Comparison Theorems on Trajectory-Harps
and
the Ideal Boundary
of a Hadamard Kähler Manifold

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軌道ハープの比較定理
と
アダマール・ケーラー多様体の理想境界

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Contents

Introduction 1

Chapter 1. Riemannian manifolds 5

1. Notations and some fundamental results 5
2. Real space forms 13
3. Geodesics and parallel displacements 17
4. Circles 32

Chapter 2. Kähler magnetic fields 39

1. Kähler manifolds 39
2. Complex space forms 43
3. Magnetic fields 61

Chapter 3. Comparison theorems on magnetic Jacobi fields 69

1. Magnetic Jacobi fields 69
2. Magnetic Jacobi field on complex space forms 74
3. Comparison theorems on magnetic Jacobi fields 86

Chapter 4. Comparison theorems on trajectory-harps 99

1. Trajectory-harps 99
2. Trajectory-harps on a complex space form 105
3. Comparison theorems on string-lengths and string-cosines of trajectory-harps 117
4. Volumes of trajectory-balls 132
5. Comparison theorems on zenith angles and lengths of sector-arcs 139

Chapter 5. Ideal boundary of a Hadamard Kähler manifold 151
1. Hadamard manifold 151
2. Asymptotic behaviors of trajectories on a Hadamard manifold 164
3. Magnetic exponential maps on Hadamard manifolds 168
4. Trajectory-horn 174
5. Trajectories and its ideal boundary on a Hadamard manifold 180

Chapter A. Appendix 187
1. Dual linear maps 187
2. Inverse mapping theorem 188
3. Connectedness and compactness 192

Refrences 199

Index 201

Index of Theorems 203

Introduction

One of the typical research methods in the study of Riemannian manifolds is to investigate properties of geodesics on these manifolds. Many geometers obtained various results on manifolds by investigating some properties on geodesics. If we give some of the most fundamental results, we have Hopf-Rinow theorem, Rauch's comparison theorem on Jacobi fields, Toponogov's theorem on geodesic triangles and so on.

The supervisor of the author T. Adachi considered to progress this study on Riemannian manifolds with some additional geometric structures. If we study some properties on curves associated with a geometric structure on a Riemannian manifold, is it possible to get the feature of this structure and properties of the underlying manifold? In this context, he began to study Kähler manifolds by using trajectories for Kähler magnetic fields. We say a smooth curve of unit speed to be a trajectory for a Kähler magnetic field if its velocity vector and its acceleration vector form a complex line in the tangent space at each point and if the norm of acceleration vector is constant along this curve. Since geodesics are curves without accelerations, we may say that trajectories are generalizations of geodesics and are closely related with the complex structure of the underlying Kähler manifold. Though many studies on Kähler manifolds are based on complex geometry and not on real geometry, from curve-theoretic point of view, this idea on studying Kähler manifolds by making use of trajectories seems to be quite natural.

In this paper, we study Kähler manifolds of negative sectional curvature, more precisely, study the relationship between trajectories and ideal boundaries of Hadamard

Kähler manifolds, which are simply connected Kähler manifolds of non-positive curvature. Sectional curvatures of Riemannian manifolds give sufficiently precise information on these manifolds. When they are flat, that is, they have null sectional curvatures for all tangent 2-planes, they are quotients of Euclidean spaces. When a Riemannian manifold is of positive sectional curvature then it is compact and its fundamental group is finite by Myers Theorem. On the other hand, when a Riemannian manifold is of non-positive sectional curvature then its universal covering space is diffeomorphic to a Euclidean space by Cartan-Hadamard Theorem. Since this does not tells on topology of manifolds of non-positive curvature, and as their geodesic flows on their unit tangent bundles are of hyperbolic type and have many interesting properties, many geometers are interested in such manifolds. In 1973, Eberlein and O'Neill [**15**] introduced the notion of ideal boundaries of Hadamard manifolds. This boundary consists of asymptotic classes of geodesic rays (geodesic half-lines). Here, two geodesic rays of unit speed are said to be asymptotic if the distance between them is bounded. For a Hadamard manifold, we can define some different kinds of boundaries, this ideal boundary, analytic boundary and so on. This geometric boundary shows many properties of the interior part, the Hadamard manifold itself. For example, the Tit metric on the ideal boundary shows the flatness of the Hadamard manifold ([**9, 12**]). Therefore, we are interested in asymptotic behaviors of trajectories on a Hadamard Kähler manifold. Our main result shows that when the strength of a Kähler magnetic field is less than the square root of the absolute value of the upper bound of sectional curvatures of the underlying Kähler manifold its trajectories form the same ideal boundary.

We here give the organization of this paper. In Chapter 1, we introduce some notations, give some basic notions concerning Riemannian manifolds, and review some fundamental results. In Chapter 2, we describe Kähler manifolds, especially complex space forms which are simply connected Kähler manifolds of constant holomorphic sectional curvature. Also, we give definitions of trajectories for Kähler magnetic fields. We note that a closed 2-form on a Riemannian manifold is said to be a magnetic field because it can be regarded as a generalization of static magnetic field on a Euclidean 3-space. Though we use physical terms, important thing is that trajectories are curves showing a complex line spanned by velocity vector at each point. In order to study properties of trajectories, we study in Chapter 3 magnetic Jacobi fields which are obtained as differentials of variations of trajectories. Just like Rauch's comparison theorem on Jacobi fields plays quite an important role in the study of geodesics, comparison theorems on magnetic Jacobi fields play important role in the study of trajectories. We give explicit expressions on magnetic Jacobi fields on complex space forms and estimate norms of magnetic Jacobi fields on general Kähler manifolds by comparing them to those on complex space forms. The core parts of this paper are chapter 4 and chapter 5. To study the relationship between trajectories and geodesics we consider trajectory-harps in Chapter 4, which are variations of geodesics associated with trajectories. We regard a trajectory-segment and a geodesic segment joining two ends of the trajectory-segment as a correspondence of a geodesic triangle. By applying Rauch's comparison theorem, we give comparison theorems on trajectory-harps. We study string-lengths of trajectory-harps, which are lengths of geodesic segments joining origins and other points of trajectories, and zenith angles of trajectory-harps, which are lengths of curves formed by initial vectors of geodesic segments. We show that trajectory-harps on a Kähler manifold of large sectional curvature are "shorter" and "fatter" than those on a Kähler manifold of small sectional curvature. In Chapter 5, we study asymptotic behaviors of trajectory half-lines. By applying a comparison theorem on string-lengths of trajectory-harps, we can show under the condition on strengths of Kähler magnetic fields that every magnetic exponential map is a diffeomorphism and that every trajectory-half line converges to some point in the ideal boundary. In order to study the relationship between asymptotic behaviors of geodesics and those of trajectories, we introduce trajectory-horns, which are variations of trajectories associated with geodesics. Corresponding to comparison theorems on trajectory-harps, we give estimates on tube-lengths, lengths of trajectory segments joining the origin and other points of geodesics, and on embouchure angles, which are lengths of curves formed by initial vectors of trajectory segments. With the aid of these comparison theorems, we can show that trajectories have the same properties as of geodesics under the condition on strengths of Kähler magnetic fields.

Here, the author would like to express his sincere gratitude to his supervisor, Professor Toshiaki Adachi for his instructive advice and useful suggestions on his thesis. In this thesis, we use some results due to him without refering his papers. Without his help, this thesis would not have reached to its present form. Also, the author thanks to Professor Hideya Hashimoto (Meijo University) for his advice on preparing the author's paper [**22**] in *Current Developments in Differential Geometry and its Related Fields*. Special express the author's hearty thanks to the members of Ban BunTane Scholarship organization. He does not think that he could obtain his degree successfully in three years without their help. In addition, the author would like to thanks to his colleagues on their help and support. Finally, the author is indebted to his family for their continuous support and encouragement.

CHAPTER 1

Riemannian manifolds

In this chapter, we introduce some notations and give some notions which are quite familiar in the field of differential geometry. After recalling some fundamental results on Riemannian manifolds, we give explicit formulas of circles on real space forms.

1. Notations and some fundamental results

Let M be an m-dimensional C^∞–manifold. We denote by $\pi_M : TM \to M$ its tangent bundle. At each point $p \in M$ we take an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$ on the tangent space T_pM at p. For a local coordinate neighborhood $\big(U,\ \varphi = (x_1, \dots, x_m)\big)$ around $p \in M$ and i, j with $1 \le i, j \le m$, we define a function $g_{ij} : U \to \mathbb{R}$ by

$$g_{ij}(q) = g_q\Big(\Big(\frac{\partial}{\partial x_i}\Big)_q, \Big(\frac{\partial}{\partial x_j}\Big)_q\Big).$$

Here, the vector field $\dfrac{\partial}{\partial x_i}$ on U is defined by

$$\Big(\frac{\partial}{\partial x_i}\Big)_q f = \Big(\frac{\partial}{\partial x_i} f \circ \varphi^{-1}\Big)\big(\varphi(q)\big)$$

for each smooth function f on U. When such functions $g_{ij} : U \to \mathbb{R}$ are smooth for an arbitrary local coordinate neighborhood (U, φ), we call a family of inner products $g = \{g_p\}_{p \in M}$ a C^∞-Riemannian metric on M. In order to simplify the notation, we denote a Riemannian metric just by $\langle\ ,\ \rangle$. We say a pair $\big(M, \langle\ ,\ \rangle\big)$ of a smooth manifold and a Riemannian metric to be a Riemannian manifold.

A smooth map $\sigma : I \to M$ of an interval I is called a smooth curve on M. For a smooth curve $\sigma : I \to M$ and $a, b \in I$ with $a \le b$, we set

$$\mathrm{length}\big(\sigma|_{[a,b]}\big) = \int_a^b \|\sigma'(t)\| dt = \int_a^b \sqrt{\langle \sigma'(t), \sigma'(t)\rangle}\ dt$$

and call it the length of $\sigma|_{[a,b]}$. A continuous map $\sigma : [a,b] \to M$ is said to be a piecewise smooth curve from $\sigma(a)$ to $\sigma(b)$ if there is a division $a = t_0 < t_1 < \cdots < t_K = b$ of the interval $[a,b]$ such that $\sigma|_{(t_{i-1},t_i)}$ is a smooth curve and both $\lim_{t\downarrow t_{i-1}} \sigma'(t)$ and $\lim_{t\uparrow t_i} \sigma'(t)$ exist for $i = 1, \cdots, K$. For this curve, we set

$$\text{length}(\sigma) = \sum_{i=1}^{K} \int_{t_{i-1}}^{t_i} \|\sigma'(t)\| dt.$$

For a piecewise smooth curve $\sigma : [a,b] \to M$, we define a piecewise smooth curve $\sigma^{-1} : [a,b] \to M$ by $\sigma^{-1}(t) = \sigma(a+b-t)$. For two piecewise smooth curve $\sigma_1 : [a_1,b_1] \to M$ and $\sigma_2 : [a_2,b_2] \to M$ with $\sigma_1(b_1) = \sigma_2(a_1)$, we define a piecewise smooth curve $\sigma_1 \cdot \sigma_2 : [a_1, b_1 + b_2 - a_2] \to M$ by

$$\sigma_1 \cdot \sigma_2(t) = \begin{cases} \sigma_1(t), & \text{when } a_1 \leq t \leq b_1, \\ \sigma_2(t - b_1 + a_2), & \text{when } b_1 \leq t \leq b_1 + b_2 - a_2. \end{cases}$$

Then we have

$$\text{length}(\sigma^{-1}) = \text{length}(\sigma), \ \ \text{length}(\sigma_1 \cdot \sigma_2) = \text{length}(\sigma_1) + \text{length}(\sigma_2).$$

We call σ^{-1} the reversed curve of σ, and call $\sigma_1 \cdot \sigma_2$ the join of σ_1 and σ_2.

Given two points $p, q \in M$, we denote by $\mathcal{C}_{p,q}(M)$ the set of all piecewise smooth curves on M from p to q. When M is connected, we see $\mathcal{C}_{p,q} \neq \emptyset$ (see §A3). We define $d(p,q)$ by

$$d(p,q) = \inf\{\text{length}(\sigma) \bigm| \sigma \in \mathcal{C}_{p,q}(M)\}.$$

If $\sigma \in \mathcal{C}_{p,q}(M)$ then $\sigma^{-1} \in \mathcal{C}_{q,p}(M)$ and if $\sigma_1 \in \mathcal{C}_{p,q}(M), \sigma_2 \in \mathcal{C}_{q,r}(M)$ then $\sigma_1 \cdot \sigma_2 \in \mathcal{C}_{p,r}(M)$. Since we have a constant map $\sigma \in \mathcal{C}_{p,p}(M)$, we find that this map $d : M \times M \to \mathbb{R}$ satisfies

(1) $d(p,q) \geq 0$ and $d(p,q) = 0$ if and only if $p = q$,

(2) $d(p,q) = d(q,p)$,

(3) $d(p,q) + d(q,r) \geq d(p,r)$.

Therefore this d is a distance function on M. In case M is not connected, for two points p, q which are not contained in a same connected component, we set $d(p,q) = \infty$. On a Riemannian manifold we usually consider this distance function induced by the Riemannian metric.

We denote by $C^\infty(M)$ the set of all smooth functions on M, and by $\mathfrak{X}(M)$ the set of all C^∞–vector fields on M. We see that $\mathfrak{X}(M)$ is a vector space and is a $C^\infty(M)-$module.

Given a local coordinate neighborhood (U,φ), we see $\left\{\left(\frac{\partial}{\partial x_1}\right)_q, \cdots, \left(\frac{\partial}{\partial x_m}\right)_q\right\}$ is a basis of T_qM at each $q \in U$. Hence every vector field $X \in \mathfrak{X}(M)$ is expressed on U as $X|_U = \sum_{i=1}^{m} f_i \frac{\partial}{\partial x_i}$ with smooth functions $f_1, f_2, \cdots, f_m$ on U. For vector fields $X, Y \in \mathfrak{X}(M)$, we define their bracket $[X,Y] \in \mathfrak{X}(M)$ by

$$[X,Y]f = X(Yf) - Y(Xf)$$

for an arbitrary smooth function $f \in C^\infty(M)$. On a local coordinate neighborhood (U,φ), we note that this bracket satisfies

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0.$$

On a Riemannian manifold M, we have a unique bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X,Y) \mapsto \nabla_X Y \in \mathfrak{X}(M)$$

satisfying the following conditions for an arbitrary function $f \in C^\infty(M)$ and arbitrary vector fields $X, Y, Z \in \mathfrak{X}(M)$:

(1) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z, \quad \nabla_{fX} Y = f\nabla_X Y;$

(2) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z, \quad \nabla_X (fY) = (Xf)Y + f\nabla_X Y;$

(3) $\nabla_X Y - \nabla_Y X = [X,Y];$

(4) $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$

We call this bilinear map the *Riemannian connection* or the *Levi-Civita connection* of this Riemannian manifold M, and call $\nabla_X Y$ the covariant differentiation of Y by X.

Lemma 1.1 (Koszul formula). *For arbitrary vector fields $X, Y, Z \in \mathfrak{X}(M)$ we have*

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle \\ &\quad + \langle [X,Y],Z\rangle - \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle, \end{aligned} \tag{1.1}$$

Proof. By using the third and fourth properties of Riemannian connections we see that the right hand side turns to

$$\langle \nabla_X Y, Z\rangle + \langle \nabla_Y X, Z\rangle + \langle [X,Y], Z\rangle = \langle \nabla_X Y, Z\rangle + \langle \nabla_Y X, Z\rangle + \langle \nabla_X Y, Z\rangle - \langle \nabla_Y X, Z\rangle$$

and get the conclusion. $\square$

This lemma shows that Riemannian connection ∇ is defined by (1.1) (see Lemma A1). Thus we see that the Riemannian connection is uniquely determined.

We take a local coordinate neighborhood $\big(U, \varphi = (x_1, \dots, x_m)\big)$ of M. Since the Riemannian connection ∇ on M is determined by Lemma1.1, we find $(\nabla_X Y)|_U = \nabla_{X|_U} Y|_U$. We denote as

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{m} \Gamma_{i\,j}^{k} \frac{\partial}{\partial x_k}$$

and call the functions $\Gamma_{i\,j}^{k}$ on U the Christoffel's symbols. These satisfy $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$. If we express vector fields $X, Y \in \mathfrak{X}(M)$ as $X|_U = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x_i}$ and $Y|_U = \sum_{j=1}^{m} Y^j \frac{\partial}{\partial x_j}$, then we have

$$(\nabla_X Y)|_U = \sum_{k=1}^{m} \Big\{ \sum_{i=1}^{m} X^i \frac{\partial Y^k}{\partial x_i} + \sum_{i=1}^{m}\sum_{j=1}^{m} \Gamma_{i\,j}^{k} X^i Y^j \Big\} \frac{\partial}{\partial x_k}.$$

This expression shows that if $X_1(p) = X_2(p)$ then $(\nabla_{X_1} Y)(p) = (\nabla_{X_2} Y)(p)$. Thus we frequently denote it by $\nabla_v Y$ with $v = X_1(p) \in T_pM$. Let $\sigma : (-\epsilon, \epsilon) \to M$ be a smooth curve satisfying $\sigma(0) = p$ and $\frac{d\sigma}{dt}(0) = v$. If two vector fields Y_1, Y_2 satisfies $Y_1(\sigma(t)) = Y_2(\sigma(t))$ for $-\epsilon < t < \epsilon$, then the above expression shows that $\nabla_v Y_1 = \nabla_v Y_2$.

When N is a sub-manifold of a Riemannian manifold $\big(M, \langle\ ,\ \rangle\big)$ we can define a metric on N by considering $T_qN \subset T_qM$ at an arbitrary point $q \in N \subset M$ and by restricting $\langle\ ,\ \rangle$ on $\bigcup_{q\in N}\big(T_qN \times T_qN\big)$. We call this metric on N the *induced metric*, and call N with this induced metric a Riemannian sub-manifold of M. When real dimensions of N and M satisfy $\dim(M) = \dim(N) + 1$, we call N a real hypersurface of M. If we take a (local) unit normal vector field $\mathcal{N}$ on N in M, that is a unit vector field satisfying that $\mathcal{N}_q$ is orthogonal to all tangent vectors in T_qN at each point $q \in N$, then the Riemannian connections ${}^M\nabla$, ${}^N\nabla$ of M and N are related with each other as

$$ {}^N\nabla_X Y = {}^M\nabla_X Y - \langle {}^M\nabla_X Y, \mathcal{N}\rangle \mathcal{N} $$

for arbitrary vector fields $X, Y \in \mathfrak{X}(N)$, where we regard these vector fields as vector fields defined on $N(\subset M)$ which are tangent to N.

Generally, at an arbitrary point $q \in N$ we decompose T_qM orthogonally as $T_qM = T_qN \oplus (T_qN)^\perp$. We call $TN^\perp = \cup_{q\in N}(T_qN)^\perp$ the normal bundle of N in M. For $X, Y \in \mathfrak{X}(N)$, we consider the tangential component $({}^M\nabla_X Y)^\top$ of ${}^M\nabla_X Y$. Then $({}^M\nabla_\cdot \cdot)^\top$ satisfies four conditions of Riemannian connections on N. Hence we have $({}^M\nabla_X Y)^\top = {}^N\nabla_X Y$. We define $S : TN \times TN \to TN^\perp$ by $S(X,Y) = ({}^M\nabla_X Y)^\perp = {}^M\nabla_X Y - {}^N\nabla_X Y$, and call it the second fundamental form of N. As we have

$$ S(X,Y) - S(Y,X) = ({}^M\nabla_X Y - {}^N\nabla_X Y)^\perp = ([X,Y])^\perp = 0, $$

it is symmetric bilinear map. We call N is totally geodesic if S is the null map, that is, $S(X,Y) = 0$ for all $X, Y \in \mathfrak{X}(N)$.

We define the *Riemannian curvature tensor* $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ on M by

$$ R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, $$

with the Riemannian connection ∇. For vector fields $X, Y, Z, W \in \mathfrak{X}(M)$, the Riemannian curvature tensor satisfies following properties :

(1) $R(X,Y)Z = -R(Y,X)Z$;

(2) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$;

(3) $\langle R(X,Y)Z, W\rangle + \langle R(X,Y)W, Z\rangle = 0$;

(4) $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$.

Lemma 1.2. *For smooth functions $f \in C^\infty(M)$ and vector fields $X, Y, Z \in \mathfrak{X}(M)$, the Riemannian curvature tensor satisfies following properties :*

(1) $R(fX,Y)Z = fR(X,Y)Z,\ R(X,fY)Z = fR(X,Y)Z$;

(2) $R(X,Y)(fZ) = fR(X,Y)Z$.

Proof. (1) By the definition of the Riemannian curvature tensor, we have

$$R(fX,Y)Z = \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z.$$

Also, by properties (1), (2) of Riemannian connections, we have

$$\begin{aligned}
&\nabla_{fX}\nabla_Y Z = f\nabla_X\nabla_Y Z,\\
&\nabla_Y\nabla_{fX}Z = \nabla_Y(f\nabla_X Z) = (Yf)\nabla_X Z + f\nabla_Y\nabla_X Z,\\
&\nabla_{[fX,Y]}Z = \nabla_{f[X,Y]-(Yf)X}Z = f\nabla_{[X,Y]}Z - (Yf)\nabla_X Z.
\end{aligned}$$

Then we get

$$\begin{aligned}
R(fX,Y)Z &= f\nabla_X\nabla_Y Z - (Yf)\nabla_X Z - f\nabla_Y\nabla_X Z - f\nabla_{[X,Y]}Z + Y(f\nabla_X Z)\\
&= f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)\\
&= fR(X,Y)Z.
\end{aligned}$$

By use of the property (1) of curvature tensors, we have

$$R(X,fY)Z = -R(fY,X)Z = -fR(Y,X)Z = fR(X,Y)Z.$$

(2) By the definition of the Riemannian curvature tensor, we have

$$R(X,Y)(fZ) = \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ).$$

Also, by properties (1), (2) of Riemannian connections, we have

$$\begin{aligned}
\nabla_X\nabla_Y(fZ) &= \nabla_X\big((Yf)Z + f\nabla_Y Z\big)\\
&= (X(Yf))Z + (Yf)\nabla_X Z + (Xf)\nabla_Y Z + f\nabla_X\nabla_Y Z,\\
\nabla_Y\nabla_X(fZ) &= (Y(Xf))Z + (Xf)\nabla_Y Z + (Yf)\nabla_X Z + f\nabla_Y\nabla_X Z,\\
\nabla_{[X,Y]}(fZ) &= ([X,Y]f)Z + f\nabla_{[X,Y]}Z.
\end{aligned}$$

We hence get

$$\begin{aligned}
R(X,Y)(fZ) = &(X(Yf))Z + (Yf)\nabla_X Z + (Xf)\nabla_Y Z + f\nabla_X\nabla_Y Z\\
&- (Y(Xf))Z - (Xf)\nabla_Y Z - (Yf)\nabla_X Z - f\nabla_Y\nabla_X Z\\
&- (X(Yf))Z + (Y(Xf))Z - f\nabla_{[X,Y]}Z\\
= &f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)\\
= &fR(X,Y)Z.
\end{aligned}$$

We get the conclusion. $\square$

For smooth functions $f, g, h \in C^\infty(M)$, by Lemma1.2, we have $R(fX, gY)(hZ) = fghR(X,Y)Z$, hence we find that the curvature tensor is a $C^\infty(M)-$trilinear map. Hence for tangent vectors $u, v, w \in T_pM$ at an arbitrary point $p \in M$, we can define $R(u,v)w$.

For two linearly independent tangent vectors $v, w \in T_pM$ at $p \in M$, we set

$$\mathrm{Riem}(v,w) = \frac{\langle R(v,w)w, v\rangle}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2}$$

and call it the sectional curvature of the tangent plane spanned by v, w.

LEMMA 1.3. *Let $\langle\ ,\ \rangle$ be a Riemannian metric on a manifold M. For a positive constant λ we consider a new Riemannian metric $\langle\ ,\ \rangle' = \lambda^2\langle\ ,\ \rangle$. Then their Riemannian connections ∇, ∇' and sectional curvatures* Riem, Riem$'$ *have the following relations :*

(1) $\nabla'_X Y = \nabla_X Y$ *for arbitrary* $X, Y \in \mathfrak{X}(M)$;
(2) $\mathrm{Riem}'(v,w) = \lambda^{-2}\mathrm{Riem}(v,w)$ *for arbitrary* $v, w \in T_pM$ *at an arbitrary point* $p \in M$.

PROOF. (1) By Lemma 1.1, for arbitrary $X, Y, Z \in \mathfrak{X}(M)$ we have

$$\begin{aligned}
2\langle \nabla'_X Y, Z\rangle' &= X\langle Y,Z\rangle' + Y\langle X,Z\rangle' - Z\langle X,Y\rangle' \\
&\qquad + \langle [X,Y],Z\rangle' - \langle [X,Z],Y\rangle' - \langle [Y,Z],X\rangle' \\
&= \lambda^2 X\langle Y,Z\rangle + \lambda^2 Y\langle X,Z\rangle - \lambda^2 Z\langle X,Y\rangle \\
&\qquad + \lambda^2\langle [X,Y],Z\rangle - \lambda^2\langle [X,Z],Y\rangle - \lambda^2\langle [Y,Z],X\rangle \\
&= \lambda^2\big\{X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle \\
&\qquad + \langle [X,Y],Z\rangle - \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle\big\} \\
&= 2\lambda^2\langle \nabla_X Y, Z\rangle = 2\langle \nabla_X Y, Z\rangle'
\end{aligned}$$

Thus we have $\langle \nabla'_X Y, Z\rangle' = \langle \nabla_X Y, Z\rangle'$. It leads us to $\nabla'_X Y = \nabla_X Y$.

(2) By definition of sectional curvatures, for arbitrary $v, w \in T_pM$ we have

$$\begin{aligned}\mathrm{Riem}'(v,w) &= \frac{\langle R(v,w)w, v\rangle'}{\langle v,v\rangle'\langle w,w\rangle' - {\langle v,w\rangle'}^2} = \frac{\lambda^2\langle R(v,w)w,v\rangle}{\lambda^4\bigl\{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2\bigr\}} \\ &= \frac{\langle R(v,w)w,v\rangle}{\lambda^2\bigl\{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2\bigr\}} = \lambda^{-2}\mathrm{Riem}(v,w).\end{aligned}$$

Thus, we get the conclusion. □

At an arbitrary point $p \in M$, we take an orthonormal basis $\{e_1(p), \dots, e_m(p)\}$ on T_pM. We call a C^∞–differential m–form ω on M the *volume element*, if it satisfies

$$\omega_p\bigl(e_1(p), \dots, e_m(p)\bigr) = 1.$$

On a local coordinate neighborhood $\bigl(U, \varphi = (x_1, \dots, x_m)\bigr)$, we set a function G on U by $G = \det(g_{ij})$. Since the matrix (g_{ij}) is symmetric and positive definite, we see that G is a positive function. We then have

$$\omega = \sqrt{G}\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m$$

on U, where dx_i denotes the dual 1-form of $\dfrac{\partial}{\partial x_i}$ on U, that is, the 1-form satisfying $dx_i\Bigl(\dfrac{\partial}{\partial x_i}\Bigr) = 1$ and $dx_i\Bigl(\dfrac{\partial}{\partial x_j}\Bigr) = 0$ for $j \neq i$. Here, $\wedge$ denotes the wedge product, which is defined as

$$(\alpha\wedge\beta)(v_1, \dots, v_{k+\ell}) = \frac{1}{k!\ell!}\sum_{\tau\in\mathfrak{S}_{k+\ell}} \bigl(\mathrm{sgn}(\tau)\bigr)\alpha(v_{\tau(1)}, \dots, v_{\tau(k)})\beta(v_{\tau(k+1)}, \dots, v_{\tau(k+\ell)})$$

for a k-from α and a ℓ-form β, where $\mathrm{sgn}(\tau)$ denotes the signature of a permutation τ.

Let $(M, \langle\ ,\ \rangle)$ and $(M', \langle\ ,\ \rangle')$ be two Riemannian manifolds, and let $\varphi : M \to M'$ be a diffeomorphism. This map $\varphi : (M, \langle\ ,\ \rangle) \to (M', \langle\ ,\ \rangle')$ is called an isometry if its differential map $(d\varphi)_p : T_pM \to T_pM'$ keeps the inner product as

$$\langle v, w\rangle = \bigl\langle (d\varphi)_p(v), (df)_p(w)\bigr\rangle'$$

for arbitrary $v, w \in T_pM$ at each point $p \in M$.

2. Real space forms

When we study Riemannian manifolds, we may say that real space forms, which are standard spheres, Euclidean spaces and real hyperbolic spaces, are most basic manifolds. From the historical point of view, Euclid gave some postulates in his book "Elements". His fifth postulate is called the parallel postulate. It states that on a plane for an arbitrary line γ and an arbitrary point p which does not lie on γ there is a unique line which passes through p and that does not intersect γ. Negating this postulate Lobachevsky and some geometers independently gave "new" geometry, which are called non-Euclidean geometry. Such geometry were developed on hyperbolic planes and on standard 2-spheres.

[1] Euclidean spaces.

For a Euclidean space $\mathbb{R}^m = \{(x_1, x_2, \dots, x_m) \mid x_i \in \mathbb{R}\}$ we take its canonical inner product. That is, at an arbitrary point $p \in \mathbb{R}^m$, for arbitrary two tangent vectors $v = (v_1, v_2, \dots, v_m), w = (w_1, w_2, \dots, w_m) \in T_p\mathbb{R}^m \cong \mathbb{R}^m$, we set $\langle v, w\rangle = \sum_{i=1}^m v_i w_i$. The Riemannian connection associated with this metric is the ordinary differentiation. By using the coordinate $(x_1, x_2, \dots, x_m)$ on $\mathbb{R}^m$, we express two vector fields X, Y on $\mathbb{R}^m$ as $X_p = \displaystyle\sum_{i=1}^m a_i\big(\frac{\partial}{\partial x_i}\big)_p$, $Y_p = \displaystyle\sum_{j=1}^m b_j\big(\frac{\partial}{\partial x_j}\big)_p \in \mathfrak{X}(\mathbb{R}^m)$. Then the covariant differentiation of Y by X is given as

$$\nabla_X Y(p) = \sum_{i=1}^m \sum_{j=1}^m a_i(p) \frac{\partial b_j}{\partial x_i}(p) \Big(\frac{\partial}{\partial x_j}\Big)_p.$$

By direct computation we have $R(X, Y)Z = 0$ for arbitrary $X, Y, Z \in \mathfrak{X}(\mathbb{R}^m)$. Thus, sectional curvatures of tangent planes of a Euclidean space are zero. We say a Riemannian manifold to be flat if sectional curvatures of all tangent planes are zero. Thus, an Euclidean space is a typical example of flat manifolds. Another typical example is a flat torus T^m, which is a quotient manifold of $\mathbb{R}^m$.

[2] Standard spheres.

On a standard sphere

$$S^m[r] = \left\{x = (x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1} \;\middle|\; x_0^2 + x_1^2 + \cdots + x_m^2 = r^2\right\}$$

of radiu r, as it is a sub-manifold of $\mathbb{R}^{m+1}$, we take the metric induced by the canonical metric on $\mathbb{R}^{m+1}$. We denote by $\mathcal{N}$ the unit outward normal vector field on $S^m[r]$. That is, $\mathcal{N}_p = (1/r)p$ by regarding a point $p \in S^m[r] \subset \mathbb{R}^{m+1}$ as the position vector. This identification shows $\overline{\nabla}_X \mathcal{N} = (1/r)X$ for $X \in \mathfrak{X}(S^m[r])$, where $\overline{\nabla}$ is the Riemannian connection on $\mathbb{R}^{m+1}$. Thus, the Riemannian connections $\nabla, \overline{\nabla}$ on $S^m[r]$ and on $\mathbb{R}^{m+1}$ are related with each other as follows :

Lemma 1.4. *For arbitrary $X, Y \in \mathfrak{X}(S^m[r])$, regarding them as vector fields defined on $S^m(\subset \mathbb{R}^{m+1})$ tangent to S^m we have*

$$\nabla_X Y = \overline{\nabla}_X Y + \frac{1}{r}\,\langle X, Y\rangle \mathcal{N}.$$

Proof. As $\langle Y, \mathcal{N}\rangle = 0$, we have

$$0 = \langle \overline{\nabla}_X Y, \mathcal{N}\rangle + \langle Y, \overline{\nabla}_X \mathcal{N}\rangle = \langle \overline{\nabla}_X Y, \mathcal{N}\rangle + \frac{1}{r}\langle Y, X\rangle.$$

Since S^m is a Riemannian sub-manifold, we see $\nabla_X Y$ is obtained by removing the orthogonal component of $\overline{\nabla}_X Y$. Hence we have

$$\nabla_X Y = \overline{\nabla}_X Y - \langle \overline{\nabla}_X Y, \mathcal{N}\rangle \mathcal{N} = \overline{\nabla}_X Y + \frac{1}{r}\,\langle X, Y\rangle \mathcal{N}.$$

This completes the proof. □

Thus, we obtain

$$R(X,Y)Z = \frac{1}{r^2}\left\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\right\}$$

(see [**21**], for example). We therefore find that sectional curvatures of all tangent planes of $S^m[r]$ are $1/r^2$. Standard spheres are typical examples of positively curved Riemannian manifolds.

Since $S^m[1]$ is of constant sectional curvature 1 with respect to $\langle\ ,\ \rangle$, we have another way of getting a Riemannian manifold of constant sectional curvature $c\ (> 0)$.

On $S^m[1]$ we define a new Riemannian metric $\langle\ ,\ \rangle'$ by $\langle\ ,\ \rangle' = \dfrac{1}{c}\langle\ ,\ \rangle$. By Lemma1.3, we see that $S^m[1]$ with this new metric is of constant sectional curvature c. We shall denote this Riemannian manifold by $S^m(c)$. Trivially, the map $S^m[r] \ni p \mapsto (1/r)p \in S^m(1/r^2)$ is an isometry.

[3] Real hyperbolic spaces.

We take a real hyperbolic space

$$H^m[r] = \big\{x = (x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1} \bigm| -x_0^2 + x_1^2 + \cdots + x_m^2 = -r^2\big\}.$$

On $\mathbb{R}^{m+1}$, we define a bilinear form $⟪\ ,\ ⟫$ by

$$⟪x, y⟫ = -x_0y_0 + x_1y_1 + \cdots + x_my_m$$

for $x = (x_0, x_1, \dots, x_m),\ y = (y_0, y_1, \dots, y_m) \in \mathbb{R}^{m+1}$. At a point $p = (p_0, \dots, p_m) \in H^m[r]$, the tangent space at p is given as

$$T_pH^m[r] = \big\{(p, v) \in \{p\} \times \mathbb{R}^{m+1} \bigm| -p_0v_0 + p_1v_1 + \cdots + p_mv_m = 0\big\} \cong \mathbb{R}^m,$$

where we denote $v = (v_0, \dots, v_m)$. Therefore we have

$$\begin{aligned} ⟪v, v⟫ &= -v_0^2 + v_1^2 + \cdots + v_m^2 \\ &= -\Big(\frac{v_1p_1 + \cdots + v_mp_m}{p_0}\Big)^2 + v_1^2 + \cdots + v_m^2 \\ &\geq (v_1^2 + \cdots + v_m^2) - (v_1^2 + \cdots + v_m^2)\frac{p_1^2 + \cdots + p_m^2}{p_0^2} \\ &= (v_1^2 + \cdots + v_m^2)p_0^{-2} > 0 \end{aligned}$$

and find that the restriction of $⟪\ ,\ ⟫$ on to each tangent space of $H^m[r]$ is positive definite. Thus $⟪\ ,\ ⟫$ induces a Riemannian metric on $H^m[r]$. Though $⟪\ ,\ ⟫$ is not positive definite on $\mathbb{R}^{m+1}$(or more precisely it has signature $(1, m)$), we can define a connection $\overline{\nabla}$ by the relation (1.1). We denote by $\mathcal{N}$ an outward normal vector field of a real hyperbolic space $H^m[r]$ in $\mathbb{R}^{m+1}$ satisfying $⟪\mathcal{N}, \mathcal{N}⟫ = -1$. That is, we set $\mathcal{N}_p = (1/r)p$ by regarding a point $p \in H^m[r] \subset \mathbb{R}^{m+1}$ as the position vector. This identification shows $\nabla_X\mathcal{N} = (1/r)X$ for $X \in \mathfrak{X}(H^m[r])$. Thus, the Riemannian connection ∇ and the connection $\overline{\nabla}$ on $\mathbb{R}^{m+1}$ corresponding to the indefinit metric on $\mathbb{R}^{m+1}$ are related with each other as follows:

Lemma 1.5. *For arbitrary vector fields $X, Y \in \mathfrak{X}(H^m[r])$, we have*

$$\nabla_X Y = \overline{\nabla}_X Y - \frac{1}{r}\, \langle\!\langle X, Y \rangle\!\rangle\, \mathcal{N}.$$

Proof. As $\langle\!\langle Y, \mathcal{N} \rangle\!\rangle = 0$, we have

$$0 = \langle\!\langle \overline{\nabla}_X Y, \mathcal{N} \rangle\!\rangle + \langle\!\langle Y, \overline{\nabla}_X \mathcal{N} \rangle\!\rangle = \langle\!\langle \overline{\nabla}_X Y, \mathcal{N} \rangle\!\rangle + \frac{1}{r} \langle\!\langle Y, X \rangle\!\rangle$$

Therefore we get

$$\nabla_X Y = \overline{\nabla}_X Y - \frac{\langle\!\langle \overline{\nabla}_X Y, \mathcal{N} \rangle\!\rangle}{\langle\!\langle \mathcal{N}, \mathcal{N} \rangle\!\rangle}\, \mathcal{N} = \overline{\nabla}_X Y + \langle\!\langle \nabla_X Y, \mathcal{N} \rangle\!\rangle\, \mathcal{N} = \overline{\nabla}_X Y - \frac{1}{r}\, \langle\!\langle X, Y \rangle\!\rangle\, \mathcal{N}.$$

This completes the proof. □

Thus we obtain

$$R(X, Y)Z = \frac{-1}{r^2} \left\{ X \langle Y, Z \rangle - Y \langle X, Z \rangle \right\}$$

(see [**21**], for example). We therefore find that sectional curvatures of all tangent planes of $H^m[r]$ are $-1/r^2$. A real hyperbolic space $H^m[r]$ with this metric is a typical example of Riemannian manifolds of sectional curvature $-1/r^2$.

Since $H^m[1]$ is of constant sectional curvature -1, we have another way of getting a Riemannian manifold of constant sectional curvature c (< 0). On $H^m[1]$ we define a new Riemannian metric $\langle\ ,\ \rangle'$ by $\langle\ ,\ \rangle' = \dfrac{1}{|c|} \langle\ ,\ \rangle$, where $\langle\ ,\ \rangle$ denotes the canonical metric on $H^m[1]$. By Lemma 1.3, we see that $H^m[1]$ with this new metric is of constant sectional curvature c. We shall denote this Riemannian manifold by $H^m(c)$. Trivially, the map $H^m[r] \ni p \mapsto (1/r)p \in H^m(-1/r^2)$ is an isometry.

3. Geodesics and parallel displacements

Let $\sigma : I \to M$ be a smooth curve on a Riemannian manifold M defined on an interval I. A smooth map $I \ni t \mapsto Y(t) \in T_{\sigma(t)}M$ is said to be a vector field along σ. As we see in §1.1, we can determine a new vector field $\nabla_{\frac{d\sigma}{dt}} Y$ along σ. We call this the covariant differentiation of Y along σ. We sometimes denote this also by $\nabla_{\frac{\partial}{\partial t}} Y$.

A vector field Y along a smooth curve σ is said to be *parallel* along σ, if it satisfies $\nabla_{\frac{\partial}{\partial t}} Y \equiv 0$. On a local coordinate neighborhood $\big(U, (x_1, \dots, x_m)\big)$, a vector field Y along σ is parallel if and only if

$$\frac{dY^i}{dt} + \sum_{j=1}^{m} \sum_{k=1}^{m} \Gamma_{j\,k}^{\,i} \frac{d(x_j \circ \sigma)}{dt} Y^k \equiv 0, \tag{1.2}$$

where $Y|_{U \cap \sigma(I)} = \displaystyle\sum_{j=1}^{m} Y^j \frac{\partial}{\partial x_j}$ with smooth functions $Y^j : I \to \mathbb{R}$ $(j = 1, \dots, m)$. Since (1.2) is a linear differential equation of order 1, for an arbitrary $a \in I$ and for each $v \in T_{\sigma(a)}M$ we have a unique vector field Y along σ which is parallel along σ and that satisfies $Y(a) = v$.

Let $\sigma : I \to M$ be a smooth curve on a Riemannian manifold. For $a, b \in I$, we define a map $P_{\sigma,a}^{\;b} : T_{\sigma(a)}M \to T_{\sigma(b)}M$ in the following manner: Given $v \in T_{\sigma(a)}M$, we take a parallel vector field Y_v along σ, and define $P_{\sigma,a}^{\;b}(v) = Y_v(b)$. We call this a *parallel displacement* along σ from $\sigma(a)$ to $\sigma(b)$. As (1.2) is a linear differential equation, parallel displacements are linear maps. Moreover, we find $P_{\sigma,a}^{\;a}$ is the identity of $T_{\sigma(a)}M$ and $P_{\sigma,b}^{\;a} = \big(P_{\sigma,a}^{\;b}\big)^{-1}$. Since we have

$$\frac{d}{dt}\Big\langle Y_v(t), Y_w(t)\Big\rangle = \Big\langle \big(\nabla_{\frac{d\sigma}{dt}} Y_v\big)(t), Y_w(t)\Big\rangle + \Big\langle Y_v(t), \big(\nabla_{\frac{d\sigma}{dt}} Y_w\big)(t)\Big\rangle \equiv 0,$$

we have $\big\langle Y_v(a), Y_w(a)\big\rangle = \big\langle Y_v(b), Y_w(b)\big\rangle$. Thus, we see that $P_{\sigma,a}^{\;b}$ is a linear isomorphism preserving the inner product.

LEMMA 1.6. *Given $X, Y \in \mathfrak{X}(M)$ and a point $p \in M$, we take a smooth curve $\sigma : (-\epsilon, \epsilon) \to M$ satisfying $\sigma(0) = p$ and $\dot\sigma(0) = X(p)$. Then we have*

$$\nabla_X Y(p) = \lim_{t \to 0} \frac{1}{t} \big\{ P_{\sigma,t}^{\;0}\big(Y\big(\sigma(t)\big)\big) - Y(p) \big\}.$$

PROOF. We take a local coordinate neighborhood $\big(U, \varphi = (x_1, \dots, x_m)\big)$ around p. We denote as $X = \sum_{i=1}^{m} X^i \dfrac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^{m} Y^i \dfrac{\partial}{\partial x_i}$. We define functions $a_{ij}(t)$ and $b_{ij}(t)$ by

$$P_{\sigma,0}^{\ t}\Big(\Big(\frac{\partial}{\partial x_i}\Big)_p\Big) = \sum_{j=1}^{m} a_{ij}(t)\Big(\frac{\partial}{\partial x_i}\Big)_{\sigma(t)}, \qquad P_{\sigma,t}^{\ 0}\Big(\Big(\frac{\partial}{\partial x_i}\Big)_{\sigma(t)}\Big) = \sum_{j=1}^{m} b_{ij}(t)\Big(\frac{\partial}{\partial x_i}\Big)_p.$$

Since $P_{\sigma,0}^{\ 0}$ is the identity, we have $a_{ij}(0) = \delta_{ij}$ and $b_{ij}(0) = \delta_{ij}$, where δ_{ij} denotes the Kronecker's delta, that is, $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$. By (1.2) we have

$$\frac{da_{ij}}{dt}(t) + \sum_{\alpha=1}^{m}\sum_{\beta=1}^{m} \Gamma_{\alpha\beta}^{\ j}\big(\sigma(t)\big)\frac{d(x_\alpha \circ \sigma)}{dt}(t) a_{i\beta}(t) \equiv 0.$$

In particular, we have

$$\frac{da_{ij}}{dt}(0) + \sum_{\alpha=1}^{m} \Gamma_{\alpha i}^{\ j}(p) X^\alpha(p) = 0.$$

On the other hand, as $P_{\sigma,t}^{\ 0} = \big(P_{\sigma,0}^{\ t}\big)^{-1}$, we have $\sum_{k=1}^{m} a_{ik}(t) b_{kj}(t) = \delta_{ij}$. Differentiating this equality, we obtain

$$\sum_{k=1}^{m}\Big\{\frac{da_{ik}}{dt}(t) b_{kj}(t) + a_{ik}(t)\frac{b_{kj}}{dt}(t)\Big\} = 0.$$

In particular, we have $\dfrac{da_{ij}}{dt}(0) + \dfrac{db_{ij}}{dt}(0) = 0$. Hence we find that

$$\frac{db_{ij}}{dt}(0) = \sum_{\alpha=1}^{m} \Gamma_{\alpha i}^{\ j}(p) X^\alpha(p).$$

By using the above, we get

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}\big\{P_{\sigma,t}^{\ 0}\big(Y\big(\sigma(t)\big)\big) - Y(p)\big\} \\
&\quad = \lim_{t\to 0}\frac{1}{t}\left\{\sum_{i=1}^{m}\sum_{j=1}^{m} Y^i\big(\sigma(t)\big) b_{ij}(t)\Big(\frac{\partial}{\partial x_j}\Big)_p - \sum_{j=1}^{m} Y^j(p)\Big(\frac{\partial}{\partial x_j}\Big)_p\right\} \\
&\quad = \lim_{t\to 0}\frac{1}{t}\left\{\sum_{i=1}^{m}\sum_{j=1}^{m}\Big\{\big\{Y^i\big(\sigma(t)\big) b_{ij}(t) - Y^i(p) b_{ij}(t)\big\}\right. \\
&\qquad\qquad\qquad \left. + \big\{Y^i(p) b_{ij}(t) - Y^i(p) b_{ij}(0)\big\}\Big\}\Big(\frac{\partial}{\partial x_j}\Big)_p\right\}
\end{aligned}$$

$$= \sum_{j=1}^{m} \left\{ \frac{dY^j\big(\sigma(t)\big)}{dt}(0) + \sum_{i=1}^{m} Y^i(p)\frac{db_{ij}}{dt}(0) \right\} \Big(\frac{\partial}{\partial x_j}\Big)_p$$
$$= \sum_{j=1}^{m} \left\{ \sum_{i=1}^{m} X^i(p)\frac{\partial Y^j}{\partial x_i}(p) + \sum_{i=1}^{m}\sum_{k=1}^{m} Y^i(p)\Gamma^{\,j}_{k\,i}(p)X^k(p) \right\} \Big(\frac{\partial}{\partial x_j}\Big)_p = \nabla_X Y(p).$$

We get the conclusion. □

A smooth curve $\gamma : I \to M$ on a Riemannian manifold M defined on an interval I is called a *geodesic*, if it satisfies the differential equation $\nabla_{\dot\gamma}\dot\gamma = 0$. By using a local coordinate neighborhood $\big(U, (x_1, \ldots, x_n)\big)$ this equation turns to

$$\frac{d^2(x_k\circ\gamma)}{d^2t} + \sum_{i=1}^{m}\sum_{j=1}^{m} \Gamma^{\,k}_{i\,j}\big(\gamma(t)\big)\frac{d(x_i\circ\gamma)}{dt}\,\frac{d(x_j\circ\gamma)}{dt} \equiv 0 \qquad (k = 1, \ldots, m).$$

Since this is a system of nonlinear differential equations, for given an arbitrary $v \in T_pM$ at an arbitrary point $p \in M$ there exists a neighborhood V $(\subset TM)$ of v and a positive ϵ such that for each $w \in V$ there exists a geodesic $\gamma_w : (-\epsilon, \epsilon) \to M$ satisfying $\gamma(0) = \pi_M(w)$ and $\dfrac{d\gamma}{dt}(0) = w$. Here, ϵ depends on w. By general theory on differential equations, we know that solutions of differential equations depend smoothly on initial conditions, we therefore find that the geodesic γ_w depends smoothly on w.

LEMMA 1.7. *For a geodesic $\gamma : [a, b] \to M$, the norm function $\Big\|\dfrac{d\gamma}{dt}\Big\|$ of its verocity vectors is constant. Hence* $\mathrm{length}(\gamma) = \Big\|\dfrac{d\gamma}{dt}(a)\Big\|(b-a)$.

PROOF. We have

$$\frac{d}{dt}\Big\|\frac{d\gamma}{dt}(t)\Big\|^2 = 2\Big\langle\Big(\nabla_{\frac{d\gamma}{dt}}\frac{d\gamma}{dt}\Big)(t), \frac{d\gamma}{dt}(t)\Big\rangle = 0$$

and get the conclusion. □

For a non-zero λ, we consider a smooth curve $\sigma : (-\epsilon/|\lambda|, \epsilon/|\lambda|) \to M$ by $\sigma(t) = \gamma_w(\lambda t)$. Since $\dfrac{d\sigma}{dt}(t) = \lambda\dfrac{d\gamma_w}{ds}(\lambda t)$, we see

$$\nabla_{\frac{d\sigma}{dt}}\frac{d\sigma}{dt} = \lambda^2\nabla_{\frac{d\gamma_w}{ds}}\frac{d\gamma_w}{ds} = 0,$$

and as we have $\sigma(0) = \pi_M(\lambda w)$ and $\dfrac{d\sigma}{dt}(0) = \lambda w$, we find $\gamma_{\lambda w}(t) = \gamma_w(\lambda t)$.

Given a point $p \in M$ we define a smooth map $\exp_p : \mathcal{U} \to M$ of an open neighborhood $\mathcal{U}$ of $0_p \in T_pM$ by $\exp_p(v) = \gamma_v(1)$. As a matter of fact, as the unit tangent space $U_pM = \{u \in T_pM \mid \|u\| = 1\}$ is diffeomorphic to a sphere S^{n-1} and is compact, there is positive ϵ satisfying that γ_u is defined on $(-\epsilon, \epsilon)$ for all $u \in U_pM$. Then for every $v \in B_{\epsilon/2}(p)$ we find that γ_v is defined on a interval including $(-2, 2)$. Thus, we can take $\mathcal{U}$ so that it includes $B_{\epsilon/2}(p)$. As γ_v depends smoothly on v, we find that this map is smooth. We call this map the *exponential map* at p. We note that $\exp_p$ depends smoothly on p, that is, the map $(u, w) \mapsto \exp_{\pi_M(u)}(w)$ is smooth, because γ_v depends smoothly on v.

Lemma 1.8. *The differential* $(d\exp_p)_{0_p} : T_{0_p}(T_pM) \to T_pM$ *is the identity if we identify* $T_{0_p}(T_pM)$ *with* T_pM.

Proof. Given a tangent vector $u \in T_pM$ $(\cong \mathbb{R}^m)$, we consider a curve $c : (-\rho, \rho) \to T_pM$ given by $c(t) = tu$. Then we have $c(0) = 0_p \in T_pM$ and $\dfrac{dc}{dt}(0) = u \in T_{0_p}(T_pM)$ by regarding u as a vector in $T_{0_p}(T_pM)$ $(\cong \mathbb{R}^m)$. We hence have

$$(d\exp_p)_{0_p}(u) = \frac{d}{dt}\exp_p\big(c(t)\big)\Big|_{t=0} = \frac{d}{dt}\gamma_u(t)\big|_{t=0} = \frac{d\gamma_u}{dt}(0) = u,$$

which shows the assertion. □

Since $(d\exp_p)_{0_p}$ is invertible, by inverse mapping theorem there exists a positive δ such that the restriction $\exp_p\big|_{B_\delta(0_p)} : B_\delta(p) \to M$ of the exponential map to an open ball $B_\delta(0_p) = \{v \in T_pM \mid \|v\| < \delta\}$ in T_pM is a diffeomorphism onto an open subset $U = \exp_p\big(B_\delta(0_p)\big)$ of M. If we set $\varphi = \big(\exp_p\big|_{B_\delta(0_p)}\big)^{-1} : U \to T_pM \cong \mathbb{R}^m$, then (U, φ) is a local coordinate neighborhood around p. We call this a *normal coordinate neighborhood* centered at p.

A smooth map $\alpha : [a, b] \times (-\epsilon, \epsilon) \to M$ is said to be a variation of geodesics, if for each $s \in (-\epsilon, \epsilon)$ the map $[a, b] \ni t \mapsto \alpha(t, s) \in M$ is a geodesic. For a variation of geodesic $\alpha : [a, b] \times (-\epsilon, \epsilon) \to M$, we define a vector field Y along a geodesic γ given by $\gamma(t) = \alpha(t,0)$ by $Y(t) = \dfrac{\partial \alpha}{\partial s}(t,0)$. Then it satisfies

$$\big(\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y\big)(t) + R\big(Y(t),\dot\gamma(t)\big)\dot\gamma(t) \equiv 0. \tag{1.3}$$

As a matter of fact, as $t \mapsto \alpha(t,s)$ are geodesics, we have

$$\begin{aligned}\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y &= \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial \alpha}{\partial s} = \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial \alpha}{\partial t} \\ &= \nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial \alpha}{\partial t} + R\Big(\frac{\partial \alpha}{\partial t},\frac{\partial \alpha}{\partial s}\Big)\frac{\partial \alpha}{\partial t} = -R\Big(\frac{\partial \alpha}{\partial s},\frac{\partial \alpha}{\partial t}\Big)\frac{\partial \alpha}{\partial t}.\end{aligned}$$

A vector field Y along a geodesic γ satisfying the linear differential equation (1.3) is called a *Jacobi field*. A Jacobi filed Y along a geodesic γ is defined uniquely if we give initial condition $Y(0)$ and $\big(\nabla_{\dot\gamma}Y\big)(0)$. Therefore, the set $\mathfrak{J}a(\gamma)$ the set of all Jacobi fields along γ is a $2\dim(M)$-dimensional vector space.

Proposition 1.1 (Gauss Lemma). *We take two tangent vectors $u, v \in T_pM$ at an arbitrary point $p \in M$. Suppose the geodesic γ_u of initial vector u is defined on an interval containing $[0,\epsilon]$. Then for $0 < t < \epsilon$ we have the following by identifying $T_{tu}(T_pM)$ with T_pM :*

(1) $(d\exp_p)_{tu}(u) = \dot\gamma_u(t)$,

(2) $\big\langle (d\exp_p)_{tu}(v), \dot\gamma_u(t)\big\rangle = \langle v, u\rangle$, *in particular,* $\|(d\exp_p)_{tu}(v)\| = \|v\|$.

Proof. (1) Since the curve $c(s) = (t+s)u$ on $T_{tu}(T_pM) \cong T_pM$ satisfies $c(0) = tu$ and $\dfrac{dc}{ds}(0) = u$, we have

$$(d\exp_p)_{tu}(u) = \frac{d}{ds}\exp_p\big(c(s)\big)\big|_{s=0} = \frac{d}{ds}\gamma_u(t+s)\big|_{s=0} = \frac{d\gamma_u}{ds}(t).$$

(2) We define a smooth map $\alpha : [-\epsilon,\epsilon]\times(-\delta,\delta) \to M$ with some positive δ by $\alpha(t,s) = \exp_p\big(t(u+sv)\big)$. Then it is a variation of geodesics. We set

$$Y(t) = \frac{\partial \alpha}{\partial s}(t,0) = (d\exp_p)_{tu}(tv) = t(d\exp_p)_{tu}(v),$$

which is a Jacobi field along γ_u. It satisfies $Y(0) = 0$ and

$$\begin{aligned}\big(\nabla_{\dot\gamma_u} Y\big)(0) &= \Big(\nabla_{\frac{\partial}{\partial s}} \frac{\partial\alpha}{\partial t}\Big)(0,0)\\ &= \Big(\nabla_{\frac{\partial}{\partial s}} \big((d\mathrm{exp}_p)_{t(u+sv)}(u+sv)\big|_{t=0}\big)\Big)\Big|_{s=0}\\ &= \Big(\nabla_{\frac{\partial}{\partial s}} \big((d\mathrm{exp}_p)_{0_p}(u+sv)\big)\Big)\Big|_{s=0} = \nabla_{\frac{\partial}{\partial s}}(u+sv)\big|_{s=0} = v.\end{aligned}$$

Since we have

$$\frac{d^2}{dt^2}\Big\langle Y(t), \dot\gamma(t)\Big\rangle = \Big\langle \big(\nabla_{\dot\gamma_u}\nabla_{\dot\gamma_u} Y\big)(t), \dot\gamma_u(t)\Big\rangle = -\big\langle R\big(Y(t), \dot\gamma_u(t)\big)\dot\gamma_u(t), \dot\gamma_u(t)\big\rangle = 0,$$

because γ_u is a geodesic, we see

$$\big\langle Y(t), \dot\gamma(t)\big\rangle = \big\langle Y(0), \dot\gamma(0)\big\rangle + \big\langle\big(\nabla_{\dot\gamma_u}\big)Y(0), \dot\gamma(0)\big\rangle t = \langle v, u\rangle t.$$

Thus we obtain

$$\big\langle (d\mathrm{exp}_p)_{tu}(v), \dot\gamma_u(t)\big\rangle = \Big\langle \frac{1}{t} Y(t), \dot\gamma(t)\Big\rangle = \langle v, u\rangle t.$$

This complete the proof. □

We showed that a variation of geodesics induces a Jacobi field. On the other hand, for a Jacobi filed Y along a geodesic γ we have a variation α of geodesics satisfying $\alpha(t,0) = \gamma(t)$ and $\dfrac{\partial\alpha}{\partial s}(t,0) = Y(t)$ for all t. As a matter of fact, we set $\gamma(0) = p,\ \dot\gamma(0) = u,\ Y(0) = v$ and $\nabla_{\dot\gamma} Y(0) = w$. We take a curve $\sigma : (-\epsilon, \epsilon) \to M$ satisfying $\sigma(0) = p$ and $\sigma'(0) = v$. If we define a variation of geodesics by $\alpha(t,s) = \exp_{\sigma(s)}\big(t(u+sw)\big)$, then we have $\alpha(t,0) = \exp_p(tu) = \gamma(t)$ and

$$\frac{\partial\alpha}{\partial s}(0,0) = \Big(\frac{d}{ds}\exp_{\sigma(s)}(0_{\sigma(s)})\Big)\Big|_{s=0} = \frac{d\sigma}{ds}(0) = v.$$

Since $(d\mathrm{exp}_{\sigma(s)})_{0_{\sigma(s)}}$ is the identity, we have

$$\begin{aligned}\Big(\nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial s}\Big)(0,0) &= \Big(\nabla_{\frac{\partial\alpha}{\partial s}} \frac{\partial\alpha}{\partial t}\Big)(0,0) = \Big(\nabla_{\frac{\partial}{\partial s}} \big((d\mathrm{exp}_{\sigma(s)})_{t(u+sw)}(u+sw)\big|_{t=0}\big)\Big)\Big|_{s=0}\\ &= \Big(\nabla_{\frac{\partial}{\partial s}} \big((d\mathrm{exp}_{\sigma(s)})_{0_{\sigma(s)}}(u+sw)\big)\Big)\Big|_{s=0} = \big(\nabla_{\frac{\partial}{\partial s}}(u+sw)\big)\big|_{s=0} = w.\end{aligned}$$

Thus we see the Jacobi field $\dfrac{\partial\alpha}{\partial s}(t,0)$ along γ coincides with Y.

Let $\gamma : (a,b) \to M$ $(a < 0 < b)$ be a geodesic and $t_0 \in (a,b)$. If there is a non-trivial Jacobi field Y satisfying $Y(0) = 0$ and $Y(t_0) = 0$, we say that $\gamma(t_0)$ is a *conjugate point* of $\gamma(0)$ along γ, and say that t_0 a *conjugate value* of $\gamma(0)$ along γ. We call the minimum positive conjugate value $c_\sigma(p)$ the *first conjugate value* of p along σ.

LEMMA 1.9. *Let γ be a geodesic. Suppose $\gamma(t_0)$ is not a conjugate point of $\gamma(0)$ along γ. For arbitrary $v \in T_{\gamma(0)}M$ and $w \in T_{\gamma(t_0)}M$, there exists a unique Jacobi field Y along γ satisfying $Y(0) = v$ and $Y(t_0) = w$.*

PROOF. Since the set $\mathcal{J}a(\gamma)^0 = \{Y \in \mathcal{J}a(\gamma) \mid Y(0) = 0\}$ is an m-dimensional linear space and $\gamma(t_0)$ is not a conjugate point, we see the linear map $\mathcal{J}a(\gamma)^0 \ni Y \mapsto Y(t_0) \in T_{\gamma(t_0)}M$ is bijective, that is, it is a linear isomorphism. Thus, we have $Y_1 \in \mathcal{J}a(\gamma)^0$ satisfying $Y_1(t_0) = w$.

Let X be a Jacobi field along γ. We consider a geodesic $\tilde{\gamma}$ given by $\tilde{\gamma}(t) = \gamma(t_0 - t)$. If we set a vector field $\widetilde{X}$ along $\tilde{\gamma}$, we have

$$\begin{aligned}&\big(\nabla\tilde{\gamma}'\nabla\tilde{\gamma}'\widetilde{X}\big)(t) + R\big(\widetilde{X}(t), \tilde{\gamma}'(t)\big)\tilde{\gamma}'(t)\\&\quad = \big(\nabla_{\dot\gamma}\nabla_{\dot\gamma}X\big)(t_0 - t) - R\big(X(t_0 - t), -\dot\gamma(t_0 - t)\big)\dot\gamma(t_0 - t) = 0,\end{aligned}$$

hence $\widetilde{X}$ is a Jacobi field along $\tilde{\gamma}$. By definition of conjugate points, we find that $\gamma(0)$ is not a conjugate point of $\gamma(t_0)$ along $\tilde{\gamma}$. Thus, above argument shows that we have a Jacobi field $\widetilde{Y}_2$ along $\tilde{\gamma}$ satisfying $\widetilde{Y}(0) = 0$ and $\widetilde{Y}(t_0) = v$. Therefore, by setting $Y_2(t) = \widetilde{Y}_2(t_0 - t)$, it is a Jacobi field along γ satisfying $Y(0) = v$ and $Y(t_0) = 0$. Thus we find that the Jacobi field $Y = Y_1 + Y_2$ satisfies the desirable condition.

If we have another Jacobi field Z along γ satisfying $Z(0) = v$ and $Z(t_0) = w$, the Jacobi field $Y - Z$ satisfies $(Y - Z)(0) = 0$ and $(Y - Z)(t_0) = 0$. Since $\gamma(t_0)$ is not a conjugate point, we see $Y - Z \equiv 0$. We hence get the conclusion. □

By using Gauss Lemma (Proposition 1.1), we study the relationship between the distance function and exponential maps.

