On Holomorphic Curves Extremal for the Defect Relation

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Let f be a transcendental holomorphic curve from $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ and X be a subset of $\boldsymbol{C}^n-\{\mathbf{0}\}$ in N-subgeneral position. In this paper we shall give some results on $\delta(\boldsymbol{a},f)$ ($\boldsymbol{a}\in X$) when $N>n=2m-1$ ($m\in\boldsymbol{N}$) and the defect relation for X with respect to f is extremal.

1. Introduction

Let $f=[f_1,\cdots,f_{n+1}]$ be a holomorphic curve from $\boldsymbol{C}$ into the n-dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1,\cdots,f_{n+1}):\boldsymbol{C}\to\boldsymbol{C}^{n+1}-\{\mathbf{0}\}$$

where n is a positive integer. We use the following notations:

$$\|f(z)\|=(|f_1(z)|^2+\cdots+|f_{n+1}(z)|^2)^{1/2}$$

and for a vector $\boldsymbol{a}=(a_1,\cdots,a_{n+1})\in\boldsymbol{C}^{n+1}-\{\mathbf{0}\}$

$$\|\boldsymbol{a}\|=(|a_1|^2+\cdots+|a_{n+1}|^2)^{1/2},\quad (\boldsymbol{a},f(z))=a_1f_1(z)+\cdots+a_{n+1}f_{n+1}(z),\quad (\boldsymbol{a},f)=a_1f_1+\cdots+a_{n+1}f_{n+1}.$$

The characteristic function $T(r,f)$ of f is defined as follows (see [10]):

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta-\log\|f(0)\|.$$

We suppose throughout the paper that f is transcendental and that f is linearly non-degenerate over $\boldsymbol{C}$; namely, $f_1,\cdots,f_{n+1}$ are linearly independent over $\boldsymbol{C}$. It is well-known that f is linearly non-degenerate if and only if the Wronskian $W=W(f_1,\cdots,f_{n+1})$ of $f_1,\cdots,f_{n+1}$ is not identically equal to zero.

For meromorphic functions in the complex plane we use the standard notation of Nevanlinna theory of meromorphic functions ([4],[6]).

For $\boldsymbol{a}\in\boldsymbol{C}^{n+1}-\{\mathbf{0}\}$, we write

$$m(r,\boldsymbol{a},f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|\boldsymbol{a}\|\|f(re^{i\theta})\|}{|(\boldsymbol{a},f(re^{i\theta}))|}d\theta\quad\text{and}\quad N(r,\boldsymbol{a},f)=N(r,\frac{1}{(\boldsymbol{a},f)}).$$

We call the quantity

$$\delta(\boldsymbol{a},f)=1-\limsup_{r\to\infty}\frac{N(r,\boldsymbol{a},f)}{T(r,f)}=\liminf_{r\to\infty}\frac{m(r,\boldsymbol{a},f)}{T(r,f)}$$

the deficiency of $\boldsymbol{a}$ with respect to f. It is known that $0\le\delta(\boldsymbol{a},f)\le 1$.

We denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$, possibly outside a set of r of finite linear measure.

Let X be a subset of $\boldsymbol{C}^{n+1}-\{\mathbf{0}\}$ in N-subgeneral position ; that is to say, (i) $\#X\ge N+1$ and (ii) any $N+1$ elements of X generate $\boldsymbol{C}^{n+1}$, where N is an integer satisfying $N\ge n$.

Cartan([1], $N=n$) and Nochka([7], $N>n$) gave the following

Theorem A (Defect relation). For any q elements $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q$ of X,

$$\sum_{j=1}^{q}\delta(\boldsymbol{a}_j,f)\le 2N-n+1$$

($2N-n+1\le q\le\infty$) (see also [2] or [3]).

We are interested in holomorphic curves f extremal for the defect relation:

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N-n+1. \tag{1}$$

In [8] we proved the following theorem.

Theorem B. Suppoes that there are vectors $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ in X such that (1) holds, where $2N-n+1<q\leq\infty$. If $(n+1, 2N-n+1)=1$, then there are at least

$$[(2N-n+1)/(n+1)]+1$$

vectors $\boldsymbol{a} \in \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q\}$ satisfying $\delta(\boldsymbol{a}, f)=1$.

In [9] we improved Theorem B. Namely, we changed the condition "$(n+1, 2N-n+1)=1$" in Theorem B into a weaker condition "$N>n$ and $n=2m$" with the same conclusion, where m is a positive integer.

