

Suppose that $d(P)\geq 2$. Then, from $(*)$ we have the inequality $\varepsilon_1/n>(N-n)/n(n-1)$, from which we have $\varepsilon_1>(N-n)/(n-1)$, which contradicts (9). This means that $d(P)$ must be equal to 1.

Further, as $d(P)=1$ we have from Proposition 4-4(b), (15) and Note 1

$$(N+\varepsilon_1)/n>\theta_q=(2N-n+1-\#P)/n \geq N/n$$

so that $N-\varepsilon_1<\#P\leq N-n+1$. As $\varepsilon_1<1$ we have that $\#P=N-n+1$, so we obtain from (15) that $\theta_q=N/n$ and by Lemma 2(d), (4-a) and (11)

$$1=d(P)\geq \sum_{j\in P}\omega_q(j)\geq (N-n+1)\lambda_q \geq 1$$

since $\omega_q(j)\geq\lambda_q$ (see Remark 2-2), so that we have $\lambda_q=1/(N-n+1)=\omega_q(j)\ (j\in P)$.

By the choice of $P_0$, $1\leq d(P_0)\leq d(P)=1$ and we have $d(P_0)=1$ and $\#P_0=N-n+1$.

**Corollary 4-2.** $\lambda_q=\mu=1/(N-n+1)=\omega_q(j)\ \ (j\in P_0)$.

**Proposition 4-7.** $P_1=P_0$ and $d_q=n$.

Proof. As $\theta_q=N/n$ (Proposition 4-6) and $\omega_q(j)=1/(N-n+1)\ \ (j\in P_0)$ (Corollary 4-2),

$$\theta_q\,\omega_q(j)=N/n\,(N-n+1)<1\ \ (j\in P_0). \tag{16}$$

Next, we prove that $P_0\cap P(0)=\phi$. Suppose to the contrary that $P_0\cap P(0)\neq\phi$. Then, as $d(P_0)=1$, we have $P_0\subset P(0)$. In this case by Propositions 4-3, 4-4, 4-6 and Lemma 2-2(a)

$$\begin{aligned}n-\varepsilon_1/\theta_q<d_q&=\sum_{j\in P(0)}\omega_q(j)=\sum_{j\in P_0}\omega_q(j)+\sum_{j\in P(0)-P_0}\omega_q(j)\\ &\leq 1+\#(P(0)-P_0)/\theta_q=1+(n-1)n/N,\end{aligned}$$

which reduces to the inequality $(N-n)(n-1)/n<\varepsilon_1$. This contradicts (9). This implies that

$$P_0\cap P(0)=\phi. \tag{17}$$

(16) and (17) mean that $P_0\subset P_1$. Suppose that $\#P_1>N-n+1$. Then, by Lemma 2-2 (b), Lemma 2-2(a), Propositions 4-4 and 4-6

$$\begin{aligned}q-(2N-n+1) &=\theta_q\sum_{j\notin P(0)\cup P_0}\omega_q(j)-\theta_q(n-d_q)\\ &=q-(2N-n+1)-\sum_{j\in P_1-P_0}(1-\theta_q\,\omega_q(j))-\theta_q(n-d_q),\end{aligned}$$

that is to say,

$$\sum_{j\in P_1-P_0}(1-\theta_q\,\omega_q(j))+\theta_q(n-d_q)=0.$$

As $\theta_q\omega_q(j)<1$ for $j\in P_1$ and $d_q\le n$, it must hold that $P_1=P_0$ and $d_q=n$.

**Proposition 4-8.** For any $j\in P_0$, $\delta_n(\boldsymbol{a}_j,f)=1$.

Proof. Suppose to the contrary that

$$\min_{j\in P_0}\delta_n(\boldsymbol{a}_j,f)=\delta<1. \tag{18}$$

Now, for any positive number $\varepsilon_2$ satisfying

$$0<\varepsilon_2<\min\{(1-\frac{n}{n(N-n+1)})(1-\delta),\ \frac{(N-n)(1-\Omega)}{(N-n+1)(n+1)}\}, \tag{19}$$

we choose $s\in\boldsymbol{N}$ satisfying $S=\{1,\cdots,s\}\supset P_0$, $s\ge p$ and

$$2N-n+1-\varepsilon_2<\sum_{j=1}^{s}\delta_n(\boldsymbol{a}_j,f). \tag{20}$$

Note that $2N-n+1<s<\infty$. For this $S$ we use $\theta_s$, $\omega_s$ and $\lambda_s$ instead of $\theta$, $\omega$ and $\lambda$ in Section 2 respectively. Then the following relations hold from the results obtained above by the choice of $s$:

(4-c) $\lambda_s=\mu=1/(N-n+1)=\omega_s(j)$ for $j\in P_0$ (Corollary 4-2); (4-d) $\theta_s=N/n$ (Proposition 4-6).