LEMMA 1.10. *Suppose $\exp_p : B_r(0_p) \to \exp_p\big(B_r(0_p)\big)$ is an embedding. For $v \in B_r(0_p)$, the geodesic $\gamma_v : [0, 1] \to M$ satisfies*

$$d\big(p, \exp_p(v)\big) = \mathrm{length}(\gamma_v)\ \big(= \|v\|\big).$$

Moreover, if a smooth curve $c : [a, b] \to M$ *with* $c(a) = p$, $c(b) = \exp_p(v)$ *satisfies* $\mathrm{length}(c) = \|v\|$ *then* $c([a, b]) = \gamma_v([0, 1])$*, and if* γ *is a geodesic of unit speed with* $\gamma(0) = p$ *and* $\gamma(\|v\|) = \exp_p(v)$*, then* $\gamma_v(t) = \gamma(\|v\|t)$ *holds for* $t \in [0, 1]$*. In particular,* $\exp_p\big(B_r(0_p)\big)$ *is a distance-ball* $B_r(p) = \{q \mid d(p, q) < r\}$*.*

Proof. First we take a smooth curve $c : [a, b] \to M$ in $B := \exp_p\big(B_r(0_p)\big)$ satisfying $c(a) = p$ and $c(b) = q := \exp_p(v)$. Then we have a smooth curve $\mu : [a, b] \to B_r(0_p)$ with $c(t) = \exp_p\big(\mu(t)\big)$. We shall show $\mathrm{length}(c) \geq \|v\|$. If there are $a < t_1 < t_2 < \cdots < t_K < b$ with $c(t_j) = p$, we divid c into $K + 1$ curves $c|_{[a,t_1]}, c|_{[t_1,t_2]}, \ldots, c|[t_K, b]$. We may hence suppose $c(t) \neq p$ for $t > a$. We set $\rho(t) = \|\mu(t)\|$ for $a \leq t \leq b$ and $u(t) = \mu(t)/\rho(t) \in U_pM$ for $a < t \leq b$. By Gauss Lemma (Proposition 1.1), we find

$$\big\|(d\exp_p)_{\mu(t)}(u(t))\big\| = \|\dot{\gamma}_{u(t)}(r(t))\| = 1,$$

$$\Big\langle (d\exp_p)_{\mu(t)}(u(t)), (d\exp_p)_{\mu(t)}\Big(\frac{du}{dt}(t)\Big)\Big\rangle = \Big\langle u(t), \frac{du}{dt}(t)\Big\rangle = \frac{1}{2}\frac{d}{dt}\langle u(t), u(t)\rangle = 0.$$

Therefore, as $\dfrac{d\mu}{dt}(t) = \dfrac{d\rho}{dt}(t)u(t) + \rho(t)\dfrac{du}{dt}(t)$, we have

$$\begin{aligned}
\Big\|\frac{dc}{dt}(t)\Big\| &= \Big\|(d\exp_p)_{\mu(t)}\Big(\frac{d\mu}{dt}(t)\Big)\Big\| = \Big\|(d\exp_p)_{\mu(t)}\Big(\frac{d\rho}{dt}(t)u(t) + \rho(t)\frac{du}{dt}(t)\Big)\Big\| \\
&= \Big\{\Big|\frac{d\rho}{dt}(t)\Big|^2\big\|(d\exp_p)_{\mu(t)}(u(t))\big\|^2 + \rho(t)^2\Big\|(d\exp_p)_{\mu(t)}\Big(\frac{du}{dt}(t)\Big)\Big\|^2\Big\}^{1/2} \\
&\geq \Big|\frac{d\rho}{dt}(t)\Big| = \Big|\Big\langle\frac{d\mu}{dt}(t), u(t)\Big\rangle\Big| = \frac{1}{\rho(t)}\Big|\Big\langle\frac{d\mu}{dt}(t), \mu(t)\Big\rangle\Big| \\
&= \Big|\frac{d}{dt}\|\mu(t)\|\Big| = \Big|\frac{d\rho}{dt}(t)\Big|.
\end{aligned} \tag{1.4}$$

Hence we find

$$\begin{aligned}
\mathrm{length}(c) &= \int_a^b \Big\|\frac{dc}{dt}(t)\Big\|\, dt \geq \int_a^b \Big|\frac{d\rho}{dt}(t)\Big|\, dt \\
&\geq \Big|\int_a^b \frac{d\rho}{dt}(t)\, dt\Big| = \big|\rho(b) - \rho(a)\big| = \|v\|.
\end{aligned} \tag{1.5}$$

Next we take a curve $c : [a, b] \to M$ satisfying $c(a) = p$ and $c(b) = q$ which does not contained in B. Then there is $t_0 >$ with $a < t_0 < b$ satisfying $c(t) \in \exp_p\big(B_{(r+\|v\|)/2}(0_p)\big)$ for $a \leq t < t_0$ and $c(t_0) \notin \exp_p\big(B_{(r+\|v\|)/2}(0_p)\big)$. By the above argument we have length$(c|_{[a,t_0]}) \geq (r + \|v\|)/2 > \|v\|$. We hence find $d(p,q) \geq \|v\|$. Since length$(\gamma_v) = \|v\|$, we obtain $d(p,q) = \|v\|$.

Next we take a smooth curve $c : [a,b] \to M$ with $c(a) = p$ and $c(b) = q$ which satisfies length$(c) = d(p,q)(= \|v\|)$. Then we find that $c([a,b])$ is contained in B and equalities hold in (1.4) and (1.5). Thus we see $u(t) \equiv u \in U_pM$ and $\frac{d\rho}{dt} \geq 0$. This means $c([a,b]) = \gamma_u([0,\|v\|])$. Since $\exp_p : B_r(0_p) \to B$ is bijective, we have $v = \|v\|u$ and $c([a,b]) = \gamma_v([0,1])$.

Finally we study $B_r(p)$. It is clear that $B \subset B_r(p)$ by the assertion we showed in the above. On the other hand, if we suppose that we have a point $x \in B_r(p) \setminus B$ then for every smooth curve $c : [a,b] \to M$ with $c(a) = p$ and $c(b) = x$ there is t_0 with $a < t_0 < b$ satisfying $c(t) \in B$ and $c(t_0) \notin B$. Since length$(c|_{[a,t_0]}) \geq r$ as we see in the above and length$(c|_{[t_0,b]} \geq d\big(c(t_0), x\big) > 0$, we find $d(p,x) > r$. We hence get the conclusion. □

An open set W in M is called *uniformly normal neighborhood* if there exists positive δ such that the following conditions hold at each point $q \in W$:

i) $\exp_q : B_q(0_q) \to \exp_q\big(B_q(0_q)\big)$ is an embedding;
ii) $W \subset \exp_q\big(B_q(0_q)\big)$.

Lemma 1.11 (Existence of uniformly normal neighborhoods). *Given a point $p \in M$ and a neighborhood U of p, there exists a uniformly normal neighborhood W of p contained in U.*

Proof. We take a neighbourhood V of p whose closure $\overline{V}$ is contained in U. Since $TM|_{\overline{V}} = \bigcup_{q\in\overline{V}}\big\{u \in T_qM \;\big|\; \|u\| = 1\big\}$ is compact, there is positive ϵ satisfying that γ_u is defined on $(-\epsilon,\epsilon)$ for all $u \in TM|_{\overline{V}}$. Therefore we see that $\exp_q : B_\epsilon(0_q) \to M$ is defined for each $q \in V$. By setting $\mathcal{E} = \big\{u \in TM|_V \;\big|\; \|u\| < \epsilon\big\}$, we define a smooth map $\mathit{\Phi} : \mathcal{E} \to M \times M$ by $\mathit{\Phi}(u) = \big(\pi_M(u), \exp_{\pi_M(u)}(u)\big)$.

For a normal coordinate neighborhood $\big(U', \varphi = (x_1,\dots,x_m)\big)$ of M centered at p with $U' \subset V$, we have a corresponding coordinate neighborhood of TM given by

$\big(\widetilde{U'}, \tilde{\varphi} = (x_1, \dots, x_n; \xi_1, \dots, \xi_m)\big)$ which is defined as $\widetilde{U'} = TM|_{U'}$ and

$$\widetilde{U'} \ni \sum_{j=1}^{m} \xi_j \Big(\frac{\partial}{\partial x_j}\Big)_q \longmapsto \big(x_1(q), \dots, x_m(q), \xi_1, \dots, \xi_m\big) \in \varphi(U') \times \mathbb{R}^m.$$

Since we have

$$\pi_M\Big(\xi_j\Big(\frac{\partial}{\partial x_j}\Big)_q\Big) = q \qquad \text{and} \qquad \exp_p\Big(\sum_{j=1}^{m} \xi_j\Big(\frac{\partial}{\partial x_j}\Big)_q\Big) = \varphi^{-1}\big(\xi_1, \dots, \xi_m\big),$$

We find that $(d\Phi)_{0_p} : T_{0_p}\mathcal{E} \to T_{(p,p)}\big(M \times M\big) \cong T_pM \times T_pM$ is given by the matrix

$$(D\Phi)_{0_p} = \begin{pmatrix} \dfrac{\partial x_j}{\partial x_k} & \dfrac{\partial x_j}{\partial \xi_k} \\[2ex] \dfrac{\partial(x_j \circ \exp_p)}{\partial x_k} & \dfrac{\partial(x_j \circ \exp_p)}{\partial \xi_k} \end{pmatrix} = \begin{pmatrix} E & O \\ * & E \end{pmatrix}$$

by identifying $T_{0_p}\mathcal{E} = \mathbb{R}^{2m}$ and $T_pM \times T_pM = \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$. As $(D\Phi)_{0_p}$ is invertible, by inverse mapping theorem, there is an open neighborhood $\mathcal{V}$ $\big(\subset \mathcal{E}\big)$ of 0_p such that $\Phi|_{\mathcal{V}} : \mathcal{V} \to \Phi\big(\mathcal{V}\big)$ is a diffeomorphism.

There exist an open neighborhood U'' of p and a positive ϵ' such that $\overline{U''} \subset U'$ and $\tilde{\varphi}^{-1}\big(U'' \times B_{\epsilon'}(0)\big) \subset \mathcal{V}$. Since $q \mapsto \big(\frac{\partial}{\partial x_j}\big)_q$ is smooth, we can set

$$C_* := \min\Big\{\Big\|\sum_{j=1}^{m} a_j\Big(\frac{\partial}{\partial x_j}\Big)_q\Big\| \;\Big|\; a_j \in \mathbb{R} \text{ with } \sum_{j=1}^{m} a_j^2 = 1,\ q \in \overline{U''}\Big\},$$
$$C_\sharp := \min\Big\{\Big\|\sum_{j=1}^{m} a_j\Big(\frac{\partial}{\partial x_j}\Big)_q\Big\| \;\Big|\; a_j \in \mathbb{R} \text{ with } \sum_{j=1}^{m} a_j^2 = 1,\ q \in \overline{U''}\Big\}.$$

Then for arbitrary $v \in TM|_{U''}$ we have $C_* \sum_{j=1}^{n} \xi_j(v)^2 \le \|v\| \le C_\sharp \sum_{j=1}^{n} \xi_j(v)^2$. Thus, if we put $\mathcal{U} := \bigcup_{q\in U''}\{v \in T_qM \mid \|v\| < C_*\epsilon'\}$, then it is an open neighborhood of 0_p satisfying $\mathcal{U} \subset \tilde{\varphi}^{-1}\big(U'' \times B_{\epsilon'}(0)\big)$. Since $\Phi|_{\mathcal{V}} : \mathcal{V} \to \Phi\big(\mathcal{V}\big)$ is a diffeomorphism, we can take an open neighborhood W $(\subset U'')$ of p so that $W \times W \subset \Phi(\mathcal{U})$.

We shall show that W and $\delta := C_*\epsilon'$ satisfy the conditions of uniform normal neighborhood. We set $\mathcal{W} := \bigcup_{q\in W}\{v \in T_qM \mid \|v\| < \delta\}$. As $\mathcal{W} \subset \mathcal{U} \subset \mathcal{V}$, the map $\Phi|_{\mathcal{W}} : \mathcal{W} \to \Phi\big(\mathcal{W}\big)$ is a diffeomorphism.

We take an arbitrary point $q_0 \in W$. We define a map $\psi : \exp_{q_0}\big(B_\delta(0_{q_0})\big) \to B_\delta(0_{q_0})$ so that $\big(\Phi|_{\mathcal{W}}\big)^{-1}(q_0, y) = \psi(y)$. Then, for each $v \in T_{q_0}M$ with $\|v\| < \delta$, we have

$$\psi \circ \exp_{q_0}(v) = \big(\Phi|_{\mathcal{W}}\big)^{-1}\big(q_0, \exp_{q_0}(v)\big) = \big(\Phi|_{\mathcal{W}}\big)^{-1} \circ \Phi(v) = v.$$

On the other hand, for arbitrary $y \in M$ satisfying $y = \exp_{q_0}(v)$ with some $v \in T_{q_0}M$ with $\|v\| < \delta$, we have

$$\big(q_0, \exp_{q_0} \circ \psi(y)\big) = \Phi \circ \big(\Phi|_{\mathcal{W}}\big)^{-1}(q_0, y) = (q_0, y),$$

hence have $\exp_{q_0} \circ \psi(y) = y$. Thus, $\exp_{q_0}\big|_{B_\delta(0_{q_0})} : B_\delta(0_{q_0}) \to \exp_{q_0}\big(B_\delta(0_{q_0})\big)$ is an embedding.

We take arbitrary points $q, y \in W$. Since $W \times W \in \Phi(\mathcal{U})$, we take $v = \big(\Phi|_{\mathcal{U}}\big)^{-1}(q, y) \in T_qM$. We note that $\|v\| < \delta$ and that $v \in \mathcal{W}$ because $q \in W$. As $(q, y) = \Phi(v) = \big(q, \exp_q(v)\big)$, we find that $y \in \exp_q\big(B_\delta(0_q)\big)$. Thus, we find that W is uniformly normal. $\square$

By Lemmas 1.10 and 1.11, for arbitrary points q_1, q_2 in a uniform normal neighborhood W, there is a geodesic γ joining these points whose length is less than δ.

When a Riemannian manifold M is complete, by *Hopf-Renow* theorem every geodesic can be extended unlimitedly. That is, domains of geodesics are the set $\mathbb{R}$ of all real numbers. We say a Riemannian manifold to be *geodesically complete* at $p \in M$ if every geodesic of initial poit p is defined on $\mathbb{R}$.

THEOREM 1.1 (Hopf-Renow Theorem). *For a connected Riemannian manifold M, the following conditions are mutually equivalent*:

(1) *At some point $p \in M$, the manifold M is geodesically complete*;
(2) *M is geodesically complete*;
(3) *At some point $p \in M$, for every $r > 0$, the closed ball $\overline{B_r}(p) := \{q \in M; d(p, q) \leq r\}$ is compact*;
(4) *For an arbitrary point $p \in M$ and every $r > 0$, the closed ball $\overline{B_r}(p)$ is compact*;

(5) *The distance space (M, d) is a complete distance space, that is, every Cauchy sequence in M is a convergent sequence.*

In the proof of this theorem we can show the following.

PROPOSITION 1.2. *Given two points p, q on a complete connected Riemannian manifold, there exists a geodesic γ joining p and q with* $\mathrm{length}(\gamma) = d(p, q)$.

The geodesic in Proposition 1.2 is called a *minimizing geodesic* joining p and q.

PROOF OF THEOREM 1.1. (1) $\Rightarrow$ (3). First, we show that the condition (1) guarantees that for every $q \in M$ there exists a minimizing geodesic joining p and q.

By Lemma 1.8 and by inverse mapping theorem, there is positive ϵ such that $\exp_p : B_{2\epsilon}(0_p) \to M$ is an embedding into M. By Lemma 1.10 each point $q \in B_{2\epsilon}(p) = \exp_p\big(B_{2\epsilon}(0_p)\big)$ can be joined by a unique minimal geodesic with p. We consider the case $d(p, q) \geq 2\epsilon$. Since $\overline{B_\epsilon}(p) = \exp_p\big(\overline{B_\epsilon}(0_p)\big)$ is compact, we can take $\bar{q} \in \partial\overline{B_\epsilon}(p)$ satisfying $d(p, \bar{q}) + d(\bar{q}, q) = d(p, q)$. In fact, for each positive integer j we have a smooth curve $c_j : [0, 1] \to M$ from p to q which satisfies $\mathrm{length}(c_j) < d(p, q) + \frac{1}{j}$. We take t_j so that $c_j(t) \in B_\epsilon(p)$ for $0 \leq t < t_j < 1$ and $d\big(p, c_n(t_n)\big) = \epsilon$. Then putting $q_j = c_j(t_j)$ we have

$$d(p, q_j) + d(q_j, q) \leq \mathrm{length}(c_j|_{[0,t_j]}) + \mathrm{length}(c_j|_{[t_j,1]}) = \mathrm{length}(c_j) < d(p, q) + \frac{1}{j}.$$

As $q_j \in \partial B_\epsilon(p)$ we can take a convergent subsequence $\{q_{j_k}\}_{k=1}^\infty$. If we set $\bar{q} = \lim_{k\to\infty} q_{j_k} \in \partial\overline{B_\epsilon}(p)$, we have

$$d(p, \bar{q}) + d(\bar{q}, q) = \lim_{k\to\infty}\{d(p, q_{j_k}) + d(q_{j_k}, q)\} \leq d(p, q)$$

by the above inequality. On the other hand we have $d(p, q) \leq d(p, \bar{q}) + d(\bar{q}, q)$ by the triangle inequality. Hence we have $d(p, \bar{q}) + d(\bar{q}, q) = d(p, q)$.

We take a geodesic γ of unit speed with $\gamma(0) = p$ and $\gamma(\epsilon) = \bar{q}$. Under the condition (1), this geodesic is defined on $\mathbb{R}$. We set

$$\mathcal{T} = \left\{ t \in [0, d(p, q)] \;\middle|\; \begin{array}{l} d\big(p, \gamma(t)\big) = t, \\ d\big(p, \gamma(t)\big) + d\big(\gamma(t), q\big) = d(p, q) \end{array} \right\},$$

and put $T := \sup \mathfrak{T}$. Since $\gamma|_{[0,\epsilon]}$ is the minimizing geodesic from p to $\bar{q}$, we find that $t \in [0, \epsilon]$ satisfies the conditions, hence $T \geq \epsilon$. We suppose $T < d(p, q)$ and set $q' = \gamma(T)$. By definition of T, and continuity of the distance function d, we have $d(p, q') = T$ and $d(p, q') + d(q', q) = d(p, q)$. We take a positive δ so that $2\delta < d(p, q) - T$ and that $B_{2\delta}(q')$ is contained in a uniform normal neighborhood W around q'. By the same argument when we take $\bar{q}$, by taking a sequence of smooth curve joining q' and q, we have a point $\bar{q}' \in \partial B_\delta(q')$ satisfying $d(q', \bar{q}') + d(\bar{q}', q) = d(q', q)$. By definition of T, we have a monotone increasing sequence $\{t_k\}_{k=1}^\infty \subset \mathfrak{T}$ with $\lim_{k\to\infty} t_k = T$. For sufficiently large k_0 we have $\gamma(t_{k_0}) \in B_\epsilon(q')$. We set $p' = \gamma(t_{k_0})$. Then we have $d(p, q) = d(p, p') + d(p', q) = t_{k_0} + d(p', q)$. Thus we obtain

$$\begin{aligned} d(p', q) &= d(p, q) - t_{k_0} = d(p, q') + d(q', q) - t_{k_0} \\ &= d(q', q) + T - t_{k_0} = d(q', q) + d(p', q'), \end{aligned}$$

because p' and q' are joined by a minimizing geodesic $\gamma|_{[t_{k_0}.T]}$. By the triangle inequality we have

$$\begin{aligned} d(p', \bar{q}') \leq d(p', q') + d(q', \bar{q}') &= d(p', q') + d(q', q) - d(\bar{q}', q) \\ &= d(p', q) - d(\bar{q}', q) \leq d(p', \bar{q}'), \end{aligned}$$

hence have $d(p', q') + d(q', \bar{q}') = d(p', \bar{q}')$. Since $p', q', \bar{q}' \in W$ this equality shows that the join of $\gamma|_{[t_{k_0}.T]}$ and the minimizing geodesic of unit speed from q' to $\bar{q}'$ is the minimal geodesic of unit speed from p' to $\bar{q}'$. Hence it coincides with $\gamma|_{[t_{k_0}, T+\delta]}$ because of the uniqueness of geodesics of given initial vector. We therefore have $\gamma(T+\delta) = \bar{q}'$. Again, by the triangle inequality we have

$$\begin{aligned} d(p, \bar{q}') \leq d(p, q') + d(q', \bar{q}') &= d(p, q') + d(q', q) - d(\bar{q}', q) \\ &= d(p, q) - d(\bar{q}', q) \leq d(p, \bar{q}'), \end{aligned}$$

hence have $d(p, \bar{q}') = d(p, q') + d(q', \bar{q}') = T + \delta$. Moreover, we have

$$d(p, \bar{q}') + d(\bar{q}', q) = d(p, q') + d(q', \bar{q}') + d(\bar{q}', q) = d(p, q') + d(q', q) = d(p, q).$$

Thus, we find $T + \delta \in \mathfrak{T}$, which is a contradiction. We hence find $T = d(p, q)$. This shows that γ is a minimal geodesic of unit speed joining p and q.

Next we show the condition (3) holds under the condition that for every $q \in M$ there exists a minimizing geodesic joining p and q. For an arbitrary sequence $\{q_k\}_{k=1}^{\infty} \subset \overline{B_r}(p)$, we take $u_k \in U_pM$ so that the geodesic of initial vector u_k is a minimal geodesic of unit speed from p to q_k. Since U_pM is compact and $d(p, q_k) \le r$, we can take a subsequence $\{q_{k_j}\}_{j=1}^{\infty}$ so that both $\{u_{k_j}\}_{j=1}^{\infty}$ and $\{d(p, q_{k_j})\}_{j=1}^{\infty}$ converge. We set $u = \lim_{j\to\infty} u_{k_j}$ and $d = \lim_{j\to\infty} d(p, q_{k_j})$ $(\le r)$. Since γ_u depends smoothly on u, we find $\gamma_u(d) = \lim_{j\to\infty} \gamma_{u_{k_j}}\big(d(p, q_{k_j})\big) = \lim_{j\to\infty} q_{k_j}$. Thus $\{q_{k_j}\}_{j=1}^{\infty}$ converges to $\gamma_u(d) \in \overline{B_r}(p)$. Hence $\overline{B_r}(p)$ is compact.

(3) $\Rightarrow$ (4). We take an arbitrary point $q \in M$. By triangle inequality, we see $\overline{B_r}(q) \subset \overline{B_{r+d(p,q)}}(p)$. Since $\overline{B_{r+d(p,q)}}(p)$ is compact, its closed subset $\overline{B_r}(q)$ is also compact.

(4) $\Rightarrow$ (5). We take a Caucy sequence $\{p_j\}_{j=1}^{\infty}$ in M with respect to the induced distance function d. For a positive ϵ there is a number N such that $d(p_j, p_k) < \epsilon$ for every j, k with $j, k \ge N$ the distance. If we set $R = \max\{d(p_1, p_j) \mid 2 \le j \le N\}$, we have $d(p_1, p_k) < d(p_1, p_N) + \epsilon$ for $k \ge N$. Thus we see $\{p_j\}_{j=1}^{\infty} \subset B_{R+\epsilon}(p_1)$. Since $\overline{B_{R+\epsilon}}(p_1)$ is compact, we find that $\{p_j\}_{j=1}^{\infty}$ converges to a point in $\overline{B_{R+\epsilon}}(p_1)$. Thus, M is complete.

(5) $\Rightarrow$ (2). We take a geodesic γ of unit speed. It is defined at least on the interval $(-\epsilon, \epsilon)$. We set $T_+ = \sup\{\tau \mid \gamma \text{ is defined on } (-\epsilon, \tau]\}$. If we suppose $T_+ < \infty$, we take a monotone increasing sequence $\{t_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} t_j = T$. As we have

$$d\big(\gamma(t_j), \gamma(t_k)\big) \le \mathrm{length}(\gamma|_{[t_j,t_k]}) = t_k - t_j$$

for arbitrary j, k with $j \le k$, and since $\{t_j\}_{j=1}^{\infty}$ is a convergent sequence, we find that $\{\gamma(t_j)\}_{j=1}^{\infty}$ is a Cauchy sequence in M. We put $q = \lim_{j\to\infty} \gamma(t_j)$. We take a uniform normal neighborhood W of q. There is j_0 such that if $j \ge j_0$ we have $\gamma(t_j) \in W$. We take a geodesic $\sigma : (-\epsilon', \epsilon') \to M$ of unit speed which satisfies $\sigma(0) = q$ and $\sigma\big(d(q, \gamma(t_{j_0+1}))\big) = \gamma(t_{j_0+1})$ and that lies in W. Since

$$d\big(\gamma(t_j), q\big) = \lim_{k\to\infty} d\big(\gamma(t_j), \gamma(t_k)\big) = \lim_{k\to\infty} |t_k - t_j| = T - t_j,$$

Since $\gamma(t_j) \in W$ for $j \geq j_0$, the geodesic $\gamma|_{[t_{j_0}, t_{j_0+1}]}$ is minimizing, hence

$$d\big(\gamma(t_{j_0+1}), \gamma(t_{j_0})\big) = t_{j_0+1} - t_{j_0}.$$

Thus, we have

$$d\big(q, \gamma(t_{j_0+1})\big) + d\big(\gamma(t_{j_0+1}), \gamma(t_{j_0})\big) = (T - t_{j_0+1}) + (t_{j_0+1} - t_{j_0}) = T - t_{j_0} = d\big(q, \gamma(t_{j_0})\big).$$

This means that the joined curve of $\gamma|_{[t_{j_0}, t_{j_0+1}]}$ and σ^{-1} is a minimizing geodesic joining $\gamma(t_{j_0})$ and q. If we consider a geodesic

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t), & \text{when } t \leq t_{j_0+1}, \\ \sigma(T - t), & \text{when } t_{j_0+1} \leq t < T + \epsilon', \end{cases}$$

then $\tilde{\gamma}|_{[0,T)} = \gamma$, because $\dot{\gamma}(t_{j_0+1}) = \dot{\tilde{\gamma}}(t_{j_0+1})$. This is a contradiction to the definition of T. Hence we find that γ is defined on $(-\epsilon, \infty)$. Considering γ^{-1}, we get γ is defined on $\mathbb{R}$. Thus we see M is geodesically complete. $\square$

4. Circles

A smooth curve $\gamma : I \to M$ on a Riemannian manifold M is said to be parameterized by its arc-length if it satisfies $\|\dot\gamma(t)\| = 1$ at each t, where $\dot\gamma$ denotes the differential of γ with respect to the parameter t. A smooth curve γ parameterized by its arc-length is said to be a *circle*, if it satisfies the following system of differential equations:

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma = kY, \\ \nabla_{\dot\gamma}Y = -k\dot\gamma, \end{cases} \tag{1.6}$$

with a constant $k(\geq 0)$ and a field Y of unit vectors along γ. We call the constant k its *geodesic curvature* and $\{\dot\gamma, Y\}$ its *Frenet frame*. When M is complete, every circle is defined on $\mathbb{R}$.

PROPOSITION 1.3. *A smooth curve γ parameterized by its arc-length is a circle if and only if it satisfies*

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \|\nabla_{\dot\gamma}\dot\gamma\|^2\dot\gamma = 0. \tag{1.7}$$

PROOF. First, we suppose a curve γ is a circle. Then we get

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}(kY) = -k^2\dot\gamma = -||\nabla_{\dot\gamma}\dot\gamma||^2\dot\gamma.$$

Next, we suppose a curve γ satisfies (1.7). As it is parameterized by its arc-length, we have

$$0 = \dot\gamma(\|\dot\gamma\|^2) = 2\langle\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle.$$

By (1.7), we obtain

$$\dot\gamma(\|\nabla_{\dot\gamma}\dot\gamma\|^2) = 2\langle\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle = -2\|\nabla_{\dot\gamma}\dot\gamma\|^2\langle\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle = 0,$$

hence we find that $\|\nabla_{\dot\gamma}\dot\gamma\|$ is constant along γ. We put $k = \|\nabla_{\dot\gamma}\dot\gamma\|$. When $k = 0$, for an arbitrary parallel unit vector field Y along γ, it satisfies the equations of circle (1.6). When $k > 0$, we set $Y = (1/k)\nabla_{\dot\gamma}\dot\gamma$. Then we find that (1.7) turns to

$$k\nabla_{\dot\gamma}Y + k^2\dot\gamma = 0.$$

This shows that γ satisfies the system of equations (1.6). $\square$

LEMMA 1.12. *Let γ be circles of geodesic curvature k on $(M, \langle\ ,\ \rangle)$. For a positive constant λ, we consider a new Riemannian metric $\langle\ ,\ \rangle' = \lambda^2\langle\ ,\ \rangle$. If we define a curve σ by $\sigma(t) = \gamma(t/\lambda)$. then it is a circle of geodesic curvature k/λ on $(M, \langle\ ,\ \rangle')$.*

PROOF. We put $a(t) = t/\lambda$. We then have

$$\frac{d\sigma}{dt}(t) = \frac{d}{dt}\gamma(a(t)) = a'(t)\frac{d\gamma}{dt}(a(t)) = \frac{1}{\lambda}\frac{d\gamma}{dt}(a(t)).$$

Hence have

$$\left\|\frac{d\sigma}{dt}\right\|' = \frac{1}{\lambda}\left\|\frac{d\gamma}{dt}(a(t))\right\|' = \frac{1}{\lambda}\cdot\lambda\left\|\frac{d\gamma}{dt}(a(t))\right\| = 1.$$

Thus σ is parameterized by its arc-length with respect to $\langle\ ,\ \rangle'$. We put $Y' = \dfrac{1}{\lambda}Y$. We then have $\|Y'\|' = \dfrac{1}{\lambda}\|Y\|' = \|Y\| = 1$.

By Lemma 1.3, we have

$$\nabla'_{\dot\sigma}\dot\sigma = \nabla_{\dot\sigma}\dot\sigma = \nabla_{\frac{1}{\lambda}\dot\gamma}\frac{1}{\lambda}\dot\gamma = \frac{1}{\lambda^2}\nabla_{\dot\gamma}\dot\gamma = \frac{1}{\lambda^2}kY = \frac{k}{\lambda}\times Y',$$
$$\nabla'_{\dot\sigma}Y' = \nabla_{\dot\sigma}Y' = \frac{1}{\lambda^2}\nabla_{\dot\gamma}Y = -\frac{k}{\lambda^2}\dot\gamma = -\frac{k}{\lambda}\dot\sigma,$$

hence σ is a circle of geodesic curvature k/λ. We get the conclusion. □

For the sake of later use, we here study circles on real space forms.

[1] Circles on a Euclidean space

First we study circles on a Euclidean space $\mathbb{R}^m$. Since the covariant differentiation with respect to the Riemannian connection on $\mathbb{R}^m$ is the ordinary differentiation, the equation (1.7) of a circle of geodesic curvature k turns to

$$\gamma''' + k^2\gamma' = 0.$$

Since its characteristic equation is $\lambda^3 + k^2\lambda = 0$, we find that γ is expressed as

$$\gamma(t) = A + Be^{\sqrt{-1}kt} + Ce^{-\sqrt{-1}kt} = A + B'\cos kt + C'\sin kt$$

with some $A, B, C, B', C' \in \mathbb{R}^m$. Under the initial conditions that $\gamma(0) = p \in \mathbb{R}^m$ and $\gamma'(0) = u,\ \gamma''(0) = kv$ with $u, v \in U_p\mathbb{R}^m \subset T_p\mathbb{R}^m \cong \mathbb{R}^m$, we have

$$\gamma(t) = p + \frac{1}{k}(\sin kt)u + \frac{1}{k^2}(1 - \cos kt)v.$$

Thus this circle is closed of minimal period $2\pi/k$. Here, a curve γ parameterized by its arclength is said to be closed if there is $t_0 \neq 0$ satisfying $\gamma(t+t_0)=\gamma(t)$ for all t. The minimum positive t_0 with this property is said to be the minimal period of this closed curve. When γ is not closed it is said to be open.

[2] Circles on a standard sphere

Next we study circles on a standard sphere $S^m(1)$. Regarding $S^m(1)$ as a submanifold of $\mathbb{R}^{n+1}$, we denote Riemannian connections of $S^m(1)$ and $\mathbb{R}^{m+1}$ by ∇ and $\widetilde{\nabla}$, respectively. We take a circle γ of geodesic curvature k whose Frenet frame is $\{\dot\gamma, Y\}$. We regard this curve as a curve in $\mathbb{R}^{m+1}$. For the sake of simplicity we denote it also by γ. For arbitrary vector fields $X, Y \in \mathfrak{X}\big(S^m(1)\big)$, we have $\nabla_X Y = \widetilde{\nabla}_X Y + \langle X, Y\rangle \mathcal{N}$ (see §1.3). Hence, the system of differential equations (1.6) turns to

$$\begin{cases} \widetilde{\nabla}_{\dot\gamma}\dot\gamma = kY - \gamma, \\ \widetilde{\nabla}_{\dot\gamma} Y = -k\dot\gamma. \end{cases}$$

We therefore get

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma = k\widetilde{\nabla}_{\dot\gamma}Y - \dot\gamma = -(k^2+1)\dot\gamma,$$

which is equivalent to $\gamma''' + (k^2+1)\gamma' = 0$. Since its characteristic equation is $\lambda^3 + (k^2+1)\lambda = 0$, we find that γ as a curve in $\mathbb{R}^{m+1}$ is of the form

$$\gamma(t) = A + Be^{\sqrt{-(k^2+1)}\,t} + Ce^{-\sqrt{-(k^2+1)}\,t} = A + B'\cos\big(\sqrt{k^2+1}\,t\big) + C'\cos\big(\sqrt{k^2+1}\,t\big)$$

with some $A, B, C, B', C' \in \mathbb{R}^{m+1}$. Under the initial conditions $\gamma(0) = p \in S^m(1) \subset \mathbb{R}^{m+1}$ and $\dot\gamma(0) = u,\ \nabla_{\dot\gamma}\dot\gamma(0) = kv$ with $u, v \in U_pS^m(1) \subset T_pS^m(1) \subset T_p\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1}$, which is equivalent to $\gamma(0)=p,\ \gamma'(0)=u,\ \gamma''(0)=kv+p$, we obtain

$$\gamma(t) = \frac{1}{k^2+1}(\cos\sqrt{k^2+1}t + k^2)p + \frac{1}{\sqrt{k^2+1}}\sin\sqrt{k^2+1}t\ u - \frac{k}{k^2+1}(\cos\sqrt{k^2+1}t - 1)v.$$

Thus, we find that every circle of geodesic curvature k on $S^m(1)$ is closed and of minimal period $2\pi/\sqrt{k^2+1}$.

We here make mention of circles on $S^m(c)$.

Let γ be a circle of constant geodesic curvature k on $\big(S^m(c), \langle\ ,\ \rangle'\big)$. Here, the metric $\langle\ ,\ \rangle'$ is given by $\langle\ ,\ \rangle' = \dfrac{1}{c}\langle\ ,\ \rangle$ with the canonical metric $\langle\ ,\ \rangle$ on a standard sphere $S^m(1)$. If we define a curve σ on $S^m(1)$ by $\sigma(s) = \gamma(s/\sqrt{c})$, then it is a circle of geodesic curvature $k/\sqrt{c}$ on $(S^m(1), \langle\ ,\ \rangle)$ by Lemma1.12. Hence it is expressed as

$$\begin{aligned}
\sigma(s) = &\frac{1}{(k/\sqrt{c})^2+1}\Big(\cos\sqrt{(k/\sqrt{c})^2+1}\, s + (k/\sqrt{c})^2\Big)p \\
&+ \frac{1}{\sqrt{(k/\sqrt{c})^2+1}} \sin\sqrt{(k/\sqrt{c})^2+1}\, s\, u \\
&- \frac{(k/\sqrt{c})}{(k/\sqrt{c})^2+1}\Big(\cos\sqrt{(k/\sqrt{c})^2+1}\, s - 1\Big)v \\
= &\frac{c}{k^2+c}\Big(\cos\sqrt{(k/\sqrt{c})^2+1}\, s + (k/\sqrt{c})^2\Big)p + \frac{\sqrt{c}}{\sqrt{k^2+c}} \sin\sqrt{(k/\sqrt{c})^2+1}\, s\, u \\
&- \frac{\sqrt{c}k}{k^2+c}\Big(\cos\sqrt{(k/\sqrt{c})^2+1}\, s - 1\Big)v.
\end{aligned}$$

Here, we put $t = s/\sqrt{c}$. Since $\|\dot\sigma(0)\|' = 1/\sqrt{c}\|\dot\sigma(0)\| = u/\sqrt{c}$, we have

$$\begin{aligned}
\gamma(t) = \sigma(\sqrt{c}t) = &\frac{1}{k^2+c}\big(c\cos\sqrt{k^2+c}\, t + k^2\big)p + \frac{\sqrt{c}}{\sqrt{k^2+c}} \sin\sqrt{k^2+c}\, t\, u \\
&- \frac{\sqrt{c}k}{k^2+c}\big(\cos\sqrt{k^2+c}\, t - 1\big)v \\
= &\frac{1}{k^2+c}\big(c\cos\sqrt{k^2+c}\, t + k^2\big)p + \frac{1}{\sqrt{k^2+c}} \sin\sqrt{k^2+c}\, t\, u \\
&- \frac{k}{k^2+c}\big(\cos\sqrt{k^2+c}\, t - 1\big)v.
\end{aligned}$$

As $\gamma(t) = \sigma(\sqrt{c}t)$, if $\sigma(s+s_0) = \sigma(s)$, we see

$$\gamma(t + s_0/\sqrt{c}) = \sigma(\sqrt{c}t + s_0) = \sigma(\sqrt{c}t) = \gamma(t).$$

Since σ is closed of minimal $2\pi/\sqrt{(k/\sqrt{c})^2+1}$, we see that γ is closed of minimal period $2\pi/\sqrt{k^2+c}$.

[3] Circle on a real hyperbolic space

In the third place we study circles on a real hyperbolic space $H^m(-1)$. Regarding $H^m(-1)$ as a sub-manifold of $\mathbb{R}^{m+1}$, we denote Riemannian connections of $H^m(-1)$ and $\mathbb{R}^{m+1}$ by ∇ and $\widetilde{\nabla}$, respectively. We take a circle γ of geodesic curvature k whose Frenet frame is $\{\dot\gamma, Y\}$. We regard this curve as a curve in $\mathbb{R}^{m+1}$. We use the same convention as in the case of $S^m(1)$. For arbitrary vector feilds $X, Y \in \mathfrak{X}\big(H^m(-1)\big)$, we have $\nabla_X Y = \widetilde{\nabla}_X Y - \langle X, Y\rangle \mathcal{N}$ (see §1.3). Hence the system of equations (1.6) turns to

$$\begin{cases} \widetilde{\nabla}_{\dot\gamma}\dot\gamma = kY + \gamma, \\ \widetilde{\nabla}_{\dot\gamma}Y = -k\dot\gamma. \end{cases}$$

We therefore get

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma = k\widetilde{\nabla}_{\dot\gamma}Y + \dot\gamma = -(k^2-1)\dot\gamma,$$

which is equivalent to $\gamma''' + (k^2-1)\gamma' = 0$. Since its characteristic equation is $\lambda^3 + (k^2-1)\lambda = 0$, we have three cases according to properties of its solutions. When $k > 1$ it has two pure imaginary solutions and a real solution, when $k = 1$ all its solutions are null, and when $0 \le k < 1$ it has three real solutions. Thus, we find that γ as a curve in $\mathbb{R}^{m+1}$ is of the following form

$$\gamma(t) = \begin{cases} A + Be^{\sqrt{-(k^2-1)}\,t} + Ce^{-\sqrt{-(k^2-1)}\,t} & \\ \qquad = A + B'\cos\bigl(\sqrt{k^2-1}\,t\bigr) + C'\sin\bigl(\sqrt{k^2-1}\,t\bigr), & \text{if } k > 1, \\ A + Bt + Ct^2, & \text{if } k = 1, \\ A + Be^{\sqrt{1-k^2}t} + Ce^{-\sqrt{1-k^2}t} & \\ \qquad = A + B'\cosh\bigl(\sqrt{1-k^2}\,t\bigr) + C'\sinh\bigl(\sqrt{1-k^2}\,t\bigr), & \text{if } 0 < k < 1. \end{cases}$$

with some $A, B, C, B', C' \in \mathbb{R}^{m+1}$. Under the initial conditions $\gamma(0) = p \in H^m(-1) \subset \mathbb{R}^{m+1}$ and $\dot\gamma(0) = u,\ \nabla_{\dot\gamma}\dot\gamma(0) = kv$ with $u, v \in U_pH^m(-1) \subset T_pH^m(-1) \subset T_p\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1}$, which is equivalent to $\gamma(0) = p,\ \gamma'(0) = u,\ \gamma''(0) = kv + p$, we obtain

$$\gamma(t) = \begin{cases} \dfrac{1}{k^2-1}\bigl(k^2 - \cos\sqrt{k^2-1}\;t\bigr)p + \dfrac{1}{\sqrt{k^2-1}}\sin\sqrt{k^2-1}\;t\;u & \\ \qquad + \dfrac{k}{k^2-1}\bigl(1 - \cos\sqrt{k^2-1}\;t\bigr)v, & \text{if } k > 1, \\ \Bigl(1 + \dfrac{t^2}{2}\Bigr)p + tu + \dfrac{t^2}{2}kv, & \text{if } k = 1, \\ \dfrac{1}{1-k^2}\bigl(\cosh\sqrt{1-k^2}\;t - k^2\bigr)p + \dfrac{1}{\sqrt{1-k^2}}\sinh\sqrt{1-k^2}\;t\;u & \\ \qquad + \dfrac{k}{1-k^2}\bigl(\cosh\sqrt{1-k^2}\;t - 1\bigr)v, & \text{if } 0 < k < 1. \end{cases}$$

Thus, we find that every circle of geodesic curvature k with $k > 1$ on $H^m(-1)$ is closed and is of length $2\pi/\sqrt{k^2-1}$. On the other hand, when $k \le 1$, every circle of geodesic curvature k on $H^m(-1)$ is unbounded.

We here make mention of circles on $H^m(c)$. Let γ be circles of geodesic curvature k on $\big(H^m(c), \langle\ ,\ \rangle'\big)$. Here, the metric $\langle\ ,\ \rangle'$ is given by $\langle\ ,\ \rangle' = \dfrac{1}{|c|}\langle\ ,\ \rangle$ with the canonical metric $\langle\ ,\ \rangle$ on a $H^m(-1)$. If we define a curve σ on $H^m(-1)$ by $\sigma(s) = \gamma(s/\sqrt{|c|})$, then it is a circle of geodesic curvature $k/\sqrt{|c|}$ on $(H^m(-1), \langle\ ,\ \rangle)$ by Lemma 1.12. Hence it is expressed as

$$\sigma(s) = \begin{cases} \dfrac{|c|}{k^2-|c|}\Big(\dfrac{k^2}{|c|} - \cos\dfrac{\sqrt{k^2-|c|}}{\sqrt{|c|}}\,s\Big)p + \dfrac{\sqrt{|c|}}{\sqrt{k^2-|c|}}\sin\dfrac{\sqrt{k^2-|c|}}{\sqrt{|c|}}\,s\,u \\ \qquad + \dfrac{\sqrt{|c|}k}{k^2-|c|}\Big(1-\cos\dfrac{\sqrt{k^2-|c|}}{\sqrt{|c|}}\,s\Big)v, & k>1, \\ \Big(1+\dfrac{s^2}{2}\Big)p + su + \dfrac{s^2}{2}\dfrac{k}{\sqrt{|c|}}v, & k=1, \\ \dfrac{|c|}{|c|-k^2}\Big(\cosh\dfrac{\sqrt{|c|-k^2}}{\sqrt{|c|}}\,s - \dfrac{k^2}{|c|}\Big)p + \dfrac{\sqrt{|c|}}{\sqrt{|c|-k^2}}\sinh\dfrac{\sqrt{|c|-k^2}}{\sqrt{|c|}}s\,u \\ \qquad + \dfrac{\sqrt{|c|}k}{|c|-k^2}\Big(\cosh\dfrac{\sqrt{|c|-k^2}}{\sqrt{|c|}}\,s - 1\Big)v, & 0<k<1. \end{cases}$$

Here, we put $t = s/\sqrt{|c|}$. Since $\|\sigma(0)\|' = 1/\sqrt{|c|}\|\dot\sigma(0)\| = u/\sqrt{|c|}$, we have

$$\gamma(t) = \begin{cases} \dfrac{|c|}{k^2-|c|}\Big(\dfrac{k^2}{|c|} - \cos\dfrac{\sqrt{k^2-|c|}}{\sqrt{|c|}}\,s\Big)p + \dfrac{1}{\sqrt{k^2-|c|}}\sin\dfrac{\sqrt{k^2-|c|}}{\sqrt{|c|}}\,s\,u \\ \qquad + \dfrac{k}{k^2-|c|}\Big(1-\cos\dfrac{\sqrt{k^2-|c|}}{\sqrt{|c|}}\,s\Big)v, & k>1, \\ \Big(1+\dfrac{s^2}{2}\Big)p + \dfrac{s}{\sqrt{|c|}}u + \dfrac{s^2}{2}\dfrac{k}{|c|}v, & k=1, \\ \dfrac{|c|}{|c|-k^2}\Big(\cosh\dfrac{\sqrt{|c|-k^2}}{\sqrt{|c|}}\,s - \dfrac{k^2}{|c|}\Big)p + \dfrac{1}{\sqrt{|c|-k^2}}\sinh\dfrac{\sqrt{|c|-k^2}}{\sqrt{|c|}}\,s\,u \\ \qquad + \dfrac{k}{|c|-k^2}\Big(\cosh\dfrac{\sqrt{|c|-k^2}}{\sqrt{|c|}}\,s - 1\Big)v, & 0<k<1. \end{cases}$$

As $\gamma(t) = \sigma(\sqrt{|c|}t)$, if $\sigma(s+s_0) = \sigma(s)$, we see

$$\gamma(t + s_0/\sqrt{|c|}) = \sigma(\sqrt{|c|t + s_0}) = \sigma(\sqrt{|c|}t) = \gamma(t).$$

Since σ is closed of minimal period $2\pi\Big/\sqrt{\big(k/\sqrt{|c|}\big)^2+1}$, we see γ is closed of minimal period $2\pi/\sqrt{k^2+|c|}$.

CHAPTER 2

Kähler magnetic fields

Our attempt is to study Kähler manifolds by using some smooth curves associated with their complex structure. We define a family of smooth curves so that it include all geodesics. In this section we introduce the notion of magnetic fields so that we can define such a nice family of curves.

1. Kähler manifolds

We shall start by giving the definition of Kähler manifolds.

Let M be a Hausdorff topological space. We call M a *complex manifold* of complex dimension n if we have a family $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in A}$ of pairs of an open set U_α of M and a homeomorphism $\psi_\alpha U_\alpha \to \psi(U_\alpha)$ onto an open subset ψ_α of $\mathbb{C}^n$ satisfying the following conditions:

i) $M = \bigcup_{\alpha\in A} U_\alpha$;

ii) When $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$\psi_\beta \circ \psi_\alpha^{-1} : \mathbb{C}^n \supset \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$$

is holomorphic isomorphism.

We call (U_α, ψ_α) a holomorphic local coordinate neighborhood. If we denote $\psi(p) = \big(z_1(p), \dots, z_n(p)\big)$ the family of these functions $\{z_1, \dots, z_n\}$ is called a holomorphic coordinate system.

We denote as $z_j = x_j + \sqrt{-1}y_j$ by using two real functions x_j, y_j on U_α. Then, with $\big\{\big(U_\alpha, (x_1, y_1, \dots, x_n, y_n)\big)\big\}_{\alpha\in A}$ we see that M is a real $2n$-dimensional real analytic manifold. Hence

$$\Big(\frac{\partial}{\partial x_1}\Big)_q, \Big(\frac{\partial}{\partial y_1}\Big)_q, \dots, \Big(\frac{\partial}{\partial x_n}\Big)_q, \Big(\frac{\partial}{\partial y_n}\Big)_q$$

are a basis of T_qM at $q \in U_\alpha$. We define an linear isomorphism $J_q : T_qM \to T_qM$ by

$$J_q\Big(\frac{\partial}{\partial x_j}\Big)_q = \Big(\frac{\partial}{\partial y_j}\Big)_q, \quad J_q\Big(\frac{\partial}{\partial y_j}\Big)_q = -\Big(\frac{\partial}{\partial x_j}\Big)_q \qquad (1 \le j \le n).$$

We note that this linear isomorphism does not depend on the choice of holomorphic coordinates. As a matter of fact, if $\big(U_\beta, (w_1, \dots, w_n)\big)$ is also a holomorphic coordinate around q, we denote as $w_j = u_j + \sqrt{-1}v_j$ by using real functions u_j, v_j. By Caucy-Riemann equations, we have

$$\frac{\partial x_j}{\partial u_k} = \frac{\partial y_j}{\partial v_k}, \quad \frac{\partial y_j}{\partial u_k} = -\frac{\partial x_j}{\partial v_k} \qquad (1 \le j,\ k \le n)$$

on $U_\alpha \cap U_\beta$. As we have

$$\begin{aligned}
\Big(\frac{\partial}{\partial u_k}\Big)_q &= \sum_{j=1}^n \Big\{ \Big(\frac{\partial x_j}{\partial u_k}\Big)(q)\, \Big(\frac{\partial}{\partial x_j}\Big)_q + \Big(\frac{\partial y_j}{\partial u_k}\Big)(q)\, \Big(\frac{\partial}{\partial y_j}\Big)_q \Big\}, \\
\Big(\frac{\partial}{\partial v_k}\Big)_q &= \sum_{j=1}^n \Big\{ \Big(\frac{\partial x_j}{\partial v_k}\Big)(q)\, \Big(\frac{\partial}{\partial x_j}\Big)_q + \Big(\frac{\partial y_j}{\partial v_k}\Big)(q)\, \Big(\frac{\partial}{\partial y_j}\Big)_q \Big\},
\end{aligned}$$

we find

$$\begin{aligned}
J_q\Big(\frac{\partial}{\partial u_k}\Big)_q &= \sum_{j=1}^n \Big\{ \Big(\frac{\partial x_j}{\partial u_k}\Big)(q)\, \Big(\frac{\partial}{\partial y_j}\Big)_q - \Big(\frac{\partial y_j}{\partial u_k}\Big)(q)\, \Big(\frac{\partial}{\partial x_j}\Big)_q \Big\} \\
&= \sum_{j=1}^n \Big\{ \Big(\frac{\partial y_j}{\partial v_k}\Big)(q)\, \Big(\frac{\partial}{\partial y_j}\Big)_q + \Big(\frac{\partial x_j}{\partial v_k}\Big)(q)\, \Big(\frac{\partial}{\partial x_j}\Big)_q \Big\} = \Big(\frac{\partial}{\partial v_k}\Big)_q, \\
J_q\Big(\frac{\partial}{\partial v_k}\Big)_q &= \sum_{j=1}^n \Big\{ \Big(\frac{\partial x_j}{\partial v_k}\Big)(q)\, \Big(\frac{\partial}{\partial y_j}\Big)_q - \Big(\frac{\partial y_j}{\partial v_k}\Big)(q)\, \Big(\frac{\partial}{\partial x_j}\Big)_q \Big\} \\
&= \sum_{j=1}^n \Big\{ -\Big(\frac{\partial y_j}{\partial u_k}\Big)(q)\, \Big(\frac{\partial}{\partial y_j}\Big)_q - \Big(\frac{\partial x_j}{\partial u_k}\Big)(q)\, \Big(\frac{\partial}{\partial x_j}\Big)_q \Big\} = -\Big(\frac{\partial}{\partial u_k}\Big)_q.
\end{aligned}$$

Thus, the definition of J_q does not depend on the choice of holomorphic coordinates. Clearly, we have $J_q^2 = -\mathrm{Id}_{T_qM}$. We define an endomorphism $J : TM \to TM$ so that its restriction onto T_pM at each $p \in M$ is J_p, and call it the *complex structure* of M.

A Riemannian metric $\langle\ ,\ \rangle$ on a complex manifold (M, J) with complex structure J is said to be *Hermitian* if it satisfies $\langle Jv, Jw\rangle = \langle v, w\rangle$ for arbitrary $v, w \in T_pM$ at an arbitrary point $p \in M$. We say a complex manifold with Hermitian metric to be a Hermitian manifold. Given a Riemannian metric $\langle\ ,\ \rangle$ on a complex manifold (M, J), if we define $\langle\ ,\ \rangle'$ by

$$\langle v, w\rangle' = \frac{1}{2}\big\{\langle v, w\rangle + \langle Jv, Jw\rangle\big\},$$

then it is a Hermitian metric on M.

On a complex Riemannian manifold $(M, J, \langle\ ,\ \rangle)$ with complex structure J, we define a 2-form $\mathbb{B}_J$ by $\mathbb{B}_J(u, v) = \langle u, Jv\rangle$ for all $u, v \in T_pM$ at an arbitrary point $p \in M$. We say this 2-form $\mathbb{B}_J$ to be a *Kähler form.* When the Kähler form is closed, we call $\langle\ ,\ \rangle$ a Kähler metric. A complex Riemannian manifold with Kähler metric is called a *Kähler manifold*.

On a smooth manifold M, we call a linear isomorphism $J : TM \to TM$ an *almost complex structure* if it satisfies $J^2 = -\mathrm{Id}_{TM}$.

LEMMA 2.1. *Let M be a Hermitian manifold with almost complex structure J.*

(1) *When J is parallel with respect to the Riemannian connection, then M is a Kähler manifold.*

(2) *The complex structure J of a Kähler manifold M is parallel.*

PROOF. For arbitrary vector fields $X, Y, Z \in \mathfrak{X}(M)$ on M, we have

$$(d\mathbb{B}_J)(X, Y, Z) = (\nabla_X\mathbb{B}_J)(Y, Z) - (\nabla_Y\mathbb{B}_J)(X, Z) + (\nabla_Z\mathbb{B}_J)(X, Y).$$

Since we have

$$\begin{aligned}(\nabla_X\mathbb{B}_J)(Y, Z) &= \nabla_X\big(\mathbb{B}_J(Y, Z)\big) - \mathbb{B}_J(\nabla_X Y, Z) - \mathbb{B}_J(Y, \nabla_X Z)\\ &= \nabla_X\langle Y, JZ\rangle - \langle\nabla_X Y, JZ\rangle - \langle Y, J(\nabla_X Z)\rangle\\ &= \langle Y, \nabla_X(JZ)\rangle - \langle Y, J(\nabla_X Z)\rangle\\ &= \langle Y, (\nabla_X J)Z\rangle,\end{aligned}$$

we find

$$(d\mathbb{B}_J)(X, Y, Z) = \langle Y, (\nabla_X J)Z\rangle - \langle X, (\nabla_Y J)Z\rangle + \langle X, (\nabla_Z J)Y\rangle.$$

Thus if J is parallel, then $\mathbb{B}_J$ is closed.

On the other hand, we shall show that $\big\langle(\nabla_u J)v, w\big\rangle = 0$ for arbitrary vectors $u, v, w \in T_pM$ at an arbitrary point $p \in M$. We extend u, v, w to a vector fields

X, Y, Z which are defined in some neighborhood of p. That is, X, Y, Z are local vector fields satisfying $X(p) = u$, $Y(p) = v$, $Z(p) = w$. We may suppose (by using local coordinates) that $[X, Y] = [X, Z] = [X, JY] = [X, JZ] = [Y, JZ] = [JY, Z] = 0$. First, we have

$$\langle(\nabla_X J)Y, Z\rangle = \langle\nabla_X(JY), Z\rangle - \langle J(\nabla_X Y), Z\rangle = \langle\nabla_X(JY), Z\rangle + \langle\nabla_X Y, JZ\rangle.$$

By using the Koszul formula (Lemma 1.1), we have

$$\begin{aligned}
2\langle\nabla_X(JY), Z\rangle &= X\langle JY, Z\rangle + JY\langle X, Z\rangle - Z\langle X, JY\rangle \\
&\qquad + \langle[X, JY], Z\rangle - \langle[X, Z], Y\rangle - \langle[JY, Z], X\rangle \\
&= X\langle JY, Z\rangle + JY\langle X, Z\rangle - Z\langle X, JY\rangle \\
&= -\nabla_X\big(\mathbb{B}_J(Y, Z)\big) - \nabla_{JY}\big(\mathbb{B}_J(X, JZ)\big) - \nabla_Z\big(\mathbb{B}_J(X, Y)\big) \\
2\langle\nabla_X Y, JZ\rangle &= X\langle Y, JZ\rangle + Y\langle X, JZ\rangle - JZ\langle X, Y\rangle \\
&\qquad + \langle[X, Y], JZ\rangle - \langle[X, JZ], Y\rangle - \langle[Y, JZ], X\rangle \\
&= X\langle Y, JZ\rangle + Y\langle X, JZ\rangle - JZ\langle X, Y\rangle \\
&= \nabla_X\big(\mathbb{B}_J(JY, JZ)\big) + \nabla_Y\big(\mathbb{B}_J(X, Z)\big) + \nabla_{JZ}\big(\mathbb{B}_J(X, JY)\big)
\end{aligned}$$

We hence obtain

$$\begin{aligned}
2\langle(\nabla_X J)Y, Z\rangle &= -\big\{\nabla_X\big(\mathbb{B}_J(Y, Z)\big) - \nabla_Y\big(\mathbb{B}_J(X, Z)\big) + \nabla_Z\big(\mathbb{B}_J(X, Y)\big)\big\} \\
&\qquad + \big\{\nabla_X\big(\mathbb{B}_J(JY, JZ)\big) - \nabla_{JY}\big(\mathbb{B}_J(X, JZ)\big) + \nabla_{JZ}\big(\mathbb{B}_J(X, JY)\big)\big\} \\
&= -(d\mathbb{B}_J)(X, Y, Z) + (d\mathbb{B}_J)(X, JY, JZ).
\end{aligned}$$

Thus if $\mathbb{B}_J$ is closed, then we have $\langle(\nabla_u J)v, w\rangle = 0$ for arbitrary vectors $u, v, w \in T_pM$ at an arbitrary point $p \in M$. Therefore we get $(\nabla_u J)v = 0$ for arbitrary vectors u, v at an arbitrary point $p \in M$, which shows that J is parallel. $\square$

2. Complex space forms

We here give complex space forms which correspond to real space forms. Complex space forms are complex Euclidean spaces, complex projective spaces and complex hyperbolic spaces.

[1] Complex Euclidean spaces.

We denote by $\mathbb{C}$ the field of all complex numbers. An n-dimensional complex Euclidean space

$$\mathbb{C}^n = \{(z_1, z_2, \ldots, z_n) \mid z_j \in \mathbb{C},\ j = 1, 2, \ldots, n\}$$

is a direct n-product of complex lines. On this space, we have a canonical Hermitian inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ given by

$$\langle\!\langle z, w \rangle\!\rangle = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n$$

for $z = (z_1, z_2, \ldots, z_n), w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$. Here, we denote by $\bar{z}$ the complex conjugate of a complex number z, that is, if we denote as $z = x + \sqrt{-1}y$ with real numbers x, y, we set $\bar{z} = x - \sqrt{-1}y$. The canonical Riemannian metric is the real part of this Hermitian product : We set $\langle z, w\rangle = \mathrm{Re}\langle\!\langle z, w\rangle\!\rangle$. Here, for a complex number $z = x + \sqrt{-1}y$ we denote by $\mathrm{Re}(z)$ the real part of z, which means $\mathrm{Re}(z) = x$. As a Riemannian manifold, a complex Euclidean space $\mathbb{C}^n$ is isometric to a $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ by the map

$$(z_1, z_2, \ldots, z_n) \mapsto (x_1, y_1, x_2, y_2, \ldots, x_n, y_n),$$

where we denote a complex number z_j as $z_j = x_j + \sqrt{-1}y_j$ with two real numbers x_j, y_j, because we have

$$\begin{aligned}\langle\!\langle z, w\rangle\!\rangle &= \big(x_1 + \sqrt{-1}y_1\big)\big(u_1 - \sqrt{-1}v_1\big) + \cdots + \big(x_n + \sqrt{-1}y_n\big)\big(u_n - \sqrt{-1}v_n\big)\\ &= \{(x_1u_1 + y_1v_1) + \cdots + (x_nu_n + y_nv_n)\}\\ &\qquad + \sqrt{-1}\{(y_1u_1 - x_1v_1) + \cdots + (y_nu_n - x_nv_n)\},\end{aligned}$$

where we denote as $w_j = u_j + \sqrt{-1}v_j$ with $u_j, v_j \in \mathbb{R}$. Thus, the covariant differentiation with respect to the Riemannian connection is the ordinary differentiation.

As $T_p\mathbb{C}^n = \{p\} \times \mathbb{C}^n \cong \mathbb{C}^n$ at each point $p \in \mathbb{C}^n$, we define $J : T_p\mathbb{C}^n \to T_p\mathbb{C}^n$ by $Jv = \sqrt{-1}v$. Then it is a complex structure on $\mathbb{C}^n$. Since it is clearly parallel with respect to the canonical metric, we see $\mathbb{C}^n$ is a Kähler manifold.

[2] Complex projective spaces

We consider a unit sphere

$$S^{2n+1} = \bigl\{z \in \mathbb{C}^{n+1} \bigm| \langle\!\langle z, z\rangle\!\rangle = 1\bigr\} = \bigl\{z \in \mathbb{C}^{n+1} \bigm| \|z\| = 1\bigr\}$$

in a complex Euclidean space $\mathbb{C}^{n+1}$ with respect to the canonical Hermitian inner product $\langle\!\langle\ ,\ \rangle\!\rangle$. A unit circle $S^1 = \{\lambda \in \mathbb{C} \bigm| |\lambda| = 1\}$ acts on S^{2n+1} as $\lambda \cdot z := \lambda z = (\lambda z_0, \lambda z_1, \cdots, \lambda z_n)$. We denote by $\mathbb{C}P^n$ the quotient space S^{2n+1}/S^1 of S^{2n+1} under this action, and call it an n-dimensional *complex projective space*. Here, a quotient space of the action means the following. We say two points $z, w \in S^{2n+1}$ are equivalent to each other if there is $\lambda \in S^1$ with $w = \lambda z$. The quotient space is the set of all equivalence classes. We define a projection by

$$\varpi : S^{2n+1} \ni z = (z_0, z_1, \dots, z_n) \mapsto [z] = [(z_0, z_1, \dots, z_n)] \in \mathbb{C}P^n,$$

where $[z]$ denotes the equivalence class containing z. We call the pair (S^{2n+1}, ϖ) a *Hopf fibration*. We note that we can construct a complex projective space as a quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$, where $z, w \in \mathbb{C}^{n+1} \setminus \{0\}$ are equivalent to each other if and only if there is $\alpha \in \mathbb{C} \setminus \{0\}$ with $w = \alpha z$. We can hence express a point of $\mathbb{C}P^n$ as $[z_0, z_1, \dots, z_n]$ with $(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. By definition we have

$$[z_0, z_1, \dots, z_n] = [\alpha z_0, \alpha z_1, ..., \alpha z_n]$$

for an arbitrary $\alpha \in \mathbb{C} \setminus \{0\}$. We call this expression $[z_0, z_1, \dots, z_n]$ the homogeneous coordinate of $\mathbb{C}P^n$.