The main purpose of this paper is to give some results on $\delta(\boldsymbol{a}, f)$ when the equality (1) holds under the following condition:

$$N>n=2m-1 \quad \text{and} \quad (N+1, m)=1 \quad (m\in\mathbf{N}).$$

2 Preliminaries

Let $f=[f_1,\cdots,f_{n+1}]$ and X etc. be as in Section 1. Let q be an integer satisfying $2N-n+1<q<\infty$ and put $Q=\{1,2,\cdots,q\}$. Let $\{\boldsymbol{a}_j \mid j\in Q\}$ be a family of vectors in X. For a non-empty subset P of Q, we denote

$$V(P) = \text{the vector space spanned by } \{\boldsymbol{a}_j \mid j\in P\}, \quad d(P)=\dim V(P)$$

and we put $\mathcal{O}=\{P\subset Q \mid 0<\#P\leq N+1\}$.

For $\{\boldsymbol{a}_j \mid j\in Q\}$, let $\omega: Q\rightarrow(0,1]$ be the Nochka weight function given in [3, p.72] and θ the reciprocal number of the Nochka constant given in [3, p.72]. Then, they have the following properties:

Lemma 1 (see [3], Theorem 2.11.4).

(a) $0<\omega(j)\theta\leq 1$ for all $j\in Q$; (b) $q-2N+n-1=\theta(\sum_{j=1}^{q}\omega(j)-n-1)$;

(c) $(N+1)/(n+1)\leq\theta\leq(2N-n+1)/(n+1)$; (d) If $P\in\mathcal{O}$, then $\sum_{j\in P}\omega(j)\leq d(P)$.

Lemma 2 (see pp.109-110 in [3]). Let f, $\{\boldsymbol{a}_j \mid j\in Q\}$ and ω be as above. Then,

[I] $\sum_{j=1}^{q}\omega(j)\delta(\boldsymbol{a}_j, f)\leq n+1$; [II] $\sum_{j=1}^{q}\delta(\boldsymbol{a}_j, f)\leq 2N-n+1$.

Lemma 3. Suppose that there exists a function $\sigma: Q\rightarrow(0,1]$ which satisfies the following condition $(*)$:

$(*)$ For any $P\in\mathcal{O}$, $\sum_{j\in P}\sigma(j)\leq d(P)$.

Then, for any $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q\in X$ we have the following inequalities:

(I) $\sum_{j=1}^{q}\sigma(j)m(r, \boldsymbol{a}_j, f)\leq(n+1)T(r,f)-N(r, 1/W)+S(r,f)$;

(II) $\sum_{j=1}^{q}\sigma(j)\delta(\boldsymbol{a}_j, f)\leq n+1$.

Proof. (I) Due to the assumption $(*)$ and Proposition 1 ([8]) we can prove this lemma as in the case of Theorem 1 in [8].

(II) It is quite easy to obtain the second inequality from (I) as usual.

Definition 1 ([8], Definition 1). We put

$$\lambda=\min_{P\in\mathcal{O}} d(P)/\#P \quad \text{and} \quad \sigma(j)=\lambda \quad (j\in Q).$$

These λ and σ have the following properties.

Proposition 1 ([8], Proposition 2).

(a) $1/(N-n+1)\leq\lambda\leq(n+1)/(N+1)$; (b) For any $P\in\mathcal{O}$, $\sum_{j\in P}\sigma(j)\leq d(P)$.

By Proposition 1 and Lemma 3, we obtain the following

Lemma 4 ([8], Theorem 1 and Corollary 2). For $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q\in X$,

(I) $\sum_{j=1}^{q}m(r, \boldsymbol{a}_j, f)\leq\frac{n+1}{\lambda}T(r,f)-\frac{1}{\lambda}N(r, \frac{1}{W})+S(r,f)$;

(II) $\sum_{j=1}^{q}\delta(\boldsymbol{a}_j, f)\leq\min(2N-n+1, (n+1)/\lambda)$.

3 Extremal case for the defect relation, I : $q<\infty$

Let f, X etc. be as in Section 1 or 2 and q an integer satisfying $2N-n+1<q<\infty$. From now on throughout this section we suppose that

(i) $N>n=2m-1$ and $(N+1, m)=1$ $(m\in \boldsymbol{N})$;

(ii) there exist $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q \in X$ satisfying $\delta(\boldsymbol{a}_j, f)>0$ $(j=1, \cdots, q)$ and

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f)=2N-n+1.$$

We note that $n=2m-1$ implies $(n+1)/(2N-n+1)=m/(N-m+1)$.