By the equality (3) in the Proof of Lemma 2-4, Lemma 2-3, Remark 2-1 and (20) we obtain

$$\sum_{j=1}^{s}(1-\theta_s\omega_s(j))(1-\delta_n(\boldsymbol{a}_j,f))<\varepsilon_2$$

so that for any $j\in S$, $(1-\theta_s\omega_s(j))(1-\delta_n(\boldsymbol{a}_j,f))<\varepsilon_2$. By the definition of $\delta$, (4-c) and (4-d) given above $(1-N/n(N-n+1))(1-\delta)<\varepsilon_2$, which is a contradiction to (19). This means that $\delta_n(\boldsymbol{a}_j,f)=1$ $(j\in P_0)$.

Summarizing Propositions from 4-1 through 4-9 given above we obtain the following

**Theorem 4-1.** Suppose that (i) $N>n\ge 2$ and (ii) there an infinite number of vectors $\boldsymbol{a}_1,\boldsymbol{a}_2,\cdots\in X$ satisfying $\delta_n(\boldsymbol{a}_j,f)>0$ and

$$\sum_{j=1}^{\infty}\delta_n(\boldsymbol{a}_j,f)=2N-n+1.$$

If $\Omega<1$, then (I) $X(0)\subset\{\boldsymbol{a}_1,\boldsymbol{a}_2,\cdots\}$ and $\#X(0)=N$; (II) there is a subset $P$ of $\boldsymbol{N}$ satisfying

$$\#P=N-n+1,\quad d(P)=1,\quad \delta_n(\boldsymbol{a}_j,f)=1\ (j\in P)\quad\text{and}\quad X(0)\cap\{\boldsymbol{a}_j\mid j\in P\}=\phi.$$

## 5 Extremal case III : $n=2m$ and $q<\infty$

Let $f=[f_1,\cdots,f_{n+1}]$, $X$ etc. be as in Section 1 and let $q$ be an integer satisfying $2N-n+1<q<\infty$, $Q=\{1,\cdots,q\}$, $V(P)$, $d(P)$, $\mathcal{O}$, $\omega$, $\theta$ and $\lambda$ etc. as in Section 2.

From now on throughout this section we suppose that

(i) $N>n=2m$ $(m\in\boldsymbol{N})$ and (ii) there exist $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q\in X$ satisfying

$$\sum_{j=1}^{q}\delta_n(\boldsymbol{a}_j,f)=2N-n+1.$$

**Proposition 5-1.** $\lambda\le(n+1)/(2N-n+1)$.

In fact, we obtain this inequality from Corollary 2-3 and the assumption (ii) immediately.

Now, suppose that for some $n=2m$

$$\lambda=(n+1)/(2N-n+1). \tag{21}$$

Put $\mathcal{O}_1=\{P\in\mathcal{O}\mid d(P)/\#P=\lambda=(n+1)/(2N-n+1)\}$. Then, we have the followings.

**Proposition 5-2.** For $P \in \mathcal{O}_1$, (a) $\#P - d(P) < N - n$; (b) $\#P \leq N - m$; (c) $d(P) \leq m$.

Proof. (a) In general, we have the inequality

$$\#P - d(P) \leq N - n \tag{22}$$

(Lemma 2-1). As $P \in \mathcal{O}_1$

$$\#P - d(P) = \#P - \#P \cdot \frac{n+1}{2N-n+1} = \frac{2(N-n)}{2N-n+1} \cdot \#P. \tag{23}$$

If $\#P - d(P) = N - n$, we have from (23)

$$\#P = (2N - n + 1)/2 = N - m + 1/2,$$

which is a contradiction. We have (a) from (22).

(b) From (a) and (23) we have $\#P < N - m + 1/2$, so that $\#P \leq N - m$.