We now introduce a Riemannian metric and a complex structure on this quotient manifold. We express the tangent space T_zS^{2n+1} of a standard sphere S^{2n+1} in $\mathbb{C}^{n+1}$ as

$$T_zS^{2n+1} = \bigl\{(z, u) \in \{z\} \times \mathbb{C}^{n+1} \mid \langle z, u\rangle = 0\bigr\},$$

where $\langle z, u\rangle = \mathrm{Re}\langle\!\langle z, u\rangle\!\rangle$ denotes the canonical Riemannian metric on $\mathbb{C}^{n+1}$. We decompose it into horizontal and vertical subspaces as $T_zS^{2n+1} = \mathcal{H}_z \oplus \mathcal{V}_z$ with respect to the projection ϖ. That is, the vertical space $\mathcal{V}_z$ is the tangent line generated by the action of S^1, hence is expressed as

$$\mathcal{V}_z = \big\{(z, \sqrt{-1}az) \in T_zS^{2n+1} \mid a \in \mathbb{R}\big\},$$

and the horizontal space $\mathcal{H}_z$ is the orthogonal complement of the vertical space, hence is expressed as

$$\mathcal{H}_z = \big\{(z, u) \in T_zS^{2n+1} \mid \langle\!\langle z, u\rangle\!\rangle = 0\big\}.$$

The action of S^1 onto S^{2n+1} induces an action on TS^{2n+1}, which is given as $(z, v) \mapsto (\lambda z, \lambda v)$ for an arbitrary $\lambda \in S^1 \subset \mathbb{C}$.

The horizontal subspace $\mathcal{H}_z$ is a complex subspace of $T_z\mathbb{C}^{n+1}$. That is to say, for a horizontal tangent vector $(z, v) \in \mathcal{H}_z$ we see $\sqrt{-1}\cdot(z, v) = (z, \sqrt{-1}v)$ is also contained in $\mathcal{H}_z$. Identifying $T_{\varpi(z)}\mathbb{C}P^n$ with $\mathcal{H}_z$ at each point $z \in S^{2n+1}$, we define $J : T_{\varpi(z)}\mathbb{C}P^n \to T_{\varpi(z)}\mathbb{C}P^n$ by $Jd\varpi((z, v)) = d\varpi((z, \sqrt{-1}v))$. Since we have $\lambda\sqrt{-1}v = \sqrt{-1}\lambda v$ for an arbitrary $\lambda \in S^1 \subset \mathbb{C}$, we find that J is well defined. As $J^2 = -I$ is clear, we see this J is a complex structure on $\mathbb{C}P^n$.

We define a Riemannian metric of $\mathbb{C}P^n$ by

$$\big\langle [z, u], [z, v]\big\rangle = \mathrm{Re}\langle\!\langle u, v\rangle\!\rangle,$$

where $(z, u), (z, v) \in \mathcal{H}_z$. We note that if $(w, u'), (w, v') \in \mathcal{H}_z$ satisfy $[(w, u')] = [(z, u)]$, $[(w, v')] = [(z, v)]$. We have $\lambda \in S^1$ with $u' = \lambda u, v' = \lambda v$. Thus, we have

$$\mathrm{Re}\langle\!\langle u', v'\rangle\!\rangle = \mathrm{Re}\langle\!\langle \lambda u, \lambda v\rangle\!\rangle = \mathrm{Re}(|\lambda|^2\langle\!\langle u, v\rangle\!\rangle) = \mathrm{Re}\langle\!\langle u, v\rangle\!\rangle$$

and find that our metric is well defined. This Riemannian metric on $\mathbb{C}P^n$ is call the *Fubini-Study metric.*

We denote by $\mathcal{N}$ the outward unit normal vector field of $S^{2n+1}(1)$ in $\mathbb{C}^{n+1}$.

Lemma 2.2. *Let ∇ and $\widetilde{\nabla}$ the Riemannian connections of $\mathbb{C}P^n$ with the Fubini-Study metric and of $S^{2n+1}(1)$, respectively. For $X, Y \in \mathfrak{X}(\mathbb{C}P^n)$ we take their horizontal lifts $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(S^{2n+1}(1))$. Then the horizontal lift $\widetilde{\nabla_X Y}$ of $\nabla_X Y$ satisfies*

$$\widetilde{\nabla_X Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \langle X, JY\rangle J\mathcal{N}. \tag{2.1}$$

Hence if we denote by $\overline{\nabla}$ the Riemannian connection of $\mathbb{C}^{n+1}$, we have

$$\widetilde{\nabla_X Y} = \overline{\nabla}_{\widetilde{X}}\widetilde{Y} + \langle X, Y\rangle\mathcal{N} - \langle X, JY\rangle J\mathcal{N}. \tag{2.2}$$

Proof. By definition of Hopf fibration, we see $\widetilde{\nabla_X Y}$ is obtained by removing the vertical component of $\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}$. Since $\langle \widetilde{Y}, J\mathcal{N}\rangle = 0$ and $\mathcal{N}_{\tilde{p}}$ can be identified with the position vector $\tilde{p} \in S^{2n+1}$, by Lemma 1.4 we have

$$\begin{aligned}
0 &= \langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, J\mathcal{N}\rangle + \langle \widetilde{Y}, \widetilde{\nabla}_{\widetilde{X}}(J\mathcal{N})\rangle = \langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, J\mathcal{N}\rangle + \Big\langle \widetilde{Y}, \overline{\nabla}_{\widetilde{X}}(J\mathcal{N}) + \langle \widetilde{X}, J\mathcal{N}\rangle\mathcal{N}\Big\rangle \\
&= \langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, J\mathcal{N}\rangle + \langle \widetilde{Y}, J\overline{\nabla}_{\widetilde{X}}\mathcal{N}\rangle = \langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, J\mathcal{N}\rangle + \langle \widetilde{Y}, J\widetilde{X}\rangle = \langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, J\mathcal{N}\rangle - \langle \widetilde{X}, J\widetilde{Y}\rangle.
\end{aligned}$$

Thus we obtain

$$\widetilde{\nabla_X Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, J\mathcal{N}\rangle J\mathcal{N} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \langle X, JY\rangle J\mathcal{N}.$$

The second relation follows directly from Lemma 1.4. □

Corollary 2.1. *The complex structure J on $\mathbb{C}P^n$ is parallel with respect to the Fubini-Study metric.*

Proof. We take arbitrary vector fields $X, Y \in \mathfrak{X}(\mathbb{C}P^n)$. We denote by $\widetilde{X}, \widetilde{Y}$ their horizontal lifts. By definition of the complex structure J on $\mathbb{C}P^n$ we see that the horizontal lift $\widetilde{JY}$ of JY coincides with $\widetilde{J}\widetilde{Y}$, where $\widetilde{J}$ denotes the complex structure on $\mathbb{C}^{n+1}$. By (2.2), we have

$$\begin{aligned}
\widetilde{\nabla_X(JY)} &= \overline{\nabla}_{\widetilde{X}}(\widetilde{J}\widetilde{Y}) + \langle X, JY\rangle\mathcal{N} + \langle X, Y\rangle\widetilde{J}\mathcal{N} \\
&= \widetilde{J}\overline{\nabla}_{\widetilde{X}}\widetilde{Y} + \langle X, JY\rangle\mathcal{N} + \langle X, Y\rangle\widetilde{J}\mathcal{N} \\
&= \widetilde{J}\widetilde{\nabla_X Y} - \big\{\langle X, Y\rangle\widetilde{J}\mathcal{N} + \langle X, JY\rangle\mathcal{N}\big\} + \langle X, JY\rangle\mathcal{N} + \langle X, Y\rangle\widetilde{J}\mathcal{N} = \widetilde{J\nabla_X Y}.
\end{aligned}$$

This shows that $\nabla_X(JY) = J\nabla_X Y$. We hence find that J is parallel. □

By using the relationship on connections we can express the curvature tensor of $\mathbb{C}P^n$.

LEMMA 2.3. *The curvature tensor on a complex space form $\mathbb{C}P^n$ satisfies*

$$R(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y - \langle Y,JZ\rangle JX + \langle X,JZ\rangle JY + 2\langle X,JY\rangle JZ.$$

PROOF. For vector fields $X,Y,Z \in \mathfrak{X}(\mathbb{C}P^n)$, we denote their horizontal lifts on S^n also by X,Y,Z for simplicity. By Lemma 2.2, we find

$$\begin{aligned}\nabla_X\nabla_Y Z &= \widetilde{\nabla}_X(\nabla_Y Z) - \langle X, J\nabla_Y Z\rangle J\mathcal{N}\\ &= \widetilde{\nabla}_X\big(\widetilde{\nabla}_Y Z - \langle Y,JZ\rangle J\mathcal{N}\big) - \langle X, J\nabla_Y Z\rangle J\mathcal{N}\\ &= \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - X\langle Y,JZ\rangle J\mathcal{N} - \langle Y,JZ\rangle JX - \langle X,J\nabla_Y Z\rangle J\mathcal{N},\end{aligned} \tag{2.3}$$

where $\mathcal{N}$ denotes the outward unit normal of $S^{2n+1}(1)$ in $\mathbb{C}^{n+1}$, and $\widetilde{\nabla}$ denotes the Riemannian connection on S^{2n+1}. Here, we note

$$\widetilde{\nabla}_X(J\mathcal{N}) = \overline{\nabla}_X(J\mathcal{N}) + \langle X, J\mathcal{N}\rangle\mathcal{N} = J\overline{\nabla}_X\mathcal{N} = JX.$$

We denote by $[\ ,\ ]_P$ and $[\ ,\ ]_S$ bracket products of $\mathbb{C}P^n$ and S^{2n+1}, respectively. They are related with each other by the following equality :

$$\begin{aligned}[X,Y]_P &= \nabla_X Y - \nabla_Y X = \widetilde{\nabla}_X Y - \langle X,JY\rangle J\mathcal{N} - \widetilde{\nabla}_Y X + \langle Y,JX\rangle J\mathcal{N}\\ &= [X,Y]_S - 2\langle X,JY\rangle J\mathcal{N},\end{aligned}$$

By using this, we have

$$\begin{aligned}\nabla_{[X,Y]_P} Z &= \nabla_Z [X,Y]_P + \big[[X,Y]_P, Z\big]_P\\ &= \widetilde{\nabla}_Z[X,Y]_P - \big\langle Z, J[X,Y]_P\big\rangle J\mathcal{N} + \big[[X,Y]_P, Z\big]_P\\ &= \widetilde{\nabla}_Z\big([X,Y]_S - 2\langle X,JY\rangle J\mathcal{N}\big) + \big\langle JZ, [X,Y]_P\big\rangle J\mathcal{N} + \big[[X,Y]_P, Z\big]_P\\ &= \widetilde{\nabla}_{[X,Y]_S} Z - \big[[X,Y]_S, Z\big]_S - 2\widetilde{\nabla}_Z\big(\langle X,JY\rangle J\mathcal{N}\big)\\ &\qquad + \big\langle JZ, [X,Y]_P\big\rangle J\mathcal{N} + \big[[X,Y]_P, Z\big]_S - 2\big\langle [X,Y]_P, JZ\big\rangle J\mathcal{N}\end{aligned}$$

Applying the properties of bracket product, we have

$$\begin{aligned}
[X,Y]_P &= \widetilde{\nabla}_{[X,Y]_S}Z - \big[[X,Y]_P, Z\big]_S - 2\big[\langle X, JY\rangle J\mathcal{N}, Z\big]_S - 2\widetilde{\nabla}_Z\big(\langle X, JY\rangle J\mathcal{N}\big) \\
&\qquad + \big\langle JZ, [X,Y]_P\big\rangle J\mathcal{N} + \big[[X,Y]_P, Z\big]_S - 2\big\langle [X,Y]_P, JZ\big\rangle J\mathcal{N} \\
&= \widetilde{\nabla}_{[X,Y]_S}Z - 2\Big(\langle X, JY\rangle \widetilde{\nabla}_{J\mathcal{N}}Z - \widetilde{\nabla}_Z\big(\langle X, JY\rangle J\mathcal{N}\big)\Big) - 2\widetilde{\nabla}_Z\big(\langle X, JY\rangle J\mathcal{N}\big) \\
&\qquad - \big\langle JZ, [X,Y]_P\big\rangle J\mathcal{N} \\
&= \widetilde{\nabla}_{[X,Y]_S}Z - 2\langle X, JY\rangle\big(JZ + [J\mathcal{N}, Z]_S\big) - \big\langle JZ, [X,Y]_P\big\rangle J\mathcal{N}.
\end{aligned}$$

Here, as we have

$$\begin{aligned}
[J\mathcal{N}, Z]_S &= \widetilde{\nabla}_{J\mathcal{N}}Z - \widetilde{\nabla}_Z(J\mathcal{N}) \\
&= \overline{\nabla}_{J\mathcal{N}}Z - \overline{\nabla}_Z(J\mathcal{N}) + (\langle J\mathcal{N}, Z\rangle - \langle Z, J\mathcal{N}\rangle)\mathcal{N} = 0,
\end{aligned}$$

we find

$$\nabla_{[X,Y]_P}Z = \widetilde{\nabla}_{[X,Y]_S}Z - \big\langle JZ, [X,Y]_P\big\rangle J\mathcal{N} - 2\langle X, JY\rangle JZ \tag{2.4}$$

Therefore, by (2.3), (2.4) we obtain

$$\begin{aligned}
R(X,Y)Z &= \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - (X\langle Y, JZ\rangle - \langle X, J\nabla_Y Z\rangle)J\mathcal{N} - \langle Y, JZ\rangle JX \\
&\qquad - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z + (Y\langle X, JZ\rangle + \langle Y, J\nabla_X Z\rangle)J\mathcal{N} + \langle X, JZ\rangle JY \\
&\qquad - \widetilde{\nabla}_{[X,Y]_S}Z + \langle JZ, [X,Y]_P\rangle J\mathcal{N} + 2\langle X, JY\rangle JZ \\
&= \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]_S}Z \\
&\qquad + \big\{-X\langle Y, JZ\rangle - \langle X, J\nabla_Y Z\rangle + Y\langle X, JZ\rangle \\
&\qquad\quad + \langle Y, J\nabla_X Z\rangle + \langle JZ, [X,Y]_P\rangle\big\}J\mathcal{N} \\
&\qquad - \langle Y, JZ\rangle JX + \langle X, JZ\rangle JY + 2\langle X, JY\rangle JZ.
\end{aligned}$$

Here, as we see in [2] of §1.2, the curvature tensor $\widetilde{R}$ of a standard sphere $S^{2n+1}(1)$ is given as $\widetilde{R}(X,Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$. We hence get

$$R(X,Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y - \langle Y, JZ\rangle JX + \langle X, JZ\rangle JY + 2\langle X, JY\rangle JZ.$$

This completes the proof. □

As a consequence of Lemma 2.3 for a unit tangent vector $v \in U\mathbb{C}P^n$ we have

$$\begin{aligned}
\langle R(v, Jv)Jv, v\rangle &= \langle Jv, Jv\rangle\langle v, v\rangle - \langle v, Jv\rangle\langle Jv, v\rangle - \langle Jv, -v\rangle\langle Jv, v\rangle \\
&\qquad + \langle v, -v\rangle\langle -v, v\rangle + 2\langle v, -v\rangle\langle -v, v\rangle = 4.
\end{aligned}$$

Thus $\mathbb{C}P^n$ endowed with the Fubini-Study metric has constant holomorphic sectional curvature 4. We hence denote this Kähler manifold by $\mathbb{C}P^n(4)$. When we consider a metric on $\mathbb{C}P^n$ given by $\langle\ ,\ \rangle' = \dfrac{4}{c}\langle\ ,\ \rangle$ with a positive c and the Fubini-Study metric, we see it has constant holomorphic sectional curvature c by Lemma 1.3. We denote this Kähler manifold by $\mathbb{C}P^n(c)$. The curvature tensor R of $\mathbb{C}P^n(c)$ is hence expressed as

$$R(X,Y)Z = \frac{c}{4}\bigl\{\langle Y,Z\rangle X - \langle X,Z\rangle Y - \langle Y,JZ\rangle JX + \langle X,JZ\rangle JY + 2\langle X,JY\rangle JZ\bigr\} \tag{2.5}$$

by Lemma 2.3.

If we take unit tangent vectors $v, w \in T_p\mathbb{C}P^n(4)$ which satisfy $\langle v,w\rangle = \langle v,Jw\rangle = 0$, that is, v, w are orthonormal vectors which span a real vector subspace, we have

$$\begin{aligned}\langle R(v,w)w,v\rangle &= \langle w,w\rangle\langle v,v\rangle - \langle v,w\rangle\langle w,v\rangle - \langle w,Jw\rangle\langle Jv,v\rangle \\ &\qquad + \langle v,Jw\rangle\langle Jw,v\rangle + 2\langle v,Jw\rangle\langle Jw,v\rangle = 1.\end{aligned}$$

Hence for unit tangent vectors $v, w \in T_p\mathbb{C}P^n(c)$ which satisfy $\langle v,w\rangle = \langle v,Jw\rangle = 0$ we have $\langle R(v,w)w,v\rangle = c/4$.

Lemma 2.4. *For an arbitrary point $p \in \mathbb{C}P^n$ and an arbitrary unit tangent vector $v \in T_p\mathbb{C}P^n$ we have a totally geodesic $\mathbb{C}P^1\bigl(\subset \mathbb{C}P^n\bigr)$ satisfying $p \in \mathbb{C}P^1$ and $v \in T_p\mathbb{C}P^1$.*

Proof. Let $\varpi : S^{2n+1} \to \mathbb{C}P^n$ be a Hopf fibration of a unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$. We take a point $z \in S^{2n+1}$ and a horizontal vector $(z,u) \in \mathcal{H}_z \subset T_zS^{2n+1} \subset T_z\mathbb{C}^{n+1}$ so that they satisfy $\varpi(z) = p$ and $d\varpi(z,u) = v$. Since $\|z\| = \|u\| = 1$ and $\langle\!\langle z,u\rangle\!\rangle = 0$, we see that the subset $\widehat{N} = \{\mu z + \nu u \mid \mu,\nu \in \mathbb{C},\ |\mu|^2 + |\nu|^2 = 1\}$ of S^{2n+1} is a three dimensional standard sphere S^3 in $\mathbb{C}^2 = \mathbb{C}z \oplus \mathbb{C}u \subset \mathbb{C}^{n+1}$. Thus we find that $N = \varpi(\widehat{N})$ is $\mathbb{C}P^1$ by its construction. As $z \in \widehat{N}$ and the horizontal part of $T_z\widehat{N}$ is $\mathbb{C}u$, we see $p \in N$ and $v \in T_pN$. Since the outward normal $\mathcal{N}$ is identified with p by regarding it as a unit vector, by definition of covariant differentiations of N and $\mathbb{C}P^n$ we see N is totally geodesic in $\mathbb{C}P^n$. $\square$

LEMMA 2.5. *For arbitrary points $p, p' \in \mathbb{C}P^n$ and arbitrary unit tangent vectors $v \in T_p\mathbb{C}P^n$, $v' \in T_{p'}\mathbb{C}P^n$ we have a holomorphic isometry φ satisfying $\varphi(p) = p'$ and $d\varphi(v) = v'$.*

PROOF. We take $z, z' \in S^{2n+1} \subset \mathbb{C}^{n+1}$ so that $\varpi(z) = p$, $\varpi(z') = p'$. Also, we take horizontal vectors $(z, u) \in T_z S^{2n+1}$ and $(z', u') \in T_{z'} S^{2n+1}$ satisfying $d\varpi((z, u)) = v$ and $d\varpi((z', u')) = v'$. Then both the pairs $\{z, u\}$ and $\{z', u'\}$ are $\mathbb{C}$-linearly independent. Thus we can take two sets of orthonormal vectors $u_2, \ldots, u_n \in \mathbb{C}^{n+1}$ and $u_2', \ldots, u_n' \in \mathbb{C}^{n+1}$ so that both $\{z, u, u_2, \ldots, u_n\}$ and $\{z', u', u_2', \ldots, u_n'\}$ are $\mathbb{C}$-linearly independent and are orthonormal. If we express them by vertical vectors, then two matrices $(z, u, u_2, \ldots, u_n)$ and $(z', u', u_2', \ldots, u_n')$ are unitary. We set a unitary matrix $A \in U(n + 1)$ by

$$A = (z', u', u_2', \ldots, u_n') \cdot (z, u, u_2, \ldots, u_n)^{-1}.$$

As $A(z, u, u_2, \ldots, u_n) = (z', u', u_2', \ldots, u_n')$, this induces a $\mathbb{C}$-linear transformation $\hat{\varphi} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ satisfying $\hat{\varphi}(z) = z'$ and $\hat{\varphi}(u) = u'$. Since A is unitary, we have $\hat{\varphi} \circ \sqrt{-1} = \sqrt{-1} \circ \hat{\varphi}$ and $\hat{\varphi}(S^{2n+1}) = S^{2n+1}$. We hence find that $\hat{\varphi}$ induces a bijection $\varphi : \mathbb{C}P^n \to \mathbb{C}P^n$ satisfying $d\varphi \circ J = J \circ d\varphi$. As $\hat{\varphi}$ preserves the Hermitian inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ because A is unitary, we see that φ is an isometry. $\square$

REMARK 2.1. For arbitrary points $p, p' \in \mathbb{C}P^n$ and arbitrary unit tangent vectors $v \in T_p\mathbb{C}P^n$, $v' \in T_{p'}\mathbb{C}P^n$ we can construct an anti-holomorphic isometry φ satisfying $\varphi(p) = p'$ and $d\varphi(v) = v'$.

PROPOSITION 2.1. *A complex projective line $\mathbb{C}P^1(c)$ is isomorphic to a standard sphere $S^2(c)$.*

PROOF. We define $\varphi : \mathbb{C}P^1 \to S^2 \subset R^3$ by

$$\begin{aligned}\varphi\big(\varpi((z_0, z_1))\big) &= \big(|z_0|^2 - |z_1|^2,\ 2\mathrm{Re}(\bar{z}_0 z_1),\ 2\mathrm{Im}(\bar{z}_0 z_1)\big)\\ &= \big(\bar{z}_0 z_0 - \bar{z}_1 z_1, \bar{z}_0 z_1 + z_0 \bar{z}_1, \sqrt{-1}(-\bar{z}_0 z_1 + z_0 \bar{z}_1)\big).\end{aligned}$$

To simplify notations by considering $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$ we can express φ as

$$\varphi\big(\varpi((z_0, z_1))\big) = \big(|z_0|^2 - |z_1|^2,\ 2\bar{z}_0 z_1\big).$$

We adopt this expression in this proof. We note that

$$(|z_0|^2 - |z_1|^2)^2 + 4|\bar{z}_1 z_2|^2 = (|z_0|^2 + |z_1|^2)^2 = 1.$$

We take a horizontal vector $(z, u) \in T_z S^3 \subset T_z\mathbb{C}^2 \cong \mathbb{C}^2$ and denote $u = (u_0, u_1)$. As $\langle\!\langle z, u\rangle\!\rangle = 0$, which means $z_0\bar{u}_0 + z_1\bar{u}_1 = 0$, we have

$$\begin{aligned} d\varphi\big(d\varpi(z,u)\big) &= \big(\bar{u}_0 z_0 + \bar{z}_0 u_0 - \bar{u}_1 z_1 - \bar{z}_1 u_1,\ 2\bar{u}_0 z_1 + 2\bar{z}_0 u_1\big) \\ &= 2\big(\bar{u}_0 z_0 - \bar{z}_1 u_1,\ \bar{u}_0 z_1 + \bar{z}_0 u_1\big) \in \mathbb{R}\times\mathbb{C}. \end{aligned}$$

Thus, for two horizontal vectors $(z,u), (z,w) \in T_z S^3$ we have

$$\begin{aligned} &\big\langle d\varphi\big(d\varpi(z,u)\big), d\varphi\big(d\varpi(z,w)\big)\big\rangle \\ &\quad = 4\mathrm{Re}\big(\bar{u}_0 w_0 |z_0|^2 + u_1\bar{w}_1|z_1|^2 - \bar{u}_0\bar{w}_1 z_0 z_1 - u_1 w_0 \bar{z}_0\bar{z}_1 \\ &\qquad\qquad + \bar{u}_0 w_0|z_1|^2 + u_1\bar{w}_1|z_0|^2 + \bar{u}_0\bar{w}_1 z_0 z_1 + u_1 w_0\bar{z}_0\bar{z}_1\big) \\ &\quad = 4\mathrm{Re}\big(\bar{u}_0 w_0 + u_1\bar{w}_1\big) = 2\big(\bar{u}_0 w_0 + u_1\bar{w}_1 + u_0\bar{w}_0 + \bar{u}_1 w_1\big) \\ &\quad = 4\mathrm{Re}\big(u_0\bar{w}_0 + u_1\bar{w}_1\big) = \langle d\varpi(z,u), d\varpi(z,w)\rangle. \end{aligned}$$

Here, we note that the standard metric on $S^3(1)$ induces the metric of $\mathbb{C}P^1(4)$. Our computation shows that φ is an isometry of $\mathbb{C}P^1(4)$ to $S^2(4)$. We hence get the conclusion. □

[3] Complex hyperbolic spaces

We take a Hermitian form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathbb{C}^{n+1}$ given by

$$\langle\!\langle z, w\rangle\!\rangle = -z_0\bar{w}_0 + z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$$

for $z = (z_0, z_1, \ldots, z_n),\ w = (w_0, w_1, \ldots, w_n) \in \mathbb{C}^{n+1}$. We consider an anti-de Sitter space

$$H_1^{2n+1} = \big\{z\in\mathbb{C}^{n+1}\ \big|\ \langle\!\langle z,z\rangle\!\rangle = -1\big\} = \big\{z\in\mathbb{C}^{n+1}\ \big|\ \|z\| = -1\big\}.$$

A unit circle $S^1 = \{\lambda\in\mathbb{C}\ \big|\ |\lambda| = 1\}$ acts on H_1^{2n+1} as $\lambda\cdot z = \lambda z = (\lambda z_0, \lambda z_1, \ldots, \lambda z_n)$. We denote by $\mathbb{C}H^n$ the quotient space H_1^{2n+1}/S^1 of H_1^{2n+1} under this action, and call it an $n-$dimensional *complex hyperbolic space*. We define a projection by

$$\varpi : H_1^{2n+1} \ni z = (z_0, z_1, \ldots, z_n) \mapsto [z] = [(z_0, z_1, \ldots, z_n)] \in \mathbb{C}H^n,$$

where $[z]$ denotes the equivalence class containing z. We call the pair (H_1^{2n+1}, ϖ) a *Hopf fibration*.

With respect to the projection ϖ, we decompose the tangent space $T_zH_1^{2n+1}$ at z into horizontal and vertical subspaces as $T_zH_1^{2n+1} = \mathcal{H}_z \oplus \mathcal{V}_z$. Here, the tangent space $T_zH_1^{2n+1}$ is expressed as a subset of $T_z\mathbb{C}^{n+1}$ as

$$T_zH_1^{2n+1} = \bigl\{(z,u) \in \{z\} \times \mathbb{C}^{n+1} \mid \mathrm{Re}\,\langle\!\langle z,u\rangle\!\rangle = 0\bigr\},$$

and the vertical space $\mathcal{V}_z$ is the tangent line generated by the action of S^1, hence is expressed as

$$\mathcal{V}_z = \{(z, \sqrt{-1}az) \in T_zH_1^{2n+1} \mid a \in \mathbb{R}\},$$

and the horizontalspace $\mathcal{H}_z$ is the orthogonal complement of the vertical space, hence is expressed as

$$\mathcal{H}_z = \{(z,u) \in T_zH_1^{2n+1} \mid \langle\!\langle z,u\rangle\!\rangle = 0\}.$$

It is also clear that the action of S^1 onto H_1^{2n+1} induces an action of S^1 onto the tangent bundle TH_1^{2n+1}, which is given as $(z,u) \mapsto (\lambda z, \lambda u)$. By the same way as for the case of complex projective spaces, we can define a complex structure on $\mathbb{C}H^n$ which is induced by the canonical complex structure on $\mathbb{C}^{n+1}$. That is, identifying $T_{\varpi(z)}\mathbb{C}H^n$ with $\mathcal{H}_z$ at each point $z \in S^{2n+1}$, we define $J : T_{\varpi(z)}\mathbb{C}H^n \to T_{\varpi(z)}\mathbb{C}H^n$ by $Jd\varpi((z,w)) = d\varpi((z,\sqrt{-1}w))$. Since we have $\lambda\sqrt{-1}v = \sqrt{-1}\lambda v$ for an arbitrary $\lambda \in S^1 \subset \mathbb{C}$, we find that J is well defined. As $J^2 = -I$ clearly holds, we see this J is a complex structure on $\mathbb{C}H^n$.

We now define a Riemannian metric on $\mathbb{C}H^n$. By identifying $T_{\varpi(z)}\mathbb{C}H^n$ with $\mathcal{H}_z$ at each point $z \in H_1^{2n+1}$, we set

$$\bigl\langle d\varpi((z,u)), d\varpi((z,v))\bigr\rangle = \mathrm{Re}\,\langle\!\langle u,v\rangle\!\rangle$$

for (z,u), $(z,v) \in \mathcal{H}_z$. Since we have $\langle\!\langle \lambda v, \lambda w\rangle\!\rangle = \langle\!\langle v,w\rangle\!\rangle$ for an arbitrary $\lambda \in S^1 \subset \mathbb{C}$, and since $d\varpi((z,u)) = d\varpi((z',u'))$ if and only if there is $\mu \in S^1 \subset \mathbb{C}$ with $z' = \mu z$ and

$u' = \mu u$, we find that this form on $\mathbb{C}H^n$ is well defined. Moreover, as we have

$$\begin{aligned}\langle\langle v, v\rangle\rangle &= -|v_0|^2 + |v_1|^2 + \cdots + |v_n|^2 \\ &= |v_1|^2 + \cdots + |v_n|^2 - |v_1\bar{z}_1 + \cdots + v_n\bar{z}_n|^2|z_0|^{-2} \\ &\geq |v_1|^2 + \cdots + |v_n|^2 - (|v_1||z_1| + \cdots + |v_n||z_n|)^2|z_0|^{-2} \\ &\geq -(|v_1|^2 + \cdots + |v_n|^2)(|z_1|^2 + \cdots + |z_n|^2 - |z_0|^2)|z_0|^{-2} \\ &= (|v_1|^2 + \cdots + |v_n|^2)|z_0|^{-2} \geq 0,\end{aligned}$$

for an arbitrary $(z, v) \in \mathcal{H}_z$ because $\langle\langle z, w\rangle\rangle = 0$, we find it is positive-definite. Hence we get a Riemannian metric on $\mathbb{C}H^n$.

We denote by $\mathcal{N}$ the outward normal vector field of H_1^{2n+1} in $\mathbb{C}^{n+1}$ satisfying $\langle\langle \mathcal{N}, \mathcal{N}\rangle\rangle = -1$.

LEMMA 2.6. *Let ∇ and $\widetilde{\nabla}$ be the Riemannian connections of $\mathbb{C}H^n$ with respect to the above metric and the canonical connection of H_1^{2n+1}, respectively. For $X, Y \in \mathfrak{X}(\mathbb{C}H^n)$ we take their horizontal lifts $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(H_1^{2n+1})$. Then the horizontal lift $\widetilde{\nabla_X Y}$ of $\nabla_X Y$ satisfies*

$$\widetilde{\nabla_X Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} + \langle X, JY\rangle J\mathcal{N}. \tag{2.6}$$

Hence if we denote by $\overline{\nabla}$ the cannonical connection of $\mathbb{C}^{n+1}$, we have

$$\widetilde{\nabla_X Y} = \overline{\nabla}_{\widetilde{X}}\widetilde{Y} - \langle X, Y\rangle\mathcal{N} + \langle X, JY\rangle J\mathcal{N}, \tag{2.7}$$

PROOF. Though an anti-de Sitter space is not a real hyperbolic space, the canonical connections $\widetilde{\nabla}$ and $\overline{\nabla}$ on H_1^{2n+1} and on $\mathbb{C}^{n+1}$ are related to each other by the same relationship as that of Riemannian connections on H^n and $\mathbb{R}^{n+1}$. If we take $\widetilde{Z}, \widetilde{W} \in \mathfrak{X}(H_1^{2n+1})$ we have

$$\widetilde{\nabla}_{\widetilde{Z}}\widetilde{W} = \overline{\nabla}_{\widetilde{Z}}\widetilde{W} - \frac{\langle\langle \overline{\nabla}_{\widetilde{Z}}\widetilde{W}, \mathcal{N}\rangle\rangle}{\langle\langle \mathcal{N}, \mathcal{N}\rangle\rangle}\mathcal{N}.$$

As $\langle\langle \widetilde{W}, \mathcal{N}\rangle\rangle = 0$, we have

$$0 = \langle\langle \overline{\nabla}_{\widetilde{Z}}\widetilde{W}, \mathcal{N}\rangle\rangle + \langle\langle \widetilde{W}, \overline{\nabla}_{\widetilde{Z}}\mathcal{N}\rangle\rangle = \langle\langle \overline{\nabla}_{\widetilde{Z}}\widetilde{W}, \mathcal{N}\rangle\rangle + \langle\langle \widetilde{W}, \widetilde{Z}\rangle\rangle,$$

because we can identiy $\mathcal{N}_{\tilde{p}}$ with the position vector $\tilde{p} \in H^{2n+1}$. Thus we have

$$\widetilde{\nabla}_{\widetilde{Z}}\widetilde{W} = \overline{\nabla}_{\widetilde{Z}}\widetilde{W} - \langle\langle \widetilde{W}, \widetilde{Z}\rangle\rangle \mathcal{N}.$$

Since the horizontal lift of the covariant differentiation $\widetilde{\nabla_X Y}$ is obtained by removing the vertical component of the covariant differential $\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}$ on H^{2n+1}, we have

$$\nabla_X Y = \widetilde{\nabla}_X Y - \frac{\langle\langle \widetilde{\nabla}_X Y, J\mathcal{N}\rangle\rangle}{\langle\langle J\mathcal{N}, J\mathcal{N}\rangle\rangle} J\mathcal{N} = \widetilde{\nabla}_X Y + \langle\langle \widetilde{\nabla}_X Y, J\mathcal{N}\rangle\rangle J\mathcal{N}.$$

As $\langle\langle \widetilde{Y}, J\mathcal{N}\rangle\rangle = 0$ we have

$$\begin{aligned} 0 &= \langle\langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, J\mathcal{N}\rangle\rangle + \langle\langle \widetilde{Y}, \widetilde{\nabla}_{\widetilde{X}}(J\mathcal{N})\rangle\rangle = \langle\langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, J\mathcal{N}\rangle\rangle + \Big\langle\!\Big\langle \widetilde{Y}, \overline{\nabla}_{\widetilde{X}}(J\mathcal{N}) - \langle \widetilde{X}, J\mathcal{N}\rangle \mathcal{N}\Big\rangle \\ &= \langle\langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, J\mathcal{N}\rangle\rangle + \langle\langle \widetilde{Y}, J\overline{\nabla}_{\widetilde{X}}\mathcal{N}\rangle\rangle = \langle\langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, J\mathcal{N}\rangle\rangle + \langle\langle \widetilde{Y}, J\widetilde{X}\rangle\rangle \\ &= \langle\langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, J\mathcal{N}\rangle\rangle - \langle\langle \widetilde{X}, J\widetilde{Y}\rangle\rangle, \end{aligned}$$

hence we obtain

$$\nabla_X Y = \widetilde{\nabla}_X Y + \langle X, JY\rangle J\mathcal{N}.$$

We hence get the relationship between ∇ and $\overline{\nabla}$ by the above relationship between $\widetilde{\nabla}$ and $\overline{\nabla}$. □

Corollary 2.2. *The complex structure J on $\mathbb{C}H^n$ is parallel with respect to the canonical metric.*

Proof. We take arbitrary vector fields $X, Y \in \mathfrak{X}(\mathbb{C}H^n)$. We denote by $\widetilde{X}, \widetilde{Y}$ their horizontal lifts. By definition of the complex structure J on $\mathbb{C}H^n$, we see that the horizontal lift $\widetilde{JY}$ of JY coincides with $\widetilde{J}\widetilde{Y}$, where $\widetilde{J}$ denotes the complex structure on $\mathbb{C}^{n+1}$. By (2.7), we have

$$\begin{aligned} \widetilde{\nabla_X(JY)} &= \overline{\nabla}_{\widetilde{X}}(\widetilde{JY}) - \langle X, JY\rangle\mathcal{N} - \langle X, Y\rangle\widetilde{J}\mathcal{N} \\ &= \widetilde{J}\overline{\nabla}_{\widetilde{X}}\widetilde{Y} - \langle X, JY\rangle\mathcal{N} - \langle X, Y\rangle\widetilde{J}\mathcal{N} \\ &= \widetilde{J}\widetilde{\nabla_X Y} + \langle X, Y\rangle\widetilde{J}\mathcal{N} + \langle X, JY\rangle\mathcal{N} - \langle X, JY\rangle\mathcal{N} - \langle X, Y\rangle\widetilde{J}\mathcal{N} = \widetilde{J\nabla_X Y}. \end{aligned}$$

This shows that $\nabla_X(JY) = J\nabla_X Y$. We hence find that J is parallel. □

By using the relationship on connections we can express the curvature tensor on a complex hyperbolic space $\mathbb{C}H^n$.

LEMMA 2.7. *The curvature tensor on a complex hyperbolic space* $\mathbb{C}H^n$ *satisfies*

$$R(X,Y)Z = -\{\langle Y,Z\rangle X - \langle X,Z\rangle Y - \langle Y,JZ\rangle JX + \langle X,JZ\rangle JY + 2\langle X,JY\rangle JZ\}.$$

PROOF. For vector fields $X, Y, Z \in \mathfrak{X}(\mathbb{C}H^n)$, we denote their horizontal lifts on H^n also by X, Y, Z for simplicity. By Lemma 2.6, we find

$$\begin{aligned}\nabla_X\nabla_Y Z &= \widetilde{\nabla}_X(\nabla_Y Z) + \langle X, J\nabla_Y Z\rangle J\mathcal{N}\\ &= \widetilde{\nabla}_X\big(\widetilde{\nabla}_Y Z + \langle Y,JZ\rangle J\mathcal{N}\big) + \langle X, J\nabla_Y Z\rangle J\mathcal{N}\\ &= \widetilde{\nabla}_X\widetilde{\nabla}_Y Z + X\langle Y,JZ\rangle J\mathcal{N} + \langle Y,JZ\rangle JX + \langle X, J\nabla_Y Z\rangle J\mathcal{N},\end{aligned} \tag{2.8}$$

where $\mathcal{N}$ denotes the outward unit normal of H_1^{2n+1} in $\mathbb{C}^{n+1}$, and $\widetilde{\nabla}$ denotes the Riemannian connection on H_1^{2n+1}. Here, we note

$$\widetilde{\nabla}_X(J\mathcal{N}) = \overline{\nabla}_X(J\mathcal{N}) - \langle X,\mathcal{N}\rangle J\mathcal{N} = J\overline{\nabla}_X\mathcal{N} = JX.$$

We denote by $[\ ,\ ]_H$ and $[\ ,\ ]_{H_1}$ bracket products of $\mathbb{C}H^n$ and H_1^{2n+1},respectively. They are related with each other by the following equality :

$$\begin{aligned}[X,Y]_H &= \nabla_X Y - \nabla_Y X = \widetilde{\nabla}_X Y + \langle X,JY\rangle J\mathcal{N} - \widetilde{\nabla}_Y X - \langle Y,JX\rangle J\mathcal{N}\\ &= [X,Y]_{H_1} + 2\langle X,JY\rangle J\mathcal{N}.\end{aligned}$$

By using this, we have

$$\begin{aligned}\nabla_{[X,Y]_H}Z &= \nabla_Z[X,Y]_H + \big[[X,Y]_H, Z\big]_H\\ &= \widetilde{\nabla}_Z[X,Y]_H + \big\langle Z, J[X,Y]_H\big\rangle J\mathcal{N} + \big[[X,Y]_H,Z\big]_H\\ &= \widetilde{\nabla}_Z\big([X,Y]_{H_1} + 2\langle X,JY\rangle J\mathcal{N}\big) - \big\langle JZ,[X,Y]_H\big\rangle J\mathcal{N} + \big[[X,Y]_H,Z\big]_H\\ &= \widetilde{\nabla}_{[X,Y]_{H_1}}Z - \big[[X,Y]_{H_1},Z\big]_{H_1} + 2\widetilde{\nabla}_Z\big(\langle X,JY\rangle J\mathcal{N}\big)\\ &\qquad - \big\langle JZ,[X,Y]_H\big\rangle J\mathcal{N} + \big[[X,Y]_H,Z\big]_H\\ &= \widetilde{\nabla}_{[X,Y]_{H_1}}Z - \big[[X,Y]_H - 2\langle X,JY\rangle J\mathcal{N}, Z\big]_{H_1} + 2\widetilde{\nabla}_Z\big(\langle X,JY\rangle J\mathcal{N}\big)\\ &\qquad - \big\langle JZ,[X,Y]_H\big\rangle J\mathcal{N} + \big[[X,Y]_H,Z\big]_{H_1} + 2\big\langle [X,Y]_H, JZ\big\rangle J\mathcal{N}\\ &= \widetilde{\nabla}_{[X,Y]_{H_1}}Z + 2\big[\langle X,JY\rangle J\mathcal{N}, Z\big]_{H_1} + 2\widetilde{\nabla}_Z\big(\langle X,JY\rangle J\mathcal{N}\big) + \big\langle JZ,[X,Y]_H\big\rangle J\mathcal{N}.\end{aligned}$$

By the definition of bracket product [,], we have

$$
\begin{aligned}
&= \widetilde{\nabla}_{[X,Y]_{H_1}} Z + 2\Big(\langle X, JY\rangle \widetilde{\nabla}_{JN} Z - \widetilde{\nabla}_Z\big(\langle X, JY\rangle J\mathcal{N}\big)\Big) + 2\widetilde{\nabla}_Z\big(\langle X, JY\rangle J\mathcal{N}\big) \\
&\quad + \langle JZ, [X,Y]_H\rangle J\mathcal{N} \\
&= \widetilde{\nabla}_{[X,Y]_{H_1}} Z + 2\langle X, JY\rangle (JZ + [J\mathcal{N}, Z]_{H_1}) + \langle JZ, [X,Y]\rangle J\mathcal{N}.
\end{aligned}
$$

Here, as we have

$$
\begin{aligned}
[J\mathcal{N}, Z]_{H_1} &= \widetilde{\nabla}_{J\mathcal{N}} Z - \widetilde{\nabla}_Z (J\mathcal{N}) \\
&= \overline{\nabla}_{J\mathcal{N}} Z - \overline{\nabla}_Z (J\mathcal{N}) + (\langle J\mathcal{N}, Z\rangle - \langle Z, J\mathcal{N}\rangle)\mathcal{N} = 0,
\end{aligned}
$$

we find

$$
\nabla_{[X,Y]_H} Z = \widetilde{\nabla}_{[X,Y]_{H_1}} Z + \langle JZ, [X,Y]_H\rangle J\mathcal{N} + 2\langle X, JY\rangle JZ \tag{2.9}
$$

Therefore, by (2.8), (2.9) we obtain

$$
\begin{aligned}
R(X,Y)Z &= \widetilde{\nabla}_X \widetilde{\nabla}_Y Z + (X\langle Y, JZ\rangle + \langle X, J\nabla_Y Z\rangle) J\mathcal{N} + \langle Y, JZ\rangle JX \\
&\qquad - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - (Y\langle X, JZ\rangle + \langle Y, J\nabla_X Z\rangle) J\mathcal{N} - \langle X, JZ\rangle JY \\
&\qquad\quad - \widetilde{\nabla}_{[X,Y]_{H_1}} Z - \langle JZ, [X,Y]_H\rangle J\mathcal{N} - 2\langle X, JY\rangle JZ \\
&= \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]_{H_1}} Z \\
&\qquad + \big\{X\langle Y, JZ\rangle + \langle X, J\nabla_Y Z\rangle - Y\langle X, JZ\rangle \\
&\qquad\quad - \langle Y, J\nabla_X Z\rangle - \langle JZ, [X,Y]_H\rangle\big\} J\mathcal{N} \\
&\qquad + \langle Y, JZ\rangle JX - \langle X, JZ\rangle JY - 2\langle X, JY\rangle JZ.
\end{aligned}
$$

Here, as we see in [3] of §1.2, the curvature tensor $\widetilde{R}$ of an anti-de Sitter space H_1^{2n+1} is given as $\widetilde{R}(X,Y)Z = -\langle Y, Z\rangle X - \langle X, Z\rangle Y$. We hence get

$$
\begin{aligned}
R(X,Y)Z &= -\langle Y, Z\rangle X + \langle X, Z\rangle Y + \langle Y, JZ\rangle JX - \langle X, JZ\rangle JY - 2\langle X, JY\rangle JZ \\
&= -\{\langle Y, Z\rangle X - \langle X, Z\rangle Y - \langle Y, JZ\rangle JX + \langle X, JZ\rangle JY + 2\langle X, JY\rangle JZ\}.
\end{aligned}
$$

This completes the proof. □

As a consequence of Lemma 2.7 for a unit tangent vector $v \in U\mathbb{C}H^n$ we have

$$
\begin{aligned}
\langle R(v, Jv)Jv, v\rangle &= -\{\langle Jv, Jv\rangle\langle v, v\rangle - \langle v, Jv\rangle\langle Jv, v\rangle - \langle Jv, -v\rangle\langle Jv, v\rangle \\
&\qquad + \langle v, -v\rangle\langle -v, v\rangle + 2\langle v, -v\rangle\langle -v, v\rangle\} = -4.
\end{aligned}
$$

Thus $\mathbb{C}H^n$ endowed with the Riemannian metric through the fibration $\varpi : H_1^{2n+1} \to \mathbb{C}H^n$ has constant holomorphic sectional curvature -4. We hence denote this Kähler manifold by $\mathbb{C}H^n(-4)$. When we consider a metric on $\mathbb{C}H^n$ given by $\langle\ ,\ \rangle' = \dfrac{-4}{c}\langle\ ,\ \rangle$ with a positive c and the Riemannian metric, we see it has constant holomorphic sectional curvature c by Lemma 1.3. We denote this Kähler manifold by $\mathbb{C}H^n(c)$. The curvature tensor R of $\mathbb{C}H^n(c)$ is hence expressed as

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\big\{&\langle Y,Z\rangle X - \langle X,Z\rangle Y - \langle Y,JZ\rangle JX \\ &+ \langle X,JZ\rangle JY + 2\langle X,JY\rangle JZ\big\} \end{aligned} \tag{2.10}$$

by Lemma 2.7.

If we take unit tangent vectors $v,w \in T_p\mathbb{C}H^n(-4)$ which satisfy $\langle v,w\rangle = \langle v,Jw\rangle = 0$, that is, v,w are orthonormal vectors which span a real vector subspace, we have

$$\begin{aligned} \langle R(v,w)w,v\rangle = -\{&\langle w,w\rangle\langle v,v\rangle - \langle v,w\rangle\langle w,v\rangle - \langle w,Jw\rangle\langle Jv,v\rangle \\ &+ \langle v,Jw\rangle\langle Jw,v\rangle + 2\langle v,Jw\rangle\langle Jw,v\rangle\} = -1. \end{aligned}$$

Hence for unit tangent vectors $v,w \in T_p\mathbb{C}H^n(c)$ which satisfy $\langle v,w\rangle = \langle v,Jw\rangle = 0$ we have $\langle R(v,w)w,v\rangle = c/4$.

Lemma 2.8. *For an arbitrary point $p \in \mathbb{C}H^n$ and an arbitrary unit tangent vector $v \in T_p\mathbb{C}H^n$ we have a totally geodesic $\mathbb{C}H^1\big(\subset \mathbb{C}P^n\big)$ satisfying $p \in \mathbb{C}H^1$ and $v \in T_p\mathbb{C}H^1$.*

Proof. Let $\varpi : H_1^{2n+1} \to \mathbb{C}P^n$ be a Hopf fibration of an anti-de Sitter space H_1^{2n+1} in $\mathbb{C}^{n+1}$. We take a point $z \in H_1^{2n+1}$ and a horizontal vector $(z,u) \in \mathcal{H}_z \subset T_zH_1^{2n+1} \subset T_z\mathbb{C}^{n+1}$ so that they satisfy $\varpi(z) = p$ and $d\varpi(z,u) = v$. Since $\langle\!\langle z,z\rangle\!\rangle = -1$, $\|u\| = 1$ and $\langle\!\langle z,u\rangle\!\rangle = 0$, we see that the subset $\widehat{N} = \{\mu z + \nu u \mid \mu,\nu \in \mathbb{C},\ -|\mu|^2 + |\nu|^2 = -1\}$ of H_1^{2n+1} is an anti-de Sitter space in $\mathbb{C}^2 = \mathbb{C}z \oplus \mathbb{C}u \subset \mathbb{C}^{n+1}$. As a matter of fact, we have

$$\langle\!\langle \mu z + \nu u, \mu z + \nu u\rangle\!\rangle = |\mu|^2\langle\!\langle z,z\rangle\!\rangle + |\nu|^2\langle\!\langle u,u\rangle\!\rangle = -|\mu|^2 + |\nu|^2 = -1.$$

Thus we find that $N = \varpi(\widehat{N})$ is $\mathbb{C}H^1$ by its construction. As $z \in \widehat{N}$ and the horizontal part of $T_z\widehat{N}$ is $\mathbb{C}u$, we see $p \in N$ and $v \in T_pN$. Since the outward normal $\mathcal{N}$ is identified with p by regarding it as a position vector, by definition of covariant differentiations of N and $\mathbb{C}H^n$ we see N is totally geodesic in $\mathbb{C}H^n$. □

Lemma 2.9. *For arbitrary points $p, p' \in \mathbb{C}H^n$ and arbitrary unit tangent vectors $v \in T_p\mathbb{C}H^n$, $v' \in T_{p'}\mathbb{C}H^n$ we have a holomorphic isometry φ of $\mathbb{C}H^n$ satisfying $\varphi(p) = p'$ and $d\varphi(v) = v'$.*

Proof. We take $z, z' \in H_1^{2n+1} \subset \mathbb{C}^{n+1}$ so that $\varpi(z) = p,\ \varpi(z') = p'$. Also, we take horizontal vectors $(z, u) \in T_z H_1^{2n+1}$ and $(z', u') \in T_{z'} H_1^{2n+1}$ satisfying $d\varpi((z, u)) = v$ and $d\varpi((z', u')) = v'$. Then both the pairs $\{z, u\}$ and $\{z', u'\}$ are $\mathbb{C}$-linearly independent. Thus we can take two sets of orthonormal vectors $u_2, \dots, u_n \in \mathbb{C}^{n+1}$ and $u_2', \dots, u_n' \in \mathbb{C}^{n+1}$ so that both $\{z, u, u_2, \dots, u_n\}$ and $\{z', u', u_2', \dots, u_n'\}$ are $\mathbb{C}$-linearly independent and satisfies that $\langle\langle z, z\rangle\rangle = \langle\langle z', z'\rangle\rangle = -1,\ \langle\langle u_j, u_j\rangle\rangle = \langle\langle u_j', u_j'\rangle\rangle = 1,\ \langle\langle z, u_j\rangle\rangle = \langle\langle u_j, u_k\rangle\rangle = 0$ and $\langle\langle z', u_j'\rangle\rangle = \langle\langle u_j', u_k'\rangle\rangle = 0\,(j \neq k)$. If we express them by vertical vectors, then two matrices $(z, u, u_2, \dots, u_n)$ and $(z', u', u_2', \dots, u_n')$ are matrices in $U(n+1, 1)$ (that is, "unitary" matrices with respect to $\langle\langle\ ,\ \rangle\rangle$). We set a matrix $A \in U(n+1, 1)$ by

$$A = (z', u', u_2', \dots, u_n') \cdot (z, u, u_2, \dots, u_n)^{-1}.$$

As $A(z, u, u_2, \dots, u_n) = (z', u', u_2', \dots, u_n')$, this induces a $\mathbb{C}$-linear transformation $\hat{\varphi} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ satisfying $\hat{\varphi}(z) = z'$ and $\hat{\varphi}(u) = u'$. Since $A \in U(n+1, 1)$, we have $\hat{\varphi} \circ \sqrt{-1} = \sqrt{-1} \circ \hat{\varphi}$ and $\hat{\varphi}(H_1^{2n+1}) = H_1^{2n+1}$. As a matter of fact, if we denote $A = (a_{ij})$ we have

$$\begin{aligned}
&- a_{00}\bar{a}_{00} + \sum_{\ell=1}^{n} a_{\ell 0}\bar{a}_{\ell 0} = -1, \qquad -a_{0j}\bar{a}_{0j} + \sum_{\ell=1}^{n} a_{\ell j}\bar{a}_{\ell j} = 1 \quad (j \geq 1), \\
&- a_{0j}\bar{a}_{0k} + \sum_{\ell=1}^{n} a_{\ell j}\bar{a}_{\ell k} = 0 \quad (0 \leq j, k \leq n,\ j \neq k).
\end{aligned}$$

For $w = (w_0, \dots, w_n) \in H_1^{2n+1}$ we have $Aw = \bigl(\sum_j a_{0j}z_j, \dots, \sum_j a_{nj}z_j\bigr)$, hence have

$$\begin{aligned}
\langle\langle Aw, Aw\rangle\rangle &= -\sum_{j,k} a_{0j}z_j\bar{a}_{0k}\bar{z}_k + \sum_{j,k} a_{1j}z_j\bar{a}_{1k}\bar{z}_k + \cdots + \sum_{j,k} a_{nj}z_j\bar{a}_{0nk}\bar{z}_k \\
&= \sum_{j,k} \bigl\{-a_{0j}\bar{a}_{0k} + a_{1j}\bar{a}_{1k} + \cdots a_{nj}\bar{a}_{0nk}\bigr\} z_j\bar{z}_k \\
&= -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = -1.
\end{aligned}$$

We hence find that $\hat{\varphi}$ induces a bijection $\varphi : \mathbb{C}H^n \to \mathbb{C}H^n$ satisfying $d\varphi \circ J = J \circ d\varphi$. Since we have

$$\begin{aligned}\langle\!\langle Au, Av\rangle\!\rangle &= -\sum_{j,k} a_{0j}u_j\bar{a}_{0k}\bar{v}_k + \sum_{j,k} a_{1j}u_j\bar{a}_{1k}\bar{v}_k + \cdots + \sum_{j,k} a_{nj}u_j\bar{a}_{0nk}\bar{v}_k \\ &= \sum_{j,k}\big\{-a_{0j}\bar{a}_{0k} + a_{1j}\bar{a}_{1k} + \cdots a_{nj}\bar{a}_{0nk}\big\}u_j\bar{v}_k \\ &= -u_0\bar{v}_0 + u_1\bar{v}_1 + \cdots + u_n\bar{v}_n = \langle\!\langle u, v\rangle\!\rangle,\end{aligned}$$

$\hat{\varphi}$ preserves the Hermitian inner product $\langle\!\langle\ ,\ \rangle\!\rangle$. Thus, we find that φ is an isometry. $\square$

REMARK 2.2. For arbitrary points $p, p' \in \mathbb{C}H^n$ and arbitrary unit tangent vectors $v \in T_p\mathbb{C}H^n$, $v' \in T_{p'}\mathbb{C}H^n$ we can construct an anti-holomorphic isometry φ satisfying $\varphi(p) = p'$ and $d\varphi(v) = v'$.

PROPOSITION 2.2. *A complex projective line $\mathbb{C}H^1(c)$ is isomorphic to a real hyperbolic space $H^2(c)$.*

PROOF. We define $\varphi : \mathbb{C}H^1 \to H^2 \subset R^3$ by

$$\begin{aligned}\varphi\big(\varpi((z_0, z_1))\big) &= \big(|z_0|^2 + |z_1|^2,\ 2\mathrm{Re}(\bar{z}_0 z_1),\ 2\mathrm{Im}(\bar{z}_0 z_1)\big) \\ &= \big(\bar{z}_0 z_0 + \bar{z}_1 z_1, \bar{z}_0 z_1 + z_0\bar{z}_1, \sqrt{-1}(-\bar{z}_0 z_1 + z_0\bar{z}_1)\big).\end{aligned}$$

To simplify notations by considering $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$ we can express φ as

$$\varphi\big(\varpi((z_0, z_1))\big) = \big(|z_0|^2 + |z_1|^2,\ 2\bar{z}_0 z_1\big).$$

We adopt this expression in this proof. We note that

$$-(|z_0|^2 + |z_1|^2)^2 + 4|\bar{z}_1 z_2|^2 = -(|z_0|^2 - |z_1|^2)^2 = -1.$$

We take a horizontal vector $(z, u) \in T_z H_1^3 \subset T_z\mathbb{C}^2 \cong \mathbb{C}^2$ and denote $u = (u_0, u_1)$. As $\langle\!\langle z, u\rangle\!\rangle = 0$, which means $-z_0\bar{u}_0 + z_1\bar{u}_1 = 0$, we have

$$\begin{aligned}d\varphi\big(d\varpi(z, u)\big) &= \big(\bar{u}_0 z_0 + \bar{z}_0 u_0 + \bar{u}_1 z_1 + \bar{z}_1 u_1,\ 2\bar{u}_0 z_1 + 2\bar{z}_0 u_1\big) \\ &= 2\big(\bar{u}_0 z_0 + \bar{z}_1 u_1,\ \bar{u}_0 z_1 + \bar{z}_0 u_1\big) \in \mathbb{R} \times \mathbb{C}.\end{aligned}$$

Thus, for two horizontal vectors $(z,u),(z,w)\in T_zH_1^3$ we have

$$\begin{aligned}
&\big\langle d\varphi\big(d\varpi(z,u)\big), d\varphi\big(d\varpi(z,w)\big)\big\rangle \\
&\quad = 4\mathrm{Re}\big(-\bar u_0 w_0|z_0|^2 - u_1\bar w_1|z_1|^2 - \bar u_0\bar w_1 z_0 z_1 - u_1 w_0\bar z_0\bar z_1 \\
&\qquad\qquad + \bar u_0 w_0|z_1|^2 + u_1\bar w_1|z_0|^2 + \bar u_0\bar w_1 z_0 z_1 + u_1 w_0\bar z_0\bar z_1\big) \\
&\quad = 4\mathrm{Re}\big(-\bar u_0 w_0 + u_1\bar w_1\big) = 2\big(-\bar u_0 w_0 + u_1\bar w_1 - u_0\bar w_0 + \bar u_1 w_1\big) \\
&\quad = 4\mathrm{Re}\big(-u_0\bar w_0 + u_1\bar w_1\big) = \langle d\varpi(z,u), d\varpi(z,w)\rangle.
\end{aligned}$$

Here, we note that the canonical form on H_1^3 induces the metric of $\mathbb{C}H^1(-4)$. Our computation shows that φ is an isometry of $\mathbb{C}H^1(-4)$ to $H^2(-4)$. We hence get the conclusion. $\square$

3. Magnetic fields

A closed 2-form on a Riemannian manifold is said to be a *magnetic field*. In order to explain why we say closed 2-forms to be magnetic fields, we here recall static magnetic fields on a Euclidean 3-space $\mathbb{R}^3$. On $\mathbb{R}^3$, a vector valued function $\vec{B} = (B_1, B_2, B_3) : \mathbb{R}^3 \to \mathbb{R}^3$ is said to be a *static magnetic field* under the action of constant current if it satisfies the Gauβ's law

$$\mathrm{div}(\vec{B}) := \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} = 0.$$

Here, (x_1, x_2, x_3) is the ordinary orthonormal coordinate system of $\mathbb{R}^3$. We regard $\vec{B}$ as a 1-form $B_1 dx_1 + B_2 dx_2 + B_3 dx_3$. When we treat with magnetic fields in physics, we need to consider the orientation which is called the right-hand system or left-hand system. Therefore by use of duality we identifiy this 1-form with the 2-form

$$\mathbb{B} = B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2.$$

Since we have

$$\begin{aligned} d\mathbb{B} &= \frac{\partial B_1}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial B_2}{\partial x_2} dx_2 \wedge dx_3 \wedge dx_1 + \frac{\partial B_3}{\partial x_3} dx_3 \wedge dx_1 \wedge dx_2 \\ &= \Big(\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} \Big) dx_1 \wedge dx_2 \wedge dx_3, \end{aligned}$$

we find that the Gauβ's law is equivalent to the property that $\mathbb{B}$ is closed.

Example 2.1. We take an orientable Riemann surface M and denote its volume form by $dvol_M$ (see §1.1). Every 2-form is expressed as $f dvol_M$ with some function f on M and is closed. We call this a *surface magnetic field*.

Example 2.2. We take a Kähler manifold (see §2.2 for definition). Since its Kähler form $\mathbb{B}_J$ is closed, we see its constant multiple is also closed. For a constant $k \in \mathbb{R}$ we denote as $\mathbb{B}_k = k\mathbb{B}_J$ and call it a *Kähler magnetic field*.

Example 2.3. Let M be a real hypersurface, a real submanifold of dimension $2\dim_{\mathbb{C}}(\widetilde{M}) - 1$ in a Kähler manifold $\widetilde{M}$, where $\dim_{\mathbb{C}}(\widetilde{M})$ denotes the complex dimension of $\widetilde{M}$. We define $\phi : TM \to TM$ by $\phi(v) = Jv + \langle v, J\mathcal{N} \rangle \mathcal{N}$, where $\mathcal{N}$ is a unit normal vector field on M in $\widetilde{M}$. If we define a 2-form $\mathbb{F}$ on M by $\mathbb{F}_\phi$ by $\mathbb{F}(v,w) = \langle v, \phi w\rangle$ then it is closed (see [**11**]). We say a constant multiple of this form to be a Sasakian magnetic field.

Under the influence of a static magnetic field $\vec{B}$ on $\mathbb{R}^3$, a charged particle of mass m and electricity e which moves in $\mathbb{R}^3$ gets a Lorentz force $e\vec{v}\times\vec{B}$ if we denote its velocity vector by $\vec{v}$. Here, $\times$ denotes the vector product in $\mathbb{R}^3$. Therefore, its equation of motion is given as

$$m\frac{d}{dt}\vec{v} = e\vec{v}\times\vec{B}. \tag{2.11}$$

If we set $\vec{v} = {}^t(v_1, v_2, v_3)$ we have

$$\vec{v}\times\vec{B} = (v_2B_3 - v_3B_2, v_3B_1 - v_1B_3, v_1B_2 - v_2B_1) = \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Thus, by using a skew symmetric matrix

$$\varOmega = \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix},$$

we see (2.11) turns to $m\dfrac{d}{dt}\vec{v} = e\varOmega\vec{v}$.