Proposition 2. $\lambda \leq m/(N-m+1)$.

In fact, we obtain this inequality from Lemma 4 (II) and the assumption (ii) immediately.

Here we give a remark.

Remark 1. (a) If $\lambda < m/(N-m+1)$, then $\lambda=\min_{1\leq j\leq q} \omega(j)$ and $\theta\omega(j)<1$ $(j\in P_o)$ for an element $P_o\in \mathcal{O}$ satisfying $\lambda=d(P_o)/\#P_o$.

(b) If $\lambda=m/(N-m+1)$, then $\omega(j)=1/\theta=\lambda$ $(j=1,\cdots,q)$.

In fact, the first assertion of (a) is given in the proof of Proposition 2.4.4([3],p.68) and by the definition of $\omega(j)$ ([3], p.72.). For the second assertion of (a), as $\omega(j)=\lambda$ $(j\in P_o)$ and $m/(N-m+1)\leq 1/\theta$, we have the conclusion.

(b) See the definition of $\omega(j)$ ([3],p.72).

(I) **The case when** $\lambda<m/(N-m+1)$.

By Proposition 1(a) we have $m\geq 2$, so that $n=2m-1\geq 3$. By Theorem 2 in [8] there are at least $[(2N-n+1)/(n+1)]+1$ vectors $\boldsymbol{a}\in\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q\}$ satisfying $\delta(\boldsymbol{a}, f)=1$.

(II) **The case when** $\lambda=m/(N-m+1)$.

We note that $\omega(j)=\lambda$ $(j\in Q)$ by Remark 1(b). Put

$$\mathcal{O}_0=\{P\in\mathcal{O} \mid d(P)/\#P=\lambda=m/(N-m+1)\}.$$

It is easy to see that $\mathcal{O}_0$ is non-empty and finite.

Proposition 3. For any $P\in\mathcal{O}_0$, $\#P=N-m+1$ and $d(P)=m$.

Proof. Let P be in $\mathcal{O}_0$. Then, $\#P=(N-m+1)d(P)/m$ and so we have the inequality

$$\#P-d(P)=d(P)(N-n)/m\leq N-n$$

by (2.4.3) in [3], p.68 and $n=2m-1$. This implies that $d(P)\leq m$ and $\#P\leq N-m+1$. By the definition of λ and the assumption $(N+1, m)=1$, we have the conclusion:

$$\#P=N-m+1 \quad \text{and} \quad d(P)=m.$$

Proposition 4. $\#\mathcal{O}_0\geq 2$.

Proof. Suppose to the contrary that $\#\mathcal{O}_0=1$. Let $\mathcal{O}_0=\{P_0\}$. Then, by Proposition 3 we have $\#P_0=N-m+1$ and $d(P_0)=m$.

Let $\mathcal{O}_1=\{P\in\mathcal{O} \mid P-P_0\neq\phi\}$. Then $\mathcal{O}_1\neq\phi$. In fact, if $\mathcal{O}_1=\phi$, any $P\in\mathcal{O}$ is a subset of P_0, so that $q=\#Q=\#P_0=N-m+1<2N-n+1<q$, which is a contradiction. Put $\lambda_1=\min_{P\in\mathcal{O}_1} d(P)/\#P$. Then, we have

$$\lambda<\lambda_1. \tag{2}$$

In fact, the inequality $\lambda\leq\lambda_1$ holds by the definition of λ. Suppose that $\lambda=\lambda_1$. Then there exists an element $P_1\in\mathcal{O}_1$ satisfying

$$d(P_1)/\#P_1=m/(N-m+1)=\lambda.$$

This means that $P_1\in\mathcal{O}_0$ and $P_1\neq P_0$, which is a contradiction. We have (2). Now put

$$\sigma(j)=\begin{cases}\lambda & (j\in P_0);\\ \lambda_1 & (j\in Q-P_0).\end{cases}$$

Then, the function $\sigma: Q \to (0, 1]$ satisfies the condition $(*)$ in Lemma 3. In fact, for any element P of $\mathcal{O}$,

(a) when $P\subset P_0$, $\sum_{j\in P}\sigma(j)=\lambda\,\#P\le(d(P)/\#P)\,\#P=d(P)$;

(b) when $P-P_0\ne\phi$, $\sum_{j\in P}\sigma(j)\le\lambda_1\#P\le(d(P)/\#P)\,\#P=d(P)$.