(c) As $P \in \mathcal{O}_1$,

$$d(P) = \frac{n+1}{2N-n+1} \#P \leq \frac{2m+1}{2(N-m)+1}(N-m) < m + \frac{1}{2},$$

so that we have $d(P) \leq m$.

**Proposition 5-3.** If $P_1, P_2 \in \mathcal{O}_1$, then $P_1 \cup P_2 \in \mathcal{O}_1$.

Proof. By the definition of $\mathcal{O}_1$,

$$d(P_1)/\#P_1 = d(P_2)/\#P_2 = (n+1)/(2N-n+1).$$

From Proposition 5-2 (a) we obtain

$$d(P_1) + d(P_2) = \lambda(\#P_1 + \#P_2) < \lambda(d(P_1) + d(P_2) + 2(N-n)),$$

which reduces to the inequality

$$d(P_1) + d(P_2) < 2(\lambda/(1-\lambda))(N-n) = n+1$$

since $\lambda = (n+1)/(2N-n+1)$. This means that

$$d(P_1) + d(P_2) \leq n. \tag{24}$$

As

$$d(P_1 \cup P_2) + d(P_1 \cap P_2) \leq d(P_1) + d(P_2). \tag{25}$$

(see p.68 in [3]), we have from (24) and (25) $d(P_1 \cup P_2) \leq n$, which implies that $\#(P_1 \cup P_2) \leq N$, so that $P_1 \cup P_2 \in \mathcal{O}$.

Next, by the definition of $\lambda$,

$$\lambda \leq d(P_1 \cup P_2)/\#(P_1 \cup P_2). \tag{26}$$

On the other hand, we have the inequality

$$\lambda \#(P_1 \cap P_2) \leq d(P_1 \cap P_2). \tag{27}$$

In fact, when $d(P_1 \cap P_2) > 0$, by the definition of $\lambda$ we have the inequality

$$\lambda \leq d(P_1 \cap P_2)/\#(P_1 \cap P_2),$$

which implies (27).

When $d(P_1 \cap P_2) = 0$, as $P_1 \cap P_2 = \phi$, we have $\#(P_1 \cap P_2) = 0$, so that (27) holds in this case.

We have (27) in any case. Therefore from (25) and (27) we have the inequality

$$\frac{d(P_1 \cup P_2)}{\#(P_1 \cup P_2)} \leq \frac{d(P_1) + d(P_2) - d(P_1 \cup P_2)}{\#P_1 + \#P_2 - \#(P_1 \cap P_2)} \leq \lambda. \tag{28}$$

(26) and (28) imply that

$$d(P_1 \cup P_2)/\#(P_1 \cup P_2) = \lambda = (n+1)/(2N-n+1),$$

which means that $P_1 \cup P_2 \in \mathcal{O}_1$.

Put

$$S_1 = \bigcup_{P \in \mathcal{O}_1} P. \tag{29}$$

Then, from Proposition 5-3 and (29) we have the following

**Proposition 5-4.** $S_1 \in \mathcal{O}_1$ and if $P \in \mathcal{O}_1$, then $P \subset S_1$.

Next, put $\mathcal{O}_2 = \{P \in \mathcal{O} \mid P - S_1 \neq \phi\}$ and $\lambda_1 = \min_{P \in \mathcal{O}_2} d(P)/\#P\ (\leq 1)$. Note that $\mathcal{O}_2 \neq \phi$. Then, we have the following

**Proposition 5-5.** $\lambda < \lambda_1$.

Proof. By the definitions of $\lambda$ and $\lambda_1$, we have $\lambda \leq \lambda_1$. Suppose that $\lambda = \lambda_1$. Then, there is a set $P \in \mathcal{O}_2$ satisfying $d(P)/\#P = \lambda$, which means that $P \in \mathcal{O}_1$.

On the other hand, as $P \in \mathcal{O}_2$, $P - S_1 \neq \phi$. This means that $P \cup S_1 \in \mathcal{O}_1$ by Proposition 5-3 and $P \cup S_1 \supsetneqq S_1$, which contradicts Proposition 5-4. This means that the inequality $\lambda < \lambda_1$ must hold.