We generalize (2.11). For a magnetic field $\mathbb{B}$ on a Riemannian manifold M, we define an endomorphism $\varOmega_{\mathbb{B}} : TM \to TM$ of the tangent bundle TM by $\langle v, \varOmega(w)\rangle = \mathbb{B}(v,w)$ for arbitrary tangent vectors $v, w \in T_pM$ at an arbitrary point $p\in M$. Since $\mathbb{B}(v,w) = -\mathbb{B}(w,v)$, we find that $\varOmega_{\mathbb{B}}$ is skew symmetric. We call a smooth curve γ parameterized by its arclength a *trajectory* if it satisfies the differential equation

$$\nabla_{\dot\gamma}\dot\gamma = \varOmega_{\mathbb{B}}(\dot\gamma).$$

When $\mathbb{B}$ is a trivial magnetic field, that is, $\mathbb{B}$ is the null 2-form and is the case that there are no influences of magnetic fields, then we find that the skew symmetric operator is null operator. Hence a smooth curve γ is a trajectory for this magnetic field if it satisfies $\nabla_{\dot\gamma}\dot\gamma = 0$, hence is a geodesic of unit speed. Therefore we may say that trajectories are generalizations of geodesics.

By the definition of the Kähler form, we find that the skew symmetric endomorphism of a Kähler magnetic field $\mathbb{B}_k$ is given as $\Omega_{\mathbb{B}_k} = kJ$. Thus, a smooth curve γ parameterized by its arclength is said to be a $\mathbb{B}_k$-*trajectory*, if it satifies $\nabla_{\dot\gamma}\dot\gamma = kJ\dot\gamma$. Generally, when the skew symmetric operator $\Omega_{\mathbb{B}}$ of a magnetic field $\mathbb{B}$ is parallel, that is, its covariant differential $\nabla\Omega_{\mathbb{B}}$ vanishes, this magnetic field is called *uniform*. Clearly, Kähler magnetic fields are uniform magnetic fields.

The geodesic maintains the property of geodesic even if the speed is changed, but we note that if we change speeds of trajectories, then they turn to trajectories of other magnetic fields. More explicitely, we have the following.

LEMMA 2.10. *Let γ be a trajectory for a Kähler magnetic field $\mathbb{B}_k$. If we change its speed to $\lambda-$times of the orignal, it can be seen as a"trajectory" for $\mathbb{B}_{\lambda k}$.*

PROOF. For a constant $\lambda(>0)$, we put $\sigma(t) = \gamma(\lambda t)$. Considering the differential, we get $\sigma'(t) = \lambda\dot\gamma(\lambda t)$. It leads us to

$$\nabla_{\sigma'}\sigma' = \lambda^2\nabla_{\dot\gamma}\dot\gamma = \lambda^2 kJ\dot\gamma = \lambda kJ(\sigma').$$

Thus we find that σ satisfies the equation of trajectories for $\mathbb{B}_{\lambda k}$, though it is not of unit speed. □

It is well known that geodesics on real space forms are expressed explicitly. Corresponding to this we give explicit representations of trajectories for Kähler magnetic fields on complex space forms.

Since complex structure J of a Kähler manifold is parallel, we have

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma = kJ\dot\gamma, \\ \nabla_{\dot\gamma}(J\dot\gamma) = -k\dot\gamma. \end{cases}$$

Hence we have the following.

LEMMA 2.11. *Let γ be a trajectory for a Kähler magnetic field. Then it is a circle of geodesic curvature $|k|$ and of Frenet frame $\{\dot\gamma, \mathrm{sgn}(k)J\dot\gamma\}$. Here, for a real number k, we denote by $\mathrm{sgn}(k)$ its signature.*

For the sake of later use we here express the velocity vectors of a trajectory γ by use of parallel displacement along γ. We set $v = \dot\gamma(0)$. We denote by $P_{\gamma,0}^{\,t} : T_{\gamma(0)}M \to T_{\gamma(t)}M$ be the paralle displacement along γ. Since the complex structure J on a Kähler manifold is parallel, we find that the vector field $\{JP_{\gamma,0}^{\,t}(v)\}_t$ along γ is parallel. As $\{P_{\gamma,0}^{\,t}(Jv)\}_t$ is also a parallel vector field along γ and $P_{\gamma,0}^{\,0}(Jv) = Jv = JP_{\gamma,0}^{\,0}(v)$, we find $JP_{\gamma,0}^{\,t}(v) = P_{\gamma,0}^{\,t}(Jv)$. Now we set a vector field X along γ by $X(t) = \cos kt\, P_{\gamma,0}^{\,t}(v) + \sin kt\, P_{\gamma,0}^{\,t}(Jv)$. We then find

$$\begin{cases} \bigl(\nabla_{\dot\gamma}X\bigr)(t) = -k\sin kt\, P_{\gamma,0}^{\,t}(v) + k\cos kt\, P_{\gamma,0}^{\,t}(Jv) \\ \qquad\qquad\quad = k\bigl\{\sin kt\, P_{\gamma,0}^{\,t}(v) + \cos kt\, JP_{\gamma,0}^{\,t}(v) = kJX(t), \\ X(0) = v, \qquad \bigl(\nabla_{\dot\gamma}X\bigr)(0) = kJv. \end{cases}$$

Thus we see X satisfies the same differential equation as of $\dot\gamma$. Hence we find

$$\dot\gamma(t) = \cos kt\, P_{\gamma,0}^{\,t}(v) + \sin kt\, P_{\gamma,0}^{\,t}(Jv). \tag{2.12}$$

Lemma 2.12. *Let M be a Kähler manifold. For a unit tangent vector $v \in UM$, there exists a unique trajectory $\gamma : (-\epsilon, \epsilon) \to M$ for $\mathbb{B}_k$ with initial vector v. If M is complete, this curve is defined on a hole real line.*

Proof. Since $\nabla_{\dot\gamma}\dot\gamma = kJ\dot\gamma$ is a linear differential equation, we get the existence and the uniqueness on trajectories by general theory on differential equations. We take the maximal interval I where γ is defined.

Suppose I is bounded from above. We set b the superimum of I. As $\|\dot\gamma\| \equiv 1$, we see the distance $d(\gamma(t_1), \gamma(t_2))$ between two points $\gamma(t_1), \gamma(t_2)$ is not greater than $|t_1 - t_2|$. Therefore the set $\{\gamma(t) \mid 0 \le t < b\}$ is bounded. Since M is complete, we have a limit point $\lim_{t\uparrow b}\gamma(t) \in M$. Becauce $\dot\gamma(t)$ is a unit tangent vector for each t, we also have a limit unit tangent vector $\lim_{t\uparrow b}\dot\gamma(t) \in UM$ in the unit tangent space at $\lim_{t\uparrow b}\gamma(t)$. Thus we find $b \in I$. Applying the theorem on local existence of solutions at $\gamma(b)$ we find γ is defined on an interval $I \cup [b, b+\epsilon)$ for some positive ϵ. As we chose I to be maximal, this is a contradiction.

If we suppose I is bounded from below, along the same lines as above we have a contradiction. Hence we get the conclusion. $\square$

LEMMA 2.13. *Let γ be a trajectory for $\mathbb{B}_k$. We define a smooth curve σ by $\sigma(t) = \gamma(t_0 - t)$ with some t_0. Then σ is a trajectory for $\mathbb{B}_{-k}$.*

PROOF. As we have $\dot\sigma(t) = -\dot\gamma(t_0 - t)$, we find

$$\nabla_{\dot\sigma}\dot\sigma = \big(\nabla_{\dot\gamma}\dot\gamma\big)(t_0 - t).$$

Since γ is a trajectory, we have $\nabla_{\dot\gamma}\dot\gamma = kJ\dot\gamma$. Hence we see that

$$\nabla_{\dot\sigma}\dot\sigma = \big(\nabla_{\dot\gamma}\dot\gamma\big)(t_0 - t) = kJ\dot\gamma(t_0 - t) = -kJ\dot\sigma.$$

we get the conclusion. $\square$

EXAMPLE 2.4. Let γ be a trajectory for $\mathbb{B}_k$ on $\mathbb{C}^n$ with initial condition $\gamma(0) = p \in \mathbb{C}^n$ and $\dot\gamma = (p, v) \in T_p\mathbb{C}^n$. We consider a subset $p + \mathbb{C}v$ in $\mathbb{C}^n$. Since $p + \mathbb{C}v \cong \mathbb{C} = \mathbb{R}^2$, we take a circle $\hat\gamma$ on $\mathbb{R}^2$ with initial condition $\hat\gamma(0) = 0, \hat\gamma'(0) = (1, 0)$ and $\hat\gamma''(0) = k(0, 1)$. If we regard this curve as a curve in $\mathbb{C}^n$ we see it satisfies $\hat\gamma(0) = p + 0v = p$, $\hat\gamma'(0) = v$ and $\hat\gamma''(0) = kJv$. Since a trajectory satisfies the same differential equation as of $\hat\gamma$ regarding as a curve in $\mathbb{C}^n$, we find $\gamma = \hat\gamma$. Thus, we have

$$\gamma(t) = p + \frac{1}{k}(\sin kt)v + \frac{1}{k}(1 - \cos kt)Jv.$$

This shows that γ is closed of length $2\pi/|k|$.

EXAMPLE 2.5. Let γ be a trajectory for $\mathbb{B}_k$ on a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c. We choose a totally geodesic $\mathbb{C}P^1(c)$ with $\gamma(0) \in \mathbb{C}P^1(c)$ and $\dot\gamma(0) \in T_{\gamma(0)}\mathbb{C}P^1(c)$ (see Lemma 2.4). If we consider a trajectory $\hat\gamma$ for $\mathbb{B}_k$ on $\mathbb{C}P^1(c)$ with $\hat\gamma(0) = \gamma(0)$ and $\dot{\hat\gamma}(0) = \dot\gamma(0)$, as $\mathbb{C}P^1(c)$ is totally geodesic, its extrinsic shape $\iota \circ \hat\gamma$ in $\mathbb{C}P^n(c)$ is a trajectory for $\mathbb{B}_k$. In view of initial conditions of γ and $\iota \circ \hat\gamma$, we find $\gamma = \iota \circ \hat\gamma$. This means that γ lies on a totally geodesic $\mathbb{C}P^1(c)$. Thus, as we see in §1.3, a trajectory γ for $\mathbb{B}_k$ is a "small" circle of radius $1/\sqrt{k^2 + c}$ on $\mathbb{C}P^1(c) = \mathbb{S}^2(c)$, hence it is closed of length $2\pi/\sqrt{k^2 + c}$.

Example 2.6. Let γ be a trajectory for $\mathbb{B}_k$ on a complex hyperbolic space $\mathbb{C}H^n(-c)$ of constant holomorphic sectional curvature $-c$. We choose a totally geodesic $\mathbb{C}H^1(-c)$ with $\gamma(0) \in \mathbb{C}H^1(-c)$ and $\dot\gamma(0) \in T_{\gamma(0)}\mathbb{C}H^1(-c)$. If we consider a trajectory $\hat\gamma$ for $\mathbb{B}_k$ on $\mathbb{C}H^1(-c)$ with $\hat\gamma(0) = \gamma(0)$ and $\dot{\hat\gamma}(0) = \dot\gamma(0)$, as $\mathbb{C}H^1(-c)$ is totally geodesic, its extrinsic shape $\iota \circ \hat\gamma$ in $\mathbb{C}H^n(-c)$ is a trajectory for $\mathbb{B}_k$. In view of initial conditions of γ and $\iota \circ \hat\gamma$, we find $\gamma = \iota \circ \hat\gamma$. This means that γ lies on a totally geodesic $\mathbb{C}H^1(-c)$. Thus, every trajectory for a Kähler magnetic field is a curve without self intersections and lies on a totally geodesic $\mathbb{C}H^1(-c) = H^2(-c)$. Features of trajectories depend on strengths of Kähler magnetic fields. When $|k| > \sqrt{|c|}$, a trajectory for $\mathbb{B}_k$ is closed of length $2\pi/\sqrt{k^2 - c}$, and when $|k| \le \sqrt{|c|}$, it is open and is unbounded.

We here note more on trajectories on complex space forms. We say two smooth curves γ_1, γ_2 on a Riemannian manifold N parameterized by their arclengths to be *congruent* to each other if there exist an isometry φ of N and a constant t_0 satisfying $\gamma_2(t) = \varphi \circ \gamma_1(t + t_0)$ for all t. When we can take $t_0 = 0$, we say that they are congruent to each other in strong sense.

Proposition 2.3. *On a complex space form $\mathbb{C}M^n(c)$, two trajectories for $\mathbb{B}_\kappa$ are congruent to each other in strong sense.*

Proof. Let γ_1, γ_2 be trajectories for $\mathbb{B}_k$. By Lemma 2.9, we have a holomorphic isometry φ on $\mathbb{C}M^n(c)$ satisifying $\varphi(\gamma_1(0)) = \gamma_2(0)$ and $d\varphi(\dot\gamma_1(0)) = \dot\gamma_2(0)$. We set $\widetilde\gamma_1 = \varphi \circ \gamma_1$. Then, as φ is an isometry, we have

$$\nabla_{\dot{\widetilde\gamma}_1} \dot{\widetilde\gamma}_1 = d\varphi(\nabla_{\dot\gamma_1}\dot\gamma_1) = d\varphi(kJ\dot\gamma_1) = kd\varphi(J\dot\gamma_1)kJ\dot{\widetilde\gamma}_1.$$

Hence, $\widetilde\gamma_1$ is a trajectory for $\mathbb{B}_k$ with initial condition $\widetilde\gamma_1(0) = \gamma_2(0), \dot{\widetilde\gamma}_1(0) = \dot\gamma_2(0)$. By Lemma 2.12, we see $\widetilde\gamma_1$ coincides with γ_2. Hence we have $\gamma_2 = \varphi \circ \gamma_1$ and get the conclusion. $\square$

On a Kähler manifold M, at a point $p \in M$ we define $\mathbb{B}_k\exp_p : T_pM \to M$ by

$$\mathbb{B}_k\exp_p(v) = \begin{cases} \gamma_{v/\|v\|}(\|v\|) & v \neq 0_p, \\ p & v = 0_p. \end{cases}$$

Here, we denote by γ_v a $\mathbb{B}_k-$trajectory with $\dot{\gamma}(0) = v$ and $0_p \in T_pM$ is origin of the vector space. We call it the *magnetic exponential map* at p. When $k = 0$, the magnetic exponential map $\mathbb{B}_0\mathrm{exp}_p$ is the ordinary exponential map exp_p.

CHAPTER 3

Comparison theorems on magnetic Jacobi fields

In order to describe the differential of magnetic exponential maps we introduce magnetic Jacobi fields in this chapter. We study magnetic Jacobi fields for Kähler magnetic fields on complex space forms, and investigate results corresponding to Rauch's comparison theorem on Jacobi fields.

1. Magnetic Jacobi fields

Let γ be a trajectory for a uniform magnetic field $\mathbb{B}$ on a Riemannian manifold M. We say a vector field Y along γ to be a *magnetic Jacobi field* for $\mathbb{B}$ if it satisfies

$$\begin{cases} \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y - \Omega_{\mathbb{B}}(\nabla_{\dot\gamma}Y) + R(Y,\dot\gamma)\dot\gamma = 0, \\ \langle \nabla_{\dot\gamma}Y, \dot\gamma\rangle = 0. \end{cases} \tag{3.1}$$

Since the first equality in (3.1) is a linear differential equation, and since we have

$$\begin{aligned} \nabla_{\dot\gamma}\langle \nabla_{\dot\gamma}Y, \dot\gamma\rangle &= \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y, \dot\gamma\rangle + \langle \nabla_{\dot\gamma}Y, \Omega_{\mathbb{B}}(\dot\gamma)\rangle \\ &= \langle \Omega_{\mathbb{B}}(\nabla_{\dot\gamma}Y), \dot\gamma\rangle - \langle R(Y,\dot\gamma)\dot\gamma, \dot\gamma\rangle - \langle \Omega_{\mathbb{B}}(\nabla_{\dot\gamma}Y), \dot\gamma\rangle = 0 \end{aligned}$$

we find that a magnetic Jacobi field Y is defined uniquely by the initial condition $Y(0), (\nabla_{\dot\gamma}Y)(0)$.

We here study the relationship between magnetic Jacobi fields and variations of trajectories. We say a smooth map $\alpha : (-\epsilon,\epsilon)\times\mathbb{R} \to M$ to be a *variation of trajectories*, if the map $\alpha(s,\cdot) : \{s\}\times\mathbb{R} \to M$ is a trajectory for $\mathbb{B}$ for each s.

LEMMA 3.1. *Let $\alpha : (-\epsilon,\epsilon)\times\mathbb{R} \to M$ be a variation of $\mathbb{B}$-trajectories.*

(1) *The vector field Y_s defined by $Y_s(t) = \dfrac{\partial\alpha}{\partial s}(s,t)$ is a magnetic Jacobi field for $\mathbb{B}$ along a trajectory $t \mapsto \alpha(s,t)$.*

(2) *On the other hand, given a magnetic Jacobi field* Y *for* $\mathbb{B}$ *along a* $\mathbb{B}$*-trajectory* γ*, there exists a variation* α *of* $\mathbb{B}$*-trajectories satisfying* $\alpha(0,t) = \gamma(t)$ *and* $\dfrac{\partial\alpha}{\partial s}(0,t) = Y(t)$.

PROOF. (1) Since $\alpha : (-\epsilon,\epsilon)\times\mathbb{R}\to M$ is a variation, we see $\alpha_s : t\mapsto \alpha(s,t)$ is a trajectory for $\mathbb{B}$ for each s. Then we have $\nabla_{\frac{\partial\alpha}{\partial t}}\dfrac{\partial\alpha}{\partial t} = \Omega_{\mathbb{B}}\dfrac{\partial\alpha}{\partial t}$. We take the differentials of both sides of this equalities on s. As $\Omega_{\mathbb{B}}$ is parallel, we get

$$\begin{aligned}\nabla_{\frac{\partial\alpha}{\partial s}}\Big(\nabla_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial t}\Big) &= \nabla_{\frac{\partial\alpha}{\partial s}}\Big(\Omega_{\mathbb{B}}\frac{\partial\alpha}{\partial t}\Big)\\ &= \Big(\nabla_{\frac{\partial\alpha}{\partial s}}\Omega_{\mathbb{B}}\Big)\Big(\frac{\partial\alpha}{\partial t}\Big)+\Omega_{\mathbb{B}}\Big(\nabla_{\frac{\partial\alpha}{\partial s}}\Big(\frac{\partial\alpha}{\partial t}\Big)\Big)\\ &= \Omega_{\mathbb{B}}\Big(\nabla_{\frac{\partial\alpha}{\partial t}}\Big(\frac{\partial\alpha}{\partial s}\Big)\Big).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\nabla_{\frac{\partial\alpha}{\partial s}}\Big(\nabla_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial t}\Big) &= \nabla_{\frac{\partial\alpha}{\partial t}}\Big(\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}\Big)+R\Big(\frac{\partial\alpha}{\partial t},\frac{\partial\alpha}{\partial s}\Big)\frac{\partial\alpha}{\partial t}\\ &= \nabla_{\frac{\partial\alpha}{\partial t}}\Big(\nabla_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)+R\Big(\frac{\partial\alpha}{\partial t},\frac{\partial\alpha}{\partial s}\Big)\frac{\partial\alpha}{\partial t}.\end{aligned}$$

Thus we find that $Y_s = \dfrac{\partial\alpha}{\partial s}(s,\cdot)$ satisfies the first equality in (3.1). Moreover, since α_s is a trajectory, we have $\Big\|\dfrac{\partial\alpha}{\partial t}\Big\| = 1$. Differentiating both sides of this equalities, we obtain

$$0 = \frac{\partial\alpha}{\partial s}\Big\langle\frac{\partial\alpha}{\partial t},\frac{\partial\alpha}{\partial t}\Big\rangle = 2\Big\langle\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t},\frac{\partial\alpha}{\partial t}\Big\rangle = 2\Big\langle\nabla_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial s},\frac{\partial\alpha}{\partial t}\Big\rangle.$$

Therefore we find that $Y_s = \dfrac{\partial\alpha}{\partial s}$ is a magnetic Jacobi field along α_s.

(2) We put $p=\gamma(0)$ and $v=\dot\gamma(0)$. We take a smooth curve $\sigma : (-\epsilon,\epsilon)\to T_pM$ satisfying $\sigma(0)=p, \dot\sigma(0)=Y(0)$, and take a smooth curve $u : (-\epsilon,\epsilon)\to TM$ satisfying $u(0)=v, \dot u(0)=\nabla_{\dot\gamma}Y(0), u(s)\in U_{\sigma(s)}M$. We define a smooth map $\alpha : (-\epsilon,\epsilon)\times\mathbb{R}\to M$ by $\alpha(s,t) = \mathbb{B}\mathrm{exp}_{\sigma(s)}\big(tu(s)\big)$. It is clear that this is a variation of trajectories for $\mathbb{B}$. We have also $\alpha(0,t) = \mathbb{B}\mathrm{exp}_p(tv) = \gamma(t)$. As $\alpha(s,0) = \mathbb{B}\mathrm{exp}_{\sigma(s)}(0) = \sigma(s)$, we find $\dfrac{\partial\alpha}{\partial s}(0,0) = \dot\sigma(0) = Y(0)$. Moreover, if we denote by γ_v the $\mathbb{B}$-trajectory with $\dot\gamma_v(0)=v$, as $\dfrac{\partial\alpha}{\partial t}(s,t) = \dot\gamma_{u(s)}(t)$, we get

$$\nabla_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial s}(0,0) = \nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}(0,0) = \nabla_{\frac{\partial\alpha}{\partial s}}\dot\gamma_{u(s)}\big|_{s=0} = \nabla_{\frac{\partial\alpha}{\partial s}}u(s)\big|_{s=0} = \dot u(0) = \nabla_{\dot\gamma}Y(0).$$

Thus $\dfrac{\partial\alpha}{\partial s}(0,t)$ satisfies the same initial condition as of $Y(t)$. This means that $Y(t) = \dfrac{\partial\alpha}{\partial s}(0,t)$. We get the conclusion. □

We study more on magnetic Jacobi fields.

Lemma 3.2. *Let Y and W be $\mathbb{B}$-Jacobi fields along a $\mathbb{B}$-trajectory γ. Then*

$$\langle \nabla_{\dot\gamma} Y, W\rangle - \langle Y, \nabla_{\dot\gamma} W\rangle + \langle Y, \Omega_{\mathbb{B}}(W)\rangle$$

is constant along γ.

Proof. By taking the differential of $\langle \nabla_{\dot\gamma} Y, W\rangle - \langle Y, \nabla_{\dot\gamma} W\rangle + \langle Y, \Omega_{\mathbb{B}}(W)\rangle$, we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\langle \nabla_{\dot\gamma} Y, W\rangle - \langle Y, \nabla_{\dot\gamma} W\rangle + \langle Y, \Omega_{\mathbb{B}} W\rangle\Big)\\
&\quad= \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} Y, W\rangle + \langle \nabla_{\dot\gamma} Y, \nabla_{\dot\gamma} W\rangle - \langle \nabla_{\dot\gamma} Y, \nabla_{\dot\gamma} W\rangle\\
&\qquad\quad - \langle Y, \nabla_{\dot\gamma}\nabla_{\dot\gamma} W\rangle + \langle \nabla_{\dot\gamma} Y, \Omega_{\mathbb{B}} W\rangle + \langle Y, \Omega_{\mathbb{B}}(\nabla_{\dot\gamma} W)\rangle\\
&\quad= \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} Y, W\rangle - \langle Y, \nabla_{\dot\gamma}\nabla_{\dot\gamma} W\rangle + \langle \nabla_{\dot\gamma} Y, \Omega_{\mathbb{B}} W\rangle + \langle Y, \Omega_{\mathbb{B}}(\nabla_{\dot\gamma} W)\rangle\\
&\quad= \langle \Omega_{\mathbb{B}}(\nabla_{\dot\gamma} Y) - R(Y,\dot\gamma)\dot\gamma, W\rangle - \langle Y, \Omega_{\mathbb{B}}(\nabla_{\dot\gamma} W) - R(W,\dot\gamma)\dot\gamma\rangle\\
&\qquad\quad + \langle \nabla_{\dot\gamma} Y, \Omega_{\mathbb{B}}(W)\rangle + \langle Y, \Omega_{\mathbb{B}}(\nabla_{\dot\gamma} W)\rangle\\
&\quad= 0.
\end{aligned}$$

We therefore get conclusion. □

A vector field Y along a geodesic is said to be a Jacobi field if it satisfies $\nabla_{\dot\gamma}\nabla_{\dot\gamma} Y + R(Y,\dot\gamma)\dot\gamma = 0$. So, we do not need the second equality in (3.1). The difference comes from the fact that we restrict trajectories to be of unit speed. Even if we change the speed of a geodesic it is still a geodesic. While, as we see in Lemma 2.10, if we change the speed of a trajectory, then it is seen as a “trajectory” for another magnetic field. We hence need the second equality in the definition of magnetic Jacobi fields.

For a trajectory γ for $\mathbb{B}$, we denote by $\mathcal{J}_\gamma$ the set of all magnetic Jacobi fields for $\mathbb{B}$ along γ. Then it is a vector space of dimension $2\dim_{\mathbb{R}}(M) - 1$, because the first equality in (3.1) is a linear differential equation. For a vector field X along γ, we set $X^\sharp = X - \langle X, \dot\gamma\rangle\dot\gamma$, which is the component orthogonal to $\dot\gamma$. In order to consider singularities of the map $\Phi : (0, r) \times U_pM \to M$ with a constant r, we take a trajectory γ for $\mathbb{B}$ with $\gamma(0) = p$. We call a constant t_c a *magnetic conjugate value* of p along γ if there is a non-trival magnetic Jacobi field Y along γ which satisfies $Y^\sharp(0) = 0$ and $Y^\sharp(t_c) = 0$. The point $\gamma(t_c)$ is said to be a *magnetic conjugate point* of p along γ. We call the minimum positive magnetic conjugate value $c_\gamma(p)$ the *first magnetic conjugate value* of p along γ. We set $c_\gamma(p) = \infty$ when there are no positive magnetic conjugate values along γ.

We now restrict ourselves to Kähler magnetic fields. Since $\Omega_{\mathbb{B}_k} = kJ$, a C^∞-vector field Y along a $\mathbb{B}_k$-trajectory is a magnetic Jacobi field for $\mathbb{B}_k$, if it satisfies

$$\begin{cases} \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y - kJ(\nabla_{\dot\gamma}Y) + R(Y, \dot\gamma)\dot\gamma = 0, \\ \langle \nabla_{\dot\gamma}Y, \dot\gamma\rangle = 0. \end{cases} \tag{3.2}$$

For a vector field X along a trajectory γ for $\mathbb{B}_k$, we divide it into three components and denote as $X = f_X\dot\gamma + g_X J\dot\gamma + X^\perp$ with smooth functions $f_X,\ g_X$ and a vector field $X^\perp$ along γ which is orthogonal to both $\dot\gamma$ and $J\dot\gamma$ at each point. We hence have $X^\sharp = g_X J\dot\gamma + X^\perp$.

Lemma 3.3. *The vector field $X^\perp$ along γ satisfies $\langle \nabla^m_{\dot\gamma} X^\perp, \dot\gamma\rangle = \langle \nabla^m_{\dot\gamma} X^\perp, J\dot\gamma\rangle = 0$ for an arbitrary positive m, where $\nabla^m_{\dot\gamma} = \overbrace{\nabla_{\dot\gamma}\cdots\nabla_{\dot\gamma}}^{m}$.*

Proof. For a vector field X along a trajectory γ for $\mathbb{B}_k$, we have $\langle X^\perp, \dot\gamma\rangle = \langle X^\perp, J\dot\gamma\rangle = 0$. Then we get

$$0 = \nabla_{\dot\gamma}\langle X^\perp, \dot\gamma\rangle = \langle \nabla_{\dot\gamma}X^\perp, \dot\gamma\rangle + k\langle X^\perp, J\dot\gamma\rangle = \langle \nabla_{\dot\gamma}X^\perp, \dot\gamma\rangle$$
$$0 = \nabla_{\dot\gamma}\langle X^\perp, J\dot\gamma\rangle = \langle \nabla_{\dot\gamma}X^\perp, J\dot\gamma\rangle - k\langle X^\perp, \dot\gamma\rangle = \langle \nabla_{\dot\gamma}X^\perp, J\dot\gamma\rangle.$$

By mathematical induction, for all integers m, we have

$$\langle \nabla^m_{\dot\gamma} X^\perp, \dot\gamma\rangle = \langle \nabla^m_{\dot\gamma} X^\perp, J\dot\gamma\rangle = 0,$$

and get the conclusion. $\square$

LEMMA 3.4. *A vector field X along a trajectory γ for $\mathbb{B}_k$ is a magnetic Jacobi field if and only if it satisfies*

$$\begin{cases} f_X' = kg_X, \\ (g_X'' + k^2 g_X)J\dot\gamma + \nabla_{\dot\gamma}\nabla_{\dot\gamma}X^\perp - kJ(\nabla_{\dot\gamma}X^\perp) + R(X,\dot\gamma)\dot\gamma = 0. \end{cases} \tag{3.3}$$

PROOF. We take a vector field $X = f_X\dot\gamma + g_X J\dot\gamma + X^\perp$ along a $\mathbb{B}_k$-trajectory γ. By taking its covariant differentiation, we have

$$\nabla_{\dot\gamma}X = (f_X' - kg_X)\dot\gamma + (kf_X + g_X')J\dot\gamma + \nabla_{\dot\gamma}X^\perp.$$

Thus, we see $\langle\nabla_{\dot\gamma}X, \dot\gamma\rangle = 0$ if and only if $f_X' = kg_X$. The second covariant derivative of X is given as

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}X = \big\{(f_X' - kg_X)' - k(kf_X + g_X')\big\}\dot\gamma + \big\{k(f_X' - kg_X) + (kf_X + g_X')'\big\}J\dot\gamma + \nabla_{\dot\gamma}\nabla_{\dot\gamma}X^\perp.$$

Under the assumption that $\langle\nabla_{\dot\gamma}X, \dot\gamma\rangle = 0$ it turns to

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}X = -k(kf_X + g_X')\dot\gamma + (kf_X' + g_X'')J\dot\gamma + \nabla_{\dot\gamma}\nabla_{\dot\gamma}X^\perp.$$

Substituting the first and the second covariant differentiations into the left hand side of the first equation in (3.2), we have

$$\begin{aligned} (\text{left hand side}) &= -k(kf_X + g_X')\dot\gamma + (kf_X' + g_X'')J\dot\gamma + \nabla_{\dot\gamma}\nabla_{\dot\gamma}X^\perp \\ &\qquad - kJ\big\{(kf_X + g_X')J\dot\gamma + \nabla_{\dot\gamma}X^\perp\big\} + R(X,\dot\gamma)\dot\gamma \\ &= (g_X'' + k^2 g_X)J\dot\gamma + \nabla_{\dot\gamma}\nabla_{\dot\gamma}X^\perp - kJ(\nabla_{\dot\gamma}X^\perp) + R(X,\dot\gamma)\dot\gamma. \end{aligned}$$

Thus, we find that a vector field X along γ satisfying $\langle\nabla_{\dot\gamma}X, \dot\gamma\rangle = 0$ is a magnetic Jacobi field if and only if it satisfies the second equation in (3.2). This completes the proof. □

REMARK 3.1. We note that $R(X,\dot\gamma)\dot\gamma$ does not have a component parallel to $\dot\gamma$ and $R(X,\dot\gamma)\dot\gamma = g_X R(J\dot\gamma,\dot\gamma)\dot\gamma + R(X^\perp,\dot\gamma)\dot\gamma$. But we can not distinguish the components of $R(J\dot\gamma,\dot\gamma)\dot\gamma, R(X^\perp,\dot\gamma)\dot\gamma$ which are parallel to and orthogonal to $J\dot\gamma$.

2. Magnetic Jacobi field on complex space forms

In this section we study magnetic Jacobi fields for Kähler magnetic fields on a complex space form $\mathbb{C}M^n(c)$ of constant holomorphic sectional curvature c. We note that the curvature tensor R on a complex space form $\mathbb{C}M^n(c)$ is expressed as

$$R(X,Y)Z = \frac{c}{4}\big\{\langle Y,Z\rangle X - \langle X,Z\rangle Y - \langle Y,JZ\rangle JX + \langle X,JZ\rangle JY + 2\langle X,JY\rangle JZ\big\}$$

for vector fields $X,Y,Z \in \mathfrak{X}(\mathbb{C}M^n(c))$. Thus, for a vector field $Y = f_Y\dot\gamma + g_Y J\dot\gamma + Y^\perp$ along a $\mathbb{B}_k$-trajectory γ, we have

$$R(Y,\dot\gamma)\dot\gamma = -R(\dot\gamma,Y)\dot\gamma = -\frac{c}{4}\big\{f_Y\dot\gamma - Y - g_Y J\dot\gamma - 2g_Y J\dot\gamma\big\} = c\Big(gJ\dot\gamma + \frac{1}{4}Y^\perp\Big).$$

Therefore, by Lemma 3.4, we find that a vector field $Y = f_Y\dot\gamma + g_Y J\dot\gamma + Y^\perp$ along γ is a magnetic Jacobi field if and only if it satisfies

$$\begin{cases} f_Y' = kg_Y, \\ g_Y'' + (k^2+c)g_Y = 0, \\ \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y^\perp - kJ(\nabla_{\dot\gamma}Y^\perp) + \dfrac{c}{4}Y^\perp = 0. \end{cases} \tag{3.4}$$

[1] Magnetic Jacobi fields on a complex Euclidean space

We study magnetic Jacobi fields along a trajectory γ for a Kähler magnetic field $\mathbb{B}_k$ on $\mathbb{C}^n$.

In this case the covariant differential is the ordinary differential. We therefore find that (3.4) turns to

$$\begin{cases} f_Y' = kg_Y, \\ g_Y'' + k^2 g_Y = 0, \\ (Y^\perp)'' - kJ(Y^\perp)' = 0. \end{cases}$$

By solving the second equality $g_Y'' + k^2 g_Y = 0$, we get

$$g_Y(t) = c_1\cos kt + c_2\sin kt$$

with some constants $c_1, c_2 \in \mathbb{R}$. As $f_Y' = kg_Y$, it leads us to

$$f_Y(t) = c_1\sin kt - c_2\cos kt + c_3$$

with a constant $c_3 \in \mathbb{R}$. Also by solving $(Y^\perp)'' - kJ(Y^\perp)' = 0$, we get

$$Y^\perp(t) = (\gamma(t), A + Be^{\sqrt{-1}kt})$$

with some vectors $A, B \in \mathbb{C}^n$ which are orthogonal to both $\dot\gamma(0)$ and $J\dot\gamma(0)$.

If we suppose $Y(0) = 0$, we then find that a magnetic Jacobi field Y on $\mathbb{C}^n$ is expressed as

$$Y(t) = a\big\{(1-\cos kt)\dot\gamma(t) + (\sin kt)J\dot\gamma(t)\big\} + \big(\gamma(t), (1 - e^{\sqrt{-1}kt})C\big) \tag{3.5}$$

with $a \in \mathbb{R}$, $C \in \mathbb{C}^n$ satisfying that C is orthogonal to both $\dot\gamma(0)$ and $J\dot\gamma(0)$.

[2] Magnetic Jacobi fields on a complex projective space

We study magnetic Jacobi fields along a trajectory γ for a Kähler magnetic field $\mathbb{B}_k$ on $\mathbb{C}P^n(c)$. By solving the second equation $g_Y'' + (k^2 + c)g = 0$ in (3.4), we get

$$g(t) = c_1 \cos\sqrt{k^2+c}\,t + c_2 \sin\sqrt{k^2+c}\,t$$

with constants $c_1, c_2 \in \mathbb{R}$. As $f_Y' = kg_Y$, it leads us to

$$f_Y(t) = \frac{k}{\sqrt{k^2+c}}\Big(c_1 \sin\sqrt{k^2+c}\,t - c_2\cos\sqrt{k^2+c}\,t\Big) + c_3$$

with a constant $c_3 \in \mathbb{R}$.

Next, we study the third equation of (3.4). Since $\nabla_{\dot\gamma}Y^\perp$ and $\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y^\perp$ are orthogonal to both $\dot\gamma$ and $J\dot\gamma$ by Lemma 3.3, if we denote by $\widetilde{Y}^\perp$ the horizontal lift of $Y^\perp$ along a horizontal lift $\tilde\gamma$ of γ through the Hopf fibration $\varpi : S^{2n+1} \to \mathbb{C}P^n$, we find by (1) of Lemma 1.3 and Lemma 2.2 that $\widetilde\nabla_{\dot{\tilde\gamma}}\widetilde{Y}^\perp$ and $\widetilde\nabla_{\dot{\tilde\gamma}}\widetilde\nabla_{\dot{\tilde\gamma}}\widetilde{Y}^\perp$ are horizontal lifts $\widetilde{\nabla_{\dot\gamma}Y^\perp}$, $\widetilde{\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y^\perp}$ of $\nabla_{\dot\gamma}Y^\perp$ and $\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y^\perp$, respectively. Here, $\widetilde\nabla$ denotes the Riemannian connection of S^{2n+1}. Hence we have $\overline\nabla_{\dot{\tilde\gamma}}\widetilde{Y}^\perp = \widetilde{\nabla_{\dot\gamma}Y^\perp}$ and $\overline\nabla_{\dot{\tilde\gamma}}\overline\nabla_{\dot{\tilde\gamma}}\widetilde{Y}^\perp = \widetilde{\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y^\perp}$ with the Riemannian connection $\overline\nabla$ on $\mathbb{C}^{n+1}$. Thus $\widetilde{Y}^\perp$ satisfies the differential equation

$$(\widetilde{Y}^\perp)'' - \sqrt{-1}k(\widetilde{Y}^\perp)' + \frac{c}{4}\widetilde{Y}^\perp = 0.$$

By solving its characteristic equation $\lambda^2 - k\sqrt{-1}\lambda + (c/4) = 0$, we have

$$\lambda_1 = \sqrt{-1}\big(k + \sqrt{k^2+c}\,\big)/2, \quad \lambda_2 = \sqrt{-1}\big(k - \sqrt{k^2+c}\,\big)/2.$$

Thus, we get

$$\widetilde{Y}^{\perp}(t) = d\varpi\Big(\tilde{\gamma}(t), e^{\sqrt{-1}kt/2}\big(A\cos\frac{\sqrt{k^2+c}\,t}{2} + B\sin\frac{\sqrt{k^2+c}\,t}{2}\big)\Big)$$

with some $A, B \in \mathbb{C}^{n+1}$ which are orthogonal to $\tilde{\gamma}(0), J\tilde{\gamma}(0), \dot{\tilde{\gamma}}(0)$ and $J\dot{\tilde{\gamma}}(0)$.

If we suppose that a magnetic Jacobi field Y satisfies $Y(0) = 0$, we then find that it is expressed as

$$(3.6)\qquad \begin{aligned} Y(t) &= a\big\{k(1-\cos\sqrt{k^2+c}\,t)\dot{\gamma}(t) + \sqrt{k^2+c}\,(\sin\sqrt{k^2+c}\,t)J\dot{\gamma}(t)\big\} \\ &\quad + d\varpi\Big(\tilde{\gamma}(t), Ce^{\sqrt{-1}kt/2}\sin\frac{1}{2}\sqrt{k^2+c}\,t\Big) \end{aligned}$$

with some $a \in \mathbb{R}$ and $C \in \mathbb{C}^{n+1}$ which is orthogonal to $\tilde{\gamma}(0), J\tilde{\gamma}(0), \dot{\tilde{\gamma}}(0)$ and $J\dot{\tilde{\gamma}}(0)$.

[3] Magnetic Jacobi fields on a complex hyperbolic space

We study magnetic Jacobi fields along a trajectory γ for a Kähler magnetic field $\mathbb{B}_k$ on $\mathbb{C}H^n(c)$. Since k^2+c is positive, zero and negative according to $|k| > \sqrt{|c|}$, $k = \pm\sqrt{|c|}$ and $|k| < \sqrt{|c|}$, we need to study separately. First, by solving the second equation $g'' + (k^2+c)g = 0$ of (3.4), we get

$$g_Y(t) = \begin{cases} c_1\cosh\sqrt{k^2+c}\,t + c_2\sinh\sqrt{k^2+c}\,t, & \text{if } |k| < \sqrt{|c|}, \\ c_1 + c_2 t, & \text{if } k = \pm\sqrt{|c|}, \\ c_1\cos\sqrt{k^2+c}\,t + c_2\sin\sqrt{k^2+c}\,t, & \text{if } |k| > \sqrt{|c|}, \end{cases}$$

with some constants $c_1, c_2 \in \mathbb{R}$. As $f_Y' = kg_Y$, it leads us to

$$f_Y(t) = \begin{cases} \dfrac{k}{\sqrt{k^2+c}}\big(c_1\sinh\sqrt{k^2+c}\;t + c_2\cosh\sqrt{k^2+c}\;t\big) + c_3, & \text{if } |k| < \sqrt{|c|}, \\ c_1 + \dfrac{c_2}{2}\,t^2 + c_3, & \text{if } k = \pm\sqrt{|c|}, \\ \dfrac{k}{\sqrt{k^2+c}}\big(c_1\sin\sqrt{k^2+c}\;t - c_2\cos\sqrt{k^2+c}\;t\big) + c_3, & \text{if } |k| > \sqrt{|c|}, \end{cases}$$

with some constant $c_3 \in \mathbb{R}$.

Next, we consider the third equation of (3.4). In the case $c = -4$, we take a horizontal lift $\widetilde{Y}^{\perp}$ of $Y^{\perp}$ along a horizontal lift $\tilde{\gamma}$ of γ. it is rewrited to the equation in $\mathbb{C}^{n+1}$ through the Hopf fibration $\varpi : H^{2n+1}(1) \to \mathbb{C}H^n(-4)$. Since both $Y^{\perp}$ and $\nabla_{\dot{\gamma}}Y^{\perp}$ are vertical to $\dot{\gamma}$ and $J\dot{\gamma}$, the horizontal lift of $Y^{\perp}$ is expressed as

$$(\widetilde{Y}^{\perp})'' - \sqrt{-1}k(\widetilde{Y}^{\perp})' - \widetilde{Y}^{\perp} = 0.$$

By solving its characteristic equation $\lambda^2 - k\sqrt{-1}\lambda - 1 = 0$, we have

$$\lambda_1 = \begin{cases} \big(\sqrt{-1}k + \sqrt{4-k^2}\,\big)/2, & k^2 < 4, \\ \sqrt{-1}k/2, & k = \pm 2, \\ \sqrt{-1}\big(k + \sqrt{k^2-4}\,\big)/2, & k^2 > 4, \end{cases}$$

$$\lambda_2 = \begin{cases} \big(\sqrt{-1}k - \sqrt{4-k^2}\,\big)/2, & k^2 < 4, \\ \sqrt{-1}k/2, & k = \pm 2, \\ \sqrt{-1}\big(k - \sqrt{k^2-4}\,\big)/2, & k^2 > 4. \end{cases}$$

Thus, we obtain

$$\widetilde{Y}^{\perp}(t) = \begin{cases} d\varpi\Big(\tilde{\gamma}(t), e^{\sqrt{-1}kt/2}\big(A\cosh\dfrac{\sqrt{4-k^2}\,t}{2} + B\sinh\dfrac{\sqrt{4-k^2}\,t}{2}\big)\Big), & |k| < 2, \\ d\varpi\big(\tilde{\gamma}(t), e^{\sqrt{-1}kt/2}(A + Bt)\big), & k = \pm 2, \\ d\varpi\Big(\tilde{\gamma}(t), e^{\sqrt{-1}kt/2}\big(A\cos\dfrac{\sqrt{k^2-4}\,t}{2} + B\sin\dfrac{\sqrt{k^2-4}\,t}{2}\big)\Big), & |k| > 2, \end{cases}$$

with $A, B \in \mathbb{C}^{n+1}$, which are orthogonal to $\tilde{\gamma}(0), J\tilde{\gamma}(0), \dot{\tilde{\gamma}}(0)$ and $J\dot{\tilde{\gamma}}(0)$. If we suppose $Y(0) = 0$, we then have magnetic Jacobi fields on $\mathbb{C}H^n$ are expressed as

$$Y(t) = \begin{cases} a\big\{k(\cosh\sqrt{4-k^2}\,t - 1)\dot{\gamma}(t) + \sqrt{4-k^2}(\sinh\sqrt{4-k^2}\,t)J\dot{\gamma}(t)\big\} & \\ \qquad + d\varpi\Big(\tilde{\gamma}(t), Ce^{\sqrt{-1}kt/2}\sinh\dfrac{\sqrt{4-k^2}\,t}{2}\Big), & k^2 < 4, \\ a\big\{2t^2\dot{\gamma}(t) + ktJ\dot{\gamma}(t)\big\} + d\varpi\big((\tilde{\gamma}(t), Ce^{\sqrt{-1}kt/2})\big), & k = \pm 2, \\ a\big\{k(1 - \cos\sqrt{k^2-4}\,t)\dot{\gamma}(t) + \sqrt{k^2-4}(\sin\sqrt{k^2-4}\,t)J\dot{\gamma}(t)\big\} & \\ \qquad + d\varpi\Big(\tilde{\gamma}(t), Ce^{\sqrt{-1}kt/2}\sin\dfrac{\sqrt{k^2-4}\,t}{2}\Big), & k^2 > 4, \end{cases}$$

with $a \in \mathbb{R}$ and $C \in \mathbb{C}^{n+1}$ which is orthogonal to $\tilde{\gamma}(0), J\tilde{\gamma}(0), \dot{\tilde{\gamma}}(0)$ and $J\dot{\tilde{\gamma}}(0)$.

In general case, we make use of a homothetic change of metrics. If we change the metric on $\mathbb{C}H^n(c)$ homothetically to $(\sqrt{|c|}/2)^2\langle\ ,\ \rangle$, then the curve $\sigma(s) = \gamma(2s/\sqrt{|c|})$ is a trajectory for $\mathbb{B}_{k'} = \mathbb{B}_{2k/\sqrt{|c|}}$ on $\mathbb{C}H^n(-4)$. If we set $Z(s) = Y(2s/\sqrt{|c|})$, it is a vector field along σ. We hence obtain

$$Z^{\perp}(s) = \begin{cases} d\varpi\Big(\tilde{\sigma}(s), e^{\sqrt{-1}k's/2}\big(A\cosh\dfrac{\sqrt{4-k'^2}\,s}{2} + B\sinh\dfrac{\sqrt{4-k'^2}\,s}{2}\big)\Big), & |k'|<2, \\ d\varpi\big(\tilde{\sigma}(s), e^{\sqrt{-1}k's/2}(A+Bs)\big), & k'=\pm 2, \\ d\varpi\Big(\tilde{\sigma}(s), e^{\sqrt{-1}k's/2}\big(A\cos\dfrac{\sqrt{k'^2-4}\,s}{2} + B\sin\dfrac{\sqrt{k'^2-4}\,s}{2}\big)\Big), & |k'|>2. \end{cases}$$

Therefore we have

$$\begin{aligned} Y^{\perp}(t) &= Z^{\perp}(\sqrt{|c|}t/2) \\ &= \begin{cases} d\varpi\Big(\tilde{\gamma}(t), e^{\frac{\sqrt{-1}kt}{2}}\big(A\cosh\dfrac{\sqrt{4-\frac{4k^2}{|c|}}\frac{\sqrt{|c|}\,t}{2}}{2} + B\sinh\dfrac{\sqrt{4-\frac{4k^2}{|c|}}\frac{\sqrt{|c|}\,t}{2}}{2}\big)\Big), & |k|<\sqrt{|c|}, \\ d\varpi\Big(\tilde{\gamma}(t), e^{\frac{\sqrt{-1}kt}{2}}(A+\dfrac{\sqrt{|c|}}{2}Bt)\Big), & k=\pm\sqrt{|c|}, \\ d\varpi\Big(\tilde{\gamma}(t), e^{\frac{\sqrt{-1}kt}{2}}\big(A\cos\dfrac{\sqrt{\frac{4k^2}{|c|}-4}\frac{\sqrt{|c|}\,t}{2}}{2} + B\sin\dfrac{\sqrt{\frac{4k^2}{|c|}-4}\frac{\sqrt{|c|}\,t}{2}}{2}\big)\Big), & |k|>\sqrt{|c|}, \end{cases} \\ &= \begin{cases} d\varpi\Big(\tilde{\gamma}(t), e^{\frac{\sqrt{-1}kt}{2}}\big(A\cosh\dfrac{\sqrt{|c|-k^2}\,t}{2} + B\sinh\dfrac{\sqrt{|c|-k^2}\,t}{2}\big)\Big), & |k|<\sqrt{|c|}, \\ d\varpi\big(\tilde{\gamma}(t), e^{\frac{\sqrt{-1}kt}{2}}(A+Bt)\big), & k=\pm\sqrt{|c|}, \\ d\varpi\Big(\tilde{\gamma}(t), e^{\frac{\sqrt{-1}kt}{2}}\big(A\cos\dfrac{\sqrt{k^2+c}\,t}{2} + B\sin\dfrac{\sqrt{k^2+c}\,t}{2}\big)\Big), & |k|>\sqrt{|c|}. \end{cases} \end{aligned}$$

We note that the horizontal lift $\widetilde{Y}^{\perp}$ satisfies $(\widetilde{Y}^{\perp})''-\sqrt{-1}k(\widetilde{Y}^{\perp})'+\frac{c}{4}\widetilde{Y}^{\perp}=0$ by Lemmas 1.3 and 2.10. We can get the expressions of $\widetilde{Y}^{\perp}$ directory by solving this differential equation.

We here consider magnetic Jacobi fields under the condition that their initials are null. By the condition $g_Y(0)=0$, we get $c_1=0$. Therefore, by the condition $f_Y(0)=0$, we find that

$$c_3 = \begin{cases} \dfrac{-kc_2}{\sqrt{|c|-k^2}}, & |k|<\sqrt{|c|}, \\ 0, & k=\pm\sqrt{|c|}, \\ \dfrac{kc_2}{\sqrt{k^2+c}}, & |k|>\sqrt{|c|}. \end{cases}$$

Also by the condition $Y^{\perp}(0) = 0$, we find that $A = 0$. Thus we get this magnetic Jacobi field Y is expressed as follows :

$$
(3.7) \qquad Y(t) = \begin{cases} a\Big\{k\big(\cosh\sqrt{|c|-k^2}\,t - 1\big)\dot\gamma(t) \\ \qquad +\sqrt{|c|-k^2}\big(\sinh\sqrt{|c|-k^2}\,t\big)J\dot\gamma(t)\Big\} \\ \qquad\qquad +d\varpi\Big(\tilde\gamma(t), Ce^{\frac{\sqrt{-1}kt}{2}}\sinh\dfrac{\sqrt{|c|-k^2}\,t}{2}\Big), & |k| < \sqrt{|c|}, \\ a\Big\{\dfrac{1}{2}kt^2\dot\gamma(t) + tJ\dot\gamma(t)\Big\} + d\varpi\Big(\tilde\gamma(t), Ce^{\frac{\sqrt{-1}kt}{2}}t\Big), & k = \pm\sqrt{|c|}, \\ a\Big\{k\big(1-\cos\sqrt{k^2+c}\,t\big)\dot\gamma(t) \\ \qquad +\sqrt{k^2+c}\big(\sin\sqrt{k^2+c}\,t\big)J\dot\gamma(t)\Big\} \\ \qquad\qquad +d\varpi\Big(\tilde\gamma(t), Ce^{\frac{\sqrt{-1}kt}{2}}\sin\dfrac{\sqrt{k^2+c}\,t}{2}\Big), & |k| > \sqrt{|c|}, \end{cases}
$$

with a constant $a \in \mathbb{R}$ and a horizontal vector $C \in \mathbb{C}^{n+1}$ which is orthogonal to $\tilde\gamma(0), J\tilde\gamma(0), \dot{\tilde\gamma}(0)$ and $J\dot{\tilde\gamma}(0)$.

In order to treat magnetic Jacobi fields on a complex space form $\mathbb{C}M^n(c)$ uniformly, we define two functions

$$
\mathfrak{s}_k(t;c),\ \mathfrak{t}_k(t;c) : [0, \pi/\sqrt{k^2+c}\,) \to [0,\infty)
$$

by

$$
\mathfrak{s}_k(t;c) = \begin{cases} (1/\sqrt{k^2+c}\,)\sin(\sqrt{k^2+c}\,t), & \text{when } k^2+c>0, \\ t, & \text{when } k^2+c=0, \\ (1/\sqrt{|c|-k^2}\,)\sinh(\sqrt{|c|-k^2}\,t), & \text{when } k^2+c<0, \end{cases}
$$

$$
\mathfrak{t}_k(t;c) = \frac{\mathfrak{s}_k'(t;c)}{\mathfrak{s}_k(t;c)} = \begin{cases} \sqrt{k^2+c}\cot(\sqrt{k^2+c}\,t), & \text{when } k^2+c>0, \\ 1/t, & \text{when } k^2+c=0, \\ \sqrt{|c|-k^2}\coth(\sqrt{|c|-k^2}\,t), & \text{when } k^2+c<0. \end{cases}
$$

Here, we regard $\pi/\sqrt{k^2+c}$ in the domains of $\mathfrak{s}_k(t;c)$ and $\mathfrak{t}_k(t;c)$ as infinity when $k^2+c \le 0$.

Proposition 3.1. *On a complex space form $\mathbb{C}M^n(c)$ of constant holomorphic sectional curvature c, every magnetic Jacobi field $Y = f_Y\dot\gamma + g_Y J\dot\gamma + Y^\perp$ along a trajectory γ for $\mathbb{B}_k$ with $Y(0)=0$ satisfies the following properties for $0 \le t < \pi/\sqrt{k^2+c}$:*

1) $|g_Y(t)| = |g_Y'(0)|\mathfrak{s}_k(t;c)$, $\|Y^\perp(t)\| = \|\nabla_{\dot\gamma}Y^\perp(0)\| \times 2\mathfrak{s}_k(t/2;c)$;
2) $g_Y'(t) = g_Y(t)\mathfrak{t}_k(t;c)$, $\langle \nabla_{\dot\gamma}Y^\perp(t), Y^\perp(t)\rangle = \|Y^\perp(t)\|^2 \times (1/2)\mathfrak{t}_k(t/2;c)$.

Proof. By (3.5), (3.6) and (3.7), we have

$$g_Y(t) = \begin{cases} a\sqrt{|c|-k^2}\sinh\sqrt{|c|-k^2}\,t, & \text{when } k^2+c<0,\\ at, & \text{when } k^2+c=0,\\ a\sqrt{k^2+c}\sin\sqrt{k^2+c}\,t, & \text{when } k^2+c>0,\end{cases}$$

$$Y^\perp(t) = \begin{cases} e^{\sqrt{-1}kt/2}\sinh\big(\sqrt{|c|-k^2}\,t/2\big)\ E(t), & \text{when } k^2+c<0,\\ e^{\sqrt{-1}kt/2}t\ E(t), & \text{when } k^2+c=0,\\ e^{\sqrt{-1}kt/2}\sin\big(\sqrt{k^2+c}\,t/2\big)\ E(t), & \text{when } k^2+c>0,\end{cases}$$

with some constant a and a parallel vector field E along γ whose initial $E(0)$ is orthogonal to both $\dot\gamma(0)$ and $J\dot\gamma(0)$. By taking the differentiations of $g_Y(t)$ and $Y^\perp(t)$, we have

$$g_Y'(t) = \begin{cases} a(|c|-k^2)\cosh\sqrt{|c|-k^2}\,t, & \text{when } k^2+c<0,\\ a, & \text{when } k^2+c=0,\\ a(k^2+c)\cos\sqrt{k^2+c}\,t, & \text{when } k^2+c>0,\end{cases}$$

$$\nabla_{\dot\gamma}Y^\perp(t) = \begin{cases} \dfrac{1}{2}e^{\sqrt{-1}kt/2}\Big\{\sqrt{-1}k\sinh\tfrac12\sqrt{|c|-k^2}\,t \\ \qquad +\sqrt{|c|-k^2}\cosh\tfrac12\sqrt{|c|-k^2}\,t\Big\}E(t), & \text{when } k^2+c<0,\\ e^{\sqrt{-1}kt/2}\Big\{\dfrac{\sqrt{-1}k}{2}t+1\Big\}E(t), & \text{when } k^2+c=0,\\ \dfrac{1}{2}e^{\sqrt{-1}kt/2}\Big\{\sqrt{-1}k\sin\tfrac12\sqrt{k^2+c}\,t \\ \qquad +\sqrt{k^2+c}\cos\tfrac12\sqrt{k^2+c}\,t\Big\}E(t), & \text{when } k^2+c>0.\end{cases}$$

In particular, we have

$$g_Y'(0) = \begin{cases} a(|c|-k^2), & \text{when } k^2+c<0,\\ a, & \text{when } k^2+c=0,\\ a(k^2+c) & \text{when } k^2+c>0,\end{cases}$$

$$\nabla_{\dot\gamma}Y^{\perp}(0)=\begin{cases}\dfrac{1}{2}\sqrt{|c|-k^2}E(0), & \text{when } k^2+c<0,\\ E(0), & \text{when } k^2+c=0,\\ \dfrac{1}{2}\sqrt{k^2+c}E(0), & \text{when } k^2+c>0.\end{cases}$$

We hence get the conclusion. □

LEMMA 3.5. *These functions $\mathfrak{s}_k(t;c)$ and $\mathfrak{t}_k(t;c)$ satisfy the following properties for* $0<t<\pi/\sqrt{k^2+c}$:

(1) *If $|k_1|<|k_2|$, then $\mathfrak{t}_{k_1}(t;c)>\mathfrak{t}_{k_2}(t;c)$;*

(2) *$\mathfrak{s}_k(t;c)$ is strictly increasing (i.e. $\mathfrak{s}_k'(t;c)>0$) on $(0,\pi/2\sqrt{k^2+c})$;*

(3) *$\mathfrak{s}_k(t;c)<2\mathfrak{s}_k(t/2;c)$ when $k^2+c>0$,*

$\mathfrak{s}_k(t;c)>2\mathfrak{s}_k(t/2;c)$ when $k^2+c<0$;

(4) *$2\mathfrak{t}_k(t;c)<\mathfrak{t}_k(t/2;c)$ when $k^2+c>0$,*

$2\mathfrak{t}_k(t;c)>\mathfrak{t}_k(t/2;c)$ when $k^2+c<0$.

PROOF. (1) We are enough to consider the case $k_2>k_1\geq 0$. Since we have

$$\mathfrak{t}_k(t;c)=\begin{cases}\sqrt{k^2+c}\cot(\sqrt{k^2+c}\,t), & \text{when } k^2+c>0,\\ 1/t, & \text{when } k^2+c=0,\\ \sqrt{|c|-k^2}\coth(\sqrt{|c|-k^2}\,t), & \text{when } k^2+c<0,\end{cases}$$

we consider the differential of $\mathfrak{t}_k(t;c)$ with respect to k for $0<k<+\infty$.

When $k^2+c>0$, we have

$$\frac{d}{dk}\mathfrak{t}_k(t;c)=\frac{k\big\{\sin(2\sqrt{k^2+c}\,t)-2\sqrt{k^2+c}\,t\big\}}{2\sqrt{k^2+c}\sin(\sqrt{k^2+c}\,t)^2}.$$

Here, fixing $t(>0)$ we put $F_t(k)=\sin(2\sqrt{k^2+c}\,t)-2\sqrt{k^2+c}\,t$. We then have

$$\begin{aligned}\frac{d}{dk}F_t(k)&=\frac{-2kt}{\sqrt{k^2+c}}+\frac{2kt\cos(2\sqrt{k^2+c}\,t)}{\sqrt{k^2+c}}\\ &=\frac{2kt}{\sqrt{k^2+c}}\big\{\cos(2\sqrt{k^2+c}\,t)-1\big\}<0.\end{aligned}$$

and $F_t(0)=\sin(2\sqrt{c}t)-2\sqrt{c}t<0$ for $t>0$. Thus, we find that $F_t(k)$ is a monotone decreasing negative function when $0<k<\infty$ for each $t>0$. Therefore, we have $\mathfrak{t}_k(t;c)$ is monotone decreasing with respect to k for $0<k<\infty$.

When $k^2+c<0$,

$$\frac{d}{dk}\mathfrak{t}_k(t;c)=\frac{k\big\{t\sqrt{|c|-k^2}-(1/2)\sinh(2\sqrt{|c|-k^2}\,t)\big\}}{\sqrt{|c|-k^2}\sinh(\sqrt{|c|-k^2}\,t)^2}.$$

Here, fixing $t(>0)$ we put $G_t(k)=t\sqrt{|c|-k^2}-(1/2)\sinh(2\sqrt{|c|-k^2}\,t)$. We then have

$$\begin{aligned}\frac{d}{dk}G_t(k)&=\frac{-2kt}{\sqrt{|c|-k^2}}+\frac{2kt\cosh(2\sqrt{|c|-k^2}\,t)}{\sqrt{|c|-k^2}}\\&=\frac{2kt}{\sqrt{|c|-k^2}}\big\{\cosh(2\sqrt{|c|-k^2}\,t)-1\big\}>0,\end{aligned}$$

and $G_t(\sqrt{|c|})=0$. Thus, we find that $\dfrac{d}{dk}\mathfrak{t}_k(t;c)$ is a monotone increasing negative function when $0<k<\sqrt{|c|}$ for each $t>0$. Therefore, we have $\mathfrak{t}_k(t;c)$ is monotone decreasing with respect to k for $0<k<\sqrt{|c|}$.

Moreover, by de L'Hopital's rule we have

$$\lim_{k\downarrow\sqrt{|c|}}\mathfrak{t}_k(t;c)=\lim_{k\downarrow\sqrt{|c|}}\frac{\sqrt{k^2+c}}{\tan\sqrt{k^2+c}\,t}=\lim_{k\downarrow\sqrt{|c|}}\frac{\cos^2\sqrt{k^2+c}\,t}{t}=\frac{1}{t},$$

$$\lim_{k\uparrow\sqrt{|c|}}\mathfrak{t}_k(t;c)=\lim_{k\uparrow\sqrt{|c|}}\frac{\sqrt{|c|-k^2}}{\tanh\sqrt{|c|-k^2}\,t}=\lim_{k\uparrow\sqrt{|c|}}\frac{\cosh^2\sqrt{|c|-k^2}\,t}{t}=\frac{1}{t}.$$

Therefore we get the conclusion.

(2) We have

$$\mathfrak{s}_k'(t;c)=\begin{cases}\cosh(\sqrt{|c|-k^2}\,t), & \text{when } k^2+c<0,\\ 1, & \text{when } k^2+c=0,\\ \cos(\sqrt{k^2+c}\,t), & \text{when } k^2+c>0.\end{cases}$$

Thus we see $\mathfrak{s}_k'(t;c)>0$ for $0<t<\pi/\big(2\sqrt{k^2+c}\big)$ when $k^2+c>0$ and for $0<t<\infty$ when $k^2+c\le 0$. Hence, we find that $\mathfrak{s}_k(t;c)$ is strictly increasing on that interval.

(3) We define a smooth function $F(t;c):\big(0,\pi/\sqrt{k^2+c}\big)\to\mathbb{R}$ by

$$F(t;c)=\mathfrak{s}_k(t;c)\big/\big(2\mathfrak{s}_k(t/2;c)\big).$$

When $k^2+c>0$, we have

$$\begin{aligned}F(t;c)=\frac{\mathfrak{s}_k(t;c)}{2\mathfrak{s}_k(t/2;c)}&=\frac{\big(2/\sqrt{k^2+c}\big)\sin(\sqrt{k^2+c}\,t/2)\cos(\sqrt{k^2+c}\,t/2)}{\big(2/\sqrt{k^2+c}\big)\sin(\sqrt{k^2+c}\,t/2)}\\&=\cos(\sqrt{k^2+c}\,t/2).\end{aligned}$$

Hence we see $0 < F(t;c) < 1$ for $0 < t < \pi/\sqrt{k^2+c}$, which shows $\mathfrak{s}_k(t;c) < 2\mathfrak{s}_k(t/2;c)$ for $0 < t < \pi/\sqrt{k^2+c}$.