By Lemma 3 and the assumption (ii), we obtain the inequality

$$\sum_{j=1}^{q}\sigma(j)\delta(a_j,f)\le n+1=\sum_{j=1}^{q}\lambda\delta(a_j,f)$$

from which we obtain the inequality

$$0<(\lambda_1-\lambda)\sum_{j\in Q-P_0}^{q}\delta(a_j,f)=\sum_{j\in Q-P_0}(\sigma(j)-\lambda)\delta(a_j,f)\le 0.$$

This is a contradiction. This implies that $\#\mathcal{O}_0\ge 2$.

Proposition 5. Let P_1 and P_2 be in $\mathcal{O}_0$. Then, $P_1=P_2$ or $P_1\cap P_2=\phi$.

Proof. Suppose that $P_1\cap P_2\ne\phi$. Then, from the inequality

$$d(P_1\cup P_2)+d(P_1\cap P_2)\le d(P_1)+d(P_2) \tag{3}$$

(see [3], p.68) and by Proposition 3 we obtain the inequality $d(P_1\cup P_2)\le 2m-1=n$, which implies that $\#(P_1\cup P_2)\le N$ so that $P_1\cup P_2\in\mathcal{O}$.

Next, by the definition of λ, we have the inequality

$$\lambda\,\#(P_1\cap P_2)\le d(P_1\cap P_2) \tag{4}$$

since by the definition of λ we obtain the inequality $\lambda\le d(P_1\cap P_2)/\#(P_1\cap P_2)$.

We note that $P_1\cap P_2\in\mathcal{O}$ since $0<\#(P_1\cap P_2)\le N-m+1\le N$.

From the definition of λ, (3) and (4) we have the inequality

$$\lambda\le\frac{d(P_1\cup P_2)}{\#(P_1\cup P_2)}\le\frac{d(P_1)+d(P_2)-d(P_1\cap P_2)}{\#P_1+\#P_2-\#(P_1\cap P_2)}\le\lambda,$$

which implies that

$$d(P_1\cap P_2)\,/\,\#(P_1\cap P_2)=\lambda=m/(N-m+1),$$

so that $P_1\cap P_2\in\mathcal{O}_0$. By Proposition 3 $\#P_1=\#P_2=\#(P_1\cap P_2)=N-m+1$, which implies that $P_1=P_2$.

Put $Q_o=\cup_{P_j\in\mathcal{O}_0}P_j$. Then, by Propositions 3 and 5 for a positive integer a_o, $\#Q_o=a_o(N-m+1)$.

Proposition 6. $Q=Q_o$.

Proof. Suppose that $Q_o\subsetneqq Q$. Put $\mathcal{O}_2=\{P\in\mathcal{O}\mid P-Q_o\ne\phi\}$. Then, $\mathcal{O}_2$ is a non-empty, finite set.

Put $\lambda_2=\min_{P\in\mathcal{O}_2}d(P)/\#P$. Then, we have that

$$\lambda<\lambda_2. \tag{5}$$

In fact the inequality $\lambda\le\lambda_2$ holds by the definition of λ. Suppose that $\lambda=\lambda_2$. Then, there exists an element $P\in\mathcal{O}_2$ satisfying

$$d(P)/\#P=\lambda=m/(N-m+1),$$

which implies that $P\in\mathcal{O}_0$. This is a contradiction. We have (5). We define $\sigma_2: Q\to(0, 1]$ by

$$\sigma_2(j) = \begin{cases} \lambda & (j \in Q_0); \\ \lambda_2 & (j \in Q - Q_0). \end{cases}$$

Then, this function σ_2 satisfies the condition $(*)$ of Lemma 3. In fact, let P be any element of $\mathcal{O}$.

(a) When $P \subset Q_o$, $\Sigma_{j \in P}\, \sigma_2(j) = \lambda\ \#P \leq (d(P)/\#P)\ \#P = d(P)$;

(b) When $P - Q_o \neq \phi$, $\Sigma_{j \in P}\, \sigma_2(j) \leq \lambda_2 \#P \leq (d(P)/\#P)\ \#P = d(P)$.