**Definition 5-1.**

$$\tau(j) = \begin{cases} \lambda & (j \in S_1); \\ \lambda_1 & (j \in Q - S_1). \end{cases}$$

Then, we have the following

**Proposition 5-6.** For any $P \in \mathcal{O}$, $\sum_{j \in P} \tau(j) \leq d(P)$.

Proof. (i) When $P \subset S_1$, by the definition of $\lambda$ we have the inequality

$$\sum_{j \in P} \tau(j) = \lambda \cdot \#P \leq \frac{d(P)}{\#P} \cdot \#P = d(P).$$

(ii) When $P - S_1 \neq \phi$, by the definition of $\lambda_1$ and by Proposition 5-5, we have the inequality

$$\sum_{j \in P} \tau(j) \leq \lambda_1 \cdot \#P \leq \frac{d(P)}{\#P} \cdot \#P = d(P).$$

**Theorem 5-1.** Under the hypothesis (i) and (ii) given in the first part of this section, there exist at least $[(2N-n+1)/(n+1)] + 1$ vectors $\boldsymbol{a} \in \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q\}$ satisfying $\delta_n(\boldsymbol{a}, f) = 1$.

Proof. By Proposition 5-1, $\lambda \leq (n+1)/(2N-n+1)$. We want to prove that $\lambda < (n+1)/(2N-n+1)$ under the hypothesis (i) and (ii). Suppose to the contrary that the equality (21) holds for some $n = 2m$. Then, from Lemma 2-9 and the hypothesis (ii) we have the inequality for $\tau$ given in Definition 5-1:

$$\sum_{j=1}^{q} \tau(j)\delta_n(\boldsymbol{a}_j, f) \leq n+1 = \sum_{j=1}^{q} \frac{n+1}{2N-n+1} \delta_n(\boldsymbol{a}_j, f)$$

and so we have the inequality

$$\sum_{j \in Q - S_1} \left(\tau(j) - \frac{n+1}{2N-n+1}\right) \delta_n(\boldsymbol{a}_j, f) \leq 0.$$

As $\tau(j) - (n+1)/(2N-n+1) > 0$ for $j \in Q - S_1$ by Proposition 5-4 and the definition of $\tau(j)$,

$$\delta_n(\boldsymbol{a}_j, f) = 0 \quad (j \in Q - S_1).$$

This means that by the assumption (ii) and Proposition 5-2(b)

$$2N - n + 1 = \sum_{j \in S_1} \delta_n(\boldsymbol{a}_j, f) \leq \#S_1 \leq N - m,$$

which is a contradiction. This shows that (21) does not occur. That is to say, $\lambda$ must satisfy the inequality $\lambda<(n+1)/(2N-n+1)$. By the definition of $\lambda$, there exists a subset $P_o$ of $Q$ satisfying $\lambda=d(P_o)/\#P_o$. Then, by Remark 2-2 (a) and Corollary 2-2, we have our theorem as

$$\#P_o=d(P_o)/\lambda>d(P_o)(2N-n+1)/(n+1)\geq(2N-n+1)/(n+1).$$

## 6 Extremal case IV : $n=2m$ and $q=\infty$

Let $f=[f_1,\cdots,f_{n+1}]$, $X$ etc. be as in Sections 1,2 and 5. From Theorem A, it is easy to see that the set $\{a\in X \mid \delta_n(a,f)>0\}$ is at most countable and

$$\sum_{a\in X}\delta_n(a,f)\leq 2N-n+1.$$

In this part we consider holomorphic curves with an infinitc number of vectors $a_j\in X$ such that

$$\delta_n(a_j,f)>0 \quad (j=1,2,3,\cdots).$$

We put

$$N=\{1,2,3,\cdots\} \text{ (the set of positive integers)};\quad Y=\{a_j \mid j\in N\};\quad \mathcal{O}_\infty=\{P\subset N \mid 0<\#P\leq N+1\}$$

and for any finite and non-empty subset $P$ of $N$, we use $V(P)$ and $d(P)$ as in Section 2.

**Definition 6-1.** We put $\mu=\min_{P\in\mathcal{O}_\infty}d(P)/\#P$ and $\tau(j)=\mu$ $(j\in N)$.

Note that the set $\{d(P)/\#P \mid P\in\mathcal{O}_\infty\}$ is a finite set.