When $k^2+c<0$, we have

$$F(t;c) = \frac{\bigl(2/\sqrt{|c|-k^2}\,\bigr)\sinh(\sqrt{|c|-k^2}\;t/2)\cosh(\sqrt{|c|-k^2}\;t/2)}{\bigl(2/\sqrt{|c|-k^2}\,\bigr)\sinh(\sqrt{k^2+c}\,t/2)}$$

$$= \cosh(\sqrt{|c|-k^2}\;t/2) > 1.$$

Hence we find that $\mathfrak{s}_k(t;c) < 2\mathfrak{s}_k(t/2;c)$ for $t>0$.

(4) We define a smooth function $G(t;c) : \bigl(0, \pi/\sqrt{k^2+c}\,\bigr) \to \mathbb{R}$ by

$$G(t;c) = 2\mathfrak{t}_k(t;c)/\mathfrak{t}_k(t/2;c).$$

When $k^2+c>0$, we have

$$G(t;c) = \frac{2\cos(\sqrt{k^2+c}\;t)}{\sin(\sqrt{k^2+c}\;t)} \times \frac{\sin(\sqrt{k^2+c}\;t/2)}{\cos(\sqrt{k^2+c}\;t/2)} = \frac{2\cos^2(\sqrt{k^2+c}\;t/2)-1}{\cos^2(\sqrt{k^2+c}\;t/2)}$$
$$= 2 - \frac{1}{\cos^2(\sqrt{k^2+c}\;t/2)}$$

As $0 < \cos^2(\sqrt{k^2+c}\;t/2) < 1$, we have $G(t;c) < 1$, which shows $2\mathfrak{t}_k(t;c) < \mathfrak{t}_k(t/2;c)$.

When $k^2+c<0$, we have

$$G(t;c) = \frac{2\cosh(\sqrt{|c|-k^2}\;t)}{\sinh(\sqrt{|c|-k^2}\;t)} \times \frac{\cosh(\sqrt{|c|-k^2}\;t/2)}{\sinh(\sqrt{|c|-k^2}\;t/2)}$$
$$= \frac{\cosh^2(\sqrt{|c|-k^2}\;t/2 + \sinh^2(\sqrt{|c|-k^2}\;t/2)}{\cosh^2(\sqrt{|c|-k^2}\;t/2)}$$
$$= 1 + \tanh^2(\sqrt{|c|-k^2}\;t/2) > 1$$

Hence we have $2\mathfrak{t}_k(t;c) > \mathfrak{t}_k(t/2;c)$ for $t>0$. □

For the sake of later use, we here note more on these functions $\mathfrak{s}_k(t;c)$, $\mathfrak{t}_k(t;c)$. We have

$$\mathfrak{t}_{k/2}(t;c) = \begin{cases} \sqrt{k^2+4c}/2\cot\bigl(\sqrt{k^2+4c}/2\;t\bigr), & \text{when } k^2+4c>0,\\ 1/t, & \text{when } k^2+4c=0,\\ \sqrt{4|c|-k^2}/2\coth\bigl(\sqrt{4|c|-k^2}/2\;t\bigr), & \text{when } k^2+4c<0,\end{cases}$$

and $\mathfrak{t}_{k/2}(t;c) = (1/2)\mathfrak{t}_k(t/2;4c)$.

In order to estimate norms of magnetic Jacobi fields on a complex space form $\mathbb{C}M^n(c)$ we need the following technical inequalities.

LEMMA 3.6. *For constants β, δ and positive constants B, D, we have*

$$\min\Big\{\frac{\beta}{B}, \frac{\delta}{D}\Big\} \leq \frac{\beta+\delta}{B+D} \leq \max\Big\{\frac{\beta}{B}, \frac{\delta}{D}\Big\}.$$

PROOF. Without loss of generality we may suppose $\dfrac{\beta}{B} \leq \dfrac{\delta}{D}$. Since B, D are positive, we have $\beta D \leq \delta B$. As we have $\delta(B+D) - D(\beta+\delta) = \delta B - \beta D \geq 0$ and $D, B+D$ are positive, we get $\dfrac{\delta}{D} \geq \dfrac{\beta+\delta}{B+D}$, which shows the second inequality. Similarly, as we have $(\beta+\delta)B - \beta(B+D) = \delta B - \beta D \geq 0$, and $B, B+D$ are positives, we get $\dfrac{\beta+\delta}{B+D} \geq \dfrac{\delta}{B}$, which shows the first inequality. □

PROPOSITION 3.2. *On a complex space form $\mathbb{C}M^n(c)$ with $n \geq 2$, every magnetic Jacobi field $Y = f_Y\dot\gamma + g_Y J\dot\gamma + Y^\perp$ along a trajectory γ for $\mathbb{B}_k$ with $Y(0) = 0$ satisfies the following properties on $Y^\sharp = g_Y J\dot\gamma + Y^\perp$.*

(1) *When $k^2 + c > 0$, we have*

$$\|\nabla_{\dot\gamma} Y^\sharp(0)\|\mathfrak{s}_k(t;c) \;\leq\; \|Y^\sharp(t)\| \;\leq\; \|\nabla_{\dot\gamma} Y^\sharp(0)\| \times 2\mathfrak{s}_k(t/2;c),$$

$$\mathfrak{t}_k(t;c) \leq \frac{\langle \nabla_{\dot\gamma} Y^\sharp(t), Y^\sharp(t)\rangle}{\|Y^\sharp(t)\|^2} \leq \frac{1}{2}\,\mathfrak{t}_k(t/2;c)$$

for $0 \leq t < \pi/\sqrt{k^2+c}$.

(2) *When $k^2 + c = 0$, we have*

$$\|Y^\sharp(t)\| \;=\; \|\nabla_{\dot\gamma} Y^\sharp(0)\| t, \quad \frac{\langle \nabla_{\dot\gamma} Y^\sharp(t), Y^\sharp(t)\rangle}{\|Y^\sharp(t)\|^2} = \frac{1}{t}$$

for $t \geq 0$.

(3) *When $k^2 + c < 0$, we have*

$$\|\nabla_{\dot\gamma} Y^\sharp(0)\| \times 2\mathfrak{s}_k(t/2;c) \;\leq\; \|Y^\sharp(t)\| \;\leq\; \|\nabla_{\dot\gamma} Y^\sharp(0)\|\mathfrak{s}_k(t;c),$$

$$\frac{1}{2}\,\mathfrak{t}_k(t/2;c) \leq \frac{\langle \nabla_{\dot\gamma} Y^\sharp(t), Y^\sharp(t)\rangle}{\|Y^\sharp(t)\|^2} \leq \mathfrak{t}_k(t;c)$$

for $t \geq 0$.

PROOF. Since $Y^\sharp = g_Y J\dot\gamma + Y^\perp$, we have $\nabla_{\dot\gamma}Y^\sharp = -kg_Y\dot\gamma + g_Y' J\dot\gamma + \nabla_{\dot\gamma}Y^\perp$. By the conditon $Y(0)=0$, that is, $f_Y(0)=g_Y(0)=0$ and $Y^\perp(0)=0$, we see $\nabla_{\dot\gamma}Y^\sharp(0) = g_Y'(0)J\dot\gamma(0)+\nabla_{\dot\gamma}Y^\perp(0)$. As $Y^\perp$ is orthogonal to both $\dot\gamma$ and $J\dot\gamma$, we have $\langle\nabla_{\dot\gamma}Y^\perp, J\dot\gamma\rangle = 0$ by Lemma 3.3. We hence have $\|\nabla_{\dot\gamma}Y^\sharp(0)\|^2 = |g_Y'(0)|^2 + \|\nabla_{\dot\gamma}Y^\perp(0)\|^2$. As $J\dot\gamma$ and $Y^\perp$ are perpendicular to each other, we obtain $\|Y^\sharp(t)\|^2 = \{g_Y(t)\}^2 + \|Y^\perp(t)\|^2$. Thus by Proposition 3.1 we have

$$\|Y^\sharp(t)\|^2 = |g_Y'(0)|^2\mathfrak{s}_k(t;c)^2 + 4\|\nabla_{\dot\gamma}Y^\perp(0)\|^2\mathfrak{s}_k(t/2;c)^2.$$

By Lemma 3.5 (3), we have

$$\begin{aligned}
&\|Y^\sharp(t)\|^2 = \big\{|g_Y'(0)|^2 + \|\nabla_{\dot\gamma}Y^\perp(0)\|^2\big\}t^2 = \|\nabla_{\dot\gamma}Y^\sharp(0)\|^2 t^2, && \text{when } k^2+c=0,\\
&\|\nabla_{\dot\gamma}Y^\sharp(0)\|^2\mathfrak{s}_k(t;c)^2 \le \|Y^\sharp(t)\|^2 \le 4\|\nabla_{\dot\gamma}Y^\sharp(0)\|^2\mathfrak{s}_k(t/2;c)^2 && \text{when } k^2+c>0,\\
&4\|\nabla_{\dot\gamma}Y^\sharp(0)\|^2\mathfrak{s}_k(t/2;c)^2 \le \|Y^\sharp(t)\|^2 \le \|\nabla_{\dot\gamma}Y^\sharp(0)\|^2\mathfrak{s}_k(t;c)^2 && \text{when } k^2+c<0.
\end{aligned}$$

These gurantee the estimate on $\|Y^\sharp(t)\|$.

As we have $\langle\nabla_{\dot\gamma}Y^\sharp(t), Y^\sharp(t)\rangle = g_Y'(t)g_Y(t) + \langle\nabla_{\dot\gamma}Y^\perp(t), Y^\perp(t)\rangle$, by applying Lemma 3.6 we get

$$\frac{\langle\nabla_{\dot\gamma}Y^\sharp(t), Y^\sharp(t)\rangle}{\|Y^\sharp(t)\|^2} = \frac{g_Y'(t)g_Y(t) + \langle\nabla_{\dot\gamma}Y^\perp(t), Y^\perp(t)\rangle}{\{g_Y(t)\}^2 + \|Y^\perp(t)\|^2} \ge \min\Big\{\frac{g_Y'(t)}{g_Y(t)}, \frac{\langle\nabla_{\dot\gamma}Y^\perp(t), Y^\perp(t)\rangle}{\|Y^\perp(t)\|^2}\Big\},$$

$$\frac{\langle\nabla_{\dot\gamma}Y^\sharp(t), Y^\sharp(t)\rangle}{\|Y^\sharp(t)\|^2} = \frac{g_Y'(t)g_Y(t) + \langle\nabla_{\dot\gamma}Y^\perp(t), Y^\perp(t)\rangle}{\{g_Y(t)\}^2 + \|Y^\perp(t)\|^2} \le \max\Big\{\frac{g_Y'(t)}{g_Y(t)}, \frac{\langle\nabla_{\dot\gamma}Y^\perp(t), Y^\perp(t)\rangle}{\|Y^\perp(t)\|^2}\Big\}.$$

Since

$$\frac{g_Y'(t)}{g_Y(t)} = \mathfrak{t}_k(t;c) \qquad \text{and} \qquad \frac{\langle\nabla_{\dot\gamma}Y^\perp(t), Y^\perp(t)\rangle}{\|\ Y^\perp(t)\ \|^2} = \frac{1}{2}\mathfrak{t}_k(t/2;c),$$

Lemma 3.5 (4) leads us to the assertion on $\langle\nabla_{\dot\gamma}Y^\sharp(t), Y^\sharp(t)\rangle/\|Y^\sharp(t)\|^2$. $\square$

3. Comparison theorems on magnetic Jacobi fields

In this section we estimate norms of magnetic Jacobi fields. Our results correspond to Rauch's comparison theorem on Jacobi fields. First we recall this theorem.

THEOREM 3.1 (Rauch's comparison theorem). *Let M_1, M_2 be two Riemannian manifolds and $\sigma_1 : [0,T] \to M_1$, $\sigma_2 : [0,T] \to M_2$ be geodesics of unit speed. We set $p_i = \sigma_i(0)$ $(i = 1,2)$. Assume* $\dim(M_1) \geq \dim(M_2)$ *and*

$$\max\bigl\{\mathrm{Riem}(v, \dot\sigma_1(t)) \bigm| v \in T_{\sigma_1(t)}M_1,\ v \perp \dot\sigma_1(t)\bigr\}$$
$$\leq \min\bigl\{\mathrm{Riem}(w, \dot\sigma_2(t)) \bigm| w \in T_{\sigma_2(t)}M_2,\ w \perp \dot\sigma_2(t)\bigr\}.$$

We assume that T is not greater than the first conjugate value $c_{\sigma_2}(p_2)$ of p_2 along σ_2.

We then have the following.

(1) $c_{\sigma_1}(p_1) \geq T$.

(2) *If a Jacobi field $Y_1^\perp$ along σ_1 which is orthogonal to $\dot\sigma_1$ and a Jacobi field $Y_2^\perp$ along σ_2 which is orthogonal to $\dot\sigma_2$ satisfy $Y_1^\perp(0) = 0$, $Y_2^\perp(0) = 0$ and $\|\nabla_{\dot\sigma_1} Y_1^\perp(0)\| = \|\nabla_{\dot\sigma_2} Y_2^\perp(0)\|$, then the following assertions hold :*

 (a) *the function $t \mapsto \|Y_1^\perp(t)\|/\|Y_2^\perp(t)\|$ is monotone increasing for $0 < t < T$;*

 (b) $\dfrac{\langle \nabla_{\dot\sigma_1} Y_1^\perp(t), Y_1^\perp(t)\rangle}{\|Y_1^\perp(t)\|^2} \geq \dfrac{\langle \nabla_{\dot\sigma_2} Y_2^\perp(t), Y_2^\perp(t)\rangle}{\|Y_2^\perp(t)\|^2}$ *for $0 < t < T$;*

 (c) $\|Y_1^\perp(t)\| \geq \|Y_2^\perp(t)\|$ *for $0 < t < T$.*

 Moreover, if there exists t_0 with $0 < t_0 < c_{\sigma_2}(p_2)$ such that equality holds in the inequality in (b) or in (c), then we have

 i) *equalities hold in (b) and (c) for $0 < t \leq t_0$;*

 ii) $\mathrm{Riem}\bigl(\dot\sigma_1(t), Y_1^\perp(t)\bigr) = \mathrm{Riem}\bigl(\dot\sigma_2(t), Y_2^\perp(t)\bigr)$ *for $0 < t \leq t_0$;*

 iii) *$Y_1^\perp(t)/\|Y_1^\perp(t)\|$ is parallel along σ_1 and $Y_2^\perp(t)/\|Y_2^\perp(t)\|$ is parallel along σ_2 for $0 < t \leq t_0$.*

REMARK 3.2. Under the assumption on sectional curvatures for the case $T = c_{\sigma_2}(p_2)$ we find $c_{\sigma_1}(p_1) \geq c_{\sigma_2}(p_2)$.

Let γ be a trajectory for $\mathbb{B}_k$ and T be a constant satisfying $0 \leq T \leq c_\gamma(\gamma(0))$, where $c_\gamma(\gamma(0))$ denotes the first magnetic conjugate value of $\gamma(0)$ along γ. For a vector field

X along γ which is orthogonal to $\dot\gamma$, denoting X as $X = g_X J\dot\gamma + X^\perp$, we define its index $Ind_0^T(X)$ by

$$Ind_0^T(X) = \int_0^T \Big\{ {g_X'}^2 - k^2 g_X^2 + \langle \nabla_{\dot\gamma} X^\perp - kJX^\perp,\ \nabla_{\dot\gamma} X^\perp\rangle - \langle R(X,\dot\gamma)\dot\gamma,\ X\rangle \Big\} dt.$$

Along the same way as for ordinary Jacobi fields we have the following.

Lemma 3.7. *Let $Y = f_Y\dot\gamma + g_Y J\dot\gamma + Y^\perp$ be a magnetic Jacobi field along γ. Then for $Y^\sharp = g_Y J\dot\gamma + Y^\perp$ we have $Ind_0^T(Y^\sharp) = \langle \nabla_{\dot\gamma} Y^\sharp(T), Y^\sharp(T)\rangle - \langle \nabla_{\dot\gamma} Y^\sharp(0), Y^\sharp(0)\rangle$.*

Proof. By direct computation we have

$$\begin{aligned}
Ind_0^T(Y^\sharp) &= \int_0^T \Big\{ {g_Y'}^2 - k^2 g_Y^2 + \langle \nabla_{\dot\gamma} Y^\perp - kJY^\perp, \nabla_{\dot\gamma} Y^\perp\rangle - \langle R(Y^\sharp,\dot\gamma)\dot\gamma, Y^\sharp\rangle \Big\} dt \\
&= \int_0^T \Big\{ (g_Y g_Y')' - g_Y(g_Y'' + k^2 g_Y) + \frac{d}{dt}\langle \nabla_{\dot\gamma} Y^\perp - kJY^\perp, Y^\perp\rangle \\
&\quad - \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} Y^\perp - kJ\nabla_{\dot\gamma} Y^\perp, Y^\perp\rangle - \langle R(Y^\sharp,\dot\gamma)\dot\gamma, Y^\sharp\rangle \Big\} dt \\
&= \int_0^T \Big\{ (g_Y g_Y')' - g_Y(g_Y'' + k^2 g_Y) + \frac{d}{dt}\langle \nabla_{\dot\gamma} Y^\perp, Y^\perp\rangle \\
&\quad - \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} Y^\perp - kJ\nabla_{\dot\gamma} Y^\perp, Y^\perp\rangle - \langle R(Y^\sharp,\dot\gamma)\dot\gamma, Y^\sharp\rangle \Big\} dt \\
&= \Big[g_Y g_Y' + \langle \nabla_{\dot\gamma} Y^\perp, Y^\perp\rangle \Big]_0^T \\
&\quad - \int_0^t \Big\{ g_Y(g_Y'' + k^2 g_Y) + \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} Y^\perp - kJ\nabla_{\dot\gamma} Y^\perp, Y^\perp\rangle + \langle R(Y^\sharp,\dot\gamma)\dot\gamma, Y^\sharp\rangle \Big\} dt
\end{aligned}$$

Here, for a vector field $X = g_X J\dot\gamma + X^\perp$ which is orthogonal to $\dot\gamma$, as $\nabla_{\dot\gamma} X = -kg_X\dot\gamma + g_X' J\dot\gamma + \nabla_{\dot\gamma} X^\perp$, we have $\langle \nabla_{\dot\gamma} X, X\rangle = g_X g_X' + \langle \nabla_{\dot\gamma} X^\perp, X^\perp\rangle$. Since $J\nabla_{\dot\gamma} Y^\perp$ and $\nabla_{\dot\gamma}\nabla_{\dot\gamma} Y^\perp$ are orthogonal to $J\dot\gamma$, continuing calculation by make use of the second equation in (3.3) in Lemma 3.4, we have

$$\begin{aligned}
&= \Big[\langle \nabla_{\dot\gamma} Y^\sharp, Y^\sharp\rangle \Big]_0^T \\
&\quad - \int_0^T \Big\langle \big(g_Y'' + k^2 g_Y\big) J\dot\gamma + \nabla_{\dot\gamma}\nabla_{\dot\gamma} Y^\perp - kJ\nabla_{\dot\gamma} Y^\perp + R(Y^\sharp,\dot\gamma)\dot\gamma, Y^\sharp \Big\rangle dt \\
&= \Big[\langle \nabla_{\dot\gamma} Y^\sharp, Y^\sharp\rangle \Big]_0^T = \langle \nabla_{\dot\gamma} Y^\sharp(T), Y^\sharp(T)\rangle - \langle \nabla_{\dot\gamma} Y^\sharp(0), Y^\sharp(0)\rangle.
\end{aligned}$$

□

Lemma 3.8. *Let Y be a magnetic Jacobi field along a trajectory γ for $\mathbb{B}_k$ on a Kähler manifold M satisfying $Y(0) = 0$ and X be a vector field along γ which is orthogonal to $\dot\gamma$ and satisfies $X(0) = 0$. If we have no magnetic conjugate points of p along $\gamma([0, T])$ and $X(T) = Y^\sharp(T)$, then we have $Ind_0^T(X) \geq Ind_0^T(Y^\sharp)$. The equality holds if and only if $X \equiv Y^\sharp$.*

Proof. We put $n = \dim_{\mathbb{C}} M$, the complex dimension M. We choose linearly independent magnetic Jacobi field $Y_1, ..., Y_{2n-1}$ along γ so that $Y_i(0) = 0$ $(1 \leq i \leq 2n-1)$. Since we have no magnetic conjugate points of $\gamma(0)$, which means $T \leq c_\gamma(\gamma(0))$, we have that $Y_1^\sharp(t), ..., Y_{2n-1}^\sharp(t)$ are also linearly independent for $0 < t < T$. Hence we take smooth functions $\tau_1, ..., \tau_{2n-1}$ so that it satisfies $X = \sum_{i=1}^{2n-1} \tau_i Y_i^\sharp$. As usual, we denote as $X = g_X J\dot\gamma + X^\perp$ and $Y_i^\sharp = g_i J\dot\gamma + Y_i^\perp$. Then we have $g_X = \sum_{i=1}^{2n-1} \tau_i g_i$, $X^\perp = \sum_{i=1}^{2n-1} \tau_i Y_i^\perp$. We hence have

$$
\begin{aligned}
Ind_0^T(X) &= \int_0^T \Big\{ {g'_X}^2 - k^2 g_X^2 + \langle \nabla_{\dot\gamma} X^\perp - kJX^\perp, \nabla_{\dot\gamma} X^\perp \rangle - \langle R(X, \dot\gamma)\dot\gamma, X \rangle \Big\} dt \\
&= \int_0^T \Big\{ \Big(\sum_{i=1}^{2n-1} (\tau'_i g_i + \tau_i g'_i) \Big)^2 - k^2 \Big(\sum_{i=1}^{2n-1} \tau_i g_i \Big)^2 \\
&\quad + \Big\langle \sum_{i=1}^{2n-1} \tau'_i Y_i^\perp + \tau_i \nabla_{\dot\gamma} Y_i^\perp - k\tau_i JY_i^\perp, \sum_{j=1}^{2n-1} \tau'_j Y_j^\perp + \tau_j \nabla_{\dot\gamma} Y_j^\perp \Big\rangle \\
&\quad - \Big\langle R\Big(\sum_{i=1}^{2n-1} \tau_i (g_i J\dot\gamma + Y_i^\perp), \dot\gamma \Big)\dot\gamma, \sum_{j=1}^{2n-1} \tau_j (g_j J\dot\gamma + Y_j^\perp) \Big\rangle \Big\} dt \\
&= \int_0^T \Big\{ \Big(\sum_{i=1}^{2n-1} \tau'_i g_i \Big)^2 + \Big(\sum_{i=1}^{2n-1} \tau_i g'_i \Big)^2 + 2 \sum_{i,j=1}^{2n-1} (\tau'_i \tau_j g_i g'_j) - k^2 \Big(\sum_{i=1}^{2n-1} \tau_i g_i \Big)^2 \\
&\quad + \Big\| \sum_{i=1}^{2n-1} \tau'_i Y_i^\perp \Big\|^2 + \Big\langle \sum_{i=1}^{2n-1} \tau'_i Y_i^\perp, \sum_{j=1}^{2n-1} \tau_j \nabla_{\dot\gamma} Y_i^\perp \Big\rangle + \Big\langle \sum_{i=1}^{2n-1} \tau_i \nabla_{\dot\gamma} Y_i^\perp, \sum_{j=1}^{2n-1} \tau'_j Y_j^\perp \Big\rangle \\
&\quad - \Big\langle \sum_{i=1}^{2n-1} k\tau_i JY_i^\perp, \sum_{j=1}^{2n-1} \tau_j \nabla_{\dot\gamma} Y_j^\perp \Big\rangle - \Big\langle \sum_{i=1}^{2n-1} k\tau_i JY_i^\perp, \sum_{j=1}^{2n-1} \tau'_j Y_j^\perp \Big\rangle + \Big\| \sum_{i=1}^{2n-1} \tau_i \nabla_{\dot\gamma} Y_i^\perp \Big\|^2 \\
&\quad - \Big\langle R\Big(\sum_{i=1}^{2n-1} \tau_i (g_i J\dot\gamma + Y_i^\perp), \dot\gamma \Big)\dot\gamma, \sum_{j=1}^{2n-1} \tau_j (g_j J\dot\gamma + Y_j^\perp) \Big\rangle \Big\} dt
\end{aligned}
$$

By making partial integration on $2\int_0^T \sum_{i,j=1}^{2n-1} \tau_i'\tau_j g_i g_j' dt$ and $\int_0^T \sum_{i,j=1}^{2n-1} \tau_i'\tau_j \langle Y_i^\perp, \nabla_{\dot\gamma} Y_j^\perp\rangle dt$, we have

$$2\sum_{i,j=1}^{2n-1}\int_0^T \tau_i'\tau_j g_i g_j' dt = \sum_{i,j=1}^{2n-1}\int_0^T \tau_i'\tau_j g_i g_j' dt + \Big[\sum_{i,j=1}^{2n-1}\tau_i\tau_j g_i g_j'\Big]_0^T$$
$$- \sum_{i,j=1}^{2n-1}\int_0^T \tau_i\tau_j' g_i g_J' dt - \sum_{i,j=1}^{2n-1}\tau_i\tau_j(g_i'g_j' + g_i g_j'')dt,$$

and

$$\sum_{i,j=1}^{2n-1}\int_0^T \tau_i'\tau_j\langle Y_i^\perp, \nabla_{\dot\gamma}Y_j^\perp\rangle dt = \Big[\sum_{i,j=1}\tau_i\tau_j\langle Y_i^\perp, \nabla_{\dot\gamma}Y_j^\perp\rangle\Big]_0^T - \sum_{i,j=1}^{2n-1}\int_0^T \tau_i\tau_j'\langle Y_i^\perp, \nabla_{\dot\gamma}Y_j^\perp\rangle dt$$
$$- \sum_{i,j=1}^{2n-1}\int_0^T \tau_i\tau_j\big(\langle \nabla_{\dot\gamma}Y_i^\perp, \nabla_{\dot\gamma}Y_j^\perp\rangle + \langle Y_i^\perp, \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y_j^\perp\rangle\big)dt.$$

Hence we have

$$Ind_0^T(X) = \Big[\sum_{i,j=1}^{2n-1}\tau_i\tau_j\big(g_i g_j' + \langle Y_i^\perp, \nabla_{\dot\gamma}Y_j^\perp\rangle\big)\Big]_0^T + \int_0^T \Big\|\sum_{i=1}^{2n-1}\tau_i' Y_i^\sharp\Big\|^2 dt$$
$$+ \int_0^T \Big(\sum_{i,j=1}^{2n-1}\tau_i'\tau_j\big(g_i g_j' - g_i' g_j - \langle \nabla_{\dot\gamma}Y_i^\perp, Y_j^\perp\rangle + \langle Y_i^\perp, \nabla_{\dot\gamma}Y_j^\perp\rangle - k\langle Y_i^\perp, JY_j^\perp\rangle\big)\Big)dt$$
$$- \int_0^T \Big\{\sum_{i,j=1}^{2n-1}\tau_i\tau_j\Big(g_i(g_j'' + k^2 g_j) + \langle Y_i^\perp, \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y_j^\perp\rangle - k\langle Y_i^\perp, J\nabla_{\dot\gamma}Y_j^\perp\rangle$$
$$+ \langle R\big((g_i J\dot\gamma + Y_i^\perp), \dot\gamma\big)\dot\gamma, (g_j J\dot\gamma + Y_j^\perp)\rangle\Big)\Big\}dt$$

By Lemma 3.2, we see that

$$g_i g_j' - g_i' g_j - \langle \nabla_{\dot\gamma}Y_i^\perp, Y_j^\perp\rangle + \langle Y_i^\perp, \nabla_{\dot\gamma}Y_j^\perp\rangle - k\langle Y_i^\perp, JY_j^\perp\rangle$$
$$= \langle \nabla_{\dot\gamma}Y_i^\sharp, Y_j^\sharp\rangle - \langle Y_i^\sharp, \nabla_{\dot\gamma}Y_j^\sharp\rangle + \langle Y_i^\sharp, kJY_j^\sharp\rangle$$
$$= \langle \nabla_{\dot\gamma}Y_i, Y_j\rangle - \langle Y_i, \nabla_{\dot\gamma}Y_j\rangle + \langle Y_i, kJY_j\rangle$$

is constant along γ. Hence it equals to 0, because its initial value is 0. Thus, we have

$$Ind_0^T(X) = \Big\langle \sum_{i=1}^{2n-1}\tau_i(g_i' J\dot\gamma + \nabla_{\dot\gamma}Y_i^\perp), \sum_{j=1}^{2n-1}\tau_j(g_j J\dot\gamma + Y_j^\perp)\Big\rangle\Big|_{t=0}^{t=T} + \int_0^T \Big\|\sum_{i=1}^{2n-1}\tau_i' Y_i^\sharp\Big\|^2 dt$$
$$- \int_0^T \Big\langle \sum_{i=1}^{2n-1}\tau_i\Big((g_i'' + k^2 g_i)J\dot\gamma + \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y_i^\perp - kJ\nabla_{\dot\gamma}Y_i^\perp + R(Y_i^\sharp, \dot\gamma)\dot\gamma\Big), \sum_{j=1}^{2n-1}\tau_j Y_j^\sharp\Big\rangle dt.$$

Since $\sum_{i=1}^{2n-1} \tau_i(T)Y_i(0) = 0$ and $\sum_{i=1}^{2n-1} \tau_i(T)Y_i(T) = X(T) = Y(T)$, we have $Y = \sum_{i=1}^{2n-1} \tau_i(T)Y_i$. As $X(0) = \sum_{i=1}^{2n-1} \tau_i(0)Y_i^{\perp}(0) = 0$ and the definition of the magnetic Jacobi field, we get

$$Ind_0^T(X) = \Big\langle \sum_{i=1}^{2n-1} \tau_i(\nabla_{\dot\gamma} Y_i^{\sharp}), \sum_{j=1}^{2n-1} \tau_j Y_j^{\sharp} \Big\rangle \Big|_{t=0}^{t=T} + \int_0^T \Big\| \sum_{i=1}^{2n-1} \tau_i' Y_i^{\sharp} \Big\|^2 dt.$$

By Lemma 3.7, we see

$$Ind_0^T(X) = Ind_0^T(Y^{\sharp}) + \int_0^T \Big\| \sum_{i=1}^{2n-1} \tau_i' Y_i^{\sharp} \Big\|^2 dt \geq Ind_0^T(Y^{\sharp}).$$

Since the last inequality holds if and only if all τ_i' vanish along γ, we obtain the conclusion. □

We here give estimates on norms of vertical components of magnetic Jacobi fields. First, we give an estimate from below. We define a function $\mathfrak{c}_k(t;c) : [0, \pi/\sqrt{k^2+c}\,] \to \mathbb{R}$ by

$$\mathfrak{c}_k(t;c) = \begin{cases} \cos(\sqrt{k^2+c}\,t), & k^2+c>0, \\ \ 1, & k^2+c=0, \\ \cosh(\sqrt{|c|-k^2}\,t), & k^2+c<0, \end{cases}$$

which is the differential of $\mathfrak{s}_k(t;c)$. As usual, we regard $\pi/\sqrt{k^2+c}$ as infinity when $k^2+c \leq 0$.

Theorem 3.2. *Let γ be a trajectory for $\mathbb{B}_k$ on a Kähler manifold M and ℓ be a positive number with $\ell \leq \pi/\sqrt{k^2+c}$. We suppose sectional curvatures satisfy* $\max\big\{\mathrm{Riem}(v, \dot\gamma(t)) \bigm| v \in T_{\gamma(t)}M,\ v \perp \dot\gamma(t)\big\} \leq c$ *with some constant c for $0 < t \leq \ell$. We then have the following.*

(1) $c_{\gamma}(\gamma(0)) \geq \ell$.

(2) *Every magnetic Jacobi field Y along a trajectory γ for $\mathbb{B}_k$ with $Y(0) = 0$ satisfies the following properties for $0 < t < \ell$:*

  (a) *The function $t \mapsto \|Y^{\sharp}(t)\|/\mathfrak{s}_k(t;c)$ is monotonic increasing*;

  (b) $\|Y^{\sharp}(t)\| \geq \|\nabla_{\dot\gamma} Y^{\sharp}(0)\|\, \mathfrak{s}_k(t;c)$;

  (c) $\big\langle \nabla_{\dot\gamma} Y^{\sharp}(t), Y^{\sharp}(t) \big\rangle \geq \|Y^{\sharp}(t)\|^2 \mathfrak{t}_k(t;c)$.

Moreover, if there exists t_0 with $0 < t_0 < \ell$ such that one of the equality holds in the inequalities in (b) and (c), then we have

i) *Both of the equalities hold in (b) and (c) for $0 < t \le t_0$;*

ii) *The magnetic Jacobi field Y is $Y^\sharp(t) = \pm\|\nabla_{\dot\gamma}Y^\sharp(0)\|\mathfrak{s}_k(t;c)J\dot\gamma(t)$;*

iii) *The sectional curvature satisfies $\langle R(J\dot\gamma,\dot\gamma)\dot\gamma, J\dot\gamma\rangle = c$ for $0 \le t \le t_0$.*

Remark 3.3. Under the assumption on sectional curvatures for the case $T = \pi/\sqrt{k^2+c}$ we find $c_\gamma(\gamma(0)) \ge \pi/\sqrt{k^2+c}$.

Corollary 3.1. *If sectional curvatures of a Kähler manifold M satisfy $\mathrm{Riem}^M \le c$ with some constant c, then the first magnetic conjugate value $c_k(M)$ satisfies $c_k(M) \ge \pi/\sqrt{k^2+c}$. In particular, when $\mathrm{Riem}^M \le c \le 0$ and $k^2+c \le 0$ there are no magnetic conjugate points for $\mathbb{B}_k$ on M.*

Proof of Theorem 3.2. We take a complex space form $\widehat{M} = \mathbb{C}M^n(4c)$ and a trajectory $\hat\gamma$ for $\mathbb{B}_k$ on $\widehat{M}$. We denote by $P_\gamma^t : T_{\gamma(t)}M \to T_{\gamma(0)}M$ and $\widehat{P}_{\hat\gamma}^t : T_{\hat\gamma(0)}\widehat{M} \to T_{\hat\gamma(t)}\widehat{M}$ parallel transformations along γ and $\hat\gamma$, respectively. Let $I : T_{\gamma(0)}M \to T_{\hat\gamma(0)}\widehat{M}$ be a holomorphic linear isometry which preserves the inner product and satisfies $I\big(\dot\gamma(0)\big)= \dot{\hat\gamma}(0)$ and $I\big(J\dot\gamma(0)\big)= J\dot{\hat\gamma}(0)$. We define a vector field $\widehat{X}$ along $\hat\gamma$ by $\widehat{X}(t) = \widehat{P}_{\hat\gamma}^t \circ I \circ P_\gamma^t\big(Y^\perp(t)\big)$. As I preserves the inner product and the complex structure J is parallel, we find that $\langle \widehat{X}, \dot{\hat\gamma}\rangle = \langle \widehat{X}, J\dot{\hat\gamma}\rangle = 0$. Thus we find that $\widehat{X}$ satisfies $\widehat{X} = \widehat{X}^\perp$.

We take a positive number T with $T \le \ell$. By the condition on sectional curvatures, we have

$$\begin{aligned}
Ind_0^T(Y^\sharp) &= \int_0^T \Big\{ {g'_Y}^2 - k^2 g_Y^2 + \langle \nabla_{\dot\gamma}Y^\perp - kJY^\perp, \nabla_{\dot\gamma}Y^\perp\rangle - \langle R(Y^\sharp,\dot\gamma)\dot\gamma, Y^\sharp\rangle \Big\} dt \\
&\ge \int_0^T \Big\{ {g'_Y}^2 - k^2 g_Y^2 + \langle \nabla_{\dot\gamma}Y^\perp - kJY^\perp, \nabla_{\dot\gamma}Y^\perp\rangle - c\|Y^\sharp\|^2 \Big\} dt
\end{aligned} \tag{3.8}$$

As $\|Y^\sharp\|^2 = g_Y^2 + \|Y^\perp\|^2$, we have

$$\begin{aligned}
Ind_0^T(Y^\sharp) &= \int_0^T \Big\{ {g_Y'}^2 - k^2 g_Y^2 + \langle \nabla_{\dot\gamma} Y^\perp - kJY^\perp, \nabla_{\dot\gamma} Y^\perp\rangle - c(g_Y^2 + \|Y^\perp\|^2)\Big\} dt \\
&= \int_0^T \{ {g_Y'}^2 + g_Y^2(-k^2 - c)\} dt + \int_0^T \{\langle \nabla_{\dot{\hat\gamma}} \widehat{X} - kJ\widehat{X}, \nabla_{\dot{\hat\gamma}} \widehat{X}\rangle - c\|\widehat{X}\|^2\} dt \\
&= Ind_0^T(g_Y J\dot{\tilde\gamma}) + Ind_0^T(\widehat{X}),
\end{aligned}$$

where $\tilde\gamma$ denots a trajectory for $\mathbb{B}_k$ on $\mathbb{C}M^1(c)$.

Since $T \le \pi/\sqrt{k^2+c}$, we can take a magnetic Jacobi field $\widetilde{Y} = \tilde f \dot{\tilde\gamma} + \tilde g J\dot{\tilde\gamma}$ for $\mathbb{B}_k$ along $\tilde\gamma$ satisfying $\tilde g(0) = 0,\ \tilde g(T) = g_Y(T)$ and a magnetic Jacobi field $\widehat{Y}$ for $\mathbb{B}_k$ along $\hat\gamma$ satisfying $\widehat{Y} = \widehat{Y}^\perp,\ \widehat{Y}(0) = 0$ and $\widehat{Y}(T) = \widehat{X}(T)$.

By lemma 3.8 we have $Ind_0^T(\widehat{X}) \ge Ind_0^T(\widehat{Y})$. Therefore by Proposition 3.1 and Lemma 3.7, we obtain

$$\begin{aligned}
\langle \nabla_{\dot\gamma} Y^\sharp(T), Y^\sharp(T)\rangle &= Ind_0^T(Y^\sharp) \ge Ind_0^T(g_Y J\dot{\tilde\gamma}) + Ind_0^T(\widehat{X}) \\
&\ge Ind_0^T(\tilde g J\dot{\tilde\gamma}) + Ind_0^T(\widehat{Y}) \\
&= \langle \tilde g'(T) J\dot{\tilde\gamma}(T), \tilde g(T) J\dot{\tilde\gamma}(T)\rangle + \langle \nabla_{\dot{\hat\gamma}} \widehat{Y}(T), \widehat{Y}(T)\rangle \\
&= \langle \tilde g(T)\mathfrak{t}_k(T;c) J\dot{\tilde\gamma}(T), \tilde g(T) J\dot{\tilde\gamma}(T)\rangle + \langle \nabla_{\dot{\hat\gamma}} \widehat{Y}(T), \widehat{Y}(T)\rangle \\
&= |\tilde g(T)|^2\, \mathfrak{t}_k(T;c) + \|\widehat{Y}(T)\|^2 \times \frac{1}{2}\mathfrak{t}_k(T/2;4c) \qquad (3.9) \\
&= |\tilde g(T)|^2 \mathfrak{t}_k(T;c) + \|\widehat{Y}(T)\|^2 \mathfrak{t}_{k/2}(T;c) \\
&\ge |g_Y(T)|^2 \mathfrak{t}_k(T;c) + \|\widehat{X}(T)\|^2 \mathfrak{t}_k(T;c) \\
&= |g_Y(T)|^2 \mathfrak{t}_k(T;c) + \|Y^\perp(T)\|^2 \mathfrak{t}_k(T;c) \\
&= \|Y^\sharp(T)\|^2 \mathfrak{t}_k(T;c).
\end{aligned}$$

Since T is an arbitrary positive number with $T < \ell$, we have

$$\langle \nabla_{\dot\gamma} Y^\sharp(t), Y^\sharp(t)\rangle \ge \|Y^\sharp(t)\|^2 \mathfrak{t}_k(t;c)$$

for $0 < t < \ell$.

We here consider a function $h(t) = \|\nabla_{\dot\gamma} Y^\sharp(0)\|^2 \mathfrak{s}_k^2(t;c)$. It satisfies

$$\begin{aligned}
h'(t) &= 2\|\nabla_{\dot\gamma} Y^\sharp(0)\|^2 \mathfrak{s}_k(t;c)\mathfrak{c}_k(t;c) = \|\nabla_{\dot\gamma} Y^\sharp(0)\|^2 \mathfrak{s}_k(2t;c), \\
h''(t) &= 2\|\nabla_{\dot\gamma} Y^\sharp(0)\|^2 \mathfrak{c}_k(2t;c).
\end{aligned}$$

Thus we have

$$\begin{aligned} \frac{d}{dt}\Big(\frac{\|Y^\sharp(t)\|^2}{h(t)}\Big)\Big|_{t=T} &= \frac{1}{h(T)}\Big(2\langle\nabla_{\dot\gamma}Y^\sharp(T), Y^\sharp(T)\rangle - \|Y^\sharp(T)\|^2\frac{h'(T)}{h(T)}\Big) \\ &= \frac{2}{h(T)}\big(\langle\nabla_{\dot\gamma}Y^\sharp(T), Y^\sharp(T)\rangle - \|Y^\sharp(T)\|^2\mathfrak{t}_k(T;c)\big)\geq 0. \end{aligned} \tag{3.10}$$

As $\mathfrak{s}_k(T;c) > 0$, this shows $\frac{d}{dt}\Big(\frac{\|Y^\sharp(t)\|}{\mathfrak{s}_k(t;c)}\Big)\Big|_{t=T} > 0$. Since T is an arbitrary positive number with $T < \ell$, we find that the function $t \mapsto \|Y^\sharp(t)\|/\mathfrak{s}_k(t;c)$ is monotone increasing for $0 < t < \ell$.

As the function h satisfies $h(0) = h'(0) = 0$ and $h''(0) = 2\|\nabla_{\dot\gamma}Y^\sharp(0)\|^2$, and as $Y^\sharp(0) = 0$, we have the following by de l'Hopital's rule:

$$\begin{aligned} \lim_{t\downarrow 0}\frac{\|Y^\sharp(t)\|^2}{h(t)} &= \lim_{t\downarrow 0}\frac{2\langle\nabla_{\dot\gamma}Y^\sharp(t), Y^\sharp(t)\rangle}{h'(t)} \\ &= 2\lim_{t\downarrow 0}\frac{\langle\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y^\sharp(t), Y^\sharp(t)\rangle + \|\nabla_{\dot\gamma}Y^\sharp(t)\|^2}{h''(t)} = 1. \end{aligned}$$

Since $\frac{d}{dt}\Big(\frac{\|Y^\sharp(t)\|^2}{h(t)}\Big)\Big|_{t=T} > 0$ for every T with $0 < T < \ell$, we get $\|Y^\sharp(t)\|^2/h(t) \geq 1$ and obtain $\|Y^\sharp(t)\| \geq \|\nabla_{\dot\gamma}Y^\sharp(0)\|\mathfrak{s}_k(t;c)$ for $0 \leq t < \ell$.

We now consider the case that equalities hold. First we consider the case that the equality holds in (2-b). We suppose $\|Y^\sharp(t_0)\| = \|\nabla_{\dot\gamma}Y^\sharp(0)\|\mathfrak{s}_k(t_0;c)$ at some t_0 with $0 < t_0 < \ell$. We then find that $\|Y^\sharp(t_0)\|^2/h(t_0) = 1$. As $\|Y^\sharp(0)\|^2/h(0) = 1$ and $\|Y^\sharp(t)\|^2/h(t)$ is monotone increasing, we have $\|Y^\sharp(t)\| = \|\nabla_{\dot\gamma}Y^\sharp(0)\|\mathfrak{s}_k(t;c)$ for $0 < t \leq t_0$. Since $\|Y^\sharp(t)\|^2 = \|\nabla_{\dot\gamma}Y^\sharp(0)\|^2\mathfrak{s}_k^2(t;c)$, by considering the differentiations of both sides we have

$$\frac{2\langle\nabla_{\dot\gamma}Y^\sharp(t), Y^\sharp(t)\rangle}{\|Y^\sharp(t)\|^2} = \frac{2\|\nabla_{\dot\gamma}Y^\sharp(0)\|^2\mathfrak{s}_k(t;c)\mathfrak{c}_k(t;c)}{\|\nabla_{\dot\gamma}Y^\sharp(0)\|^2\mathfrak{s}_k^2(t;c)},$$

which shows that

$$\langle\nabla_{\dot\gamma}Y^\sharp(t), Y^\sharp(t)\rangle = \|Y^\sharp(t)\|^2\mathfrak{t}_k(t;c).$$

Therefore we may only consider the case that the equality holds in (2-c). We suppose $\langle\nabla_{\dot\gamma}Y^\sharp(t_0), Y^\sharp(t_0)\rangle = \|Y^\sharp(t_0)\|^2\mathfrak{t}_k(t_0;c)$ at some t_0 with $0 \leq t_0 < \ell$. Then we see equalities hold in (3.9) with $T = t_0$. The third inequality in (3.9) should be an equality. As we have $\mathfrak{t}_{k/2}(t;c) > \mathfrak{t}_k(t;c)$ for $0 < t < \pi/\sqrt{k^2+c}$ by Lemma 3.5, this shows $\widehat{Y}(t_0) = 0$. Since every non-trivial magnetic Jacobi field along $\hat{\gamma}$ which is orthogonal to both $\dot{\hat{\gamma}}$ and $J\dot{\hat{\gamma}}$ does not vanish for $0 < t < \pi/\sqrt{k^2 + c}$, we obtain $\widehat{Y} \equiv 0$. As the second inequality in (3.9) should be an equality, we have $Ind_0^{t_0}(\widehat{X}) = Ind_0^{t_0}(\widehat{Y})$ and $Ind_0^{t_0}(g_Y J\dot{\hat{\gamma}}) = Ind_0^{t_0}(\tilde{g}J\dot{\hat{\gamma}})$. Hence by Lemma 3.8, we find $\widehat{X} \equiv \widehat{Y} \equiv 0$ and $\tilde{g} \equiv g_Y$. We therefore obtain $Y^{\perp} \equiv 0$. Since the first inequality in (3.9) should be an equality, which means that in (3.8) the inequality on $Ind_0^{t_0}(Y^{\sharp})$ should be an equality, we find $\langle R(Y^{\sharp}, \dot{\gamma})\dot{\gamma}, Y^{\sharp}\rangle \equiv c\|Y^{\sharp}\|^2$ for $0 < t \leq t_0$. As we showed that $Y^{\perp} = 0$, by the expressions of magnetic Jacobi fields on a complex sapce form $\mathbb{C}M^1(c)$, we get

$$Y^{\sharp}(t) = \pm\|\nabla_{\dot{\gamma}}Y^{\sharp}(0)\|\mathfrak{s}_k(t;c)J\dot{\gamma}(t).$$

We therefore get the conclusion. □

Next, we give an estimate of norms of magnetic fields from above.

Theorem 3.3. *Let γ be a trajectory for $\mathbb{B}_k$ on a Kähler manifold M and ℓ be a positive number with $\ell \leq c_\gamma(\gamma(0))$. We suppose sectional curvatures satisfy*

$$\min\big\{\mathrm{Riem}(v, \dot{\gamma}(t)) \bigm| v \in T_{\gamma(t)}M,\ v \perp \dot{\gamma}(t)\big\} \geq c \text{ for } 0 < t \leq \ell.$$

We then have the following.

(1) $c_\gamma(\gamma(0)) \leq 2\pi/\sqrt{k^2 + 4c}$.

(2) *Every magnetic Jacobi field Y along a trajectory γ for $\mathbb{B}_k$ with $Y(0) = 0$ satisfies the following properties for $0 < t < \ell$:*

  (a) *The function $t \mapsto \|Y^{\sharp}(t)\|/\mathfrak{s}_{k/2}(t;c)$ is monotonic decreasing;*

  (b) $\|Y^{\sharp}(t)\| \leq \|\nabla_{\dot{\gamma}}Y^{\sharp}(0)\|\ \mathfrak{s}_{k/2}(t;c)$;

  (c) $\langle\nabla_{\dot{\gamma}}Y^{\sharp}(t), Y^{\sharp}(t)\rangle \leq \|Y^{\sharp}(t)\|^2\mathfrak{t}_{k/2}(t;c)$.

Moreover, if there exists t_0 with $0 < t_0 < \ell$ such that one of the equality holds in the inequalities in (b) and (c), then we have

  i) *Both of the equalities hold in (b) and (c) for $0 < t \leq t_0$;*

  ii) *The magnetic Jacobi field Y is of the form*
  $Y^{\sharp}(t) = Y^{\perp}(t) = \|\nabla_{\dot{\gamma}}Y^{\perp}(0)\|\mathfrak{s}_{k/2}(t;c)\big\{\cos(kt/2)E(t) + \sin(kt/2)JE(t)\big\}$
  with a parallel vector field E satisfying $E(0) = \nabla_{\dot{\gamma}}Y^{\perp}(0)/\|\nabla_{\dot{\gamma}}Y^{\perp}(0)\|$;

iii) *The sectional curvature satisfy* $\langle R(Y^\perp,\dot\gamma)\dot\gamma, Y^\perp\rangle = c\|Y^\perp\|^2$ *for* $0\le t\le t_0$.

REMARK 3.4. Under the assumption on sectional curvatures for the case $\ell = c_\gamma(\gamma(0))$ we find $c_\gamma(\gamma(0)) \le 2\pi/\sqrt{k^2+4c}$.

PROOF OF THEOREM 3.3. We take a trajectory $\tilde\gamma$ for $\mathbb{B}_k$ on $\mathbb{C}M^1(c)$ and a trajectory $\hat\gamma$ for $\mathbb{B}_k$ on $\mathbb{C}M^n(4c)$. We denote by $P^t_\gamma : T_{\gamma(t)}M \to T_{\gamma(0)}M$ and $\widehat{P}^t_{\hat\gamma} : T_{\hat\gamma(0)}\widehat{M} \to T_{\hat\gamma(t)}\widehat{M}$ the parallel transformations along γ and $\hat\gamma$, respectively. Let $I : T_\gamma(0)M \to T_{\hat\gamma(0)}\widehat{M}$ be a holomorphic linear isometry which preserves the inner product and satisfies $I\big(\dot\gamma(0)\big)= \dot{\hat\gamma}(0)$ and $I\big(J\dot\gamma(0)\big)= J\dot{\hat\gamma}(0)$. For an arbitrary positive T with $T\le\ell$, we take a magnetic Jacobi field $\widehat{Y}$ for $\mathbb{B}_k$ along $\hat\gamma$ which satisfies $\widehat{Y}(0)=0$ and $\widehat{Y}(T) = \widehat{P}^T_{\hat\gamma}\circ I\circ P^T_\gamma(Y^\perp(T))$. We also take a magnetic Jacobi field $\tilde f\dot{\tilde\gamma} + \tilde g J\dot{\tilde\gamma}$ for $\mathbb{B}_k$ along $\tilde\gamma$ satisfying $\tilde g(0)=0$ and $\tilde g(T) = g_Y(T)$. We define a vector field X along γ by $X(t) = \tilde g(t)J\dot\gamma(t) + \big(\widehat{P}^t_{\dot{\hat\gamma}}\circ I\circ P^t_\gamma\big)^{-1}\big(\widehat{Y}(t)\big)$. We then have

$$\begin{aligned} X(T) &= \tilde g(T)J\dot\gamma(T) + \big(\widehat{P}^T_{\hat\gamma}\circ I\circ P^T_\gamma\big)^{-1}\big(\widehat{Y}(T)\big)\\ &= g_Y(T)J\dot\gamma(T) + Y^\perp(T) = Y^\sharp(T). \end{aligned}$$

Since $\mathfrak{t}_{k/2}(T;c) = 1/2\mathfrak{t}_k(T/2;4c)$, by Lemma 3.5 we obtain

$$\begin{aligned} \|Y^\sharp(T)\|^2\mathfrak{t}_{k/2}(T;c) &= |g_Y(T)|^2\mathfrak{t}_{k/2}(T;c) + \|Y^\perp(T)\|^2\mathfrak{t}_{k/2}(T;c)\\ &\ge |g_Y(T)|^2\mathfrak{t}_k(T;c) + \|Y^\perp(T)\|^2\mathfrak{t}_{k/2}(T;c)\\ &= |g_Y(T)|^2\mathfrak{t}_k(T;c) + \|\widehat{Y}(T)\|^2\frac12\mathfrak{t}_k(T/2;4c)\\ &= \big\langle \tilde g(T)\mathfrak{t}_k(T;c)J\dot{\tilde\gamma}, \tilde g(T)J\dot{\tilde\gamma}\big\rangle + \|\widehat{Y}(T)\|^2\frac12\mathfrak{t}_k(T/2;4c). \end{aligned}\tag{3.11}$$

Thus we find $\|Y^\sharp(T)\|^2\mathfrak{t}_{k/2}(T;c) \ge \langle\tilde g'(T)J\dot{\tilde\gamma}, \tilde g(T)J\dot{\tilde\gamma}\rangle + \langle\nabla_{\hat\gamma}\widehat{Y}(T), \widehat{Y}(T)\rangle$ by Proposition 3.1. Since $\tilde g(0)=0$, $\widehat{Y}(0)=0$, by Lemma 3.7 we continue our calculation and get

$$\begin{aligned} &\langle\tilde g'(T),\tilde g(T)\rangle + \langle\nabla_{\hat\gamma}\widehat{Y}(T),\widehat{Y}(T)\rangle\\ &= Ind_0^T(\tilde gJ\dot{\tilde\gamma}) + Ind_0^T(\widehat{Y}(T))\\ &= \int_0^T\Big\{\tilde g'^2 - k^2\tilde g^2 + \langle\nabla_{\dot{\hat\gamma}}\widehat{Y} - kJ\widehat{Y}, \nabla_{\dot{\hat\gamma}}\widehat{Y}\rangle - c(\tilde g^2 + \|\widehat{Y}\|^2)\Big\}dt\\ &= \int_0^T\Big\{\tilde g'^2 - k^2\tilde g^2 + \langle\nabla_{\dot{\hat\gamma}}\widehat{Y} - kJ\widehat{Y}, \nabla_{\dot{\hat\gamma}}\widehat{Y}\rangle - c(\|X\|^2)\Big\}dt. \end{aligned}$$

By the condition on sectional curvatures and by Lemma 3.8, we obtain

$$\begin{aligned}(3.12)\qquad &\geq \int_0^T \Big\{\tilde{g}'^2 - k^2\tilde{g}^2 + \langle \nabla_{\dot{\gamma}}\widehat{Y} - kJ\widehat{Y}, \nabla_{\dot{\gamma}}\widehat{Y}\rangle - \langle R(X,\dot{\gamma})\dot{\gamma}, X\rangle\Big\}dt \\ &= Ind_0^T(X) \geq Ind_0^T(Y^\sharp) = \langle \nabla_{\dot{\gamma}}Y^\sharp(T), Y^\sharp(T)\rangle.\end{aligned}$$

Thus we get $\|Y^\sharp(T)\|^2\mathfrak{t}_{k/2}(T;c) \geq \langle \nabla_{\dot{\gamma}}Y^\sharp(T), Y^\sharp(T)\rangle$ for arbitrary T with $0 < T \leq \ell$, which is the assertion (2–c).

We consider a function $h(t) = \|\nabla_{\dot{\gamma}}Y^\sharp(0)\|^2\mathfrak{s}_{k/2}^2(t;c)$. It satisfies

$$h'(t) = 2\|\nabla_{\dot{\gamma}}Y^\sharp(0)\|^2\mathfrak{s}_{k/2}(t;c)\mathfrak{c}_{k/2}(t;c) = \|\nabla_{\dot{\gamma}}Y^\sharp(0)\|^2\mathfrak{s}_{k/2}(2t;c),$$

$$h''(t) = 2\|\nabla_{\dot{\gamma}}Y^\sharp(0)\|^2\mathfrak{c}_{k/2}(2t;c).$$

Since we have

$$\begin{aligned}\frac{d}{dt}\Big(\frac{\|Y^\sharp(t)\|^2}{h(t)}\Big)\Big|_{t=T} &= \frac{1}{h(T)}\Big(2\langle \nabla_{\dot{\gamma}}Y^\sharp(T), Y^\sharp(T)\rangle - \|Y^\sharp(T)\|^2\frac{h'(T)}{h(T)}\Big) \\ &= \frac{2}{h(T)}\big(\langle \nabla_{\dot{\gamma}}Y^\sharp(T), Y^\sharp(T)\rangle - \|Y^\sharp(T)\|^2\mathfrak{t}_{k/2}(T;c)\big) \leq 0\end{aligned}$$

for arbitrary T with $0 < T < \ell$, we find that the function $t \mapsto \|Y^\sharp(t)\|/\mathfrak{s}_{k/2}(t;c)$ is monotonic decreasing.

As the function h satisfies $h(0) = h'(0) = 0$ and $h''(0) = 2\|\nabla_{\dot{\gamma}}Y^\sharp(0)\|^2$, and as $Y^\sharp(0) = 0$, we have the following by de l'Hopital's rule:

$$\begin{aligned}\lim_{t\downarrow 0}\frac{\|Y^\sharp(t)\|^2}{h(t)} &= \lim_{t\downarrow 0}\frac{2\langle \nabla_{\dot{\gamma}}Y^\sharp(t), Y^\sharp(t)\rangle}{h'(t)} \\ &= 2\lim_{t\downarrow 0}\frac{\langle \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}Y^\sharp(t), Y^\sharp(t)\rangle + \|\nabla_{\dot{\gamma}}Y^\sharp(t)\|^2}{h''(t)} = 1.\end{aligned}$$

Since $\frac{d}{dt}\Big(\frac{\|Y^\sharp(t)\|^2}{h(t)}\Big)\Big|_{t=T} \leq 0$ for every T with $0 < T < \ell$, we get $\|Y^\sharp(t)\|^2/h(t) \leq 1$ and obtain $\|Y^\sharp(t)\| \leq \|\nabla_{\dot{\gamma}}Y^\sharp(0)\|\mathfrak{s}_{k/2}(t;c)$ for $0 \leq t < \ell$.

We now consider the case that equalities hold. First we consider the case that the equality holds in (2-c). That is, we suppose $\langle \nabla_{\dot{\gamma}}Y^\sharp(t_0), Y^\sharp(t_0)\rangle = \|Y^\sharp(t_0)\|^2\mathfrak{t}_{k/2}(t_0;c)$ at some t_0 with $0 < t_0 < \ell$. Then we find that equalities hold in (3.11) and (3.12) with $T = t_0$. The second inequality in (3.12) should be an equality. Since $Ind_0^{t_0}(X) = Ind_0^{t_0}(Y^\sharp)$, we have $X = Y^\sharp$ by Lemma 3.8. Moreover the inequality in (3.11) should be an equality. As $\mathfrak{t}_{k/2}(t_0;c) \geq \mathfrak{t}_k(t_0;c)$, we have $g_Y(t_0) = 0$. It leads us to $\tilde{g} \equiv 0$ because $\tilde{f}\dot{\tilde{\gamma}} + \tilde{g}J\dot{\tilde{\gamma}}$ is a magnetic Jacobi field. By the structure of X, we have $X = X^\perp$.

Therefore we find that $Y^\sharp = X$ is orthogonal to $J\dot\gamma$. Since the first inequality in (3.12) should be an equality, we have that $\langle R(X,\dot\gamma)\dot\gamma, X\rangle \equiv c\|X\|^2$ for $0 < t \le t_0$. This shows $\langle R(X,\dot\gamma)\dot\gamma, X\rangle = \langle R(Y^\perp,\dot\gamma)\dot\gamma, Y^\perp\rangle \equiv c\|Y^\perp\|^2$.

By the expressions of magnetic Jacobi fields on $\mathbb{C}M^n(4c)$, we get

$$Y^\sharp(t) = Y^\perp(t) = \|\nabla_{\dot\gamma} Y^\perp(0)\|\mathfrak{s}_{k/2}(t;c)\big\{\cos(kt/2)E(t) + \sin(kt/2)JE(t)\big\},$$

where we express $e^{\sqrt{-1}kt}E(t)$ as $\cos(kt/2)E(t) + \sin(kt/2)JE(t)$. Next, we consider the case that the equality holds in (2-b). We suppose $\|Y^\sharp(t_0)\| = \|\nabla_{\dot\gamma} Y^\sharp(0)\|\mathfrak{s}_{k/2}(t_0;c)$ at some t_0 with $0 < t_0 < \ell$. Then we find that $\|Y^\sharp(t_0)\|^2/h(t_0) = 1$ for $0 < t \le t_0$. As $\|Y^\sharp(0)\|^2/h(0) = 1$ and $\|Y^\sharp(t)\|^2/h(t)$ is monotone decreasing, we have $\|Y^\sharp(t)\| = \|\nabla_{\dot\gamma} Y^\sharp(0)\|\mathfrak{s}_{k/2}(t;c)$ for $0 < t \le t_0$. Since $\|Y^\sharp(t)\|^2 = \|\nabla_{\dot\gamma} Y^\sharp(0)\|^2\mathfrak{s}^2_{k/2}(t;c)$, by considering the differentiation on both sides we have

$$\frac{2\langle\nabla_{\dot\gamma} Y^\sharp(t), Y^\sharp(t)\rangle}{\|Y^\sharp(t)\|^2} = \frac{2\|\nabla_{\dot\gamma} Y^\sharp(0)\|^2\mathfrak{s}_{k/2}(t;c)\mathfrak{c}_{k/2}(t;c)}{\|\nabla_{\dot\gamma} Y^\sharp(0)\|^2\mathfrak{s}^2_{k/2}(t;c)},$$

which shows that

$$\langle\nabla_{\dot\gamma} Y^\sharp(t), Y^\sharp(t)\rangle = \|Y^\sharp(t)\|^2\mathfrak{t}_{k/2}(t;c).$$

We get the conclusion. □

In view of our proofs of Theorems 3.2, 3.3, we study magnetic Jacobi fields by decomposing them into components parallel to $J\dot\gamma$ and components orthogonal to both $\dot\gamma$ and $J\dot\gamma$ for each trajectory γ. Therefore, we can not compare magnetic Jacobi fields on two general Kähler manifolds. We should note that our proof stands for ordinary Jacobi fields on Kähler manifolds.

CHAPTER 4

Comparison theorems on trajectory-harps

In this chapter we study trajectories for Kähler magnetic fields in connection with geodesics. A trajectory-harp consists of a trajectory and a variation of geodesics. We compare trajectory-harps on a general Kähler manifold with those on a complex space form, and give some results corresponding to Toponogov's theorem on triangles.