By Lemma 3 and the assumption (ii) we have the inequality

$$\sum_{j=1}^{q} \sigma_2(j)\,\delta(\boldsymbol{a}_j, f) \leq n+1 = \sum_{j=1}^{q} \lambda\,\delta(\boldsymbol{a}_j, f),$$

which reduces to

$$0 < (\lambda_2 - \lambda) \sum_{j \in Q - Q_o} \delta(\boldsymbol{a}_j, f) = \sum_{j \in Q - Q_o} (\sigma_2(j) - \lambda)\,\delta(\boldsymbol{a}_j, f) \leq 0,$$

which is a contradiction. This implies that $Q = Q_o$ must hold.

Summarizing the results obtained above in this section, we have the following

Theorem 1. Suppose that

(i) $N > n = 2m-1$ and $(N+1, m) = 1$ $(m \in \boldsymbol{N})$;

(ii) there exist $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q \in X$ $(2N-n+1 < q < \infty)$ satisfying $\delta(\boldsymbol{a}_j, f) > 0$ $(j = 1, \cdots, q)$ and

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N - n + 1.$$

Then, either (I) of (II) given below holds:

(I) There are at least $[(2N-n+1)/(n+1)] + 1$ vectors $\boldsymbol{a} \in \{\boldsymbol{a}_j \mid j \in Q\}$ satisfying $\delta(\boldsymbol{a}, f) = 1$.

(II) q is divisible by $N-m+1$ and $Q = \cup_{\nu=1}^{p} P_\nu$, where $p = q/(N-m+1)$, the sets $P_1, \cdots, P_p$ are mutually disjoint and $\# P_\nu = N-m+1$, $d(P_\nu) = m$ $(\nu = 1, \cdots, p)$.

Example

When $m = 1$ and $N > n = 2m-1 = 1$, if

$$\delta(\boldsymbol{a}_j, f) > 0 \ (j = 1, \cdots, q) \quad \text{and} \quad \sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N,$$

then, $\lambda = 1/N$ and q is divisible by N. Further, put $p = q/N$, then Q is divided into p mutually disjoint subsets P_ν $(\nu = 1, \cdots, p)$ satisfying the followings:

(i) $\# P_\nu = N$ and $d(P_\nu) = 1$ $(\nu = 1, \cdots, p)$.

We suppose without loss of generality that

$$P_\nu = \{N(\nu - 1) + 1, \cdots, N\nu\} \quad (\nu = 1, \cdots, p).$$

Then,

(ii) $\Sigma_{\nu=1}^{p}\, \delta(\boldsymbol{a}_{N\nu}, f) = 2$.

In fact, noting that for $j, k \in P_\nu$ there exists a non-zero constant c such that $(\boldsymbol{a}_j, f) = c(\boldsymbol{a}_k, f)$ as $d(P_\nu) = 1$ $(\nu = 1, \cdots, p)$, so that $\delta(\boldsymbol{a}_j, f) = \delta(\boldsymbol{a}_k, f)$ and we immediately have (i) and (ii) from Theorem 1.

Remark 2. (a) When $p \geq 3$, we obtain $\theta = N$ by Remark 1(b).

(b) According to the example given in [5], there does not always exist $j \in Q$ such that $\delta(\boldsymbol{a}_j, f) = 1$ in this case.

4 Extremal case for the defect relation, II : $q=\infty$

Let f, X etc. be as in Section 1 or 2. Suppose that

$$\sum_{a\in X}\delta(a,f)=2N-n+1.$$

Then, it is easy to see that the set $Y=\{a\in X\mid \delta(a,f)>0\}$ is at most countable and

$$\sum_{a\in Y}\delta(a,f)=2N-n+1.$$

We treated the case when Y is a finite set in Section 3. In this section, we suppose that Y is not finite. Let $Y=\{a_j\mid j\in \boldsymbol{N}\}$. Then,

$$\sum_{j=1}^{\infty}\delta(a_j,f)=2N-n+1. \tag{6}$$

We put $\mathcal{O}_\infty=\{P\subset \boldsymbol{N}\mid 0<\#P\leq N+1\}$, where $\boldsymbol{N}$ is the set of positive integers, and for any non-empty finite subset P of $\boldsymbol{N}$ we use $V(P)$ and $d(P)$ as in Section 2.