As in Section 3, we have the following

(6-a) $1/(N-n+1)\leq\mu\leq(n+1)/(N+1)$. (6-b) For any $P\in\mathcal{O}_\infty$, $\sum_{j\in P}\tau(j)\leq d(P)$.

(6-c) (Lemma 4-1) $\sum_{j=1}^{\infty}\delta_n(a_j,f)\leq(n+1)/\mu$.

From now on throughout this section we suppose that

(i) $N>n=2m$ $(m\in N)$;

(ii) there exists a countable subset $Y=\{a_1,a_2,a_3,\cdots\}\subset X$ satisfying $\delta_n(a_j,f)>0$ and

$$\sum_{j=1}^{\infty}\delta_n(a_j,f)=2N-n+1. \tag{30}$$

**Proposition 6-1.** $\mu\leq(n+1)/(2N-n+1)$.

We obtain this inequality from (6-c) and the assumption (ii) immediately.

Now, suppose that for some $n=2m$

$$\mu=(n+1)/(2N-n+1). \tag{31}$$

Put $\mathcal{P}=\{P\in\mathcal{O}_\infty \mid d(P)/\#P=(n+1)/(2N-n+1)\}$. Then, we can prove the following propositions as in Propositions 5-2 and 5-3.

**Proposition 6-2.** For $P\in\mathcal{P}$, (a) $\#P-d(p)<N-n$; (b) $\#P\leq N-m$; (c) $d(P)\leq m$.

**Proposition 6-3.** (a) If $P_1,P_2\in\mathcal{P}$, then $P_1\cup P_2\in\mathcal{P}$. (b) $\#\mathcal{P}<\infty$.

Proof of (b). Suppose to the contrary that $\mathcal{P}=\{P_1,P_2,\cdots,P_i,\cdots\}$, where $P_i\neq P_j$ if $i\neq j$. Then, $\#(\cup_{i=1}^{\infty}P_i)=\infty$, and so there exists a positive number $\nu$ satisfying

$$N+1<\#(\bigcup_{i=1}^{\nu}P_i). \tag{32}$$

On the other hand, by (a) of this proposition $\cup_{i=1}^{\nu}P_i\in\mathcal{P}\subset\mathcal{O}_\infty$ so that $\#(\cup_{i=1}^{\nu}P_i)\leq N+1$, which contradicts (32). This means that $\#\mathcal{P}<\infty$.

Put

$$T_1=\bigcup_{P\in\mathcal{P}}P. \tag{33}$$

Then, from Proposition 6-3 and (33) we easily have the following

**Proposition 6-4.** $T_1 \in \mathcal{P}$ and if $P \in \mathcal{P}$, then $P \subset T_1$.

Next, put $\mathcal{P}_1 = \{P \in \mathcal{O}_\infty \mid P - T_1 \neq \phi\}$ and $\lambda_1 = \min_{P \in \mathcal{P}_1} d(P)/\#P\ (\leq 1)$. Note that $\mathcal{P}_1 \neq \phi$. Then, we have the following as in Proposition 5-5.

**Proposition 6-5.** $\mu < \lambda_1$.

For any positive number $0 < \varepsilon < \min(1, \mu - \lambda_1)$, we choose a number $q \in \boldsymbol{N}$ satisfying $Q = \{1, 2, \cdots, q\} \supset T_1$, $2N - n + 1 < q < \infty$ and

$$2N - n + 1 - \varepsilon < \sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f). \tag{34}$$

For this $Q$, we use $\theta_q$, $\omega_q$, $\lambda_q$ and $\mathcal{O}_q$ instead of $\theta$, $\omega$, $\lambda$ and $\mathcal{O}$ in Section 2 respectively.

By the choice of $q$,

$$\mu = \lambda_q = (n+1)/(2N - n + 1). \tag{35}$$

**Definition 6-2.**

$$\tau_q(j) = \begin{cases} \mu & (j \in T_1); \\ \lambda_1 & (j \in Q - T_1). \end{cases}$$

Then, we have the following as in Proposition 5-6.

**Proposition 6-6.** For any $P \in \mathcal{O}_q$, $\Sigma_{j \in P} \tau_q(j) \leq d(P)$.

The following inequality holds as in Lemma 2-9.