We recall Toponogov's comparison theorem, which is a powerful global generalization of Rauch's comparison theorem. Given three distinct points p_1, p_2, p_3 on a Riemannian manifold M, we take geodesic segments $\gamma_i : [0, \ell_i] \to M$ $(i = 1, 2, 3)$ joining p_{i+1} and p_{i+2}, where indices are considered by modulo 3. We call the triangle $\varDelta(p_1p_2p_3)$ formed by $\gamma_1, \gamma_2, \gamma_3$ a geodesic triangle. Set $\alpha_i = \angle\big(-\dot\gamma_{i+1}(\ell_{i+1}), \dot\gamma_{i+2}(0)\big)$, the angle between $-\dot\gamma_{i+1}(\ell_i)$ and $\dot\gamma_{i+2}(0)$.

Theorem 4.1 (Toponogov's compariosn theorem). *Let M be a complete Riemannian manifold. We suppose that sectional curvatures satisfy* $\mathrm{Riem}^M \geq c$ *with some constant c. We set $\mathbb{R}M^2(c)$ a 2-dimensional real space form of constant sectional curvature c. Let $\varDelta(p_1p_2p_3)$ be a geodesic triangle in M. We suppose γ_1, γ_2 are minimal, and suppose $\ell_3 = length(\gamma_3) \leq \pi/\sqrt{c}$, when $c > 0$. Then we have $\ell \leq 2\pi/\sqrt{c}$ and there exists a geodesic triangle $\varDelta(\tilde{p_1}\tilde{p_2}\tilde{p_3})$ in $\mathbb{R}M^2(c)$ such that $length(\gamma_i) = lenth(\tilde\gamma_i)$ $(1 = 1, 2, 3)$ and $\alpha_1 \geq \tilde\alpha_1$, $\alpha_2 \geq \tilde\alpha_2$. Here, we set $\ell := \ell_1 + \ell_2 + \ell_3$. If $\ell < 2\pi/\sqrt{c}$, the triangle in $\mathbb{R}M^2(c)$ is uniquely determined up to isometries. Moreover, if there exists a geodesic triangle with $\ell = 2\pi/\sqrt{c}$, then M is congruent to $S^m(c)$.*

1. Trajectory-harps

Let (M, J) be a complete Kähler manifold with complex structure J. Since M is complete, as we see in Lemma 2.12, every trajectory for Kähler magnetic fields is defined on the whole line $\mathbb{R}$. For a trajectory $\gamma : \mathbb{R} \to M$, we call its restriction $\gamma|_{[T_1,T_2]}$ to a closed interval $[T_1, T_2]$ a trajectory-segment, and call its restriction $\gamma|_{[T,\infty)}$ or $\gamma|_{(-\infty,T]}$ to a closed unbounded interval $[T,\infty)$ or $(-\infty,T]$ a trajectory half-line. For the sake of simplicity, we sometimes call $\gamma : [0,T] \to M$ with $0 < T \leq \infty$ a trajectory. This means that it is a trajectory-segment when T is finite and that it is a trajectory half-line when T is infinity. Given a trajectory $\gamma : [0,T] \to M$ for a non-trivial Kähler magnetic field $\mathbb{B}_k$ satisfying $\gamma(t) \neq \gamma(0)$ for $0 < t < T$, we say a smooth variation $\alpha_\gamma : [0,T] \times \mathbb{R} \to M$ of geodesics to be a *trajectory-harp* associated with γ if it satisfies the following conditions:

i) $\alpha_\gamma(t,0) = \gamma(0)$,

ii) when $t = 0$, the curve $s \mapsto \alpha_\gamma(0,s)$ is the geodesic of initial vector $\dot\gamma(0)$,

iii) when $t > 0$, the curve $s \mapsto \alpha_\gamma(t,s)$ is the geodesic of unit speed joining $\gamma(0)$ and $\gamma(t)$.

We call the geodesic segment $s \mapsto \alpha_\gamma(t,s)$ from $\gamma(0)$ to $\gamma(t)$ the *string* of α_γ at t, and call the trajectory γ its *arch*.

When $\gamma([0,T])$ is contained in the ball $B_{\iota_p}(p)$ of radius ι_p of injectivity at $p = \gamma(0)$ centered at p, joining $\gamma(t)$ and $\gamma(0)$ by the unique minimal geodesic, we can get a trajectory-harp. Thus we have the following.

Lemma 4.1. *When a trajectory γ satisfies that its image $\gamma([0,T])$ is contained in the ball $B_{c_p}(p)$ of radius c_p of the minimum of conjugate values along geodesics emanating from $p = \gamma(0)$, we can construct a trajectory-harp associated with γ.*

Proof. When p and $\gamma(t)$ is joined by a unique minimal geodesic for $0 < t < \epsilon$, in particular when $\gamma([0,\epsilon])$ is contained in the ball $B_{\iota_p}(p)$, by using these minimal geodesics we can construct a trajectory-harp for $0 \leq t \leq \epsilon$. Here, the initial vector $\dfrac{\partial\alpha_\gamma}{\partial s}(\epsilon,0)$ of the geodesic $s \mapsto \alpha_\gamma(\epsilon,s)$ is given as $\lim_{t\uparrow\epsilon} \dfrac{\partial\alpha_\gamma}{\partial s}(t,0)$, even if $\gamma(\epsilon)$ is a cut point of p, that is, there are at least two minimal geodesics joining p and $\gamma(\epsilon)$. We suppose $\gamma(\epsilon)$ is not a conjugate point of p along the geodesic $s \mapsto \alpha_\gamma(\epsilon,s)$. Then we have a Jacobi field Y along this geodesic with $Y(0)=0$ and $Y\big(\ell_\gamma(\epsilon)\big)=\dot\gamma(\epsilon)$. Since $\dfrac{\partial\alpha_\gamma}{\partial t}(\epsilon,\cdot)=\lim_{t\uparrow\epsilon}\dfrac{\partial\alpha_\gamma}{\partial t}(t,\cdot)$ is a Jacobi field along the geodesic satisfying the same condition, we have $\dfrac{\partial\alpha_\gamma}{\partial t}(\epsilon,s)=Y(s)$. Thus, if we take a variation $\alpha':(-\delta,\delta)\times\mathbb{R}\to M$ $(\delta<\epsilon)$ of geodesics generated by Y, we have $\alpha'|_{(-\delta,0)\times\mathbb{R}}=\alpha_\gamma|_{(\epsilon-\delta,\epsilon)\times\mathbb{R}}$. Thus we can extend α_γ beyond this cut point. □

For $t>0$, we denote by $\ell_\gamma(t)$ the length of the geodesic segment $s\mapsto\alpha_\gamma(t,s)$ of $\gamma(0)$ to $\gamma(t)$. We set $\ell_\gamma(0)=0$. We call $\ell_\gamma(t)$ to be the *string-length* at $\gamma(t)$. As trajectory γ is parameterized by its arc-length, it is clear that it satisfies $\ell_\gamma(t)\le t$ for $0<t\le T$. We set $\delta_\gamma(t):=\Big\langle\dot\gamma(t),\dfrac{\partial\alpha_\gamma}{\partial s}\big(t,\ell_\gamma(t)\big)\Big\rangle$, which is the cosine of the angle formed by the tangent vector of the string at t and the tangent vector of trajectory at t, and call it its *string-cosine* at $\gamma(t)$.

We here study some fundamental properties of string-lengths and string-cosines.

Lemma 4.2. *For a trajectory-harp α_γ associated with a trajectory $\gamma:[0,T]\to M$ for a Kähler magnetic field on M, its string-length and its string-cosine satisfy $\ell_\gamma'(t)=\delta_\gamma(t)$ for $0<t<T$. We hence have $\lim_{t\downarrow 0}\ell_\gamma'(t)=\delta_\gamma(0)$.*

Proof. We define $\hat\alpha:[0,T]\times\mathbb{R}\to M$ by $\hat\alpha(t,u)=\alpha\big(t,\ell_\gamma(t)u\big)$. We then have $\hat\alpha(t,0)=\gamma(0)$ and

$$\ell_\gamma^2(t)=\Big\|\frac{\partial\hat\alpha}{\partial u}(t,u)\Big\|^2=\int_0^1\Big\|\frac{\partial\hat\alpha}{\partial u}(t,u)\Big\|^2\,du.$$

Considering the differential of this function, we have $\dfrac{d}{dt}\big(\ell_\gamma^2(t)\big)=2\ell_\gamma(t)\ell_\gamma'(t)$ and

$$\begin{aligned}\frac{d}{dt}\big(\ell_\gamma^2(t)\big)&=\frac{d}{dt}\int_0^1\Big\langle\frac{\partial\hat\alpha}{\partial u}(t,u),\frac{\partial\hat\alpha}{\partial u}(t,u)\Big\rangle du=\int_0^1\frac{d}{dt}\Big\langle\frac{\partial\hat\alpha}{\partial u}(t,u),\frac{\partial\hat\alpha}{\partial u}(t,u)\Big\rangle du\\&=2\int_0^1\Big\langle\Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\hat\alpha}{\partial u}\Big)(t,u),\frac{\partial\hat\alpha}{\partial u}(t,u)\Big\rangle du=2\int_0^1\Big\langle\Big(\nabla_{\frac{\partial}{\partial u}}\frac{\partial\hat\alpha}{\partial t}\Big)(t,u),\frac{\partial\hat\alpha}{\partial u}(t,u)\Big\rangle du\\&=2\int_0^1\Big\{\frac{d}{du}\Big\langle\frac{\partial\hat\alpha}{\partial t}(t,u),\frac{\partial\hat\alpha}{\partial u}(t,u)\Big\rangle-\Big\langle\frac{\partial\hat\alpha}{\partial t}(t,u),\Big(\nabla_{\frac{\partial}{\partial u}}\frac{\partial\hat\alpha}{\partial u}\Big)(t,u)\Big\rangle\Big\}du.\end{aligned}$$

As $u \mapsto \hat{\alpha}_\gamma(t,u)$ is a geodesic for each t and $\frac{\partial\hat{\alpha}}{\partial t}(t,0)=0$ because $\hat{\alpha}_\gamma(t,0)=\gamma(0)$, we have

$$\begin{aligned}
&= 2\int_0^1 \frac{d}{du}\Big\langle \frac{\partial\hat{\alpha}}{\partial t}(t,u), \frac{\partial\hat{\alpha}}{\partial u}(t,u)\Big\rangle du = 2\Big\langle \frac{\partial\hat{\alpha}}{\partial t}(t,u), \frac{\partial\hat{\alpha}}{\partial u}(t,u)\Big\rangle\Big|_0^1 \\
&= 2\Big\langle \frac{\partial\hat{\alpha}}{\partial t}(t,1), \frac{\partial\hat{\alpha}}{\partial u}(t,1)\Big\rangle = 2\Big\langle \dot{\gamma}(t), \frac{\partial\hat{\alpha}}{\partial u}(t,u)\Big|_{u=1}\Big\rangle \\
&= 2\Big\langle \dot{\gamma}(t), \ell_\gamma(t)\frac{\partial\alpha}{\partial s}(t,s)\Big|_{s=\ell_\gamma(t)}\Big\rangle = 2\ell_\gamma(t)\Big\langle \dot{\gamma}(t), \frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&= 2\ell_\gamma(t)\delta_\gamma(t).
\end{aligned}$$

Since $\ell_\gamma(t)>0$ for $t>0$ and δ_γ is smooth with respect to t by definition, we get the conclusion. □

Lemma 4.3. *For a trajectory γ for $\mathbb{B}_k$ on a Kähler manifold, we have the following properties* : $\ell_\gamma(0)=0,\ \delta_\gamma(0)=1,\ \lim_{t\downarrow 0}\delta'_\gamma(t)=0$ *and* $\lim_{t\downarrow 0}\delta''_\gamma(t)=-k^2/4$.

Proof. By definitions of string-lengths and of string-cosines, we get $\ell_\gamma(0)=0$ and $\delta_\gamma(0)=\Big\langle \dot{\gamma}(0),\ \frac{\partial\alpha}{\partial s}(0,0)\Big\rangle = 1$. For the third equality, we compute the differential of the string-cosine. Since $s \mapsto \alpha_\gamma(t,s)$ is a geodesic for each t, we have

$$\begin{aligned}
\delta'_\gamma(t) &= \Big\langle \nabla_{\dot{\gamma}}\dot{\gamma}(t), \frac{\partial\alpha}{\partial s}(t,\ell_\gamma(t))\Big\rangle + \Big\langle \dot{\gamma}(t), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&\quad + \ell'_\gamma(t)\Big\langle \dot{\gamma}(t), \Big(\nabla_{\frac{\partial}{\partial s}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&= \Big\langle \nabla_{\dot{\gamma}}\dot{\gamma}(t), \frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big)\Big\rangle + \Big\langle \dot{\gamma}(t), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&= \Big\langle kJ\dot{\gamma}(t), \frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big)\Big\rangle + \Big\langle \dot{\gamma}(t), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&= k\Big\langle J\dot{\gamma}(t), \frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big)\Big\rangle + \Big\langle \dot{\gamma}(t), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle.
\end{aligned}$$

As $\Big\|\frac{\partial\alpha}{\partial s}(t,s)\Big\| = 1$, by taking the differentiation of both sides of this equality we get $\Big\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial s}\Big\rangle \equiv 0$. Since we have $\dot{\gamma}(0) = \frac{\partial\alpha}{\partial s}(0,0)$, we get the third equality in the following manner:

$$\begin{aligned}\lim_{t\downarrow 0}\delta'_\gamma(t) &= \lim_{t\downarrow 0}\Big\{k\Big\langle J\dot\gamma(t), \frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big)\Big\rangle + \Big\langle \dot\gamma(t), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle\Big\}\\
&= k\Big\langle J\dot\gamma(0), \frac{\partial\alpha}{\partial s}(0,0)\Big\rangle + \Big\langle \dot\gamma(0), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)(0,0)\Big\rangle\\
&= k\Big\langle J\frac{\partial\alpha}{\partial s}(0,0), \frac{\partial\alpha}{\partial s}(0,0)\Big\rangle + \Big\langle \frac{\partial\alpha}{\partial s}(0,0), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)(0,0)\Big\rangle\\
&= 0.\end{aligned}$$

To get the fourth equality, we need to compute the differential of $\delta'_\gamma(t)$. As we see

$$\delta'_\gamma(t) = k\Big\langle J\dot\gamma(t), \frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big)\Big\rangle + \Big\langle \dot\gamma(t), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle,$$

we have

$$\begin{aligned}\delta''_\gamma(t) &= k\Big\{\Big\langle J\nabla_{\dot\gamma}\dot\gamma(t), \frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big)\Big\rangle\\
&\qquad + \Big\langle J\dot\gamma(t), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big) + \ell'_\gamma(t)\Big(\nabla_{\frac{\partial}{\partial s}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle\Big\}\\
&\quad + \Big\langle \nabla_{\dot\gamma}\dot\gamma(t), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle\\
&\quad + \Big\langle \dot\gamma(t), \Big(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big) + \ell'_\gamma(t)\Big(\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle\\
&= -k^2\Big\langle \dot\gamma(t), \frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big)\Big\rangle + 2k\Big\langle J\dot\gamma(t), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle\\
&\quad + \Big\langle \dot\gamma(t), \Big(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle + \ell'_\gamma(t)\Big\langle \dot\gamma(t), \Big(\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle.\end{aligned}$$

Since $\gamma(t) = \alpha\big(t,\ell_\gamma(t)\big)$, we see $\dot\gamma(t) = \dfrac{\partial\alpha}{\partial t}\big(t,\ell_\gamma(t)\big) + \ell'_\gamma(t)\dfrac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big)$. By definition of trajectories, we have

$$\begin{aligned}kJ\dot\gamma(t) &= \nabla_{\dot\gamma}\dot\gamma(t)\\
&= \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial t}\Big)\big(t,\ell_\gamma(t)\big) + \ell'_\gamma(t)\Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big) + \ell''_\gamma(t)\frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big)\\
&\quad + \ell'_\gamma(t)\Big\{\Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big) + \ell'_\gamma(t)\Big(\nabla_{\frac{\partial}{\partial s}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\}\\
&= \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial t}\Big)\big(t,\ell_\gamma(t)\big) + 2\ell'_\gamma(t)\Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big) + \ell''_\gamma(t)\frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big).\end{aligned}$$

Since $\dfrac{\partial\alpha}{\partial t}(t,0) = 0$, we have $\nabla_{\frac{\partial\alpha}{\partial t}}\dfrac{\partial\alpha}{\partial t}(0,0) = 0$. We therefore obtain

$$kJ\dot\gamma(0) = 2\lim_{t\downarrow 0}\Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big),$$

because $\lim_{t\downarrow 0}\ell'_\gamma(t) = \delta_\gamma(0) = 1$ and $\lim_{t\downarrow 0}\ell''_\gamma(t) = \lim_{t\downarrow 0}\delta'_\gamma(t) = 0$. As we have $\left\|\dfrac{\partial\alpha}{\partial s}\right\| = 1$, we see

$$0 = \frac{d}{dt}\Big\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial s}\Big\rangle = \Big\|\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big\|^2 + \Big\langle \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial s}\Big\rangle. \tag{4.1}$$

Thus, by noticing that $s \mapsto \alpha_\gamma(t,s)$ is a geodesic for each t, we get

$$\begin{aligned}
\lim_{t\downarrow 0}\delta''_\gamma(t) &= \lim_{t\downarrow 0}\Big\{-k^2\Big\langle\dot\gamma(t), \frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big)\Big\rangle + 2k\Big\langle J\dot\gamma(t), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&\qquad + \Big\langle\dot\gamma(t), \Big(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&\qquad + \ell'_\gamma(t)\Big\langle\dot\gamma(t), \Big(\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle\Big\} \\
&= -k^2\Big\langle\dot\gamma(0), \frac{\partial\alpha}{\partial s}(0,0)\Big\rangle + k\Big\langle J\dot\gamma(0), 2\lim_{t\downarrow 0}\Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&\qquad + \Big\langle\dot\gamma(0), \lim_{t\downarrow 0}\Big(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&\qquad + \Big(\lim_{t\downarrow 0}\ell'_\gamma(t)\Big)\Big\langle\dot\gamma(0), \lim_{t\downarrow 0}\Big(\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&= -k^2\langle\dot\gamma(0),\dot\gamma(0)\rangle + k\langle J\dot\gamma(0), kJ\dot\gamma(0)\rangle \\
&\qquad + \Big\langle\frac{\partial\alpha}{\partial s}(0,0), \lim_{t\downarrow 0}\Big(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&\qquad + \Big\langle\dot\gamma(0), \lim_{t\downarrow 0}\Big(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&\qquad - \Big\langle\dot\gamma(0), \lim_{t\downarrow 0}R\Big(\frac{\partial\alpha}{\partial t}\big(t,\ell_\gamma(t)\big), \frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big)\Big)\frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&= -k^2 + k^2 + \lim_{t\downarrow 0}\Big\langle\frac{\partial\alpha}{\partial s}\big(t,\ell_\gamma(t)\big), \Big(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\rangle \\
&\qquad - \Big\langle\dot\gamma(0), R\Big(\frac{\partial\alpha}{\partial t}(0,0), \frac{\partial\alpha}{\partial s}(0,0)\Big)\frac{\partial\alpha}{\partial s}(0,0)\Big\rangle.
\end{aligned}$$

By using (4.1) we obtain

$$\begin{aligned}
&= -\lim_{t\downarrow 0}\Big\|\Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\Big\|^2 - \Big\langle\dot\gamma(0), R\Big(\frac{\partial\alpha}{\partial t}(0,0), \dot\gamma(0)\Big)\dot\gamma(0)\Big\rangle \\
&= -\Big\|\frac{k}{2}J\dot\gamma(0)\Big\|^2 \\
&= -\frac{k^2}{4}.
\end{aligned}$$

This completes the proof. $\square$

2. Trajectory-harps on a complex space form

We say two trajectory-harps $\alpha_{\gamma_1}, \alpha_{\gamma_2} : [0,T]\times\mathbb{R} \to M$ on a Kähler manifold M to be congruent to each other if there is an isometry φ of M satisfying $\alpha_{\gamma_2}(t,s) = \varphi\circ\alpha_{\gamma_1}(t,s)$ for all $(t,s) \in [0,T] \times \mathbb{R}$. On a complex space form $\mathbb{C}M^n(c)$ of constant holomorphic sectional curvature c, as we see in Proposition 2.3 two trajectories γ_1, γ_2 for a Kähler magnetic field $\mathbb{B}_k$ are congruent to each other in strong sense. Since each trajectory lies on some totally geodesic complex line $\mathbb{C}M^1(c)$, we see that the trajectory-harp associated with this trajectory lies on this $\mathbb{C}M^1(c)$. Therefore we find the following.

Proposition 4.1. *On a complex space form $\mathbb{C}M^n(c)$ two trajectory-harps associated with trajectories $\gamma_1, \gamma_2 : [0,T] \to \mathbb{C}M^n(c)$ for $\mathbb{B}_k$ are congruent to each other.*

By this proposition, string-lengths and string-cosines for trajectory-harps for a Kähler magnetic field $\mathbb{B}_k$ on $\mathbb{C}M^n(c)$ do not depend on the choice of trajectory-harps. We therefore denote them by $\ell_k(t;c)$ and $\delta_k(t;c)$. These functions are given in the following.

[1] Trajectory-harps on a complex Euclidean space

On a complex Euclidean space $\mathbb{C}^n$, as the covariant differentiation is the ordinary differentiation, a trajectory γ for $\mathbb{B}_k$ is a circle of radius $1/|k|$ in the sense of Euclidean geometry, hence is closed of length $2\pi/|k|$.

Proposition 4.2. *For $0 \le t \le 2\pi/|k|$, we have the following:*

(1) *The string-length is given as $\ell_k(t;0) = (2/|k|)\sin(|k|t/2)$;*

(2) *Every trajectory for $\mathbb{B}_k$ emanating from an arbitrary point $p \in \mathbb{C}^n$ and the corresponding chord make the angle $\theta_k(t;0) = |k|t/2$. Hence the string-cosine is given as $\delta_k(t;0) = \cos(|k|t/2)$.*

Proof. Let γ be a trajectory for $\mathbb{B}_k$. We set $p = \gamma(0)$. We consider a circular arc $\gamma([0,t])$, which is inferior when $0 < t \le \pi/|k|$ and is superior when $\pi/|k| \le t < 2\pi/|k|$. The geodesic-segment joining $\gamma(0)$ and $\gamma(t)$ is a sub-tense for this circular arc. We take a triangle of vertices $p, \gamma(t)$ and the center o of the circle. As it is an isosceles triangle with $d(p,o) = d(\gamma(t), o)$, we find that the distance between $\gamma(0)$ and $\gamma(t)$ is $(2/|k|)\sin(|k|t/2)$. Since the angle $\theta_k(t;0)$ coincides with the angle of circumference over $\gamma([0,t])$, hence is equal to the half of the angle $|k|t$ of the sector of circular arc $\gamma([0,t])$. $\square$

As we see in Lemma 2.13, when γ is a trajectory for $\mathbb{B}_k$ then the curve $\tilde{\gamma}$ defined by $\tilde{\gamma}(s) = \gamma(t-s)$ is a trajectory for $\mathbb{B}_{-k}$. If a geodesic σ satisfies $\sigma(0) = \gamma(0)$ and $\sigma(\ell) = \gamma(r)$, we see that the angle between $\dot{\gamma}(0)$ and $\dot{\sigma}(0)$ coincides with the angle between $\dot{\tilde{\gamma}}(0)$ and $\dot{\tilde{\sigma}}(0)$, where $\tilde{\sigma}$ is given by $\tilde{\sigma}(s) = \sigma(\ell - s)$. Thus $\theta_k(t;0)$ also shows the angle between $\dot{\gamma}(0)$ and $\dot{\sigma}(0)$.

Lemma 4.4. *The functions $\ell_k(t;0)$ and $\delta_k(t;0)$ satisfy the following properties:*

(1) $\ell_k(t;0) = \ell_{-k}(t;0)$, $\delta_k(t;0) = \delta_{-k}(t;0)$;

(2) *The function $\ell_k(\cdot\,;0) : [0, 2\pi/|k|] \to \mathbb{R}$ is monotone increasing in the interval $[0, \pi/|k|]$ and is monotone decreasing in the interval $[\pi/|k|, 2\pi/|k|]$;*

(3) *The function $\delta_k(\cdot\,;0) : [0, 2\pi/|k|] \to \mathbb{R}$ is monotone decreasing.*

Proof. The first assertion is trivial by their expressions in Proposition 4.2. Since $\ell_k(t;0) = (2/|k|)\sin(|k|t/2)$, we have

$$\frac{d}{dt}\ell_k(t;0) = \cos(|k|t/2) = \delta_k(t;0), \qquad \frac{d}{dt}\delta_k(t;0) = -(|k|t/2)\sin(|k|t/2).$$

We get the conclusion. $\square$

Proposition 4.3. *The string-lengths and the string-cosines on $\mathbb{C}^n$ with respect to $|k|$ satisfies the following:*

(1) *The function $\ell_k(\cdot;0) : [0, \pi/|k|] \to \mathbb{R}$ is monotone decreasing with respect to $|k|$;*

(2) *The function $\delta_k(\cdot;0) : [0, \pi/|k|] \to \mathbb{R}$ is monotone decreasing with respect to $|k|$.*

PROOF. We are enough to study the case $k > 0$.

By differentiating of the string-length $\ell_k(t;0)$ and the string-cosine $\delta_k(t;0)$, we have

$$\frac{d}{dk}\ell_k(t;0) = -\frac{t}{k^2}\cos\frac{kt}{2} < 0, \qquad \frac{d}{dk}\delta_k(t;0) = -\frac{t}{2}\sin\frac{kt}{2} < 0.$$

We hence get the conclusion. $\square$

[2] Trajectory-harps on a complex projective space

We take a trajectory γ for $\mathbb{B}_k$ on a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c. As we see in §2.3, a trajectory γ for $\mathbb{B}_k$ is a "small" circle of radius $1/\sqrt{k^2+c}$ on a totally geodesic $\mathbb{C}P^1(c) = S^2(c)$, hence it is closed of length $2\pi/\sqrt{k^2+c}$. We note that the injectivity radius $\iota_{\mathbb{C}P^n(c)}$ of $\mathbb{C}P^n(c)$ is equal to $\pi/\sqrt{c}$.

PROPOSITION 4.4. *For $0 < t < 2\pi/\sqrt{k^2+c}$, we have the following*:

(1) *The string-length is given by* $\sqrt{k^2+c}\sin\bigl(\sqrt{c}\ell_k(t;c)/2\bigr) = \sqrt{c}\sin\bigl(\sqrt{k^2+c}\,t/2\bigr)$;

(2) *The string-cosine is given by*

$$\delta_k(t;c) = \frac{\sqrt{k^2+c}\cos\bigl(\sqrt{k^2+c}\,t/2\bigr)}{\sqrt{k^2+c\cos^2\bigl(\sqrt{k^2+c}\,t/2\bigr)}}.$$

PROOF. Since every trajectory lies on some totally geodesic $\mathbb{C}P^1(c)$, we are enough to study the case $n=1$. As we see in §2.3, $\mathbb{C}P^1(c)$ is isomorphic to $S^2(c)$, we hence use the expression of trajectories on $S^2(c) \subset \mathbb{R}^3$. We take a trajectory γ for $\mathbb{B}_k$ with $\gamma(0) = p \in S^2 \subset \mathbb{R}^3$, $\dot\gamma(0) = u \in T_pS^2 \subset T_p\mathbb{R}^3 \cong \mathbb{R}^3$ and $\nabla_{\dot\gamma}\dot\gamma(0) + c\gamma(0) = v$. Then, if we regard γ as a curve in $\mathbb{R}^3$, it is expressed as

$$\gamma(t) = \frac{1}{k^2+c}\bigl(k^2 + c\cos\sqrt{k^2+c}\,t\bigr)p + \frac{1}{\sqrt{k^2+c}}\bigl(\sin\sqrt{k^2+c}\,t\bigr)u + \frac{1}{k^2+c}\bigl(1-\cos\sqrt{k^2+c}\,t\bigr)v.$$

Also we take a geodesic σ on $S^2(c)$ with $\sigma(0) = p$ and $\dot\sigma(0) = u' \in T_pS^2 \subset T_p\mathbb{R}^3 \cong \mathbb{R}^3$, which is expressed as $\sigma(t) = \cos\sqrt{c}\,tp + \frac{1}{\sqrt{c}}\sin\sqrt{c}\,tu'$ as a curve in $\mathbb{R}^3$. We take an arbitrary positive r with $r < 2\pi/\sqrt{k^2+c}$ and suppose $\gamma(r) = \sigma(\ell_k(r;c))$. Since p is orthogonal to u, v and u', and p, u, v span $\mathbb{R}^3$, this equality shows

$$k^2 + c\cos\sqrt{k^2+c}\,r = (k^2+c)\cos\sqrt{c}\ell_k(r;c), \tag{4.2}$$

$$\frac{1}{\sqrt{c}}\sin\bigl(\sqrt{c}\ell_k(r;c)\bigr)u' = \frac{1}{\sqrt{k^2+c}}\bigl(\sin\sqrt{k^2+c}\,r\bigr)u + \frac{1}{k^2+c}\bigl(1-\cos\sqrt{k^2+c}\,r\bigr)v. \tag{4.3}$$

Applying the double angle formula to (4.2), we have

$$k^2 + c\bigl(1 - 2\sin^2(\sqrt{k^2+c}\,r/2)\bigr) = (k^2+c)\bigl(1-2\sin^2(\sqrt{c}\ell_k(r;c)/2)\bigr),$$

which shows the first assertion because $r < 2\pi/\sqrt{k^2+c}$ and $\ell_k(r;c) \le r$. As u is orthogonal to v, and $\langle u',\ u\rangle = \delta_k(r;c)$, $\|u\| = 1$, by taking the inner products of both sides of (4.3) with u, we get

$$\frac{\delta_k(r;c)}{\sqrt{c}}\sin\bigl(\sqrt{c}\ell_k(r;c)\bigr) = \frac{\sin\sqrt{k^2+c}\,r}{\sqrt{k^2+c}}.$$

Again, by using the double angle formula and the first assertion, we get

$$\begin{aligned}\delta_k(r;c) &= \frac{\sqrt{c}\sin\sqrt{k^2+c}\,r}{\sqrt{k^2+c}\sin\bigl(\sqrt{c}\ell_k(r;c)\bigr)}\\ &= \frac{\sqrt{c}\sin\bigl(\sqrt{k^2+c}\,r/2\bigr)\cos\bigl(\sqrt{k^2+c}\,r/2\bigr)}{\sqrt{k^2+c}\sin\bigl(\sqrt{c}\ell_k(r;c)/2\bigr)\cos\bigl(\sqrt{c}\ell_k(r;c)/2\bigr)}\\ &= \frac{\cos\bigl(\sqrt{k^2+c}\,r/2\bigr)}{\cos\bigl(\sqrt{c}\ell_k(r;c)/2\bigr)} = \frac{\sqrt{k^2+c}\cos\bigl(\sqrt{k^2+c}\,r/2\bigr)}{\sqrt{k^2+c\cos^2\bigl(\sqrt{k^2+c}\,r/2\bigr)}}.\end{aligned}$$

This completes the proof. □

LEMMA 4.5. *When $c > 0$, the functions $\ell_k(t;c)$ and $\delta_k(t;c)$ satisfy the following properties:*

(1) $\ell_k(t;c) = \ell_{-k}(t;c)$, $\delta_k(t;c) = \delta_{-k}(t;c)$;

(2) *The function $\ell_k(\cdot\,;c) : \bigl[0, 2\pi/\sqrt{k^2+c}\bigr] \to \mathbb{R}$ is monotone increasing in the interval $\bigl[0, \pi/\sqrt{k^2+c}\bigr]$ and is monotone decreasing in the interval $\bigl[\pi/\sqrt{k^2+c}, 2\pi/\sqrt{k^2+c}\bigr]$;*

(3) *The function $\delta_k(\cdot\,;c) : \bigl[0, 2\pi/\sqrt{k^2+c}\bigr] \to \mathbb{R}$ is monotone decreasing.*

Proof. The first assertion is trivial by their expressions in Proposition 4.4. Since we have

$$\frac{d}{dt}\ell_k(t;c) = \delta_k(t;c) = \frac{\sqrt{k^2+c}\cos\bigl(\sqrt{k^2+c}\,t/2\bigr)}{\sqrt{k^2+c\cos^2\bigl(\sqrt{k^2+c}\,t/2\bigr)}}$$

and

$$\frac{d}{dt}\delta_k(t;c) = \frac{d}{dt}\left(\frac{\sqrt{k^2+c}\cos\bigl(\sqrt{k^2+c}\,t/2\bigr)}{\sqrt{k^2+c\cos^2\bigl(\sqrt{k^2+c}\,t/2\bigr)}}\right) = -\frac{k^2(k^2+c)\sin(\sqrt{k^2+c}\,t/2)}{2\bigl\{k^2+c\cos^2(\sqrt{k^2+c}\,t/2)\bigr\}^{3/2}}.$$

We get the conclusion. □

Proposition 4.5. *The string-lengths and the string-cosines on $\mathbb{C}P^n(c)$ satisfies the following*:

(1) *The function $\ell_k(t;c)$ is monotone decreasing with respect to $|k|$;*

(2) *The function $\delta_k(t;c)$ is monotone decreasing with respect to $|k|$.*

Proof. We are enough to study the case $k > 0$.

By differentiating of the string-length $\ell_k(t;c)$ and the string-cosine $\delta_k(t;c)$, we have

$$\frac{d}{dk}\ell_k(t;c) = \frac{tk\bigl(\cos(\sqrt{k^2+c}\,t/2)-1\bigr)}{(k^2+c)\cos\bigl(\sqrt{c}\ell_k(t;c)/2\bigr)} < 0,$$

and

$$\frac{d}{dk}\delta_k(t;c) = -\frac{k\sin\bigl(\sqrt{k^2+c}\,t/2\bigr)\Bigl(k^2\sqrt{k^2+c}\,t + c\sin\bigl(\sqrt{k^2+c}\,t\bigr)\Bigr)}{2\sqrt{k^2+c}\Bigl(k^2+c\cos\bigl(\sqrt{k^2+c}\,t/2\bigr)^2\Bigr)^{3/2}} < 0.$$

We hence get the conclusion. □

[3] Trajectory-harps on a complex hyperbolic space

We take a trajectory γ for $\mathbb{B}_k$ on a complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c. As we see in §2.3, a trajectory γ for a Kähler magnetic field is a curve without self-intersections and lies on a totally geodesic $\mathbb{C}H^1(c) = H^2(c)$. In particular, when $|k| > \sqrt{|c|}$, a trajectory is a circle of radius $1/\sqrt{k^2+c}$ and when $|k| \leq \sqrt{|c|}$, it is open and is unbounded.

PROPOSITION 4.6. *When $c<0$, for $0<r<2\pi/\sqrt{k^2+c}$, we have the following :*
(1) *The string-length $\ell_k(r;c)$ satisfies the following relations ;*

$$\begin{cases} \sqrt{|c|-k^2}\,\sinh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)= \sqrt{|c|}\,\sinh\bigl(\sqrt{|c|-k^2}\;r/2\bigr), & \text{if } |k|<\sqrt{|c|},\\ 2\sinh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)= \sqrt{|c|}r, & \text{if } k=\pm\sqrt{|c|},\\ \sqrt{k^2+c}\,\sinh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)= \sqrt{|c|}\,\sin\bigl(\sqrt{k^2+c}\,r/2\bigr), & \text{if } |k|>\sqrt{|c|}. \end{cases}$$

(2) *The string-cosine $\delta_k(r;c)$ is given by*

$$\delta_k(r;c)=\begin{cases} \dfrac{\sqrt{|c|-k^2}\cosh\bigl(\sqrt{|c|-k^2}\;r/2\bigr)}{\sqrt{|c|\cosh^2\bigl(\sqrt{|c|-k^2}\;r/2\bigr)-k^2}}, & \text{if } |k|<\sqrt{|c|},\\ \dfrac{2}{\sqrt{|c|r^2+4}}, & \text{if } k=\pm\sqrt{|c|},\\ \dfrac{\sqrt{k^2+c}\;\cos\bigl(\sqrt{k^2+c}\,r/2\bigr)}{\sqrt{k^2+c\cos^2\bigl(\sqrt{k^2+c}\,r/2\bigr)}}, & \text{if } |k|>\sqrt{|c|}. \end{cases}$$

PROOF. Since every trajectory lies on some totally geodesic $\mathbb{C}H^1$, we may consider the case $n=1$.

First we study the case $|c|>k^2$. On $\mathbb{C}H^1(c)=H^2(c)$, we take trajectory γ for $\mathbb{B}_k$ with $\gamma(0)=p\in H^2\subset\mathbb{R}^3, \dot\gamma(0)=u\in T_pH^2\subset T_p\mathbb{R}^3\cong\mathbb{R}^3$ and $\nabla_{\dot\gamma}\dot\gamma(0)-|c|\gamma(0)=v$. Then if we regard it as a curve in $\mathbb{R}^3$, it is expressed as

$$\begin{aligned}\gamma(t)=&\frac{1}{|c|-k^2}\bigl(|c|\cosh\sqrt{|c|-k^2}\;t-k^2\bigr)p+\frac{1}{|c|-k^2}\bigl(\cosh\sqrt{|c|-k^2}\;t-1\bigr)v\\ &+\frac{1}{\sqrt{|c|-k^2}}\bigl(\sinh\sqrt{|c|-k^2}\;t\bigr)u.\end{aligned}$$

Also we take a geodesic σ on $H^2(c)$ with $\sigma(0)=p$ and $\dot\sigma(0)=u'\in T_pH^2\subset T_p\mathbb{R}^3\cong\mathbb{R}^3$, which is expressed as $\sigma(t)=\cosh\sqrt{|c|}\;tp+\dfrac{1}{\sqrt{|c|}}\sinh\sqrt{|c|}\;tu'$ as a curve in $\mathbb{R}^3$. We take an arbitrary positive r with $r<2\pi/\sqrt{k^2+c}$ and suppose $\gamma(r)=\sigma(\ell_k(r;c))$. Since p is orthogonal to u,v and u', and p,u,v span $\mathbb{R}^3$, this equality shows

$$|c|\cosh\sqrt{|c|-k^2}\;r-k^2=(|c|-k^2)\cosh\bigl(\sqrt{|c|}\ell_k(r;c)\bigr), \tag{4.4}$$

$$
\begin{aligned}
(4.5)\qquad \frac{1}{\sqrt{|c|}}\sinh\bigl(\sqrt{|c|}\ell_k(r;c)\bigr)u' &= \frac{1}{|c|-k^2}\bigl(\cosh\sqrt{|c|-k^2}\,r-1\bigr)v \\
&\quad + \frac{1}{\sqrt{|c|-k^2}}\bigl(\sinh\sqrt{|c|-k^2}\,r\bigr)u.
\end{aligned}
$$

Applying the double angle formula to (4.4), we have

$$\sqrt{|c|-k^2}\sinh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)=\sqrt{|c|}\sinh\bigl(\sqrt{|c|-k^2}\,r/2\bigr).$$

As u is orthogonal to v, $\langle u,u'\rangle=\delta_k(r;c)$ and $\|u\|=1$, by taking inner products of both sides of (4.5) with u, we get

$$
\begin{aligned}
\delta_k(r;c) &= \frac{\sqrt{|c|}\sinh\bigl(\sqrt{|c|-k^2}\,r\bigr)}{\sqrt{|c|-k^2}\sinh\bigl(\sqrt{|c|}\ell_k(r;c)\bigr)} \\
&= \frac{\sqrt{|c|}\sinh\bigl(\sqrt{|c|-k^2}\,r/2\bigr)\cosh\bigl(\sqrt{|c|-k^2}\,r/2\bigr)}{\sqrt{|c|-k^2}\sinh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)\cosh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)} \\
&= \frac{\cosh\bigl(\sqrt{|c|-k^2}\,r/2\bigr)}{\cosh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)} = \frac{\sqrt{|c|-k^2}\cosh\bigl(\sqrt{|c|-k^2}\,r/2\bigr)}{\sqrt{|c|\cosh^2\bigl(\sqrt{|c|-k^2}\,r/2\bigr)-k^2}}.
\end{aligned}
$$

Next we study the case $|c|=k^2$. On $\mathbb{C}H^1(c)=H^2(c)$, we take a trajectory γ for $\mathbb{B}_k$ with $\gamma(0)=p, \dot\gamma(0)=u$ and $\nabla_{\dot\gamma}\dot\gamma(0)-|c|\gamma(0)=v$. If we regard it as a curve in $\mathbb{R}^3$, it is expressed as

$$\gamma(t)=\Bigl(1+\frac{|c|t^2}{2}\Bigr)p+\frac{t^2}{2}v+tu.$$

Also we take a geodesic σ on $H^2(c)$ with $\sigma(0)=p$ and $\dot\sigma(0)=u'\in T_pH^2\subset T_p\mathbb{R}^3\cong\mathbb{R}^3$, which is expressed as $\sigma(t)=\bigl(\cosh\sqrt{|c|}\ t\bigr)p+\dfrac{1}{\sqrt{|c|}}\bigl(\sinh\sqrt{|c|}\ t\bigr)u'$ as a curve in $\mathbb{R}^3$. We take an arbitrary positive r and suppose $\gamma(r)=\sigma(\ell_k(r;c))$. Since p is orthogonal to u,v and u', and p,u,v span $\mathbb{R}^3$, this equality shows

$$(4.6)\qquad 1+\frac{|c|r^2}{2}=\cosh\bigl(\sqrt{|c|}\ell_k(r;c)\bigr),$$

$$(4.7)\qquad \frac{1}{\sqrt{|c|}}\sinh\bigl(\sqrt{|c|}\ell_k(r;c)\bigr)u'=\frac{r^2}{2}v+ru.$$

Applying the double angle formula to (4.6), we have

$$2\sinh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)=\sqrt{|c|}r.$$

As u is orthogonal to v, $\langle u, u'\rangle = \delta_k(r;c)$ and $\|u\| = 1$, by taking inner products of both sides of (4.7) with u, we get

$$\begin{aligned}\delta_k(r;c) &= \frac{\sqrt{|c|}\, r}{\sinh\bigl(\sqrt{|c|}\ell_k(r;c)\bigr)} \\ &= \frac{\sqrt{|c|}\, r}{2\sinh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)\cosh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)} \\ &= \frac{1}{\cosh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)} = \frac{2}{\sqrt{4+|c|r^2}}.\end{aligned}$$

Finally we study the case $|c| < k^2$. On $\mathbb{C}H^1(c) = H^2(c)$, we take a trajectory γ for $\mathbb{B}_k$ with $\gamma(0) = p, \dot\gamma(0) = u$ and $\nabla_{\dot\gamma}\dot\gamma(0) - |c|\gamma(0) = v$. If we regard it as a curve in $\mathbb{R}^3$, it is expressed as

$$\begin{aligned}\gamma(t) = \frac{1}{k^2-|c|}&\bigl(k^2 - |c|\cos\sqrt{k^2-|c|}\ t\bigr)p \\ &+ \frac{1}{k^2-|c|}\bigl(1-\cos\sqrt{k^2-|c|}\ t\bigr)v \\ &\quad + \frac{1}{\sqrt{k^2-|c|}}\bigl(\sin\sqrt{k^2-|c|}\ t\bigr)u.\end{aligned}$$

Also we take a geodesic σ on $H^2(c)$ with $\sigma(0) = p$ and $\dot\sigma(0) = u' \in T_pH^2 \subset T_p\mathbb{R}^3 \cong \mathbb{R}^3$, which is expressed as $\sigma(t) = \bigl(\cosh\sqrt{|c|}\ t\bigr)p + \dfrac{1}{\sqrt{|c|}}\bigl(\sinh\sqrt{|c|}\ t\bigr)u'$ as a curve in $\mathbb{R}^3$. We take an arbitrary positive r and suppose $\gamma(r) = \sigma(\ell_k(r;c))$. Since p is orthogonal to u, v and u', and p, u, v span $\mathbb{R}^3$, this equality shows

$$k^2 - |c|\cos\sqrt{k^2-|c|}\ r = (k^2-|c|)\cosh(\sqrt{|c|}\ell_k(r;c)), \tag{4.8}$$

$$\begin{aligned}\frac{1}{\sqrt{|c|}}\sinh\bigl(\sqrt{|c|}\ell_k(r;c)\bigr)u' = \frac{1}{k^2-|c|}&\bigl(1-\cos\sqrt{k^2-|c|}\ r\bigr)v \\ &+ \frac{1}{\sqrt{k^2-|c|}}\bigl(\sin\sqrt{k^2-|c|}\ r\bigr)u.\end{aligned} \tag{4.9}$$

Applying the double angle formula to (4.8), we have

$$\sqrt{k^2+c}\sinh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr) = \sqrt{|c|}\sin\bigl(\sqrt{k^2+c}\, r/2\bigr).$$

As u is orthogonal to v, $\langle u, u'\rangle = \delta_k(r;c)$ and $\|u\| = 1$, by taking inner products of both sides of (4.7) with u, we get

$$\begin{aligned}
\delta_k(r;c) &= \frac{\sqrt{|c|}\sin\sqrt{k^2-|c|}\, r}{\sqrt{k^2-|c|}\sinh\bigl(\sqrt{|c|}\ell_k(r;c)\bigr)} \\
&= \frac{\sqrt{|c|}\sin\bigl(\sqrt{k^2-|c|}\, r/2\bigr)\cos\bigl(\sqrt{k^2-|c|}\, r/2\bigr)}{\sqrt{k^2-|c|}\sinh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)\cosh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)} \\
&= \frac{\cos\bigl(\sqrt{k^2-|c|}\, r/2\bigr)}{\cosh\bigl(\sqrt{|c|}\ell_k(r;c)/2\bigr)} = \frac{\sqrt{k^2+c}\cos\bigl(\sqrt{k^2+c}\,/2\bigr)}{\sqrt{k^2+c\cos^2\bigl(\sqrt{k^2+c}\,r/2\bigr)}}.
\end{aligned}$$

This completes the proof. $\square$

We here show some properties of string-lengths and string-cosines of trajectory-harps on a complex hyperbolic space.

Lemma 4.6. *When $c < 0$, the functions $\ell_k(\cdot\ ;c)$ and $\delta_k(\cdot\ ;c)$ satisfy the following properties.*

(1) $\ell_k(t;c) = \ell_{-k}(t;c)$, $\delta_k(t;c) = \delta_{-k}(t;c)$;

(2) *The function $\ell_k(t;c) : \bigl[0, 2\pi/\sqrt{k^2+c}\,\bigr] \to \mathbb{R}$ is monotone increasing in the interval $\bigl[0, \pi/\sqrt{k^2+c}\,\bigr]$ and is monotone decreasing in the interval $\bigl[\pi/\sqrt{k^2+c}, 2\pi/\sqrt{k^2+c}\,\bigr]$. Here, when we regard $\pi/\sqrt{k^2+c}$ and $2\pi/\sqrt{k^2+c}$ as infinity, and do not consider the interval $[\pi/\sqrt{k^2+c}, 2\pi/\sqrt{k^2+c}]$;*

(3) *The function $\delta_k(t;c) : \bigl[0, 2\pi/\sqrt{k^2+c}\,\bigr] \to \mathbb{R}$ is monotone decreasing.*

Proof. The first assertion is trivial by their expressions in Proposition 4.6. Since we have

$$\frac{d}{dt}\ell_k(t;c) = \delta_k(t;c) = \begin{cases}
\dfrac{\sqrt{|c|-k^2}\cosh\bigl(\sqrt{|c|-k^2}\ t/2\bigr)}{\sqrt{|c|\cosh^2\bigl(\sqrt{|c|-k^2}\ t/2\bigr)-k^2}}, & \text{if } |k| < \sqrt{|c|}, \\[2ex]
\dfrac{2}{\sqrt{4+|c|t^2}}, & \text{if } |k| = \sqrt{|c|}, \\[2ex]
\dfrac{\sqrt{k^2+c}\ \cos\bigl(\sqrt{k^2+c}\ t/2\bigr)}{\sqrt{k^2+c\cos^2\bigl(\sqrt{k^2+c}\ t/2\bigr)}}, & \text{if } |k| > \sqrt{|c|},
\end{cases}$$

and

$$\frac{d}{dt}\delta_k(t;c) = \begin{cases} -\dfrac{k^2(|c|-k^2)\sinh\big(\sqrt{|c|-k^2}\,t/2\big)}{2\big(k^2+|c|\cosh^2(\sqrt{|c|-k^2}\,t/2)\big)^{3/2}}, & \text{if } |k|<\sqrt{|c|}, \\ -\dfrac{2|c|t}{(4+|c|t^2)^{3/2}}, & \text{if } |k|=\sqrt{|c|}, \\ -\dfrac{k^2(k^2+c)\sin(\sqrt{k^2+c}\,t/2)}{2\big(k^2+c\cos^2(\sqrt{k^2+c}\,t/2)\big)^{3/2}}, & \text{if } |k|>\sqrt{|c|}. \end{cases}$$

We get the conclusion. □

Proposition 4.7. *The string-lengths and the string-cosines on $\mathbb{C}H^n(c)$ satisfies the following*:

(1) $\ell_k(t;c)$ *is monotone decreasing with respect to* $|k|$;

(2) $\delta_k(t;c)$ *is monotone decreasing with respect to* $|k|$.

Proof. By the expressions of ℓ_k or δ_k, we are enough to study the case $k \geq 0$.

(1) When $|k| < \sqrt{|c|}$, the string-length $\ell_k(t;c)$ satisfies

$$\sqrt{|c|-k^2}\sinh\big(\sqrt{|c|}\ell_k(t;c)/2\big) = \sqrt{|c|}\sinh\big(\sqrt{|c|-k^2}\,t/2\big).$$

Differentiating both sides of this equality, we have

$$\frac{d}{dk}\ell_k(t;c) = \frac{tk\big(1-\cosh(\sqrt{|c|-k^2}\,t/2)\big)}{(|c|-k^2)\cosh\big(\sqrt{|c|}\ell_k(t;c)/2\big)} < 0.$$

Similarly, when $|k| > \sqrt{|c|}$, the string-length $\ell_k(t;c)$ satisfies

$$\sqrt{k^2+c}\sinh\big(\sqrt{|c|}\ell_k(t;c)/2\big) = \sqrt{|c|}\sin\big(\sqrt{k^2+c}\,t/2\big).$$

We hence have

$$\frac{d}{dk}\ell_k(t;c) = \frac{\cos\big(\sqrt{k^2+c}\,t/2\big)-1}{(k^2+c)\cosh(\sqrt{|c|}\ell_k(t;c)/2)} < 0.$$

Next we consider that the case $|k| = \sqrt{|c|}$. By de l'Hopital's rule, we have

$$\lim_{k\uparrow\sqrt{|c|}} \frac{\sinh\big(\sqrt{|c|-k^2}\,t/2\big)}{\sqrt{|c|-k^2}} = \lim_{k\uparrow\sqrt{|c|}} \frac{t}{2}\cosh\big(\sqrt{|c|-k^2}\,t/2\big) = \frac{t}{2}$$

$$\lim_{k\downarrow\sqrt{|c|}} \frac{\sin(\sqrt{k^2+c}\,t/2)}{\sqrt{k^2+c}} = \lim_{k\downarrow\sqrt{|c|}} \frac{t}{2}\cos\big(\sqrt{k^2+c}\,t/2\big) = \frac{t}{2}.$$

We find

$$\lim_{k\uparrow\sqrt{|c|}}\frac{2}{\sqrt{|c|}}\sinh^{-1}\left(\frac{\sqrt{|c|}\sinh\bigl(\sqrt{|c|-k^2}\,t/2\bigr)}{\sqrt{|c|-k^2}}\right)$$

$$=\lim_{k\downarrow\sqrt{|c|}}\frac{2}{\sqrt{|c|}}\sinh^{-1}\left(\frac{\sqrt{|c|}\sin\bigl(\sqrt{k^2+c}\,t/2\bigr)}{\sqrt{k^2+c}}\right)$$

$$=\frac{2}{\sqrt{|c|}}\sinh^{-1}\left(\frac{\sqrt{|c|}}{2}\,t\right),$$

we find that $\ell_k(t;c)$ is monotone decreasing with respect to k.

(2) When $|k|<\sqrt{|c|}$, the string-cosine expressed as

$$\delta_k(t;c)=\frac{\sqrt{|c|-k^2}\cosh\bigl(\sqrt{|c|-k^2}\;t/2\bigr)}{\sqrt{|c|\cosh^2\bigl(\sqrt{|c|-k^2}\;t/2\bigr)-k^2}}.$$

By direct computation, we have

$$\frac{d}{dk}\delta_k(t;c)=-\frac{k\sin\bigl(\sqrt{|c|-k^2}\;t/2\bigr)\Bigl(k^2\sqrt{|c|-k^2}\;t+|c|\sin\bigl(\sqrt{|c|-k^2}\;t\bigr)\Bigr)}{2\sqrt{|c|-k^2}\Bigl(k^2+|c|\cos\bigl(\sqrt{|c|-k^2}\;t/2\bigr)^2\Bigr)^{3/2}}<0.$$

Similarly, when $|k|>\sqrt{|c|}$, the string-cosine expressed as

$$\delta_k(t;c)=\frac{\sqrt{k^2+c}\cos\bigl(\sqrt{k^2+c}\;t/2\bigr)}{\sqrt{k^2+c\cos^2\bigl(\sqrt{k^2+c}\;t/2\bigr)}}$$

Then we have

$$\frac{d}{dk}\delta_k(t;c)=-\frac{k\sin\bigl(\sqrt{k^2+c}\;t/2\bigr)\Bigl(k^2\sqrt{k^2+c}\;t+c\sin\bigl(\sqrt{k^2+c}\;t\bigr)\Bigr)}{2\sqrt{k^2+c}\Bigl(k^2+c\cos\bigl(\sqrt{k^2+c}\;t/2\bigr)^2\Bigr)^{3/2}}<0.$$

Next we consider the case $|k|=\sqrt{|c|}$. Since we have

$$\frac{|c|-k^2}{|c|\cosh^2\bigl(\sqrt{|c|-k^2}\,t/2\bigr)-k^2}=\frac{|c|-k^2}{|c|\sinh^2\bigl(\sqrt{|c|-k^2}\,t/2\bigr)+|c|-k^2},$$

by applying de l'Hopital's rule, we have

$$\lim_{k\uparrow\sqrt{|c|}}\frac{|c|\sinh^2\bigl(\sqrt{|c|-k^2}\,t/2\bigr)}{|c|-k^2}=\lim_{k\uparrow\sqrt{|c|}}\frac{(|c|t/2)\sinh\bigl(\sqrt{|c|-k^2}\,t\bigr)}{2\sqrt{|c|-k^2}}$$

$$=\lim_{k\uparrow\sqrt{|c|}}\frac{|c|t^2\cosh\bigl(\sqrt{|c|-k^2}\,t\bigr)}{4}=\frac{|c|t^2}{4}.$$

Therefore we have

$$\begin{aligned}\lim_{k\uparrow\sqrt{|c|}}\delta_k(t;c) &= \lim_{k\uparrow\sqrt{|c|}}\frac{\sqrt{|c|-k^2}\cosh\bigl(\sqrt{|c|-k^2}\,t/2\bigr)}{\sqrt{|c|\cosh^2\bigl(\sqrt{|c|-k^2}\,t/2\bigr)-k^2}}\\ &= \frac{1}{\sqrt{(|c|t^2)/4+1}}\\ &= \frac{2}{\sqrt{|c|t^2+4}}.\end{aligned}$$

Similarly, as we have

$$\frac{k^2+c}{k^2+c\cos^2\bigl(\sqrt{k^2+c}\,t/2\bigr)} = \frac{k^2+c}{k^2+c-c\sin^2\bigl(\sqrt{k^2+c}\,t/2\bigr)},$$

by applying de l'Hopital's rule, we have

$$\begin{aligned}\lim_{k\downarrow\sqrt{|c|}}\frac{|c|\sin^2\bigl(\sqrt{k^2+c}\,t/2\bigr)}{k^2+c} &= \lim_{k\downarrow\sqrt{|c|}}\frac{(|c|t/2)\sin\bigl(\sqrt{k^2+c}\,t\bigr)}{2\sqrt{k^2+c}}\\ &= \lim_{k\downarrow\sqrt{|c|}}\frac{|c|t^2\cos\bigl(\sqrt{k^2+c}\,t\bigr)}{4} = \frac{|c|t^2}{4}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\lim_{k\downarrow\sqrt{|c|}}\delta_k(t;c) &= \lim_{k\downarrow\sqrt{|c|}}\frac{\sqrt{k^2+c}\cos\bigl(\sqrt{k^2+c}\,t/2\bigr)}{\sqrt{k^2+c\cos^2\bigl(\sqrt{k^2+c}\,t/2\bigr)}}\\ &= \frac{1}{\sqrt{(|c|t^2)/4+1}}\\ &= \frac{2}{\sqrt{|c|t^2+4}}.\end{aligned}$$

We find that $\delta_k(t;c)$ is monotone decreasing correspond to k. We get the conclusion.

□

Summarizing Propositions 4.2, 4.4 and 4.6 up we have the following.

Proposition 4.8. *For $0<t<2\pi/\sqrt{k^2+c}$, we have*

(1) $\mathfrak{s}_0\bigl(\ell_k(t;c)/2;c\bigr)=\mathfrak{s}_k(t/2;c)$.

(2) $\delta_k(t;c) = \pm\sqrt{\dfrac{1-(k^2+c)\mathfrak{s}_k(t/2;c)^2}{1-c\mathfrak{s}_k(t/2;c)^2}}$, *where the double sign takes positive when $0\le t\le \pi/\sqrt{k^2+c}$ and takes negative when $\pi/\sqrt{k^2+c}<t\le 2\pi/\sqrt{k^2+c}$.*

3. Comparison theorems on string-lengths and string-cosines of trajectory-harps

In this section we give estimates of string-lengths and of string-cosines of trajectory-harps on a general Kähler manifold M under some condition on sectional curvatures.

For a trajectory-harp α_γ associated with a trajectory $\gamma : [0,T] \to M$, we set

$$R_\gamma = \sup\{t \in [0,T] \mid \delta_\gamma(\tau) > 0 \text{ for } 0 < \tau < t\}$$

and call it the *maximal arch-length* of α_γ. On a complex space form $\mathbb{C}M^n(c)$, for a trajectory-harp α_γ associated with a trajectory $\gamma : [0,T] \to M$ for $\mathbb{B}_k$, the maximal arch-length is given as

$$R(k;c) = \begin{cases} \pi/\sqrt{k^2+c}, & \text{if } k^2+c>0, \\ \infty, & \text{if } k^2+c \le 0, \end{cases}$$

if $T \ge R(k;c)$.

We define a positive $T_\gamma(c)$ as follows. If there is t_* satisfying $0 < t_* \le T$ and $\ell_\gamma(t_*) = \ell_k\big(R(k;c);c\big)$ we set $T_\gamma(c) = \min\{t_*\}$ and in other case we set $T_\gamma(c) = T$. Since $\ell_\gamma(t) \le t$, we see $T_\gamma(c) \ge \min\{T, \ell_k\big(R(k;c);c\big)\}$. We note that

$$\ell_k\big(R(k;c);c\big) = \begin{cases} \big(2/\sqrt{c}\big)\sin^{-1}\sqrt{c/(k^2+c)}, & \text{if } c>0, \\ 2/|k|, & \text{if } c=0, \\ \big(2/\sqrt{|c|}\big)\sinh^{-1}\sqrt{|c|/(k^2+c)}, & \text{if } c<0 \text{ and } k^2+c>0, \\ \infty, & \text{if } c<0 \text{ and } k^2+c\le 0. \end{cases}$$

Given a trajectory-harp α_γ for $\mathbb{B}_k$ associated with a trajectory $\gamma : [0,T] \to M$, for $0 \le a < b \le T$, we set $\mathcal{HB}_\gamma(a,b) = \{\alpha_\gamma(t,s) | a \le t \le b, 0 \le s \le \ell_\gamma(t)\}$ and call it the *harp-body* of α_γ. We denote $\mathcal{HB}_\gamma(0,b)$ by $\mathcal{HB}_\gamma(b)$. By Lemmas 4.4, 4.5 and 4.6, the function $\ell_k(\cdot\,;c) : \big[0, \pi/\sqrt{k^2+c}\,\big] \to \mathbb{R}$ is monotone increasing. We hence define a function $\tau_k(\cdot\,;c) : \big[0, \ell_k(\pi/\sqrt{k^2+c};c)\,\big] \to \mathbb{R}$ as the inverse function of $\ell_k(\cdot\,;c)$.

First we give estimates from below when sectional curvatures are bounded from above.

THEOREM 4.2. *Let α_γ be a trajectory-harp associated with a trajectory $\gamma : [0,T] \to M$ for a non-trivial Kähler magnetic field $\mathbb{B}_k$ on a Kähler manifold M. Suppose sectional curvatures of planes tangent to its harp-body $\mathcal{HB}_\gamma(T)$ are not greater than a constant c. We then have the following :*

(1) $\ell_\gamma(t) \geq \ell_k(t;c)$ *for* $0 < t \leq \min\{R_\gamma, 2R(k;c)\}$.
(2) $\delta_\gamma(t) \geq \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$ *for* $0 < t \leq T_\gamma(c)$.

In particular, we have $R_\gamma \geq T_\gamma(c)$ and $R(k;c) \geq T_\gamma(c)$.

Moreover, we have the following when equalities hold in the above assertion.

(1) *If $\delta_\gamma(t_0) = \delta_k\big(\tau_k(\ell_\gamma(t_0);c);c\big)$ at some t_0 with $0 < t_0 \leq T_\gamma(c)$, then we have*
   1) *the derivatives of string-cosines satisfy $\delta'_\gamma(t_0) = \delta'_k\big(\tau_k(\ell_\gamma(t_0);c);c\big)$.*
   2) *the vector $\dfrac{\partial\alpha_\gamma}{\partial t}(t_0,s)$ is parallel to $J\dfrac{\partial\alpha_\gamma}{\partial s}(t_0,s)$ for $0 \leq s \leq \ell_\gamma(t_0)$.*
   3) *the sectional curvature $\mathrm{Riem}\Big(\dfrac{\partial\alpha_\gamma}{\partial t}(t_0,s), \dfrac{\partial\alpha_\gamma}{\partial s}(t_0,s)\Big)$ of the tangent plane spanned by $\dfrac{\partial\alpha_\gamma}{\partial t}(t_0,s)$ and $\dfrac{\partial\alpha_\gamma}{\partial s}(t_0,s)$ is equal to c for $0 < s \leq \ell_\gamma(t_0)$.*
(2) *If $\ell_\gamma(t_0) = \ell_k(t_0;c)$ at some t_0 with $0 < t_0 \leq T_\gamma(c)$, then the harp-body $\mathcal{HB}_\gamma(t_0)$ is totally geodesic, holomorphic and of constant sectional curvature c. In particular, we have $\ell_\gamma(t) = \ell_k(t;c)$ and $\delta_\gamma(t) = \delta_k(t;c)$ for $0 \leq t \leq t_0$.*

REMARK 4.1. We note that $\ell_k(t;c)$ is defined for $0 \leq t \leq 2R(k;c)$.

REMARK 4.2. It is likely that $R_\gamma \geq R(k;c)$ holds under the assumption of Theorem 4.2. But our result does not guarantees this.