Further we put

$$\mu=\min_{P\in\mathcal{O}_\infty} d(P)/\#P \quad\text{and}\quad \sigma(j)=\mu \quad (j\in\boldsymbol{N}).$$

Note that the set $\{d(P)/\#P\mid P\in\mathcal{O}_\infty\}$ is a finite set. We have the followings:

(4-a) $1/(N-n+1)\leq\mu\leq(n+1)/(N+1)$;

(4-b) For any $P\in\mathcal{O}_\infty$, $\Sigma_{j\in P}\,\sigma(j)\leq d(P)$;

(4-c) (the inequality (12) in [8]) $\Sigma_{j=1}^{\infty}\delta(a_j,f)\leq(n+1)/\mu$.

From (6) and (4-c), we have the following inequality;

$$\mu\leq(n+1)/(2N-n+1). \tag{7}$$

Under these circumstances, we have the following

Lemma 5. If $\mu<(n+1)/(2N-n+1)$, then there are at least $[(2N-n+1)/(n+1)]+1$ vectors $a\in Y$ satisfying $\delta(a,f)=1$. (As for the proof of this lemma, see the proof of Theorem 3 in [8], p.144-p.146.)

Next, we consider the case $\mu=(n+1)/(2N-n+1)$.

From now on throughout this section we suppose that

(i) $N>n=2m-1$ and $(N+1,m)=1$, where $m\in\boldsymbol{N}$;

(ii) $\Sigma_{j=1}^{\infty}\delta(a_j,f)=2N-n+1$.

Then, note that $\mu=(n+1)/(2N-n+1)=m/(N-m+1)$.

We put $\mathcal{F}_0=\{P\in\mathcal{O}_\infty\mid d(P)/\#P=m/(N-m+1)\}$, which is not empty in this case. As in the case of Proposition 3, we have the following

Proposition 7. For any $P\in\mathcal{F}_0$, $\#P=N-m+1$ and $d(P)=m$.

Proposition 8. $\#\mathcal{F}_0\geq 2$.

Proof. Suppose that $\#\mathcal{F}_0=1$. Let $\mathcal{F}_0=\{P_0\}$. Then, by Proposition 7, we have $\#P_0=N-m+1$ and $d(P_0)=m$. Let $\mathcal{F}_1=\{P\in\mathcal{O}_\infty\mid P-P_0\neq\phi\}$. Then, $\mathcal{F}_1\neq\phi$ since $\#P_0=N-m+1<\infty$. As the set $\{d(P)/\#P\mid P\in\mathcal{F}_1\}$ is finite, we put $\mu_1=\min_{P\in\mathcal{F}_1}d(P)/\#P$. Then,

$$\mu<\mu_1. \tag{8}$$

In fact, the inequality $\mu\leq\mu_1$ holds by the definition of μ. Suppose that $\mu=\mu_1$. Then, there exists an element $P_1\in\mathcal{F}_1$ satisfying $d(P_1)/\#P_1=\mu_1=\mu$, which means that $P_1\in\mathcal{F}_0$ and $P_1\neq P_0$. This is a contradiction to our assumption. We have (8). Now, let ε be any number satisfying

$$0<\varepsilon<1-\mu/\mu_1 \tag{9}$$

and $P_1\in\mathcal{F}_1$ satisfying $d(P_1)/\#P_1=\mu_1$. We choose a positive integer q satisfying

(4-*d*) $P_0\cup P_1\subset Q=\{1, 2,\cdots, q\}$; (4-*e*) $\sum_{j=1}^{q}\delta(\boldsymbol{a}_j, f)>2N-n+1-\varepsilon$

and $2N-n+1<q<\infty$. For this Q, we use θ_q, ω_q and λ_q instead of θ, ω and λ in Section 2 respectively. By the choice of q in (4-*d*), $\mu=\lambda_q$ and by Remark 1(b) for $j\in Q$

$$\omega_q(j)=\mu=m/(N-m+1) \tag{10}$$

and so we have from (4-*e*)

$$\sum_{j=1}^{q}\omega_q(j)\delta(\boldsymbol{a}_j, f)>n+1-\varepsilon\mu. \tag{11}$$

Put

$$\sigma(j)=\begin{cases}\mu & (j\in P_0)\\ \mu_1 & (j\in Q-P_0).\end{cases}$$

Then, the function $\sigma: Q\to(0, 1]$ satisfies the condition $(*)$ in Lemma 3 as in the case of Proposition 4. By Lemma 3, (10) and (11) we obtain the inequality

$$\sum_{j=1}^{q}\sigma(j)\delta(\boldsymbol{a}_j, f)\le n+1<\sum_{j=1}^{q}\mu\delta(\boldsymbol{a}_j, f)+\varepsilon\mu,$$

which reduces to the inequality

$$(\mu_1-\mu)\sum_{j\in Q-P_0}\delta(\boldsymbol{a}_j, f)<\varepsilon\mu. \tag{12}$$