**Proposition 6-7.** $\Sigma_{j=1}^{q} \tau_q(j) \delta_n(\boldsymbol{a}_j, f) \leq n+1$.

**Theorem 6-1.** Under the hypothesis (i) and (ii) in this section, there are at least

$$[(2N - n + 1)/(n+1)] + 1$$

vectors $\boldsymbol{a}$ in $Y$ satisfying $\delta_n(\boldsymbol{a}, f) = 1$.

Proof. There is an element $P_o$ of $\mathcal{O}_\infty$ satisfying $\mu = d(P_o)/\#P_o$. By the assuption (ii) of this theorem and (6-c) we obtain $\mu \leq (n+1)/(2N - n + 1)$. We want to prove that $\mu < (n+1)/(2N - n + 1)$. Suppose to the contrary that (31) holds. Here we use the same notations given between Propositions 6-5 and 6-6. From (34) and the euality (3) in the proof of Lemma 2-4 we have the inequality

$$n + 1 < \sum_{j=1}^{q} \omega_q(j) \delta_n(\boldsymbol{a}_j, f) + \varepsilon / \theta_q. \tag{36}$$

From Proposition 6-7 and (36) we have

$$\sum_{j=1}^{q} (\tau_q(j) - \omega_q(j)) \delta_n(\boldsymbol{a}_j, f) < \varepsilon / \theta_q.$$

As

$$\omega_q(j) = \lambda_q = (n+1)/(2N - n + 1) = 1/\theta_q \quad (j = 1, \cdots, q)$$

by (31),(35) and Remark 2-2(b), we have the inequality

$$(\lambda_1 - \mu) \sum_{j \in Q - T_1} \delta_n(\boldsymbol{a}_j, f) < \varepsilon / \theta_q.$$

By Proposition 6-2(b) and (34), we obtain the inequality

$$\theta_q (\lambda_1 - \mu)(N - m + 1 - \varepsilon) < \varepsilon,$$

so that we have

$$\lambda_1 - \mu < \theta_q (N - m)(\lambda_1 - \mu) < \varepsilon < \lambda_1 - \mu$$

since $\varepsilon < 1$ and $\theta_q (N - m) > 1$. This is a contradiction. This implies that the inequality $\mu < (n+1)/(2N - n + 1)$ must hold.

Now, we shall prove that $\delta_n(\boldsymbol{a}_j, f)=1$ $(j\in P_o)$. Suppose to the contrary that $\min_{j\in P_o}\delta_n(\boldsymbol{a}_j, f)=\delta<1$. For any positive number $\varepsilon_1$ satisfying

$$0<\varepsilon_1<(1-\mu(2N-n+1)/(n+1))(1-\delta), \tag{37}$$

we choose $q\in N$ satisfying $Q=\{1,2,\cdots,q\}\supset P_o$, $2N-n+1<q<\infty$ and

$$2N-n+1-\varepsilon_1<\sum_{j=1}^{q}\delta_n(\boldsymbol{a}_j, f). \tag{38}$$

For this $Q$, we use $\theta_q$, $\lambda_q$ and $\omega_q$ instead of $\theta$, $\lambda$ and $\omega$ respectively. By the choice of $q$, $\mu=\lambda_q$.

By Corollary 2-2(I), Remark 2-1 and (3) we obtain

$$\sum_{j=1}^{q}\delta_n(\boldsymbol{a}_j, f)+\sum_{j=1}^{q}(1-\theta_q\omega_q(j))(1-\delta_n(\boldsymbol{a}_j, f))\leq 2N-n+1. \tag{39}$$

From (38) and (39) we have

$$\sum_{j=1}^{q}(1-\theta_q\omega_q(j))(1-\delta_n(\boldsymbol{a}_j, f))<\varepsilon_1. \tag{40}$$

By Remark 2-2(a), for $j\in P_o$

$$\omega_q(j)=\mu=\lambda_q \tag{41}$$

and by Lemma 2-2(c)

$$\mu<(n+1)/(2N-n+1)\leq 1/\theta_q. \tag{42}$$

From (40), (41) and (42) for some $j\in P_o$

$$(1-\mu(2N-n+1)/(n+1))(1-\delta)\leq(1-\theta_q\mu)(1-\delta)<\varepsilon_1,$$

which contradicts (37). This means that $\delta$ must be equal to 1. As

$$(2N-n+1)/(n+1)\leq((2N-n+1)/(n+1))d(P_o)<\#P_o,$$

we have our theorem.

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