To show this we need the following local estimates.

LEMMA 4.7. *Let α_γ be a trajectory-harp associated with a trajectory $\gamma : [0,T] \to M$ for a non-trivial Kähler magnetic field $\mathbb{B}_k$ on M. When we take a positive $\hat{k}$ satisfying $|k| < \hat{k}$, there exists positive ϵ such that the following properties hold for $0 < t \leq \epsilon$:*

$$\delta''_\gamma(t) < 0, \quad \delta'_{\hat{k}}(t;c) < \delta'_\gamma(t) < 0, \quad \delta_{\hat{k}}(t;c) < \delta_\gamma(t), \quad \ell_{\hat{k}}(t;c) < \ell_\gamma(t).$$

Proof. We define a smooth function F by $F(t) = \delta'_\gamma(t) - \delta'_{\hat{k}}(t;c)$. By Lemma 4.3 we have

$$F(0) = \delta'_\gamma(0) - \delta'_{\hat{k}}(0;c) = 0 - 0 = 0,$$
$$F'(0) = \delta''_\gamma(0) - \delta''_{\hat{k}}(0;c) = -k^2/4 - (-\hat{k}^2/4) = (\hat{k}^2 - k^2)/4 > 0.$$

If we apply Taylor's theorem to F, we see there exists a small positive ϵ_1 satisfying

$$F(t) = F(0) + F'(0)t + 0(t^2) > 0$$

for $0 < t \leq \epsilon_1$. We have $\delta'_\gamma(t) > \delta'_{\hat{k}}(t;c)$ for $0 < t \leq \epsilon_1$.

We define another function G by $G(t) = \delta_\gamma(t) - \delta_{\hat{k}}(t;c)$. We then have

$$G(t) = \int_0^t F(s)ds + G(0).$$

Since Lemma 4.3 guarantees

$$G(0) = \delta_\gamma(0) - \delta_{\hat{k}}(0;c) = 1 - 1 = 0,$$

we have $G(t) > 0$ for $0 < t \leq \epsilon_1$.

We define one more function H by $H(t) = \ell_\gamma(t) - \ell_{\hat{k}}(t;c)$. We then have

$$H(t) = \int_0^t G(s)ds + H(0)$$

by Lemma 4.2. As $H(0) = \ell_\gamma(0) - \ell_{\hat{k}}(0;c) = 0$, we have $H(t) > 0$ for $0 < t \leq \epsilon_1$.

As $\delta''_\gamma(0) = -k^2/4 < 0$ and δ_γ is smooth, there is a positive ϵ_2 satisfying that $\delta''_{\hat{k}}(t;c) < 0$ for $0 \leq t \leq \epsilon_2$. Since $\delta'_\gamma(0) = 0$ we see $\delta'_\gamma(t) < 0$ for $0 \leq t \leq \epsilon_2$. By choosing $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, we get the conclusion. $\square$

Proof of Theorem 4.2. We take a positive $\hat{k}$ so that $|k| < \hat{k}$. First we study near the origin. By Lemma 4.7 we see $\delta_{\hat{k}}(t;c) < \delta_\gamma(t)$ and $\ell_{\hat{k}}(t;c) < \ell_\gamma(t)$ for $0 < t \leq \epsilon$. Since $\tau_{\hat{k}}(\cdot\,;c)$ is monotone increasing, we have $\tau_{\hat{k}}\big(\ell_\gamma(t);c\big) > \tau_{\hat{k}}\big(\ell_{\hat{k}}(t;c);c\big) = t$. As $\delta_{\hat{k}}(\cdot;c)$ is monotone decreasing, we find $\delta_\gamma(t) > \delta_{\hat{k}}(t;c) > \delta_{\hat{k}}\big(\tau_{\hat{k}}(\ell_\gamma(t);c);c\big)$.

We take a maximal positive $T_{\hat{k}}$ $(\leq T)$ so that the conditions

i) $\ell_\gamma(t) \leq \ell_{\hat{k}}(R(\hat{k};c);c)$,

ii) $\delta_\gamma(t) \geq \delta_{\hat{k}}\big(\tau_{\hat{k}}(\ell_\gamma(t);c);c\big)$

hold for $0 \le t \le T_{\hat{k}}$. The above study guarantees that $T_{\hat{k}}$ is positive.

We shall show that if $T_{\hat{k}} < T$ then we have $\ell_\gamma(T_{\hat{k}}) = \ell_{\hat{k}}(R(\hat{k};c);c)$. To do this we suppose $T_{\hat{k}} < T$ and $\ell_\gamma(T_{\hat{k}}) < \ell_{\hat{k}}(R(\hat{k};c);c)$. As $\ell_\gamma, \delta_\gamma, \ell_{\hat{k}}(\cdot;c)$ and $\delta_{\hat{k}}(\cdot;c)$ are smooth, by the maximality of $T_{\hat{k}}$ we see $\delta_\gamma(T_{\hat{k}}) = \delta_{\hat{k}}\big(\tau_{\hat{k}}(\ell_\gamma(T_{\hat{k}};c);c);c\big)$. We study the derivative of δ_γ at $T_{\hat{k}}$. By direct computation we have

$$\frac{d\delta_\gamma}{dt}(T_{\hat{k}}) = k\Big\langle J\dot\gamma(T_{\hat{k}}), \frac{\partial\alpha_\gamma}{\partial s}\big(T_{\hat{k}}, \ell_\gamma(T_{\hat{k}})\big)\Big\rangle + \Big\langle \dot\gamma(T_{\hat{k}}), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}\Big)\big(T_{\hat{k}}, \ell_\gamma(T_{\hat{k}})\big)\Big\rangle. \tag{4.10}$$

By definition of δ_γ, we have

$$\begin{aligned}
1 &= \Big\|\frac{\partial\alpha_\gamma}{\partial s}\big(T_{\hat{k}}, \ell_\gamma(T_{\hat{k}})\big)\Big\|^2 \\
&\ge \Big\langle \dot\gamma(T_{\hat{k}}), \frac{\partial\alpha_\gamma}{\partial s}\big(T_{\hat{k}}, \ell_\gamma(T_{\hat{k}})\big)\Big\rangle^2 + \Big\langle J\dot\gamma(T_k), \frac{\partial\alpha_\gamma}{\partial s}\big(T_{\hat{k}}, \ell_\gamma(T_{\hat{k}})\big)\Big\rangle^2 \\
&= \big\{\delta_\gamma(T_{\hat{k}})\big\}^2 + \Big\langle J\dot\gamma(T_k), \frac{\partial\alpha_\gamma}{\partial s}\big(T_{\hat{k}}, \ell_\gamma(T_{\hat{k}})\big)\Big\rangle^2,
\end{aligned}$$

hence the first term of the right-hand side of (4.10) satisfies

$$k\Big\langle J\dot\gamma(T_{\hat{k}}), \frac{\partial\alpha_\gamma}{\partial s}\big(T_{\hat{k}}, \ell_\gamma(T_{\hat{k}})\big)\Big\rangle \ge -|k|\sqrt{1-\{\delta_\gamma(T_{\hat{k}})\}^2}.$$

As $|k| < \hat{k}$ and $\delta_\gamma(T_{\hat{k}}) = \delta_{\hat{k}}\big(\tau_{\hat{k}}(\ell_\gamma(T_{\hat{k}});c);c\big) < 1$, we see

$$k\Big\langle J\dot\gamma(T_{\hat{k}}), \frac{\partial\alpha_\gamma}{\partial s}\big(T_{\hat{k}}, \ell_\gamma(T_{\hat{k}})\big)\Big\rangle > -\hat{k}\sqrt{1-\big\{\delta_{\hat{k}}\big(\tau_{\hat{k}}(\ell_\gamma(T_{\hat{k}});c);c\big)\big\}^2}.$$

In order to estimate the second term of the right-hand side of (4.10), we put $Z_t(s) = \dfrac{\partial\alpha_\gamma}{\partial t}(t,s)$, which is a Jacobi field along a geodesic $s \mapsto \alpha_\gamma(t,s)$. As $s \mapsto \alpha_\gamma(t,s)$ is of unit speed for each t, it satisfies $\Big\langle Z_t(s), \dfrac{\partial\alpha_\gamma}{\partial s}(t,s)\Big\rangle = 0$ for each t. This guarantees that $\Big\langle \nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_t(s), \dfrac{\partial\alpha_\gamma}{\partial s}(t,s)\Big\rangle = 0$ because

$$\begin{aligned}
0 &= \frac{d}{ds}\Big\langle Z_t(s), \frac{\partial\alpha_\gamma}{\partial s}(t,s)\Big\rangle = \Big\langle \nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_t(s), \frac{\partial\alpha_\gamma}{\partial s}(t,s)\Big\rangle + \Big\langle Z_t(s), \Big(\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}\frac{\partial\alpha_\gamma}{\partial s}\Big)(t,s)\Big\rangle \\
&= \Big\langle \nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_t(s), \frac{\partial\alpha_\gamma}{\partial s}(t,s)\Big\rangle.
\end{aligned}$$

We take a trajectory-harp $\hat\alpha_{\hat\gamma}$ associated with a trajectory $\hat\gamma : \big[0, R(\hat{k};c)\big] \to \mathbb{C}M^1(c)$ for $\mathbb{B}_{\hat{k}}$ on $\mathbb{C}M^1(c)$, and put $\widehat{Z}_t(s) = \dfrac{\partial\hat\alpha_{\hat\gamma}}{\partial t}(t,s)$, which is a Jacobi field along a geodesic $s \mapsto \hat\alpha_{\hat\gamma}(t,s)$. We note that $\mathbb{C}M^1(c)$ is congruent to a real space form of constant sectional curvature c (see Propositions 2.1, 2.2). Since we have $\ell_{\hat{k}}\big(T_{\hat{k}};c\big) < T_{\hat{k}} \le R(k;c) <$

$\pi/\sqrt{c}$, we do not have conjugate points of $\hat{\gamma}(0)$ along the geodesic $s \mapsto \alpha_{\hat{\gamma}}(T_{\hat{k}}, s)$. As $\gamma(t) = \alpha_\gamma(t, \ell_\gamma(t))$, we have

$$\dot{\gamma}(t) = \frac{\partial \alpha_\gamma}{\alpha t}\big(t, \ell_\gamma(t)\big) + \delta_\gamma(t)\frac{\partial \alpha_\gamma}{\partial s}\big(t, \ell_\gamma(t)\big) = Z_t\big(\ell_\gamma(t)\big) + \delta_\gamma(t)\frac{\partial \alpha_\gamma}{\partial s}\big(t, \ell_\gamma(t)\big).$$

By applying Rauch's comparison theorem on Jacobi fields, we have

$$\begin{aligned}
\Big\langle \dot{\gamma}(T_{\hat{k}}), \Big(\nabla_{\frac{\partial}{\partial t}} \frac{\partial \alpha_\gamma}{\partial s}\Big)\big(T_{\hat{k}}, \ell_\gamma(T_{\hat{k}})\big)\Big\rangle &= \Big\langle \dot{\gamma}(T_{\hat{k}}), \Big(\nabla_{\frac{\partial}{\partial s}} \frac{\partial \alpha_\gamma}{\partial t}\Big)\big(T_{\hat{k}}, \ell_\gamma(T_{\hat{k}})\big)\Big\rangle \\
&= \Big\langle Z_{T_{\hat{k}}}\big(\ell_\gamma(T_{\hat{k}})\big) + \delta_\gamma(T_{\hat{k}})\frac{\partial \alpha_\gamma}{\partial s}\big(T_{\hat{k}}, \ell_\gamma(T_{\hat{k}})\big), \nabla_{\frac{\partial}{\partial s}} Z_{T_{\hat{k}}}\big(\ell_\gamma(T_{\hat{k}})\big)\Big\rangle \\
&= \Big\langle Z_{T_{\hat{k}}}\big(\ell_\gamma(T_{\hat{k}})\big), \nabla_{\frac{\partial \alpha_\gamma}{\partial s}} Z_{T_{\hat{k}}}\big(\ell_\gamma(T_{\hat{k}})\big)\Big\rangle \\
&= \|Z_{T_{\hat{k}}}\big(\ell_\gamma(T_{\hat{k}})\big)\|^2 \times \frac{\big\langle Z_{T_{\hat{k}}}\big(\ell_\gamma(T_{\hat{k}})\big), \nabla_{\frac{\partial \alpha_\gamma}{\partial s}} Z_{T_{\hat{k}}}\big(\ell_\gamma(T_{\hat{k}})\big)\big\rangle}{\|Z_{T_{\hat{k}}}\big(\ell_\gamma(T_{\hat{k}})\big)\|^2} \\
&\geq \|Z_{T_{\hat{k}}}\big(\ell_\gamma(T_{\hat{k}})\big)\|^2 \times \frac{\big\langle \widehat{Z}_{\tau_{\hat{k}}(\ell_\gamma(T_{\hat{k}});c)}\big(\ell_\gamma(T_{\hat{k}})\big), \nabla_{\frac{\partial \alpha_{\hat{\gamma}}}{\partial s}} \widehat{Z}_{\tau_{\hat{k}}(\ell_\gamma(T_{\hat{k}});c)}\big(\ell_\gamma(T_{\hat{k}})\big)\big\rangle}{\|\widehat{Z}_{\tau_{\hat{k}}(\ell_\gamma(T_{\hat{k}});c)}\big(\ell_\gamma(T_{\hat{k}})\big)\|^2} \\
&= \frac{\|Z_{T_{\hat{k}}}\big(\ell_\gamma(T_{\hat{k}})\big)\|^2}{\|\widehat{Z}_{\tau_{\hat{k}}(\ell_\gamma(T_{\hat{k}});c)}\big(\ell_\gamma(T_{\hat{k}})\big)\|^2} \\
&\qquad \times \Big\langle \widehat{Z}_{\tau_{\hat{k}}(\ell_\gamma(T_{\hat{k}});c)}\big(\ell_\gamma(T_{\hat{k}})\big), \nabla_{\frac{\partial \hat{\alpha}_{\hat{\gamma}}}{\partial s}} \widehat{Z}_{\tau_{\hat{k}}(\ell_\gamma(T_{\hat{k}});c)}\big(\ell_\gamma(T_{\hat{k}})\big)\Big\rangle.
\end{aligned}$$

As we have $Z_t(\ell_\gamma(t)) = \dot{\gamma}(t) - \delta_\gamma(t)\frac{\partial \alpha_\gamma}{\partial s}\big(t, \ell_\gamma(t)\big)$, the Jacobi field Z_t satisfies $Z_t(0) = 0$ and $\|Z_t\big(\ell_\gamma(t)\big)\|^2 = 1 - \{\delta_\gamma(t)\}^2$, because we have

$$\begin{aligned}
1 = \|\dot{\gamma}(t)\|^2 &= \Big\| Z_t\big(\ell_\gamma(t)\big) + \delta_\gamma(t)\frac{\partial \alpha_\gamma}{\partial s}(t, \ell_\gamma(t))\Big\|^2 \\
&= \|Z_t\big(\ell_\gamma(t)\big)\|^2 + 2\delta_\gamma(t)\Big\langle Z_t\big(\ell_\gamma(t)\big), \frac{\partial \alpha_\gamma}{\partial s}(t, \ell_\gamma(t))\Big\rangle + \delta_\gamma^2(t)\,\Big\|\frac{\partial \alpha_\gamma}{\partial s}\big(t, \ell_\gamma(t)\big)\Big\|^2 \\
&= \|Z_t\big(\ell_\gamma(t)\big)\|^2 + \delta_\gamma^2(t).
\end{aligned}$$

By same computation we have $\big\|\widehat{Z}_t\big(\ell_{\hat{k}}(t;c)\big)\big\|^2 = 1 - \{\delta_{\hat{k}}(t;c)\}^2$.

As $\delta_\gamma\big(T_{\hat{k}}\big) = \delta_{\hat{k}}\big(\tau_{\hat{k}}(\ell_\gamma(T_{\hat{k}}); c); c\big)$, we have $\big\|\widehat{Z}_{T_{\hat{k}}}\big(\ell_{\hat{k}}(T_{\hat{k}}; c)\big)\big\| = \big\|Z_{\tau_\gamma(\ell_{\hat{k}}(T_{\hat{k}};c))}\big(\ell_{\hat{k}}(T_{\hat{k}}; c)\big)\big\|$. Continuing our computation on the second term of the right-hand side of (4.10), we have

$$\begin{aligned}\Big\langle \dot\gamma(T_{\hat k}), \nabla_{\frac{\partial}{\partial t}} \frac{\partial \alpha_\gamma}{\partial s}\big(T_{\hat k}, \ell_\gamma(T_{\hat k})\big)\Big\rangle &\ge \Big\langle \widehat{Z}_{\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)}\big(\ell_\gamma(T_{\hat k})\big), \nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}} \widehat{Z}_{\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)}\big(\ell_\gamma(T_{\hat k})\big)\Big\rangle \\ &= \Big\langle \dot{\hat\gamma}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)\big), \nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}} \frac{\partial\hat\alpha_{\hat\gamma}}{\partial t}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c), \ell_\gamma(T_{\hat k})\big)\Big\rangle \\ &= \Big\langle \dot{\hat\gamma}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)\big), \nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial t}} \frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c), \ell_\gamma(T_{\hat k})\big)\Big\rangle.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\frac{d\delta_\gamma}{dt}(T_{\hat k}) &> -\hat k\sqrt{1-\{\delta_{\hat k}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)\big)\}^2} \\ &\qquad + \Big\langle \dot{\hat\gamma}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)\big), \nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial t}} \frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c), \ell_\gamma(T_{\hat k})\big)\Big\rangle \\ &= \hat k\Big\langle J\dot{\hat\gamma}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)\big), \frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c), \ell_\gamma(T_{\hat k})\big)\Big\rangle \\ &\qquad + \Big\langle \dot{\hat\gamma}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)\big), \nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial t}} \frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c), \ell_\gamma(T_{\hat k})\big)\Big\rangle \\ &= \frac{d}{dt}\delta_{\hat k}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c);c\big) = \frac{d}{du}\delta_{\hat k}\big(\tau_{\hat k}(\ell_\gamma(u);c);c\big)\Big|_{u=T_{\hat k}}.\end{aligned}$$

By the maximality of $T_{\hat k}$, we find that it is a contradiction, because we have $\delta_\gamma(T_{\hat k}) > \delta_{\hat k}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c);c\big)$ for $0 \le T_{\hat k} < \epsilon_3$. Thus, we find that if $T_{\hat k} < T$ then $\ell_\gamma(T_{\hat k}) = \ell_{\hat k}(R(\hat k;c);c)$ holds. This means that either $T_{\hat k} = T$ holds or $T_{\hat k} < T$ and $\ell_\gamma(T_{\hat k}) = \ell_{\hat k}(R(\hat k;c)\,;c)$ holds.

We take a monotone decreasing sequence $\{\hat k_j\}_{j=1}^\infty$ with $\hat k_j > |k|$ and $\lim_{j\to\infty}\hat k_j = |k|$. Taking a subsequence, if we need, the above argument shows that one of the following conditions holds for all j:

1) $T_{\hat k_j} = T$,
2) $T_{\hat k_j} < T$ and $\ell_\gamma(T_{\hat k_j}) = \ell_{\hat k_j}(R(\hat k_j;c)\,;c)$.

In the first case it is clear that $\lim_{j\to\infty} T_{\hat k_j} = T$. By definition of $T_{\hat k}$ we find

$$\ell_\gamma(T) \le \lim_{j\to\infty} \ell_{\hat k_j}(R(\hat k_j;c);c) = \ell_{|k|}(R(|k|;c);c) = \ell_k(R(k;c);c),$$

and have $T_\gamma(c) = T$. In the second case, as we have $\ell'_\gamma(t) = \delta_\gamma(t) > 0$ in the interior of $\cup_j[0, T_{\hat k_j}]$ because $\delta_{\hat k}\big(\tau_{\hat k}(\ell_\gamma(t);c);c\big) > 0$, the function ℓ_γ is monotone increasing on this domain. Since the sequence $\{\ell_{\hat k_j}(R(\hat k_j;c);c)\}_{j=1}^\infty$ is monotone increasing by Lemmas 4.4, 4.5, 4.6 and Propositions 4.3, 4.5, 4.7. Because we have $\ell_\gamma(T_{\hat k_j}) = \ell_{\hat k_j}(R(\hat k_j;c);c)$, we see the sequence $\{\ell_\gamma(T_{\hat{k}_j})\}_{j=1}^\infty$ is monotone increasing. As ℓ_γ is monotone increasing on $\cup_j[0,T_{\hat{k}_j}]$, we find that $\{T_{\hat{k}_j}\}_{j=1}^\infty$ is a monotone increasing sequence. Hence $\lim_{j\to\infty} T_{\hat{k}_j}$ exists including the case $\lim_{j\to\infty} T_{\hat{k}_j} = \infty$. We set $\lim_{j\to\infty} T_{\hat{k}_j} = T_*$. We then have

$$\ell_\gamma(T_*) = \lim_{j\to\infty} \ell_\gamma(T_{\hat{k}_j}) = \lim_{j\to\infty} \ell_{\hat{k}_j}(R(\hat{k}_j;c);c) = \ell_{|k|}(R(|k|;c):c),$$

which shows $T_* = T_\gamma(c)$. In each case, as $\lim_{\hat{k}\to|k|} \delta_{\hat{k}}\big(\tau_{\hat{k}}(\ell_\gamma(t);c);c\big)= \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$ for each t, we obtain $\delta_\gamma(t) \ge \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$ for $0 \le t \le T_\gamma(c)$. Since $\delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$ > 0 for $0 \le t < T_\gamma(c)$, we see $R_\gamma \ge T_\gamma(c)$.

We next compare $\ell_\gamma(t)$ and $\ell_k(t;c)$. For a positive $\hat{k}$ with $|k| < \hat{k}$ we take a maximal positive $S_{\hat{k}} \le \min\{T, R(\hat{k};c)\}$ satisfying $\ell_\gamma(t) \ge \ell_{\hat{k}}(t;c)$ for $0 < t \le S_{\hat{k}}$. We shall show that $S_{\hat{k}} = \min\{T, R(\hat{k};c)\}$. If we suppose $S_{\hat{k}} < \min\{T, R(\hat{k};c)\}$, by the maximality of $S_{\hat{k}}$, we get $\ell_\gamma(S_{\hat{k}}) = \ell_{\hat{k}}(S_{\hat{k}};c)$. Since $\ell_{\hat{k}}(S_{\hat{k}};c) \le \ell_{\hat{k}}(R(\hat{k};c);c) < \ell_k(R(k;c);c)$, we find $S_{\hat{k}} < T_\gamma(c)$. Hence by the above argument on string-cosines, we have $\delta_\gamma(S_{\hat{k}}) \ge \delta_k\big(\tau_k(\ell_\gamma(S_{\hat{k}});c);c\big)$. By Propositions 4.3, 4.5 and 4.7, we have

$$\begin{aligned}\delta_\gamma(S_{\hat{k}}) &\ge \delta_k\big(\tau_k(\ell_\gamma(S_{\hat{k}});c);c\big)= \delta_k\big(\tau_k(\ell_{\hat{k}}(S_{\hat{k}};c);c);c\big)\\ &\ge \delta_k\big(\tau_k(\ell_k(S_{\hat{k}};c);c);c\big)= \delta_k(S_{\hat{k}};c) > \delta_{\hat{k}}(S_{\hat{k}};c).\end{aligned}$$

By the maximality of $S_{\hat{k}}$, we see that it is a contradiction. We hence have $S_{\hat{k}} = \min\{T, R(\hat{k};c)\}$. We take a monotone decreasing sequence $\{\hat{k}_j\}_{j=1}^\infty$ satisfying $\hat{k}_j > |k|$ and $\lim_{j\to\infty} \hat{k}_j = |k|$. Since $\lim_{j\to\infty} R(\hat{k}_j;c) = R(k;c)$, when $T < R(k;c)$ we may suppose $T < R(\hat{k}_j;c)$. Thus we have $S_{\hat{k}_j} = T$ in this case. As $\lim_{j\to\infty} \ell_{\hat{k}_j}(t;c) = \ell_k(t;c)$, we have $\ell_\gamma(t) \ge \ell_k(t;c)$ for $0 \le t \le T$ in this case. When $T \ge R(k;c)$, as $R(k;c) > R(\hat{k}_j;c)$, we have $S_{\hat{k}_j} = R(\hat{k}_j;c)$. In this case, we obtain $\ell_\gamma(t) \ge \ell_k(t;c)$ for $0 \le t \le R(k;c) = \lim_{j\to\infty} R(\hat{k}_j;c)$. In this case, if $R(k;c) < R_\gamma \le 2R(k;c)$, then for $R(k;c) < t \le R_\gamma$ we have $\ell_\gamma(t) > \ell_\gamma(R(\hat{k};c)) \ge \ell_k(R(k;c);c) \ge \ell_k(t;c)$.

We now check the relationship between $R(k;c)$ and $T_\gamma(c)$. When $T \ge R(k;c)$ we have $\ell_\gamma(t) \ge \ell_k(t;c)$ for $0 \le t \le R(k;c)$. Thus we have $T_\gamma(c) \le R(k;c)$.

Next we study the case that equalities hold. First we study the case that $\delta_\gamma(t_0) = \delta_k\big(\tau_k(\ell_\gamma(t_0);c);c\big)$ holds. Along the same lines as above estimate on the derivative of

δ_γ, by taking a trajectory-harp $\hat\alpha_{\hat\gamma}$ associated with a trajectory $\hat\gamma$ for $\mathbb{B}_k$ on $\mathbb{C}M^n(c)$ and by changing $T_{\hat k}$ to t_0, we have

$$\begin{aligned}\frac{d\delta_\gamma}{dt}(t_0) &= k\Big\langle J\dot\gamma(t_0), \frac{\partial\alpha_\gamma}{\partial s}\big(t_0, \ell_\gamma(t_0)\big)\Big\rangle + \Big\langle \dot\gamma(t_0), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}\Big)\big(t_0, \ell_\gamma(t_0)\big)\Big\rangle \\ &\geq -|k|\sqrt{1-\delta_\gamma(t_0)^2} \\ &\qquad + \Big\langle \dot{\hat\gamma}\big(\tau_{\hat k}(\ell_\gamma(t_0);c)\big), \Big(\nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial t}}\frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}\Big)\big(\tau_k(\ell_\gamma(t_0);c), \ell_\gamma(t_0)\big)\Big\rangle \\ &= \frac{d\delta_k}{dt}\big(\tau_k(\ell_\gamma(t_0);c);c\big) = \frac{d\delta_k}{du}\big(\tau_k(\ell_\gamma(u);c);c\big)\Big|_{u=t_0}.\end{aligned} \tag{4.11}$$

If we suppose $\frac{d\delta_\gamma}{dt}(t_0) > \frac{d\delta_k}{du}\big(\tau_k(\ell_\gamma(u);c);c\big)\Big|_{u=t_0}$, we have

$$\frac{d\delta_\gamma}{dt}(t) > \frac{d\delta_k}{du}\big(\tau_k(\ell_\gamma(u);c);c\big)\Big|_{u=t}$$

for $t_0-\varepsilon \leq t \leq t_0$ with some positive ε. As $\delta_\gamma(t) \geq \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$, we have

$$\begin{aligned}\delta_\gamma(t_0) &= \int_{t_0-\varepsilon}^{t_0} \frac{d\delta_\gamma}{dt}(t)\,dt + \delta_\gamma(t_0-\varepsilon) > \int_{t_0-\varepsilon}^{t_0} \frac{d\delta_k}{du}\big(\tau_k(\ell_\gamma(u);c);c\big)\,du + \delta_\gamma(t_0-\varepsilon) \\ &\geq \int_{t_0-\varepsilon}^{t_0} \frac{d\delta_k}{du}\big(\tau_k(\ell_\gamma(u);c);c\big)\,du + \delta_k\big(\tau_k(\ell_\gamma(t_0-\varepsilon);c);c\big) = \delta_k\big(\tau_k(\ell_\gamma(t_0);c);c\big).\end{aligned}$$

Hence we see $\frac{d\delta_\gamma}{dt}(t_0) = \frac{d\delta_k}{du}\big(\tau_k(\ell_\gamma(u);c);c\big)\Big|_{u=t_0}$. Thus, we find that the equality holds in the inequality in (4.11). This means that both of the following equalities hold:

$$k\Big\langle J\dot\gamma(t_0), \frac{\partial\alpha_\gamma}{\partial s}\big(t_0, \ell_\gamma(t_0)\big)\Big\rangle = -|k|\sqrt{1-\delta_\gamma(t_0)^2}. \tag{4.12}$$

$$\begin{aligned}&\Big\langle \dot\gamma(t_0), \Big(\nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}\Big)\big(t_0, \ell_\gamma(t_0)\big)\Big\rangle \\ &\qquad = \Big\langle \dot{\hat\gamma}\big(\tau_{\hat k}(\ell_\gamma(t_0);c)\big), \Big(\nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial t}}\frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}\Big)\big(\tau_k(\ell_\gamma(t_0);c), \ell_\gamma(t_0)\big)\Big\rangle.\end{aligned} \tag{4.13}$$

The equality (4.12) shows that $\frac{\partial\alpha_\gamma}{\partial s}\big(t_0, \ell_\gamma(t_0)\big)$ is contained in the complex line in $T_{\gamma(t_0)}M$ spanned by $\dot\gamma(t_0)$. Since $\dot\gamma(t_0) = Z_{t_0}\big(\ell_\gamma(t_0)\big) + \delta_\gamma(t_0)\frac{\partial\alpha_\gamma}{\partial s}\big(t_0, \ell_\gamma(t_0)\big)$ with the Jacobi field $Z_{t_0}(s) = \frac{\partial\alpha_\gamma}{\partial t}(t_0, s)$, this means that $Z_{t_0}\big(\ell_\gamma(t_0)\big)$ is parallel to $J\frac{\partial\alpha_\gamma}{\partial s}\big(t_0, \ell_\gamma(t_0)\big)$ because $Z_{t_0}\big(\ell_\gamma(t_0)\big)$ is orthogonal to $\frac{\partial\alpha_\gamma}{\partial s}\big(t_0, \ell_\gamma(t_0)\big)$. As we used Rauch's comparison theorem to obtain (4.11), the equality (4.13) and Rauch's comparison theorem (Theorem 3.1) gurantee that $Z_{t_0}(s)/\|Z_{t_0}(s)\|$ is parallel along the geodesic $s \mapsto \alpha_\gamma(t_0, s)$ for $0 \le s \le \ell_\gamma(t_0)$ and that $\mathrm{Riem}\Big(\dfrac{\partial\alpha_\gamma}{\partial t}(t_0, s), \dfrac{\partial\alpha_\gamma}{\partial s}(t_0, s)\Big) \equiv c$ for $0 \le s \le \ell_\gamma(t_0)$.

We finally study the case that $\ell_\gamma(t_0) = \ell_k(t_0; c)$ holds. We have $t_0 = \tau_k\big(\ell_\gamma(t_0); c\big)$. As we have $\delta_\gamma(t) \ge \delta_k\big(\tau_k(\ell_\gamma(t); c); c\big)$ for $0 \le t \le T_\gamma(c)$, we find

$$t_0 = \int_0^{t_0} \frac{d}{dt}\tau_k\big(\ell_\gamma(t); c\big)\, dt = \int_0^{t_0} \frac{\delta_\gamma(t)}{\delta_k\big(\tau_k(\ell_\gamma(t); c); c\big)}\, dt \ge \int_0^{t_0} dt = t_0.$$

Thus we obtain $\delta_\gamma(t) = \delta_k\big(\tau_k(\ell_\gamma(t); c); c\big)$ for $0 \le t \le t_0$. Hence we find by the above argument that $\mathrm{Riem}\Big(\dfrac{\partial\alpha_\gamma}{\partial t}(t, s), \dfrac{\partial\alpha_\gamma}{\partial s}(t, s)\Big) \equiv c$ and that $\dfrac{\partial\alpha_\gamma}{\partial t}(t, s) = \psi(t, s) J \dfrac{\partial\alpha_\gamma}{\partial s}(t, s)$ for $0 \le s \le \ell_\gamma(t)$ and $0 \le t \le t_0$ with a smooth function ψ. We then obtain

$$\begin{aligned}
&\nabla_{\frac{\partial\alpha_\gamma}{\partial s}} \frac{\partial\alpha_\gamma}{\partial s} = 0,\\
&\nabla_{\frac{\partial\alpha_\gamma}{\partial s}} \frac{\partial\alpha_\gamma}{\partial t} = \frac{\partial\psi}{\partial s} J \frac{\partial\alpha_\gamma}{\partial s},\\
&\nabla_{\frac{\partial\alpha_\gamma}{\partial t}} \frac{\partial\alpha_\gamma}{\partial t} = \frac{\partial\psi}{\partial t} J \frac{\partial\alpha_\gamma}{\partial s} + \psi J \nabla_{\frac{\partial\alpha_\gamma}{\partial s}} \frac{\partial\alpha_\gamma}{\partial t} = \frac{\partial\psi}{\partial t} J \frac{\partial\alpha_\gamma}{\partial s} - \psi \frac{\partial\psi}{\partial s} \frac{\partial\alpha_\gamma}{\partial s}.
\end{aligned}$$

Hence $\mathcal{HB}_\gamma(t_0)$ is totally geodesic. This completes the proof. □

Next we give estimates from above under a condition that sectional curvatures of underlying manifolds are bounded from below along the same lines as in the proof of Theorem 4.2. For a trajectory-harp α_γ associated with a trajectory $\gamma : [0, T] \to M$, by putting $p = \gamma(0)$ we set

$$C_\gamma = \sup\{t \in [0, T] \mid \ell_\gamma(\tau) \le c_0^{\mathcal{HB}_\gamma(T)}(p) \text{ for } 0 \le \tau \le t\},$$

where $c_0^{\mathcal{HB}_\gamma}(p)$ denotes the minimum of first conjugate values of p along geodesics $\{s \mapsto \alpha_\gamma(t, s) \mid 0 < t \le T\}$. We say α_γ is *holomorphic at its arch* if $\dfrac{\partial\alpha_\gamma}{\partial s}\big(t, \ell_\gamma(t)\big)$ is contained in the complex line spanned by $\dot\gamma(t)$ for $0 < t \le R_\gamma$. When M is an orientable Riemann surface, by regarding it as a Kähler manifold, we see every trajectory-harp is holomorphic at its arch.

Theorem 4.3. *Let α_γ be a trajectory-harp associated with a trajectory $\gamma : [0, T] \to M$ for a non-trivial Kähler magnetic field $\mathbb{B}_k$ on a Kähler manifold M. Suppose that* *it is holomorphic at its arch and sectional curvatures tangent to its harp-body $\mathcal{HB}_\gamma(T)$ are not smaller than a constant c. We then have*

$$\ell_\gamma(t) \le \ell_k(t;c) \quad \text{and} \quad \delta_\gamma(t) \le \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$$

for $0 < t \le \min\{C_\gamma,\ R_\gamma\}$.

Moreover, we have the following if equalities hold in the above estimates.

(1) *If $\delta_\gamma(t_0) = \delta_k\big(\tau_k(\ell_\gamma(t_0);c);c\big)$ holds at some t_0 with $0 < t_0 \le \min\{R_\gamma, C_\gamma\}$, then we have the following:*

a) *the derivatives of string-cosine satisfy $\delta'_\gamma(t_0) = \delta'_k\big(\tau_k(\ell_\gamma(t_0);c);c\big)$;*

b) *The vector $\dfrac{\partial\alpha_\gamma}{\partial t}(t_0,s)$ is parallel to $J\dfrac{\partial\alpha_\gamma}{\partial s}(t_0,s)$ for $0 \le s \le \ell_\gamma(t_0)$;*

c) *The sectional curvature of the tangent plane spanned by $\dfrac{\partial\alpha_\gamma}{\partial t}(t_0,s)$ and $\dfrac{\partial\alpha_\gamma}{\partial s}(t_0,s)$ is equal to c for $0 \le s \le \ell_\gamma(t_0)$.*

(2) *If $\ell_\gamma(t_0) = \ell_k(t_0;c)$ holds at t_0 with $0 < t_0 \le \min\{R_\gamma, C_\gamma\}$, then the harp-body $\mathcal{HB}_\gamma(t_0)$ is totally geodesic, holomorphic and of constant sectional curvature c. In particular, $\ell_\gamma(t) = \ell_k(t;c)$ and $\delta_\gamma(t) = \delta_k(t_0;c)$ hold for $0 \le t \le t_0$.*

To show this we need the following local estimations.

Lemma 4.8. *For a trajectory-harp α_γ associated with a trajectory $\gamma : [0,T] \to M$ for a non-trivial Kähler magnetic field $\mathbb{B}_k$ on M, and for a positive $\hat{k}$ satisfying $|k| > \hat{k}$, there exists sufficiently small positive ϵ such that the following properties hold for $0 \le t \le \epsilon$:*

$$\delta''_{\hat{k}}(t;c) < 0, \quad \delta'_\gamma(t) < \delta'_{\hat{k}}(t;c) < 0, \quad \delta_\gamma(t) < \delta_{\hat{k}}(t;c), \quad \ell_\gamma(t) < \ell_{\hat{k}}(t;c).$$

Proof. We define a smooth function F by $F(t) = \delta'_{\hat{k}}(t;c) - \delta'_\gamma(t)$. By Lemma 4.3 we have

$$F'(0) = \delta'_{\hat{k}}(0;c) - \delta'_\gamma(0) = 0 - 0 = 0,$$

$$F''(0) = \delta''_{\hat{k}}(0;c) - \delta''_\gamma(0) = -\hat{k}^2/4 - (-k^2/4) = (k^2 - \hat{k}^2)/4 > 0.$$

If we apply Taylor's theorem to F, we see that there exists a small positive ϵ_1 satisfying

$$F(t) = F(0) + F'(0)t + 0(t^2) > 0$$

for $0 < t \le \epsilon_1$. We hence have $\delta'_{\hat{k}}(t;c) > \delta'_\gamma(t)$ for $0 < t \le \epsilon_1$.

We define another function G by $G(t) = \delta_{\hat{k}}(t;c) - \delta_\gamma(t)$. We then have

$$G(t) = \int_0^t F(s)ds + G(0).$$

Since Lemma 4.3 guarantees

$$G(0) = \delta_{\hat{k}}(0;c) - \delta_\gamma(0) = 1 - 1 = 0,$$

hence we have $G(t) > 0$ for $0 < t \le \epsilon_1$.

We define one more function H by $H(t) = \ell_{\hat{k}}(t;c) - \ell_\gamma(t)$. We then have

$$H(t) = \int_0^t G(s)ds + H(0)$$

by Lemma 4.2. As $H(0) = \ell_{\hat{k}}(0;c) - \ell_\gamma(0) = 0$, we have $H(t) > 0$ for $0 < t \le \epsilon_1$.

As $\delta''_k(0) = -\hat{k}^2/4$ and $\delta_{\hat{k}}(\cdot\,;c)$ is smooth, there is a positive ϵ_2 satisfying that $\delta''_{\hat{k}}(t;c) < 0$ for $0 \le t \le \epsilon_2$. By choosing $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, we get the conclusion. □

Proof of Theorem 4.3. We take a positive $\hat{k}$ so that $0 < \hat{k} < |k|$. We estimate δ_γ and ℓ_γ by $\delta_{\hat{k}}(t;c)$ and $\ell_{\hat{k}}(t;c)$ from above, respectively.

First we study near the origin. By Lemma 4.8, we have $\ell_\gamma(t) < \ell_{\hat{k}}(t;c)$ and $\delta_\gamma(t) < \delta_{\hat{k}}(t;c)$ for $0 < t < \epsilon$. Since $\tau_k(\cdot\,;c)$ is monotone increasing, we have $\tau_{\hat{k}}\big(\ell_\gamma(t);c\big) < \tau_{\hat{k}}\big(\ell_{\hat{k}}(t;c);c\big) = t$ for $0 < t < \epsilon$. As $\delta_{\hat{k}}(\cdot\,;c)$ is monotone decreasing, we find $\delta_\gamma(t) < \delta_{\hat{k}}(t;c) < \delta_{\hat{k}}\big(\tau_{\hat{k}}(\ell_\gamma(t);c);c\big)$ for $0 < t < \epsilon$.

We take a maximal positive $T_{\hat{k}}\big(\le \min\{R_\gamma, C_\gamma\}\big)$ so that the following conditions

i) $\ell_\gamma(t) \le \ell_{\hat{k}}\big(R(\hat{k};c);c\big)$,

ii) $\delta_\gamma(t) \le \delta_{\hat{k}}\big(\tau_{\hat{k}}(\ell_\gamma(t);c);c\big)$

hold for $0 \le t \le T_{\hat{k}}$. The above argument guarantees that $T_{\hat{k}}$ is positive. We show $T_{\hat{k}} = \min\{R_\gamma, C_\gamma\}$. To do this we shall show that if we suppose $T_{\hat{k}} < \min\{R_\gamma, C_\gamma\}$ then $\ell_\gamma(t) = \ell_{\hat{k}}\big(R(\hat{k};c);c\big)$ holds. We here suppose that both $T_{\hat{k}} < \min\{R_\gamma, C_\gamma\}$ and $\ell_\gamma(t) < \ell_{\hat{k}}\big(R(\hat{k};c);c\big)$ hold. By maximality of $T_{\hat{k}}$, we have $\delta_\gamma\big(T_{\hat{k}}\big) = \delta_{\hat{k}}\big(\tau_{\hat{k}}(\ell_\gamma(T_{\hat{k}});c);c\big)$. We compute the derivative of δ_γ at $T_{\hat{k}}$:

$$\frac{d\delta_\gamma}{dt}(T_{\hat{k}}) = k\Big\langle J\dot{\gamma}(T_{\hat{k}}), \frac{\partial \alpha_\gamma}{\partial s}\big(T_{\hat{k}}, \ell_\gamma(T_{\hat{k}})\big)\Big\rangle + \Big\langle \dot{\gamma}(T_{\hat{k}}), \nabla_{\frac{\partial}{\partial t}} \frac{\partial \alpha_\gamma}{\partial s}\big(T_{\hat{k}}, \ell_\gamma(T_{\hat{k}})\big)\Big\rangle. \tag{4.14}$$

Since α_γ is holomorphic at its arch, by definition of $\delta_{\hat\gamma}(t)$, the first term of the right-hand side of (4.14) is estimated as

$$\Big\langle J\dot\gamma(T_{\hat k}), \frac{\partial\alpha_\gamma}{\partial s}\big(T_{\hat k}, \ell_\gamma(T_{\hat k})\big)\Big\rangle = -|k|\sqrt{1-\{\delta_\gamma(T_{\hat k})\}^2} < -\hat k\sqrt{1-\{\delta_\gamma(T_{\hat k})\}^2}$$

because $0<\hat k<|k|$.

In order to estimate the second term of (4.14), we take a trajectory-harp $\hat\alpha_{\hat\gamma}$ associated with a trajectory $\hat\gamma : \big[0, R(\hat k; c)\big] \to \mathbb{C}M^1(c)$ for $\mathbb{B}_{\hat k}$ on $\mathbb{C}M^1(c)$. We set $Z_t(s) = \dfrac{\partial\alpha_\gamma}{\partial t}(t,s)$ and $\widehat Z_t(s) = \dfrac{\partial\hat\alpha_{\hat\gamma}}{\partial t}(t,s)$, which are Jacobi fields along geodesics $s\mapsto\alpha_\gamma(t,s)$ and $s\mapsto\hat\alpha_{\hat\gamma}(t,s)$, respectively. Since each geodesic $s\mapsto\alpha_\gamma(t,s)$ is of unit speed, we see $\Big\langle Z_t(s), \dfrac{\partial\alpha}{\partial s}(t,s)\Big\rangle = 0$. As we have $\gamma(t) = \alpha_\gamma\big(t,\ell_\gamma(t)\big)$, we have $Z_t(\ell_\gamma(t)) = \dot\gamma(t) - \delta_\gamma(t)\dfrac{\partial\alpha_\gamma}{\partial s}\big(t,\ell_\gamma(t)\big)$. Thus the Jacobi field satisfies $Z_t(0)=0$ and $\|Z_t\big(\ell_\gamma(t)\big)\|^2 = 1-\{\delta_\gamma(t)\}^2$ because we have

$$\begin{aligned}
1 = \|\dot\gamma(t)\|^2 &= \|Z_t\big(\ell_\gamma(t)\big) + \delta_\gamma(t)\frac{\partial\alpha_\gamma}{\partial s}(t,\ell_\gamma(t))\|^2\\
&= \|Z_t\big(\ell_\gamma(t)\big)\|^2 + \{\delta_\gamma(t)\}^2\Big\|\frac{\partial\alpha_\gamma}{\partial s}\big(t,\ell_\gamma(t)\big)\Big\|^2\\
&\qquad + 2\delta_\gamma(t)\Big\langle Z_t\big(\ell_\gamma(t)\big), \frac{\partial\alpha_\gamma}{\partial s}(t,\ell_\gamma(t))\Big\rangle\\
&= \|Z_t\big(\ell_\gamma(t)\big)\|^2 + \{\delta_\gamma(t)\}^2.
\end{aligned}$$

Similarly, the Jacobi field $\widehat Z_t$ satisfies $\widehat Z_t(0) = 0$ and $\big\|\widehat Z_t\big(\ell_{\hat k}(t;c)\big)\big\|^2 = 1-\{\delta_{\hat k}(t;c)\}^2$. As $\delta_\gamma\big(T_{\hat k}\big) = \delta_{\hat k}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c);c\big)$, we have $\|Z_t\big(\ell_\gamma(T_{\hat k})\big)\| = \big\|\widehat Z_{\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)}\big(\ell_\gamma(T_{\hat k})\big)\big\|$. Since $T_{\hat k}\le C_\gamma$, by Rauch's comparison theorem, we have $\ell_\gamma\big(T_{\hat k}\big) < c_0^{\mathcal{HB}_\gamma(T)}(\gamma(0)) \le \pi/\sqrt{c}$, thus we do not have conjugate points of $\gamma(0)$ along the geodesic $s\mapsto\alpha_\gamma(T_{\hat k},s)$. By applying Rauch's comparison theorem on Jacobi fields, we have

$$\begin{aligned}
&\Big\langle \dot\gamma(T_{\hat k}), \nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}\big(T_{\hat k},\ell_\gamma(T_{\hat k})\big)\Big\rangle\\
&\quad = \Big\langle Z_{T_{\hat k}}\big(\ell_\gamma(T_{\hat k})\big) + \delta_\gamma(T_{\hat k})\frac{\partial\alpha_\gamma}{\partial s}\big(T_{\hat k},\ell_\gamma(T_{\hat k})\big), \nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_{T_{\hat k}}\big(\ell_\gamma(T_{\hat k})\big)\Big\rangle\\
&\quad = \Big\langle Z_{T_{\hat k}}\big(\ell_\gamma(T_{\hat k})\big), \nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_{T_{\hat k}}\big(\ell_\gamma(T_{\hat k})\big)\Big\rangle\\
&\quad = \big\|Z_{T_{\hat k}}\big(\ell_\gamma(T_{\hat k})\big)\big\|^2 \times \frac{\big\langle Z_{T_{\hat k}}\big(\ell_\gamma(T_{\hat k})\big), \nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_{T_{\hat k}}\big(\ell_\gamma(T_{\hat k})\big)\big\rangle}{\|Z_{T_{\hat k}}\big(\ell_\gamma(T_{\hat k})\big)\|^2}
\end{aligned}$$

$$
\begin{aligned}
&\le \big\|Z_{T_{\hat k}}\big(\ell_\gamma(T_{\hat k})\big)\big\|^2 \times \frac{\big\langle \widehat Z_{\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)}\big(\ell_\gamma(T_{\hat k})\big), \nabla_{\frac{\partial \hat\alpha_{\hat\gamma}}{\partial s}} \widehat Z_{\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)}\big(\ell_\gamma(T_{\hat k})\big)\big\rangle}{\|\widehat Z_{\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)}\big(\ell_\gamma(T_{\hat k})\big)\|^2} \\
&= \Big\langle \widehat Z_{\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)}\big(\ell_\gamma(T_{\hat k})\big), \nabla_{\frac{\partial \hat\alpha_{\hat\gamma}}{\partial s}} \widehat Z_{\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)}\big(\ell_\gamma(T_{\hat k})\big)\Big\rangle \\
&= \Big\langle \dot{\hat\gamma}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)\big), \nabla_{\frac{\partial \hat\alpha_{\hat\gamma}}{\partial s}} \frac{\partial \hat\alpha_{\hat\gamma}}{\partial t}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c), \ell_\gamma(T_{\hat k})\big)\Big\rangle \\
&= \Big\langle \dot{\hat\gamma}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c)\big), \nabla_{\frac{\partial \hat\alpha_{\hat\gamma}}{\partial t}} \frac{\partial \hat\alpha_{\hat\gamma}}{\partial s}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c), \ell_\gamma(T_{\hat k})\big)\Big\rangle.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\frac{d\delta_\gamma}{dt}(T_{\hat k}) &< -\hat k\sqrt{1-\{\delta_{\hat k}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c);c\big)\}^2} \\
&\qquad + \Big\langle \dot{\hat\gamma}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k}))\big), \nabla_{\frac{\partial \hat\alpha_{\hat\gamma}}{\partial t}} \frac{\partial \hat\alpha_{\hat\gamma}}{\partial s}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c), \ell_\gamma(T_{\hat k})\big)\Big\rangle \\
&= \frac{d\delta_{\hat k}}{dt}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c);c\big) = \frac{d\delta_{\hat k}}{du}\big(\tau_{\hat k}(\ell_\gamma(u);c);c\big)\Big|_{u=T_{\hat k}}.
\end{aligned}
$$

As we supposed $\ell_\gamma(T_{\hat k}) < \ell_{\hat k}\big(R(\hat k;c);c\big)$, by the maximality of $T_{\hat k}$, we find that it is a contradiction. Thus, if we suppose $T_{\hat k} < \min\{R_\gamma, C_\gamma\}$, then $\ell_\gamma(T_{\hat k}) = \ell_{\hat k}\big(R(\hat k;c);c\big)$. But this shows

$$
\delta_\gamma(T_{\hat k}) \le \delta_{\hat k}\big(\tau_{\hat k}(\ell_\gamma(T_{\hat k});c);c\big) = \delta_{\hat k}(R(\hat k;c);c) = 0,
$$

which tells us $T_{\hat k} \ge R_\gamma$. Thus it is again contradict to $T_{\hat k} < \min\{R_\gamma, C_\gamma\}$. We hence find that $T_{\hat k} = \min\{R_\gamma, C_\gamma\}$.

Let $\{\hat k_j\}_{j=1}^\infty$ be a monotone increasing sequence of positive constants satisfying $\hat k_j < |k|$ and $\lim_{j\to\infty} \hat k_j = |k|$. Since $\lim_{j\to\infty} \delta_{\hat k_j}\big(\tau_{\hat k_j}(\ell_\gamma(t);c);c\big) = \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$, we have $\delta_\gamma(t) \le \delta_{\hat k}\big(\tau_{\hat k}(\ell_\gamma(t);c);c)\big)$ for $0 \le t \le \min\{R_\gamma, C_\gamma\}$.

We next compare $\ell_\gamma(t)$ and $\ell_k(t;c)$. For $\hat k$ satisfying $0 < \hat k < |k|$ we take a maximal positive $S_{\hat k}\big(\le \min\{R_\gamma, C_\gamma\}\big)$ which satisfies $\ell_\gamma(t) \le \ell_{\hat k}(t;c)$ for $0 \le t \le S_{\hat k}$. We shall show that $S_{\hat k} = \min\{R_\gamma, C_\gamma\}$. To do this we suppose $S_{\hat k} < \min\{R_\gamma, C_\gamma\}$. We then have $\ell_\gamma(S_{\hat k}) = \ell_{\hat k}(S_{\hat k};c)$. Therefore, by Proposition 4.7, we find

$$
\begin{aligned}
\delta_\gamma(S_{\hat k}) &\le \delta_k\big(\tau_k(\ell_\gamma(S_{\hat k});c);c\big) < \delta_{\hat k}\big(\tau_k(\ell_\gamma(S_{\hat k});c);c\big) \\
&< \delta_{\hat k}\big(\tau_{\hat k}(\ell_\gamma(S_{\hat k});c);c\big) = \delta_{\hat k}\big(\tau_{\hat k}(\ell_{\hat k}(S_{\hat k};c);c);c\big) = \delta_{\hat k}(S_{\hat k};c).
\end{aligned}
$$

By the maximality of $S_{\hat k}$, also we find a contradiction. Thus $S_{\hat k} = \min\{R_\gamma, C_\gamma\}$. As $\lim_{\hat k\to|k|} \ell_{\hat k}(t;c) = \ell_k(t;c)$, we get $\ell_\gamma(t) \le \ell_k(t;c)$ for $0 \le t \le \min\{R_\gamma, C_\gamma\}$.

We here consider the case that $\delta_\gamma(t_0) = \delta_k\big(\tau_k(\ell_\gamma(t_0);c);c\big)$ holds. Along the same lines as in the above, by taking a trajectory-harp $\hat\alpha_{\hat\gamma}$ associated with a trajectory $\hat\gamma$ for $\mathbb{B}_k$ on $\mathbb{C}M^1(c)$, we have

$$
\begin{aligned}
\frac{d\delta_\gamma}{dt}(t_0) &= -|k|\sqrt{1-\delta_\gamma(t_0)^2} + \Big\langle Z_{t_0}\big(\ell_\gamma(t_0)\big), \nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_{t_0}\big(\ell_\gamma(t_0)\big)\Big\rangle \\
&\le -|k|\sqrt{1-\delta_k\big(\tau_k(\ell_\gamma(t_0);c);c\big)^2} \\
&\qquad + \Big\langle \widehat{Z}_{\tau_k(\ell_\gamma(t_0);c)}\big(\ell_\gamma(t_0)\big), \nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}} \widehat{Z}_{\tau_k(\ell_\gamma(t_0);c)}\big(\ell_\gamma(t_0)\big)\Big\rangle \\
&= \frac{d}{du}\delta_k\big(\tau_k(\ell_\gamma(u);c);c\big)\Big|_{u=t_0},
\end{aligned}
\tag{4.15}
$$

where $\widehat{Z}_t(s) = \dfrac{\partial\hat\alpha_{\hat\gamma}}{\partial t}(t,s)$. If we suppose $\dfrac{d\delta_\gamma}{dt}(t_0) < \dfrac{d\delta_k}{du}\big(\tau_k(\ell_\gamma(u);c);c\big)\big|_{u=t_0}$, then there is a positive ε satisfying that $\dfrac{d\delta_\gamma}{dt}(t) < \dfrac{d\delta_k}{du}\big(\tau_k(\ell_\gamma(u);c);c\big)\big|_{u=t}$ for $t_0-\varepsilon \le t \le t_0$. As we have $\delta_\gamma(t) \le \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$, we have

$$
\begin{aligned}
\delta_\gamma(t_0) &= \int_{t_0-\varepsilon}^{t_0} \frac{d\delta_\gamma}{dt}(t)\,dt + \delta_\gamma(t_0-\varepsilon) < \int_{t_0-\varepsilon}^{t_0} \frac{d\delta_k}{dt}\big(\tau_k(\ell_\gamma(t);c);c\big)\,dt + \delta_\gamma(t_0-\varepsilon) \\
&\le \int_{t_0-\varepsilon}^{t_0} \frac{d\delta_k}{dt}\big(\tau_k(\ell_\gamma(t);c);c\big)\,dt + \delta_k\big(\tau_k(\ell_\gamma(t_0-\varepsilon);c);c\big) = \delta_k\big(\tau_k(\ell_\gamma(t_0);c);c\big)
\end{aligned}
$$

Thus, we see $\dfrac{d\delta_\gamma}{dt}(t_0) = \dfrac{d\delta_k}{du}\big(\tau_k(\ell_\gamma(u);c);c\big)\big|_{u=t_0}$. Therefore (4.15) shows that

$$
\Big\langle Z_{t_0}\big(\ell_\gamma(t_0)\big), \nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_{t_0}\big(\ell_\gamma(t_0)\big)\Big\rangle = \Big\langle \widehat{Z}_{\tau_k(\ell_\gamma(t_0);c)}\big(\ell_\gamma(t_0)\big), \nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}} \widehat{Z}_{\tau_k(\ell_\gamma(t_0);c)}\big(\ell_\gamma(t_0)\big)\Big\rangle.
$$

Rauch's comparison theorem guarantees that $Z_{t_0}/\|Z_{t_0}\|$ is parallel along $s \mapsto \alpha_\gamma(t_0,s)$ and $\mathrm{Riem}\big(Z_{t_0}(s), \frac{\partial\alpha_\gamma}{\partial s}(t_0,s)\big) \equiv c$ for $0 \le s \le \ell_\gamma(t_0)$. Since α_γ is holomorphic at its arch, $Z_{t_0}\big(\ell_\gamma(t_0)\big)$ is contained in the complex line spanned by $\dot\gamma(t_0)$. Hence $Z_{t_0}\big(\ell_\gamma(t_0)\big)$ is parallel to $J\dfrac{\partial\alpha_\gamma}{\partial s}\big(t_0,\ell_\gamma(t_0)\big)$.

We finally study the case that $\ell_\gamma(t_0) = \ell_k(t_0;c)$ holds. By this condition we have $\tau_k\big(\ell_\gamma(t_0);c\big) = \tau_k\big(\ell_k(t_0;c);c\big) = t_0$. As $0 < \delta_\gamma(t) \le \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$ for $0 \le t \le t_0$, we have

$$
t_0 = \int_0^{t_0} \frac{d}{dt}\tau_k\big(\ell_\gamma(t);c\big)dt = \int_0^{t_0} \frac{\delta_\gamma(t)}{\delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)}dt \le \int_0^{t_0} dt = t_0.
$$

Hence we find $\delta_\gamma(t) = \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$ for $0 \le t \le t_0$. Thus we obtain that $\dfrac{\partial\alpha_\gamma}{\partial t}(t,s)$ is parallel to $J\dfrac{\partial\alpha_\gamma}{\partial s}(t,s)$ and $\mathrm{Riem}\Big(\dfrac{\partial\alpha_\gamma}{\partial t}(t,s), \dfrac{\partial\alpha_\gamma}{\partial s}(t,s)\Big) \equiv c$ for $0 \le t \le t_0$ and

$0 \leq s \leq \ell_\gamma(t)$. Since $s \mapsto \alpha_\gamma(t,s)$ is a geodesic of unit speed for each t, we have $\dfrac{\partial \alpha_\gamma}{\partial t}(t,s) = \psi(t,s) J \dfrac{\partial \alpha_\gamma}{\partial s}(t,s)$ with a smooth function ψ. We hence find

$$\begin{aligned}
&\nabla_{\frac{\partial \alpha_\gamma}{\partial s}} \frac{\partial \alpha_\gamma}{\partial s} = 0, \\
&\nabla_{\frac{\partial \alpha_\gamma}{\partial s}} \frac{\partial \alpha_\gamma}{\partial t} = \frac{\partial \psi}{\partial s} J \frac{\partial \alpha_\gamma}{\partial s}, \\
&\nabla_{\frac{\partial \alpha_\gamma}{\partial t}} \frac{\partial \alpha_\gamma}{\partial t} = \frac{\partial \psi}{\partial t} J \frac{\partial \alpha_\gamma}{\partial s} + \psi J \nabla_{\frac{\partial \alpha_\gamma}{\partial s}} \frac{\partial \alpha_\gamma}{\partial t} = \frac{\partial \psi}{\partial t} J \frac{\partial \alpha_\gamma}{\partial s} - \psi \frac{\partial \psi}{\partial s} \frac{\partial \alpha_\gamma}{\partial s}.
\end{aligned}$$

Hence $\mathcal{HB}_\gamma(t_0)$ is totally geodesic. $\square$

4. Volumes of trajectory-balls

As an application of comparison theorems on string-lengths of trajectory-harps, we give estimates on volumes of trajectory-balls. By using magnetic exponential map $\mathbb{B}_k\mathrm{exp}_p : T_pM \to M$ defined in §3.1, we set

$$B_r^k(p) = \big\{\mathbb{B}_k\mathrm{exp}_p(tv) \bigm| 0 \le t < r, v \in U_pM\big\}$$

and call it a *trajectory-ball* of arc-radius r centered at p. Since $\mathbb{B}_0\mathrm{exp}_p$ is the ordinary exponential map $\exp_p$, we see that $B_r^0(p)$ is a geodesic-ball of radius r.

At an arbitrary point $p \in M$, we define the $\mathbb{B}_k$-*injectivity radius* $\iota_k(p)$ at p by

$$\iota_k(p) = \sup\big\{r > 0 \bigm| \mathbb{B}_k\mathrm{exp}_p|_{B_r(0_p)} \text{ is injective}\big\},$$

where $B_r(0_p)$ $\big(\subset T_pM\big)$ is a ball of radius r centered at the origin 0_p of $T_pM \cong \mathbb{R}^{2n}$. Clearly, $\iota_0(p)$ is the ordinary injectivity radius at p. For a positive c and a constant k, we defined in §3.2 a function $\mathfrak{s}_k(t;c) : \mathbb{R} \to \mathbb{R}$ by

$$\mathfrak{s}_k(t;c) = \begin{cases} (1/\sqrt{k^2+c})\sin(\sqrt{k^2+c}\,t), & \text{when } k^2+c>0,\\ t, & \text{when } k^2+c=0,\\ (1/\sqrt{|c|-k^2})\sinh(\sqrt{|c|-k^2}\,t), & \text{when } k^2+c<0. \end{cases}$$

In order to simplify the expression of our computation, we put its derivative as $\mathfrak{c}_k(t;c)$, that is, we set a function $\mathfrak{c}_k(t;c) : \mathbb{R} \to \mathbb{R}$ by

$$\mathfrak{c}_k(t;c) = \begin{cases} \cos(\sqrt{k^2+c}\,t), & \text{when } k^2+c>0,\\ 1, & \text{when } k^2+c=0,\\ \cosh(\sqrt{|c|-k^2}\,t), & \text{when } k^2+c<0. \end{cases}$$

These functions satisfy the relation

$$(k^2+c)\{\mathfrak{s}_k(t;c)\}^2 + \{\mathfrak{c}_k(t;c)\}^2 = 1.$$

We note also

$$\mathfrak{s}_0(t;c) = \begin{cases} (1/\sqrt{c})\sin(\sqrt{c}t), & \text{when } c>0,\\ t, & \text{when } c=0,\\ (1/\sqrt{|c|})\sinh(\sqrt{|c|}t), & \text{when } c<0, \end{cases}$$

and

$$\mathfrak{c}_0(t;c) = \begin{cases} \cos(\sqrt{c}t), & \text{when } c > 0, \\ 1, & \text{when } c = 0, \\ \cosh(\sqrt{|c|}t), & \text{when } c < 0. \end{cases}$$

Therefore, these functions satisfy the following.

LEMMA 4.9. *The functions $\mathfrak{s}_0(t;c)$ and $\mathfrak{c}_0(t;c)$ satisfy*

$$\mathfrak{s}_0(t;c) = 2\mathfrak{s}_0(t/2;c)\mathfrak{c}_0(t/2;c),$$

for $0 < t < \pi/\sqrt{k^2+c}$.