As

$$\sum_{j\in Q-P_0}\delta(\boldsymbol{a}_j, f)>2N-n+1-\varepsilon-\#P_0=N-m+1-\varepsilon,$$

from (12) we have the inequality $(\mu_1-\mu)(N-m+1-\varepsilon)<\varepsilon\mu$, which reduces to the inequality

$$(\mu_1-\mu)(N-m)<\varepsilon\mu_1 \quad \text{or} \quad (1-\mu/\mu_1)(N-m)<\varepsilon,$$

which contradicts (9) as $N-m\ge 1$. This implies that $\#\mathcal{F}_0\ge 2$.

As in the case of Proposition 5, we have the following

Proposition 9. Let P_1 and P_2 be in $\mathcal{F}_0$. Then, $P_1=P_2$ or $P_1\cap P_2=\phi$.

Proposition 10. $\#\mathcal{F}_0=\infty$.

Proof. Suppose that $\#\mathcal{F}_0<\infty$ and $\mathcal{F}_0=\{P_1, P_2, \cdots, P_s\}$. Put

$$Q'=\bigcup_{j=1}^{s}P_j \quad \text{and} \quad \Delta=\sum_{j\in Q'}\delta(\boldsymbol{a}_j, f)(<2N-n+1).$$

Let $\mathcal{F}_2=\{P\in\mathcal{F}_0\mid P-Q'\neq\phi\}$. Then, $\mathcal{F}_2\neq\phi$ since $\#N=\infty$ and $\#Q'<\infty$. As the set $\{d(P)/\#P\mid P\in\mathcal{F}_2\}$ is finite, we put $\mu_2=\min_{P\in\mathcal{F}_2}d(P)/\#P$. Then,

$$\mu<\mu_2 \tag{13}$$

In fact, the inequality $\mu\le\mu_2$ holds by the definition of μ. Suppose that $\mu=\mu_2$. Then, there exists an element $F_0\in\mathcal{F}_2$ satisfying $d(F_0)/\#F_0=\mu_2=\mu$, which means that $F_0\in\mathcal{F}_0$ and $F_0-Q'\neq\phi$. This is a contradiction to our assumption. We have (13). Now, let ε be any number satisfying

$$0<\varepsilon<(1-\mu/\mu_2)(2N-n+1-\Delta) \tag{14}$$

and $F_1\in\mathcal{F}_2$ satisfying $d(F_1)/\#F_1=\mu_2$. We choose a positive integer t satisfying

(4-f) $Q'\cup F_1\subset T=\{1, 2,\cdots, t\}$; (4-g) $\sum_{j=1}^{t}\delta(\boldsymbol{a}_j, f)>2N-n+1-\varepsilon$

and $2N-n+1<t<\infty$. For this T, we use θ_t, ω_t and λ_t instead of θ, ω and λ in Section 2 respectively. By the choice of t in

(4-f) $\mu=\lambda_t$ and by Remark 1(b) for $j\in T$

$$\omega_t(j)=\mu=m/(N-n+1) \tag{15}$$

and so we have from (4-g)

$$\sum_{j=1}^{t}\omega_t(j)\delta(\boldsymbol{a}_j,f)>n+1-\varepsilon\mu. \tag{16}$$

Put

$$\sigma(j)=\begin{cases}\mu & (j\in Q')\\ \mu_2 & (j\in T-Q').\end{cases}$$

Then, the function $\sigma: T\to(0,1]$ satisfies the condition $(*)$ in Lemma 3 as in the case of Proposition 4. By Lemma 3, (15) and (16) we obtain the inequality

$$\sum_{j=1}^{t}\sigma(j)\delta(\boldsymbol{a}_j,f)\le n+1<\sum_{j=1}^{t}\mu\delta(\boldsymbol{a}_j,f)+\varepsilon\mu,$$

which reduces to the inequality

$$(\mu_2-\mu)\sum_{j\in T-Q'}\delta(\boldsymbol{a}_j,f)<\varepsilon\mu. \tag{17}$$

As

$$\sum_{j\in T-Q'}\delta(\boldsymbol{a}_j,f)>2N-n+1-\Delta-\varepsilon,$$

from (17) we have the inequality $(\mu_2-\mu)(2N-n+1-\Delta-\varepsilon)<\varepsilon\mu$, which reduces to the inequality

$$(\mu_2-\mu)(2N-n+1-\Delta)<\varepsilon\mu_2 \quad\text{or}\quad (1-\mu/\mu_2)(2N-n+1-\Delta)<\varepsilon,$$

which contradicts (14). This implies that $\#\mathcal{F}_0=\infty$.

Let $\mathcal{F}_0=\{P_1,P_2,\cdots,P_j,\cdots\}\mid P_j\in\mathcal{O}_\infty\}$ and put $\cup_{j=1}^{\infty}P_j=Q_o$. Then, we have the following

Proposition 11. $Q_o=N$.

Proof. Suppose to the contrary that $Q_o\subsetneqq N$. Put $\mathcal{F}_3=\{P\in\mathcal{O}_\infty\mid P-Q_o\neq\phi\}$, which is not empty by our assumption of this proof, and we put $\mu_3=\min_{P\in\mathcal{F}_3}d(P)/\#P$. Then,

$$\mu<\mu_3 \tag{18}$$

In fact, the inequality $\mu\le\mu_3$ holds by the definition of μ Suppose that $\mu=\mu_3$. Then, there exists an element $P\in\mathcal{F}_3$ satisfying $d(P)/\#P=\mu_3=\mu$, which means that $P\in\mathcal{F}_0$ and $P-Q_o\neq\phi$. This is a contradiction to our assumption. We have (18). Let q_o be the least number in $N-Q_o$ and let ε be any number satisfying

$$0<\varepsilon<(\mu_3/\mu-1)\delta(\boldsymbol{a}_{q_o},f) \tag{19}$$

We choose a positive integer u satisfying

(4-h) $P_1\subset U=\{1,2,\cdots,u\}$ and $u>q_o$; (4-i) $\sum_{j=1}^{u}\delta(\boldsymbol{a}_j,f)>2N-n+1-\varepsilon$

and $2N-n+1<u<\infty$. For this U, we use θ_u,ω_u and λ_u instead of θ,ω and λ in Section 2 respectively. By the choice of u in (4-h), $\mu=\lambda_u$ and by Remark 1 (b) for $j\in U$

$$\omega_u(j)=\mu=m/(N-m+1) \tag{20}$$

and so we have from (4-i)

$$\sum_{j=1}^{u}\omega_u(j)\delta(\boldsymbol{a}_j,f)>n+1-\varepsilon\mu. \tag{21}$$

Put

$$\sigma(j)=\begin{cases}\mu & (j\in Q_o\cap U)\\ \mu_3 & (j\in U-Q_o).\end{cases}$$

Then, the function $\sigma: U \to (0, 1]$ satisfies the condition $(*)$ in Lemma 3 as in the case of Proposition 4. By Lemma 3, (20) and (21) we obtain the inequality

$$\sum_{j=1}^{u}\sigma(j)\delta(a_j,f)\leq n+1<\sum_{j=1}^{u}\mu\,\delta(a_j,f)+\varepsilon\mu\,,$$

which reduces to the inequality $(\mu_3-\mu)\sum_{j\in U-Q_o}\delta(a_j,f)<\varepsilon\mu$, so that we have the inequality

$$(\mu_3-\mu)\delta(a_{q_o},f)<\varepsilon\mu \quad \text{or} \quad (\mu_3/\mu-1)\delta(a_{q_o},f)<\varepsilon\,,$$

which is a contradiction to (19). This means that $Q_o=\boldsymbol{N}$.

Summarizing Propositions from 7 through 11 obtained above in this section, we have the following

Theorem 2. Suppose that

(i) $N>n=2m-1$ and $(N+1, m)=1$, where $m\in N$;

(ii) there exist an infinite number of vectors a_j in X satisfying $\delta(a_j,f)>0$ $(j\in\boldsymbol{N})$ and

$$\sum_{j=1}^{\infty}\delta(a_j,f)=2N-n+1.$$

Then, either (I) or (II) given below holds:

(I) There are at least $[(2N-n+1)/(n+1)]+1$ vectors $a\in\{a_j\mid j\in\boldsymbol{N}\}$ satisfying $\delta(a_j,f)=1$.

(II) $N=\cup_{\nu=1}^{\infty}P_\nu$,

where $P_1, P_2,\cdots, P_\nu,\cdots$ are mutually disjoint and $\#P_\nu=N-m+1$, $d(P_\nu)=m$ $(\nu=1, 2, \cdots)$.

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