For geodesic-balls we have following estimates on their volumes studied by Bishop.

THEOREM 4.4 (Bishop's comparison theorem 1). *Let M be a Riemannian manifold of dimension m. For an arbitrary unit tangent vector $u \in UM$, we take a geodesic σ_u with $\dot\sigma_u(0) = u$. If sectional curvatures satisfy $\max\{\langle R(\dot\sigma(t), v)v, \dot\sigma(t)\rangle | v \in U_{\sigma(t)}M, v^\perp\dot\sigma(0)\} \le c$ with some constant c for $0 \le t < c_\sigma(\sigma(0))$, where $c_\sigma(\sigma(0))$ is the first conjugate value along σ. Then we have*

(1) *the function $t \mapsto \Theta(t,u)/\{\mathfrak{s}_0(t,c)\}^{m-1}$ is monotone decreasing for $0 \le t \le c_\sigma(\sigma(0))$.*
(2) *$\Theta(t,u) \ge \{\mathfrak{s}_0(t,c)\}^{m-1}$ for $0 \le t \le c_\sigma(\sigma(0))$.*

In particular, the volume of geodesic-ball $B_\ell(p)$ of radius ℓ is estimated as

$$\mathrm{vol}\big(B_\ell(p)\big) \ge \omega_{m-1}\int_0^\ell \{\mathfrak{s}_0(s;c)\}^{m-1} ds,$$

where ω_{m-1} denotes the volume of a standard unit sphere $S^{m-1}(1)$.

THEOREM 4.5 (Bishop's comparison theorem 2). *Let M be a Riemannian manifold of dimension m. For an arbitrary unit tangent vector $u \in UM$, we take a geodesic σ_u with $\dot\sigma_u(0) = u$. If sectional curvatures satisfy $\min\{\langle R(\dot\sigma(t), v)v, \dot\sigma(t)\rangle | v \in U_{\sigma(t)}M, v^\perp\dot\sigma(0)\} \ge c$ with some constant c for $0 \le t < c_\sigma(\sigma(0))$. Then we have*

(1) *the function $t \mapsto \Theta(t,u)/\{\mathfrak{s}_0(t,c)\}^{m-1}$ is monotone increasing for $0 \le t \le c_\sigma(\sigma(0))$.*

(2) $\Theta(t,u) \le \{\mathfrak{s}_0(t,c)\}^{m-1}$ *for* $0 \le t \le c_\sigma(\sigma(0))$.

In particular, the volume of geodesic-ball $B_\ell(p)$ *of radius* ℓ *is estimated as*

$$\mathrm{vol}\big(B_\ell(p)\big) \le \omega_{m-1}\int_0^\ell \{\mathfrak{s}_0(s;c)\}^{m-1}ds.$$

For a trajectory-harp α_γ associated with a trajectory $\gamma : [0,T] \to M$ for $\mathbb{B}_k$, we set

$$I_\gamma^k = \sup\{t \mid \ell_\gamma(\tau) \le \iota_0(\gamma(0)) \text{ for } 0 \le \tau \le t\},$$

and set $I_k(p) = \inf I_\gamma^k$, where infimum is taken over the set of all trajectory-harps associated with trajectories emanating from p. When $r \le \iota_0(p)$, for each trajectory $\gamma : [0,r] \to M$ with $\gamma(0) = p$ we can take a trajectory-harp associated with γ and $\ell_\gamma(\tau) \le \tau \le r$ for $0 \le \tau \le r$, we see $I_k(p) \ge \iota_0(p)$. We set $R_k(p) = \inf R_\gamma$, where infimum is taken over the set of all trajectory-harps associated with trajectories emanating from p. When sectional curvatures of M satisfy $\mathrm{Riem}^M \le c$ with some constant c, then we have $R_k(p) \ge \ell_k(R(k;c);c)$ by Theorem 4.2.

THEOREM 4.6. *Let* M *be a Kähler manifold of complex dimension* n *whose sectional curvatures satisfy* $\mathrm{Riem}^{\mathrm{M}} \le \mathrm{c}$ *with some constant* c. *For an arbitrary* r *with* $0 < r \le \min\{c_0(p), I_k(p), R_k(p)\}$, *we have*

$$\mathrm{vol}\big(B_r^k(p)\big) \ge \omega_{2n-1}\int_0^r \big\{2\mathfrak{s}_k(t/2;c)\big\}^{2n-1}\big\{1 - c\,\mathfrak{s}_k^2(t/2;c)\big\}^{n-1}\mathfrak{c}_k(t/2;c)\,dt.$$

PROOF. For an arbitrary point $q \in B_r^k(p)$ we have a trajectory $\gamma : [0,r] \to M$ satisfying $\gamma(0) = p$ and $\gamma(t_q) = q$ with some t_q with $0 \le t_q < r$. Theorem 4.2 guarantees that a trajectory-ball $B_\gamma^k(p)$ contains a geodesic-ball $B_{\ell_k(r;c)}(p)$. As $r \le \iota_k(p)$, we see $\ell_k(r;c) \le \iota_0(p)$. Hence we find that

$$\mathrm{vol}\big(B_r^k(p)\big) \ge \mathrm{vol}\big(B_{\ell_k(r;c)}(p)\big).$$

By applying Bishop's comparison Theorem 4.4 with $\ell = \ell_k(t;c)$, we obtain

$$\mathrm{vol}\big(B_r^k(p)\big) \ge \mathrm{vol}\big(B_{\ell_k(r;c)}(p)\big) \ge \omega_{2n-1}\int_0^{\ell_k(r;c)} \big\{\mathfrak{s}_0(s;c)\big\}^{2n-1}ds.$$

By Lemma 4.9, we have

$$\begin{aligned}
\omega_{2n-1}&\int_0^{\ell_k(r;c)}\big\{\mathfrak{s}_0(s;c)\big\}^{2n-1}ds\\
&=\omega_{2n-1}\int_0^{\ell_k(r;c)}\big\{2\mathfrak{s}_0(s/2;c)\cdot\mathfrak{c}_0(s/2;c)\big\}^{2n-1}ds\\
&=\omega_{2n-1}\int_0^{\ell_k(r;c)}\big\{2\mathfrak{s}_0(s/2;c)\big\}^{2n-1}\times\big\{1-c\mathfrak{s}_0(s/2;c)^2\big\}^{\frac{2n-1}{2}}ds\\
&=\omega_{2n-1}\int_0^{r}\big\{2\mathfrak{s}_0\big(\ell_k(t;c)/2;c\big)\big\}^{2n-1}\Big\{1-c\big\{\mathfrak{s}_0\big(\ell_k(t;c)/2;c\big)\big\}^2\Big\}^{\frac{2n-1}{2}}\frac{d\ell_k}{dt}dt
\end{aligned}$$

By Proposition 4.8, we have

$$\begin{aligned}
&=\omega_{2n-1}\int_0^{r}\big\{2\mathfrak{s}_k(t/2;c)\big\}^{2n-1}\Big\{1-c\big\{\mathfrak{s}_k(t/2;c)\big\}^2\Big\}^{\frac{2n-1}{2}}\sqrt{\frac{1-(k^2+c)\mathfrak{s}_k^2(t/2;c)}{1-c\mathfrak{s}_k^2(t/2;c)}}dt\\
&=\omega_{2n-1}\int_0^{r}\big\{2\mathfrak{s}_k(t/2;c)\big\}^{2n-1}\Big\{1-c\mathfrak{s}_k^2(t/2;c)\Big\}^{n-1}\Big\{1-(k^2+c)\mathfrak{s}_k^2(t/2;c)\Big\}^{1/2}dt.
\end{aligned}$$

Thus we get the conclusion. □

When M is compact, we can give another estimate on volumes of trajectory-balls by making use of the following Gromov's comparison theorem on volumes of geodesic balls.

THEOREM 4.7 (Gromov). *Let M^m be a complete Riemannian manifold whose Ricci curvatures satisfy* $\mathrm{Ricci}^M\geq(m-1)c$. *Then for arbitrary positive r,R with $r<R$, we have*

$$\mathrm{vol}\big(B_R(p)\big)/\mathrm{vol}\big(B_r(p)\big)\leq\int_0^R\big\{\mathfrak{s}_0(s;c)\big\}^{m-1}ds\Big/\int_0^r\big\{\mathfrak{s}_0(s;c)\big\}^{m-1}ds.$$

When $\sup\{d(p,q)|p,q\in M\}<\infty$, we call this constant the *diameter* of M.

THEOREM 4.8. *Let M be a compact Kähler manifold of diameter R and of complex dimension n. Suppose its sectional curvatures satisfy $c_1\leq\mathrm{Riem}^M\leq c_2$ with some constants c_1, c_2. Then at an arbitrary point $p\in M$, for an arbitrary r with $0<r\leq\max\{c_0(p),I_k(p),R_k(p)\}$, the volume of a trajectory-ball $B_r^k(p)$ of arc-radius r for a*

non-trivial Kähler magnetic field $\mathbb{B}_k$ *is estimated from below as follows*:

$$\mathrm{vol}\big(B_r^k(p)\big) \geq \frac{\mathrm{vol}(M)}{\displaystyle\int_0^R \{\mathfrak{s}_0(s;c_1)\}^{2n-1} ds} \int_0^{\ell_k(r;c_2)} \{\mathfrak{s}_0(t;c_1)\}^{2n-1}\, dt.$$

Proof. As the diameter of M is R, we have $M = B_R(p)$ for arbitrary $p \in M$. By applying Theorems 4.2, 4.7, we have

$$\mathrm{vol}\big(B_r^k(p)\big) \geq \mathrm{Vol}\big(B_{\ell_k(r;c_2)}(p)\big) \geq \frac{\mathrm{vol}(M)}{\displaystyle\int_0^R \{\mathfrak{s}_0(s;c_1)\}^{2n-1} ds} \int_0^{\ell_k(r;c_2)} \{\mathfrak{s}_0(t;c_1)\}^{2n-1}\, dt.$$

This completes the proof. $\square$

Next, we give an estimate of volumes of trajectory-balls from above.

Theorem 4.9. *Let k be a non-zero constant and M be a Kähler manifold of complex dimension n whose sectional curvatures satisfy* $\mathrm{Riem}^{\mathrm{M}} \geq \mathrm{c}$ *with some constant c. We take an arbitrary point $p \in M$ and an arbitrary r with $0 < r \leq \min\{R_k(p), C_k(p)\}$. Suppose that every trajectory-harp associated with $\gamma : [0,r] \to M$ for $\mathbb{B}_k$ with $\gamma(0) = p$ is holomorphic at its arch. Then the volume of a trajectory-ball $B_r^k(p)$ of arc-radius r for $\mathbb{B}_k$ is estimated from above as following* :

$$\mathrm{vol}\big(B_r^k(p)\big) \leq \omega_{2n-1} \int_0^r \big\{2\mathfrak{s}_k(t/2;c)\big\}^{2n-1} \big\{1 - c\mathfrak{s}_k^2(t/2;c)\big\}^{n-1} \mathfrak{c}_k(t/2;c)\, dt.$$

Proof. For an arbitrary point $q \in B_r^k(p)$ we have a trajectory γ satisfying $\gamma(0) = p$ and $\gamma(t_q) = q$ with some t_q with $0 \leq t_q < r$. As $r \leq \min\{R_k(p), C_k(p)\}$, the trajectory-ball $B_r^k(p)$ is contained in the geodesic-ball $B_{c_p}(p)$, we can take a trajectory-harp associated with γ. By Theorem 4.3 we have $d(p,q) \leq \ell_\gamma(t_q) \leq \ell_k(t_q;c) < \ell_k(r;c)$. Thus, $B_r^k(p)$ is contained in the geodesic-ball $B_{\ell_k(r;c)}(p)$ of radius $\ell_k(r;c)$. We put $\ell = \ell_k(r;c)$. We then have

$$ds = \frac{d\ell_k}{dt} dt = \frac{\mathfrak{c}_k(t/2;c)}{\sqrt{1 - c\mathfrak{s}_k^2(t/2;c)}} dt = \frac{\sqrt{1 - (k^2 + c)\mathfrak{s}^2(t/2;c)}}{\sqrt{1 - c\mathfrak{s}_k^2(t/2;c)}} dt.$$

Therefore, by applying Bishop's comparison theorem (Theorem 4.5) and by using the relation $\mathfrak{s}_0\big(\ell_k(t/2;c);c\big) = \mathfrak{s}_k(t/2;c)$, we obtain

$$\begin{aligned}
\mathrm{vol}\big(B_r^k(p)\big) &\le \mathrm{vol}\big(B_{\ell_k(r;c)}(p)\big)\\
&\le \omega_{2n-1}\int_0^{\ell_k(r;c)} \big\{\mathfrak{s}_0(s;c)\big\}^{2n-1} ds\\
&= \omega_{2n-1}\int_0^{\ell_k(r;c)} \big\{2\mathfrak{s}_0(s/2;c)\big\}^{2n-1}\big\{1-c\{\mathfrak{s}_0(s/2;c)\}^2\big\}^{\frac{2n-1}{2}} ds\\
&= \omega_{2n-1}\int_0^{r} \big\{2\mathfrak{s}_0\big(\ell_k(t;c)/2;c\big)\big\}^{2n-1}\Big\{1-c\big\{\mathfrak{s}_0\big(\ell_k(t;c)/2;c\big)\big\}^2\Big\}^{\frac{2n-1}{2}}\frac{d\ell_k}{dt} dt\\
&= \omega_{2n-1}\int_0^{r} \big\{2\mathfrak{s}_k(t/2;c)\big\}^{2n-1}\Big\{1-c\big\{\mathfrak{s}_k(t/2;c)\big\}^2\Big\}^{\frac{2n-1}{2}}\sqrt{\frac{1-(k^2+c)\mathfrak{s}_k^2(t/2;c)}{1-c\mathfrak{s}_k^2(t/2;c)}} dt\\
&= \omega_{2n-1}\int_0^{r} \big\{2\mathfrak{s}_k(t/2;c)\big\}^{2n-1}\Big\{1-c\mathfrak{s}_k^2(t/2;c)\Big\}^{n-1}\Big\{1-(k^2+c)\mathfrak{s}_k^2(t/2;c)\Big\}^{1/2} dt
\end{aligned}$$

we get the conclusion. □

For the sake of comparison, we here recall results by Bai and Adachi ([**10**]).

Proposition 4.9. *Let M be a complete Kähler manifold of complex dimension n. Suppose its sectional curvatures satisfy* $\mathrm{Riem}_M \le c$ *with some constant c. Then at an arbitrary point $p \in M$, for an arbitrary r with* $0 < r \le \max\{c_0(p), I_k(p), R_k(p)\}$, *we have*

$$\begin{aligned}
\mathrm{vol}\big(B_r^k(p)\big) &\ge \omega_{2n-1}\int_0^r \mathfrak{s}_k(t;c)\{\mathfrak{s}_{k/2}(t;c)\}^{2n-2}dt\\
&= \omega_{2n-1}\int_0^r 2^{2n-2}\mathfrak{s}_k(t;c)\{\mathfrak{s}_k(t/2;4c)\}^{2n-2}dt.
\end{aligned}$$

Bai and Adachi ([**10**]) also gave an estimate from above under a condition that sectional curvatures are bounded from below.

Proposition 4.10. *Let M be a complete Kähler manifold of complex dimension n. Suppose its sectional curvatures satisfy* $\mathrm{Riem}_M \ge c$ *with some constant c. Then at an arbitrary point $p \in M$, for an arbitrary r with* $0 < r \le \max\{c_0(p), I_k(p), R_k(p)\}$, *we have*

$$\begin{aligned}
\mathrm{vol}\big(B_r^k(p)\big) &\le \omega_{2n-1}\int_0^r \mathfrak{s}_k(t;c)\{\mathfrak{s}_{k/2}(t;c)\}^{2n-2}dt\\
&= \omega_{2n-1}\int_0^r 2^{2n-2}\mathfrak{s}_k(t;c)\{\mathfrak{s}_k(t/2;4c)\}^{2n-2}dt.
\end{aligned}$$

We note that the assumption on r for Proposition 4.9 is weaker than the assumption in Theorem 4.6. But we can not say clearly which estimate is sharper.

5. Comparison theorems on zenith angles and lengths of sector-arcs

In this section we study "fatness" of trajectory-harps. For a trajectory-harp α_γ associated with a trajectory $\gamma : [0,T] \to M$ and constants a, b with $0 \le a < b \le T$, the restriction $\alpha_\gamma^{a,b} : [a,b] \times [0, \ell_\gamma(a)] \to M$ of α_γ is said to be a *harp-sector*. We call the length $\vartheta_\gamma(a,b)$ of the curve $[a,b] \ni t \mapsto \dfrac{\partial \alpha_\gamma}{\partial s}(t,0) \in U_{\gamma(0)}M$ in the unit tangent space $U_{\gamma(0)}M$ the *zenith angle* of this harp-sector, and call the curve $[a,b] \ni t \mapsto \alpha_\gamma\big(t, \ell_\gamma(a)\big) \in M$ in M the *sector-arc* of this harp-sector. We denote by $s\ell_\gamma(a,b)$ the length of this sector-arc. We say the restriction $\alpha_\gamma|_{[a,b]\times\mathbb{R}}$ of α_γ on $[a,b] \times \mathbb{R}$ to be a *sub-harp*. We call the curve $[a,b] \ni t \mapsto \alpha_\gamma\big(t, \ell_\gamma(t)\big) \in M$ the *harp-arc* of this sub-harp. Generally, we have $\vartheta_\gamma(a,b) \ge \angle\Big(\dfrac{\partial \alpha_\gamma}{\partial s}(a,0), \dfrac{\partial \alpha_\gamma}{\partial s}(b,0)\Big)$.

Since a trajectory-harp $\hat\alpha_{\hat\gamma}$ associated with an arbitrary trajectory $\hat\gamma$ for $\mathbb{B}_k$ on a complex space form $\mathbb{C}M^n(c)$ lies on a totally geodesic $\mathbb{C}M^1(c)$, the zenith angle of a harp-sector $\hat\alpha_{\hat\gamma}^{a,b}$ is the angle between $\dfrac{\partial \hat\alpha_{\hat\gamma}}{\partial s}(a,0)$ and $\dfrac{\partial \hat\alpha_{\hat\gamma}}{\partial s}(b,0)$, hence is given by

$$\vartheta_k(a,b;c) = \cos^{-1}\delta_k(b;c) - \cos^{-1}\delta_k(a;c).$$

Therefore, the arc-length of the sector-arc is given as $\vartheta_k(a,b;c)\mathfrak{s}_k\big(\ell_k(a;c);c\big)$ if $0 \le a < b \le 2\pi/\sqrt{k^2+c}$.

We here study the relationship between zenith angles and string-cosines. When $a = 0$, we take a trajectory σ for $\mathbb{B}_{-k}$ given as $\sigma(t) = \gamma(b-t)$. Suppose we have a trajectory-harp associated with σ. Also, we suppose that the restriction $[0, \ell_\gamma(b)] \ni s \mapsto \alpha_\gamma(b,s) \in M$ is the reversed geodesic segment of the restriction $[0, \ell_\sigma(b)] \ni s \mapsto \alpha_\sigma(b,s) \in M$ of the string of α_σ at $\sigma(b)$. Such case occurs when $\gamma(b)$ is contained in $B_{\iota_0(p)}$, for example. We then have

$$\vartheta_\gamma(0,b) \ge \angle\Big(\dot\gamma(0), \frac{\partial \alpha_\gamma}{\partial s}(b,0)\Big) = \angle\Big(-\dot\gamma(0), \frac{\partial \alpha_\sigma}{\partial s}\big(b, \ell_\sigma(b)\big)\Big) = \cos^{-1}\delta_\sigma(b).$$

When $a \ne 0$, the zenith angle of a harp-sector $\alpha_\gamma^{a,b}$ is not smaller than the angle between $\dfrac{\partial \alpha_\gamma}{\partial s}(a,0)$ and $\dfrac{\partial \alpha_\gamma}{\partial s}(b,0)$, hence is estimated by

$$\vartheta_\gamma(a,b) \ge \cos^{-1}\delta_\sigma(b;c) - \cos^{-1}\delta_\sigma(a;c).$$

We now estimate zenith angles and lengths of sector-arcs under some assumption on sectional curvatures of the underlying manifold.

First we study the case that sectional curvatures are bounded from above.

Theorem 4.10. *Let α_γ be a trajectory-harp associated with a trajectory $\gamma : [0,T] \to M$ for $\mathbb{B}_k$ on a Kähler manifold M. Suppose that sectional curvatures of planes tangent to the harp-body $\mathcal{HB}_\gamma(T)$ are not greater than a constant c. Then, for arbitrary a, b with $0 \le a < b \le \min\{R_\gamma, C_\gamma\}$, by setting $\hat{a} = \tau_k\big(\ell_\gamma(a); c\big)$ and $\hat{b} = \tau_k\big(\ell_\gamma(b); c\big)$ we have the following :*

(1) *The zenith angle satisfies $\vartheta_\gamma(a,b) \le \vartheta_k(\hat{a}, \hat{b}; c)$;*
(2) *The length of the sector-arc satisfies $s\ell_\gamma(a,b) \le \vartheta_k(\hat{a},\hat{b};c)\mathfrak{s}_k\big(\ell_k(\hat{a};c);c\big)$;*
(3) *The length of the harp-arc satisfies $b - a \le \hat{b} - \hat{a}$.*

Moreover, if an equality holds in one of the above inequalities, then we have the following :

1) $\dfrac{\partial \alpha_\gamma}{\partial t}(t,s)$ *is parallel to* $J\dfrac{\partial \alpha_\gamma}{\partial s}(t,s)$ *for* $a \le t \le b,\ 0 \le s \le \ell_\gamma(t)$;
2) $\mathrm{Riem}\Big(\dfrac{\partial \alpha_\gamma}{\partial s}(t,s), \dfrac{\partial \alpha_\gamma}{\partial t}(t,s)\Big) = c$ *for* $a \le t \le b,\ 0 \le s \le \ell_\gamma(t)$;
3) *The body $\mathcal{HB}_\gamma(a,b)$ is totally geodesic and holomorphic.*

Proof. We put $Z_t(s) = \dfrac{\partial \alpha_\gamma}{\partial t}(t,s)$, which is a Jacobi field along $s \mapsto \alpha_\gamma(t,s)$. By definition, we have

$$\vartheta_\gamma(a,b) = \int_a^b \Big\| \Big(\nabla_{\frac{\partial \alpha_\gamma}{\partial t}} \frac{\partial \alpha_\gamma}{\partial s}\Big)(t,0)\Big\|\, dt = \int_a^b \Big\| \Big(\nabla_{\frac{\partial \alpha_\gamma}{\partial s}} Z_t\Big)(0)\Big\|\, dt.$$

Since the sectional curvature of the plane spanned by $Z_t(s)$ and $\dfrac{\partial \alpha_\gamma}{\partial s}(t,s)$ is not greater than c for $0 \le s \le \ell_\gamma(t)$, by Rauch's comparison theorem on Jacobi fields, we have $\|Z_t(s)\| \ge \big\|(\nabla_{\frac{\partial \alpha_\gamma}{\partial s}} Z_t)(0)\big\|\mathfrak{s}_k(s;c)$. We take a trajectory-harp $\hat{\alpha}_{\hat{\gamma}}$ associated with a trajectory $\hat{\gamma}$ for $\mathbb{B}_k$ on $\mathbb{C}M^1(c)$ and set $\widehat{Z}_t(s) = \dfrac{\partial \hat{\alpha}_{\hat{\gamma}}}{\partial t}(t,s)$. As we see $\|Z_t\big(\ell_\gamma(t)\big)\|^2 = 1 - \delta_\gamma^2(t)$ in the proof of Theorem 4.2, and as $\delta_\gamma(t) \le \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$ by the comparison theorem on string-cosines (Theorem 4.2), we have

$$\begin{aligned}\vartheta_\gamma(a,b) &\le \int_a^b \frac{\|Z_t\big(\ell_\gamma(t)\big)\|}{\mathfrak{s}_k\big(\ell_\gamma(t);c\big)}\,dt = \int_a^b \frac{\sqrt{1-\delta_\gamma(t)^2}}{\mathfrak{s}_k\big(\ell_\gamma(t);c\big)}\,dt \\ &\le \int_a^b \frac{\sqrt{1-\delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)^2}}{\mathfrak{s}_k\big(\ell_\gamma(t);c\big)}\,dt = \int_a^b \frac{\big\|\widehat{Z}_{\tau_k(\ell_\gamma(t);c)}\big(\ell_\gamma(t)\big)\big\|}{\mathfrak{s}_k\big(\ell_\gamma(t);c\big)}\,dt.\end{aligned}$$

We put $u = \tau_k\big(\ell_\gamma(t);c\big)$. We then have

$$\frac{du}{dt} = \frac{\delta_\gamma(t)}{\delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)} \ge 1.$$

Therefore, as $\hat{a} = \tau_k\big(\ell_\gamma(a);c\big), \hat{b} = \tau_k\big(\ell_\gamma(b);c\big)$, we obtain

$$\vartheta_\gamma(a,b) \le \int_{\hat{a}}^{\hat{b}} \frac{\big\|\widehat{Z}_u\big(\ell_\gamma(t)\big)\big\|}{\mathfrak{s}_k\big(\ell_k(u;c);c\big)}\,du = \int_{\hat{a}}^{\hat{b}} \Big\|\Big(\nabla_{\frac{\partial \hat{\alpha}_{\hat{\gamma}}}{\partial s}} \widehat{Z}_u\Big)(0)\Big\|du = \vartheta_{\hat{\gamma}}(\hat{a},\hat{b};c)$$

because $\ell_k(u;c) = \ell_\gamma(t)$.

Next we study the lengths of sector-arcs. By the comparison theorems on string-cosines (Theorem 4.2), we have

$$\|Z_t\big(\ell_\gamma(t)\big)\| = \sqrt{1-\delta_\gamma(t)^2} \le \sqrt{1-\delta_\gamma\big(\tau_k(\ell_\gamma(t);c);c\big)^2} = \big\|\widehat{Z}_{\tau_k(\ell_\gamma(t);c)}\big(\ell_\gamma(t)\big)\big\|.$$

As the sectional curvature of the plane spanned by $Z_t(s)$ and $\dfrac{\partial \alpha_\gamma}{\partial s}(t,s)$ is not greater than c for $0 \le s \le \ell_\gamma(t)$, by Rauch's comparison theorem we find that the function $s \mapsto \|Z_t(s)\|/\|Z_{\tau_k(\ell_\gamma(t);c)}(s)\|$ is monotone increasing. Hence we have $\|Z_t(s)\|/\|Z_{\tau_k(\ell_\gamma(t);c)}(s)\| \le 1$ for $0 \le s \le \ell_\gamma(t)$. This means that $\|Z_t(s)\| \le \big\|\widehat{Z}_{\tau_k(\ell_\gamma(t);c)}(s)\big\|$ for an arbitrary s with $0 < s \le \ell_\gamma(t)$. We therefore obtain by putting $u = \tau_k\big(\ell_\gamma(t);c\big)$ that

$$\begin{aligned} s\ell_\gamma(a,b) &= \int_a^b \|Z_t\big(\ell_\gamma(a)\big)\|\,dt \le \int_a^b \big\|\widehat{Z}_{\tau_k(\ell_\gamma(t);c)}\big(\ell_\gamma(a)\big)\big\|\,dt \\ &\le \int_{\hat{a}}^{\hat{b}} \big\|\widehat{Z}_u(\ell_\gamma(a))\big\|\frac{dt}{du}du \le \int_{\hat{a}}^{\hat{b}} \big\|\widehat{Z}_u\big(\ell_\gamma(a)\big)\big\|\,du \\ &= \int_{\hat{a}}^{\hat{b}} \big\|\widehat{Z}_u\big(\ell_{\hat{\gamma}}(\hat{a})\big)\big\|\,du = s\ell_{\hat{\gamma}}(\hat{a},\hat{b}) = \vartheta_k(\hat{a},\hat{b};c)\mathfrak{s}_k\big(\ell_k(\hat{a};c);c\big).\end{aligned}$$

At last we study the lengths of harp-arcs. Since $\|\dot\gamma\| = \|\dot{\hat\gamma}\| = 1$, we have

$$\begin{aligned} b-a &= \int_a^b \|\dot\gamma(t)\|\,dt = \int_a^b \big\|\dot{\hat\gamma}\big(\tau_k(\ell_\gamma(t);c)\big)\big\|\,dt = \int_{\hat a}^{\hat b} \|\dot{\hat\gamma}(u)\|\frac{dt}{du}du \\ &= \int_{\hat a}^{\hat b} \|\dot{\hat\gamma}(u)\|\frac{\delta_k(u;c)}{\delta_\gamma\big(\tau_\gamma(\ell_k(u;c))\big)}du \le \int_{\hat a}^{\hat b}\|\dot{\hat\gamma}(u)\|\,du = \hat b - \hat a. \end{aligned}$$

We here study the case that one of the three equalities $\vartheta_\gamma(a,b) = \vartheta_k(\hat a,\hat b;c)$, $b-a = \hat b-\hat a$ and $s\ell_\gamma(a,b) = \vartheta_k(\hat a,\hat b;c)\mathfrak{s}_k\big(\ell_k(\hat a;c);c\big)$ holds. Our proof guarantees that this holds if and only if $\delta_\gamma(t) = \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$ holds for $a \le t \le b$. Thus, by Theorem 4.2, we get the conclusion. $\square$

Corollary 4.1. *Let M be a Hadamard Kähler manifold of sectional curvature $\mathrm{Riem}^M \le c < 0$. If $|k| < \sqrt{|c|}$, then for an arbitrary trajectory half-line γ for $\mathbb{B}_k$, the trajectory-harps α_γ associated with γ satisfies*

$$\angle\Big(\frac{\partial\alpha}{\partial s}(t_1,0), \frac{\partial\alpha}{\partial s}(t_2,0)\Big) < \int_{t_1}^{t_2} \frac{\sqrt{|c|-k^2}}{\sinh\sqrt{|c|-k^2}\,t}dt$$

for all $t_2 > t_1 > 0$.

In particular, we have a limit $\lim_{t\to\infty}\dfrac{\partial\alpha_\gamma}{\partial s}(t,0) \in U_{\gamma(0)}M$ of initial vectors of harp-strings.

Proof. We take a trajectory-harp α_γ associated with γ. Since $\lim_{t\to\infty}\ell_k(t;c) = \infty$, we see γ is unbounded.

We set $Z_t(s) = \dfrac{\partial\alpha_\gamma}{\partial t}(t,s)$, which is a Jacobi field along the geodesic $s \mapsto \alpha_\gamma(t,s)$. We consider another Jacobi field $\widehat Z_t(s) = A\cosh\big(\sqrt{|c|}\ s\big)\dot{\hat\sigma}(s) + B\sinh\big(\sqrt{|c|}\ s\big)J\dot{\hat\sigma}$ along a geodesic $\hat\sigma$ on $\mathbb{C}H^1(c)$ satisfying $\widehat Z_t(0) = 0$, where $A, B \in \mathbb{R}$ constants. As $\widehat Z_t(0) = 0$, we see $A = 0$. Therefore the Jacobi field on $\mathbb{C}H^1(c)$ is of the form

$$\widehat Z_t(s) = \frac{\big\|\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}\widehat Z_t(0)\big\|}{\sqrt{|c|}}\sinh\big(\sqrt{|c|}\ s\big)J\dot{\hat\sigma}(s).$$

By Rauch's comparison theorem on Jacobi field, if $\mathrm{Riem}^M \le c < 0$, we have

$$\|Z_t(s)\| \ge \big\|\widehat Z_t(s)\big\| = \big\|\nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_t(0)\big\| \times \frac{1}{\sqrt{|c|}}\sinh\sqrt{|c|}\ s$$

if $\big\|\nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_t(0)\big\| = \big\|\nabla_{\frac{d\hat\sigma}{ds}}\widehat Z_t(0)\big\|$.

For a trajectory-harp α_γ, as $\gamma(t) = \alpha_\gamma(t, \ell_\gamma(t))$, we have

$$\dot\gamma(t) = Z_t\bigl(\ell_\gamma(t)\bigr) + \delta_\gamma(t)\frac{\partial\alpha_\gamma}{\partial s}\bigl(t, \ell_\gamma(t)\bigr).$$

Hence, we have $\|Z_t\bigl(\ell_\gamma(t)\bigr)\|^2 = 1 - \delta_\gamma(t)^2 \le 1$. Considering the case $s = \ell_\gamma(t)$, we have

$$\bigl\|\nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_t(0)\bigr\| \le \frac{\sqrt{|c|}\,\bigl\|Z_t\bigl(\ell_\gamma(t)\bigr)\bigr\|}{\sinh\bigl(\sqrt{|c|}\,\ell_\gamma(t)\bigr)} < \frac{\sqrt{|c|}}{\sinh\bigl(\sqrt{|c|}\,\ell_\gamma(t)\bigr)} \le \frac{\sqrt{|c|}}{\sinh\bigl(\sqrt{|c|}\,\ell_k(t;c)\bigr)}.$$

When $|k| < \sqrt{|c|}$, as $\sinh\Bigl(\dfrac{\sqrt{|c|}\,\ell_k(t;c)}{2}\Bigr) = \dfrac{\sqrt{|c|}\sinh\bigl(\sqrt{|c|-k^2}\,t/2\bigr)}{\sqrt{|c|-k^2}}$, we have

$$\begin{aligned}
\sinh\bigl(\sqrt{|c|}\ell_k(t;c)\bigr) &= 2\sinh\Bigl(\frac{\sqrt{|c|}\,\ell_k(t;c)}{2}\Bigr)\cosh\Bigl(\frac{\sqrt{|c|}\,\ell_k(t;c)}{2}\Bigr) \\
&= 2\sinh\Bigl(\frac{\sqrt{|c|}\,\ell_k(t;c)}{2}\Bigr)\sqrt{1+\sinh^2\Bigl(\frac{\sqrt{|c|}\,\ell_k(t;c)}{2}\Bigr)} \\
&= \frac{2\sqrt{|c|}\sinh\bigl(\sqrt{|c|-k^2}\,t/2\bigr)}{\sqrt{|c|-k^2}}\sqrt{1+\frac{|c|\sinh^2\bigl(\sqrt{|c|-k^2}\,t/2\bigr)}{|c|-k^2}}
\end{aligned}$$

We therefore obtain by noticing $|c| - k^2 < |c|$ that

$$\begin{aligned}
\int_{t_1}^{t_2}\Bigl\|\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}(t,0)\Bigr\|dt &= \int_{t_1}^{t_2}\bigl\|\nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_t(0)\bigr\|dt \le \int_{t_1}^{t_2}\frac{\sqrt{|c|}}{\sinh\sqrt{|c|}\,\ell_k(t;c)}\,dt \\
&= \int_{t_1}^{t_2}\frac{\sqrt{|c|}}{\dfrac{2\sqrt{|c|}\sinh\bigl(\sqrt{|c|-k^2}\,t/2\bigr)}{\sqrt{|c|-k^2}}\sqrt{1+\dfrac{|c|\sinh^2\bigl(\sqrt{|c|-k^2}\,t/2\bigr)}{|c|-k^2}}}\,dt \\
&< \int_{t_1}^{t_2}\frac{\sqrt{|c|-k^2}}{2\sinh(\sqrt{|c|-k^2}\,t/2)\sqrt{1+\dfrac{(|c|-k^2)\sinh^2(\sqrt{|c|-k^2}\,t/2)}{|c|-k^2}}}\,dt \\
&= \int_{t_1}^{t_2}\frac{\sqrt{|c|-k^2}}{2\sinh(\sqrt{|c|-k^2}\,t/2)\sqrt{1+\sinh^2(\sqrt{|c|-k^2}\,t/2)}}\,dt \\
&= \int_{t_1}^{t_2}\frac{\sqrt{|c|-k^2}}{\sinh\sqrt{|c|-k^2}\,t}\,dt
\end{aligned}$$

for all $t_2 > t_1 > 0$.

Since we have

$$\angle\Big(\frac{\partial\alpha_\gamma}{\partial s}(t_1,0),\,\frac{\partial\alpha_\gamma}{\partial s}(t_2,0)\Big)\le \vartheta_k(t_1,t_2)=\int_{t_1}^{t_2}\Big\|\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}(t,0)\Big\|dt,$$

we get the estimate. Our estimate shows that when $t>\log 2/2(|c|-k^2)$, we have $\sinh\big(\sqrt{|c|-k^2}\,t\big)\ge \exp\big(\sqrt{|c|-k^2}\,t\big)/4$. Hence we can estimate $\vartheta_k(t_1,t_2)$ from above as

$$\begin{aligned}\int_{t_1}^{t_2}\Big\|\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}(t,0)\Big\|dt &< \int_{t_1}^{t_2}\frac{4\sqrt{|c|-k^2}}{\exp\big(\sqrt{|c|-k^2}\,t\big)}dt\\ &\le \frac{4}{\exp\big(\sqrt{|c|-k^2}\,t_1\big)}-\frac{4}{\exp\big(\sqrt{|c|-k^2}\,t_2\big)}.\end{aligned}$$

When $t_2>t_1>\log 2/2(|c|-k^2)$, we get that $\lim_{t_1,t_2\to\infty}\angle\Big(\frac{\partial\alpha_\gamma}{\partial s}(t_1,0),\,\frac{\partial\alpha_\gamma}{\partial s}(t_2,0)\Big)=0$. Since $U_{\gamma(0)}M$ is compact, we find that this Caushy sequence $\Big\{\frac{\partial\alpha_\gamma}{\partial s}(t,0)\Big\}_{t>0}$ converges. We hence get the conclusion. $\square$

We here give an alternative estimate which is uniform with respect to k.

PROPOSITION 4.11. *Let M be a Hadamard Kähler manifold of sectional curvature $\mathrm{Riem}^M\le c<0$. If $|k|<\sqrt{|c|}$, then for an arbitrary trajectory half-line γ for $\mathbb{B}_k$, the trajectory-harps α_γ associated with γ satisfies*

$$\angle\Big(\frac{\partial\alpha}{\partial s}(t_1,0),\,\frac{\partial\alpha}{\partial s}(t_2,0)\Big)<\int_{t_1}^{t_2}\frac{2}{\sqrt{|c|}\,t^2}\,dt$$

for all $t_2>t_1>0$.

PROOF. We take a trajectory-harp α_γ associated with γ.

By the proof of Corollary 4.1, we have

$$\begin{aligned}\angle\Big(\frac{\partial\alpha}{\partial s}(t_1,0),\frac{\partial\alpha}{\partial s}(t_2,0)\Big)&=\int_{t_1}^{t_2}\Big\|\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}(t,0)\Big\|dt\\ &=\int_{t_1}^{t_2}\frac{1}{\dfrac{2\sinh\big(\sqrt{|c|-k^2}\;t/2\big)}{\sqrt{|c|-k^2}}\sqrt{1+\dfrac{|c|\sinh^2\big(\sqrt{|c|-k^2}\;t/2\big)}{|c|-k^2}}}\,dt.\end{aligned}$$

We here show

$$1+\frac{|c|\sinh^2\big(\sqrt{|c|-k^2}\;t/2\big)}{|c|-k^2}>\frac{|c|\,t^2}{4}.$$

We take a function $F(t) = 1+\dfrac{|c|\sinh^2\bigl(\sqrt{|c|-k^2}\,t/2\bigr)}{|c|-k^2} - \dfrac{|c|\,t^2}{4}$. By differentiating $F(t)$, we have

$$F'(t) = \frac{|c|}{2\sqrt{|c|-k^2}}\sinh\bigl(\sqrt{|c|-k^2}\,t\bigr) - \frac{|c|\,t}{2}.$$

$$F''(t) = \frac{|c|}{2}\bigl(\cosh(\sqrt{|c|-k^2}\,t) - 1\bigr).$$

As $F''(0) = 0$, $F'(t)$ is monotone increasing. Since $F'(0) = 0$, $F'(t) > 0$ for $t > 0$. Hence, $F(t)$ is monotone increasing. As $F(0) = 1$, we find $F(t) > 0$ for $t \geq 0$.

We take one more function $G(t) = \dfrac{2}{\sqrt{|c|-k^2}}\sinh\Bigl(\dfrac{\sqrt{|c|-k^2}\,t}{2}\Bigr) - t$. By differentiating $G(t)$, we have

$$G'(t) = \cosh\Bigl(\frac{\sqrt{|c|-k^2}\,t}{2}\Bigr) - 1 > 0,$$

for $t > 0$. As $G(0) = 0$, we find $G(t) > 0$ for $t > 0$.

We therefore find

$$\angle\Bigl(\frac{\partial\alpha}{\partial s}(t_1,0), \frac{\partial\alpha}{\partial s}(t_2,0)\Bigr) < \int_{t_1}^{t_2} \frac{1}{t\sqrt{\dfrac{|c|\,t^2}{4}}}\,dt = \int_{t_1}^{t_2} \frac{2}{\sqrt{|c|}\,t^2}\,dt.$$

We get the conclusion. □

We give an estimate in the case $|k| = \sqrt{|c|}$.

Corollary 4.2. *Let M be a Hadamard Kähler manifold of sectional curvature $\mathrm{Riem}^M \leq c < 0$. If $|k| = \sqrt{|c|}$, then for an arbitrary trajectory half-line γ for $\mathbb{B}_k$, the trajectory-harps α_γ associated with γ satisfies*

$$\angle\Bigl(\frac{\partial\alpha}{\partial s}(t_1,0), \frac{\partial\alpha}{\partial s}(t_2,0)\Bigr) < \int_{t_1}^{t_2} \frac{2}{\sqrt{|c|}\,t^2}\,dt$$

for all $t_2 > t_1 > 0$.

In particular, we have a limit $\lim_{t\to\infty} \dfrac{\partial\alpha_\gamma}{\partial s}(t,0) \in U_{\gamma(0)}M$ *of initial vectors of harp-strings.*

Proof. We take a trajectory-harp α_γ associated with γ. Since $\lim_{t\to\infty}\ell_k(t;c) = \infty$, we see γ is unbounded.

We set $Z_t(s) = \dfrac{\partial\alpha_\gamma}{\partial t}(t,s)$, which is a Jacobi field along the geodesic $s \mapsto \alpha_\gamma(t,s)$. We consider another Jacobi field $\widehat{Z}_t(s) = A\cosh\bigl(\sqrt{|c|}\ s\bigr)\dot{\hat{\sigma}}(s) + B\sinh\bigl(\sqrt{|c|}\ s\bigr)J\dot{\hat{\sigma}}$ along a geodesic $\hat{\sigma}$ on $\mathbb{C}H^1(c)$ satisfying $\widehat{Z}_t(0) = 0$, where $A, B \in \mathbb{R}$ constants. As $\widehat{Z}_t(0) = 0$, we see $A = 0$. Therefore the Jacobi field on $\mathbb{C}H^1(c)$ is of the form

$$\widehat{Z}_t(s) = \frac{\bigl\|\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}\widehat{Z}_t(0)\bigr\|}{\sqrt{|c|}}\sinh\bigl(\sqrt{|c|}\ s\bigr)J\dot{\hat{\sigma}}(s).$$

By Rauch's comparison theorem on Jacobi field, if $\mathrm{Riem}^M \le c < 0$, we have

$$\|Z_t(s)\| \ge \bigl\|\widehat{Z}_t(s)\bigr\| = \bigl\|\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}Z_t(0)\bigr\| \times \frac{1}{\sqrt{|c|}}\sinh\sqrt{|c|}\ s$$

if $\bigl\|\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}Z_t(0)\bigr\| = \bigl\|\nabla_{\frac{d\hat{\sigma}}{ds}}\widehat{Z}_t(0)\bigr\|$.

For a trajectory-harp α_γ, as $\gamma(t) = \alpha_\gamma(t, \ell_\gamma(t))$, we have

$$\dot{\gamma}(t) = Z_t\bigl(\ell_\gamma(t)\bigr) + \delta_\gamma(t)\frac{\partial\alpha_\gamma}{\partial s}\bigl(t, \ell_\gamma(t)\bigr).$$

Hence, we have $\|Z_t\bigl(\ell_\gamma(t)\bigr)\|^2 = 1 - \delta_\gamma(t)^2 \le 1$.

Considering the case $s = \ell_\gamma(t)$, we have

$$\bigl\|\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}Z_t(0)\bigr\| \le \frac{\sqrt{|c|}\ \bigl\|Z_t\bigl(\ell_\gamma(t)\bigr)\bigr\|}{\sinh\bigl(\sqrt{|c|}\ \ell_\gamma(t)\bigr)} < \frac{\sqrt{|c|}}{\sinh\bigl(\sqrt{|c|}\ \ell_\gamma(t)\bigr)} \le \frac{\sqrt{|c|}}{\sinh\bigl(\sqrt{|c|}\ \ell_k(t;c)\bigr)}.$$

When $|k| = \sqrt{|c|}$, as $\sinh\Bigl(\dfrac{\sqrt{|c|}\ \ell_k(t;c)}{2}\Bigr) = \dfrac{\sqrt{|c|}}{2}t$, we have

$$\begin{aligned}
\sinh\bigl(\sqrt{|c|}\ell_k(t;c)\bigr) &= 2\sinh\Bigl(\frac{\sqrt{|c|}\ \ell_k(t;c)}{2}\Bigr)\cosh\Bigl(\frac{\sqrt{|c|}\ \ell_k(t;c)}{2}\Bigr) \\
&= 2\sinh\Bigl(\frac{\sqrt{|c|}\ \ell_k(t;c)}{2}\Bigr)\sqrt{1 + \sinh^2\Bigl(\frac{\sqrt{|c|}\ \ell_k(t;c)}{2}\Bigr)} \\
&= \frac{\sqrt{|c|}\,t\sqrt{4 + |c|t^2}}{2}
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
\int_{t_1}^{t_2}\Bigl\|\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}(t,0)\Bigr\|dt &= \int_{t_1}^{t_2}\bigl\|\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}Z_t(0)\bigr\|dt \le \int_{t_1}^{t_2}\frac{\sqrt{|c|}}{\sinh\bigl(\sqrt{|c|}\ \ell_k(t;c)\bigr)}\,dt \\
&= \int_{t_1}^{t_2}\frac{2}{t\sqrt{4 + |c|t^2}}\,dt < \int_{t_1}^{t_2}\frac{2}{\sqrt{|c|}t^2}\,dt
\end{aligned}$$

for all $t_2 > t_1 > 0$.

Since we have

$$\angle\Big(\frac{\partial\alpha_\gamma}{\partial s}(t_1,0),\frac{\partial\alpha_\gamma}{\partial s}(t_2,0)\Big)\le \vartheta_k(t_1,t_2)=\int_{t_1}^{t_2}\Big\|\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}(t,0)\Big\|dt,$$

we get the estimate.

Since $\lim_{t_1,t_2\to\infty}\angle\Big(\frac{\partial\alpha_\gamma}{\partial s}(t_1,0),\frac{\partial\alpha_\gamma}{\partial s}(t_2,0)\Big)=0$ and $U_{\gamma(0)}M$ is compact, we find that this Caushy sequence $\Big\{\frac{\partial\alpha_\gamma}{\partial s}(t,0)\Big\}_{t>0}$ converges. We hence get the conclusion. $\square$

We shall call the geodesic half-line σ_γ of initial vector $\lim_{t\to\infty}\frac{\partial\alpha_\gamma}{\partial s}(t,0)$ the *limit string* of a trajectory-harp α_γ.

Next we study the case that sectional curvatures are bounded from below.

THEOREM 4.11. *Let α_γ be a trajectory-harp associated with a trajectory $\gamma:[0,T]\to M$ for $\mathbb{B}_k$ on a Kähler manifold M. Suppose that it is holomorphic at its arch and that sectional curvatures of planes tangent to the harp-body $\mathcal{HB}_\gamma(T)$ are not smaller than a constant c. Then, for arbitrary a,b with $0\le a<b\le\min\{R_\gamma,C_\gamma\}$, by setting $\hat a=\tau_k\big(\ell_\gamma(a);c\big)$ and $\hat b=\tau_k\big(\ell_\gamma(b);c\big)$ we have the following :*

(1) *The zenith angle satisfies $\vartheta_\gamma(a,b)\ge\vartheta_k(\hat a,\hat b;c)$;*

(2) *The length of the sector-arc satisfies $s\ell_\gamma(a,b)\ge\vartheta_k(\hat a,\hat b;c)\mathfrak{s}_k\big(\ell_k(\hat a;c);c\big)$;*

(3) *The length of the harp-arc satisfies $b-a\ge\hat b-\hat a$.*

Moreover, if an equality holds in one of the above inequalities, then we have the following :

1) $\frac{\partial\alpha_\gamma}{\partial t}(t,s)$ *is parallel to* $J\frac{\partial\alpha_\gamma}{\partial s}(t,s)$ *for* $a\le t\le b,\ 0\le s\le\ell_\gamma(t)$.

2) $\mathrm{Riem}\Big(\frac{\partial\alpha_\gamma}{\partial s}(t,s),\frac{\partial\alpha_\gamma}{\partial t}(t,s)\Big)=c$ *for* $a\le t\le b,\ 0\le s\le\ell_\gamma(t)$.

3) *The body $\mathcal{HB}_\gamma(a,b)$ is totally geodesic and is holomorphic.*

PROOF. We put $Z_t(s)=\frac{\partial\alpha_\gamma}{\partial t}(t,s)$, which is a Jacobi field along $s\mapsto\alpha_\gamma(t,s)$. By definition, we have

$$\vartheta_\gamma(a,b)=\int_a^b\Big\|\Big(\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}\Big)(t,0)\Big\|\,dt=\int_a^b\Big\|\Big(\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}Z_t\Big)(0)\Big\|\,dt.$$

Since the sectional curvature of the plane spanned by $Z_t(s)$ and $\dfrac{\partial\alpha_\gamma}{\partial s}(t,s)$ is not smaller than c for $0 \le s \le \ell_\gamma(t)$, by Rauch's comparison theorem on Jacobi fields, we have $\|Z_t(s)\| \le \|(\nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_t)(0)\|\mathfrak{s}_k(s;c)$. We take a trajectory-harp $\hat\alpha_{\hat\gamma}$ associated with a trajectory $\hat\gamma$ for $\mathbb{B}_k$ on $\mathbb{C}M^n(c)$ and set $\widehat{Z}_t(s) = \dfrac{\partial\hat\alpha_{\hat\gamma}}{\partial t}(t,s)$. As we see $\|Z_t\big(\ell_\gamma(t)\big)\|^2 = 1-\delta_\gamma^2(t)$ in the proof of Theorem 4.3, and as $\delta_\gamma(t) \le \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$ by the comparison theorem on string-cosines (Theorem 4.3), we have

$$\begin{aligned}\vartheta_\gamma(a,b) &\ge \int_a^b \frac{\|\ Z_t\big(\ell_\gamma(t)\big)\|}{\mathfrak{s}_k\big(\ell_\gamma(t);c\big)}\,dt = \int_a^b \frac{\sqrt{1-\delta_\gamma(t)^2}}{\mathfrak{s}_k\big(\ell_\gamma(t);c\big)}\,dt\\ &\ge \int_a^b \frac{\sqrt{1-\delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)^2}}{\mathfrak{s}_k\big(\ell_\gamma(t);c\big)}\,dt = \int_a^b \frac{\big\|\widehat{Z}_{\tau_k(\ell_\gamma(t);c)}\big(\ell_\gamma(t)\big)\big\|}{\mathfrak{s}_k\big(\ell_\gamma(t);c\big)}\,dt.\end{aligned}$$

We put $u = \tau_k\big(\ell_\gamma(t);c\big)$. We then have

$$\frac{du}{dt} = \frac{\delta_\gamma(t)}{\delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)} \le 1.$$

Therefore, as $\hat a = \tau_k\big(\ell_\gamma(a);c\big), \hat b = \tau_k\big(\ell_\gamma(b);c\big)$, we obtain

$$\vartheta_\gamma(a,b) \ge \int_{\hat a}^{\hat b} \frac{\big\|\widehat{Z}_u\big(\ell_\gamma(t)\big)\big\|}{\mathfrak{s}_k\big(\ell_k(u;c);c\big)}\,du = \int_{\hat a}^{\hat b} \Big\|\Big(\nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}} \widehat{Z}_u\Big)(0)\Big\| du = \vartheta_{\hat\gamma}(\hat a,\hat b;c),$$

because $\ell_k(u;c) = \ell_\gamma(t)$.

Next we study lengths of sector-arcs. By the comparison theorems on string-cosines, we have

$$\|Z_t\big(\ell_\gamma(t)\big)\| = \sqrt{1-\delta_\gamma(t)^2} \ge \sqrt{1-\delta_\gamma\big(\tau_k(\ell_\gamma(t);c);c\big)^2} = \big\|\widehat{Z}_{\tau_k(\ell_\gamma(t);c)}\big(\ell_\gamma(t)\big)\big\|.$$

As the sectional curvature of the plane spanned by $Z_t(s)$ and $\dfrac{\partial\alpha_\gamma}{\partial s}(t,s)$ is not smaller than c for $0 \le s \le \ell_\gamma(t)$, by Rauch's comparison theorem we find that $\|Z_t(s)\| \ge \big\|\widehat{Z}_{\tau_k(\ell_\gamma(t);c)}(s)\big\|$ for an arbitrary s with $0 < s \le \ell_\gamma(t)$. We therefore obtain

$$\begin{aligned}s\ell_\gamma(a,b) &= \int_a^b \|Z_t\big(\ell_\gamma(a)\big)\|\,dt \ge \int_a^b \big\|\widehat{Z}_{\tau_k(\ell_\gamma(t);c)}\big(\ell_\gamma(a)\big)\big\|\,dt \ge \int_{\hat a}^{\hat b} \big\|\widehat{Z}_u\big(\ell_\gamma(a)\big)\big\|\,du\\ &= \int_{\hat a}^{\hat b} \big\|\widehat{Z}_u\big(\ell_{\hat\gamma}(\hat a)\big)\big\|\,du = s\ell_{\hat\gamma}(\hat a,\hat b) = \vartheta_k(\hat a,\hat b;c)\mathfrak{s}_k\big(\ell_k(\hat a;c);c\big),\end{aligned}$$

where $u = \tau_k\big(\ell_\gamma(t);c\big)$.

At last we study lengths of harp-arcs. Since $\|\dot{\gamma}\| = \|\dot{\hat{\gamma}}\| = 1$, we have

$$\begin{aligned} b - a &= \int_a^b \|\dot{\gamma}(t)\|\,dt = \int_a^b \big\|\dot{\hat{\gamma}}\big(\tau_k(\ell_\gamma(t);c)\big)\big\|\,dt \\ &= \int_{\hat{a}}^{\hat{b}} \big\|\dot{\hat{\gamma}}(u)\big\| \frac{\delta_k(u;c)}{\delta_\gamma\big(\tau_\gamma(\ell_k(u;c))\big)} du \geq \int_{\hat{a}}^{\hat{b}} \big\|\dot{\hat{\gamma}}(u)\big\|\,du = \hat{b} - \hat{a}. \end{aligned}$$

We here study the case that one of the three equalities $\vartheta_\gamma(a,b) = \vartheta_k(\hat{a},\hat{b};c)$, $s\ell_\gamma(a,b) = \vartheta_k(\hat{a},\hat{b};c)\mathfrak{s}_k\big(\ell_k(\hat{a};c);c\big)$ and $b-a = \hat{b}-\hat{a}$ holds. Our proof guarantees that this holds if and only if $\delta_\gamma(t) = \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big)$ holds for $a \leq t \leq b$. Thus, as we see in the proof of Theorem 4.3, we get the conclusion. $\square$

CHAPTER 5

Ideal boundary of a Hadamard Kähler manifold

In this chapter we study asymptotic behaviors of unbounded trajectories on a Hadamard Kähler manifold, a simply connected Kähler manifold of non-positive curvature.

1. Hadamard manifold

First we study the topology of a Riemannian manifold of non-positive curvature.

Theorem 5.1 (Cartan-Hadamard). *Let M be a complete Riemannian manifold of non-positive curvature. At an arbitrary point $p \in M$, the exponential map* $\exp_p$: $T_pM \to M$ *is a covering map. Hence the universal covering space of M is diffeomorphic to $\mathbb{R}^m$, where m is the real dimension of M.*

Corollary 5.1. *A complete simply connected Riemannian manifold of non-positive curvature is diffeomorphic to a Euclidean space.*

A map $\varphi : M \to N$ between Riemannian manifolds is said to be a local isometry if each point $p \in M$ has a neighborhood U such that the restriction $\varphi|_U : U \to N$ is an isometry onto an open subset $\varphi(U)$ of N.

In order to show those results, we need the following.

Proposition 5.1. *Let M and N be m-dimensional connected Riemannian manifolds. Suppose M is complete. If $\varphi : M \to N$ is a local isometry, then it is a covering map. That is, φ is a surjective continuous map such that for each $q \in N$ there exists an open neighborhood V of q satisfying the following conditions:*

i) $\varphi^{-1}(V) = \bigcup_{\lambda\in\Lambda} U_\lambda$ *is a disjoint union of open sets in M;*

ii) $\varphi|_{U_\lambda} : U_\lambda \to V$ *is an isometry for each* λ.

Proof. First, we show φ is surjective. We take an arbitrary $p \in M$ and put $\varphi(p) = q$. Since N is complete, for each $q' \in N$ we have a geodesic ρ on N satisfying $\rho(0) = q$ and $\rho(t_0) = q'$ with some t_0. As φ is a local isometry between Riemannian manifolds of same dimension, the differential map $(d\varphi)_p : T_{p_\lambda}M \to T_qN$ is a linear isometry. we put $u = \big((d\varphi)_p\big)^{-1}\big(\dot\rho(0)\big)$ and take a geodesic σ on M with $\dot\sigma(0) = u$. Since φ is a local isometry, we see $\varphi\circ\sigma$ is a geodesic on N. As $(\varphi\circ\sigma)'(0) = (d\varphi)_p\big(\dot\sigma(0)\big) = \dot\rho(0)$, we find that $\rho = \varphi\circ\sigma$. Therefore we have $q' = \rho(t_0) = \varphi\big(\sigma(t_0)\big)$ and find that φ is surjective.

For an arbitrary $q \in N$, we take a small positive r satisfying $r < \iota_0(q)$. We put $\varphi^{-1}(q) = \{p_\lambda\}_{\lambda\in\Lambda}$. Since φ is a local isometry, every geodesic σ on M with $\sigma(0) = p$ is mapped to a geodesic $\varphi\circ\sigma$ on N with $\varphi\circ\sigma(0) = q$. Thus we have $\varphi\big(B_r(p_\lambda)\big) \subset B_r(q)$, and have

$$\exp_q \circ (d\varphi)_{p_\lambda} = \varphi\circ\exp_{p_\lambda} : B_r(0_{p_\lambda}) \to B_r(q).$$

$$\begin{tikzcd} B_r(0_{p_\lambda}) \arrow[r, "(d\varphi)_{p_\lambda}"] \arrow[d, "\exp_{p_\lambda}"'] & B_r(0_q) \arrow[d, "\exp_q"] \\ B_r(p_\lambda) \arrow[r, "\varphi"'] & B_r(q) \end{tikzcd}$$

Here, as $(d\varphi)_{p_\lambda} : T_{p_\lambda}M \to T_qN$ is a linear isometry, we see $(d\varphi)_{p_\lambda} : B_r(0_{p_\lambda}) \to B_r(0_q)$ is bijective. Since $r < \iota_0(q)$, the map $\exp_q : B_r(0_q) \to B_r(q)$ is bijective, and $\exp_{p_\lambda} : B_r(0_{p_\lambda}) \to B_r(p_\lambda)$ is surjective. Thus, for each $q' \in B_r(q)$, by taking $u \in B_r(0_{p_\lambda})$ with $q' = \exp_q\circ(d\varphi)_{p_\lambda}(u)$, we find $q' = \varphi\big(\exp_{p_\lambda}(u)\big)$. This shows $\varphi|_{B_r(p_\lambda)} : B_r(p_\lambda) \to B_r(q)$ is surjective. Also, if $p', p'' \in B_r(p_\lambda)$ satisfiy $\varphi(p') = \varphi(p'')$, then taking $u', u'' \in B_r(0_{p_\lambda})$ satisfying $\exp_{p_\lambda}(u') = p'$ and $\exp_{p_\lambda}(u'') = p''$, as $\exp_q\circ(d\varphi)_{p_\lambda}(u') = \exp_q\circ(d\varphi)_{p_\lambda}(u'') = \varphi(p')$, we see $u' = u''$, which shows $p' = p''$. Thus $\varphi|_{B_r(p_\lambda)} : B_r(p_\lambda) \to B_r(q)$ is injective Therefore we find that $\varphi|_{B_r(p_\lambda)} : B_r(p_\lambda) \to B_r(q)$ is an isometry.

As we see $\varphi\big(B_r(p_\lambda)\big) \subset B_r(q)$, we have $\bigcup_{\lambda\in\Lambda} B_r(p_\lambda) \subset \varphi^{-1}\big(B_r(q)\big)$. On the other hand, for $p' \in \varphi^{-1}\big(B_r(q)\big)$ we set $q' = \varphi(p') \in B_r(q)$ and take $v \in B_r(0_q)$ with $\exp_q(v) = q'$. Then $\rho(t) = \exp_q(1-t)v$ is a geodesic from q' to q. If we set $\sigma(t) = \exp_{p'}\big(t(d\varphi)_{p'}^{-1}(\dot\rho(0))\big)$, then it is a geodesic satisfying $\sigma(0) = p'$. Since $\|\dot\rho(0)\| = \|v\| < r$ and we have $\exp_{q'}\circ(d\varphi)_{p'} = \varphi\circ\exp_{p'} : B_r(0_{p'}) \to B_r(q')$ because the above argument holds for arbitrary q, we see $\varphi\bigl(\sigma(1)\bigr) = \rho(1) = q$. This means that $\sigma(1) = p_{\lambda_0}$ with some $\lambda_0 \in \Lambda$. As $d\bigl(\sigma(1), p'\bigr) < r$, we find $p' \in B_r(p_{\lambda_0})$. Thus we have $\varphi^{-1}\bigl(B_r(q)\bigr) \subset \bigcup_{\lambda\in\Lambda} B_r(p_\lambda)$, and get $\bigcup_{\lambda\in\Lambda} B_r(p_\lambda) = \varphi^{-1}\bigl(B_r(q)\bigr)$.

We finally show that $B_r(p_\lambda) \cap B_r(p_{\lambda'}) = \emptyset$ if $\lambda \neq \lambda'$. Suppose there exists $p' \in B_r(p_\lambda) \cap B_r(p_{\lambda'})$. We then have $u_1 \in B_r(0_{p_\lambda})$ and $u_2 \in B_r(0_{p_{\lambda'}})$ satisfying $\exp_{p_\lambda}(u_1) = p' = \exp_{p_{\lambda'}}(u_2)$. Considering geodesic segments $\sigma_1 = \exp_{p_\lambda}(tu_1)$ joining p_λ and p' and $\sigma_2 = \exp_{p_{\lambda'}}(tu_2)$ joining $p_{\lambda'}$ and p', we see $\varphi\circ\sigma_1$ and $\varphi\circ\sigma_2$ are geodesic segments joining q and $\varphi(p')$ which are contained in $B_r(q)$. Since $r < \iota_0(q)$, we have $\varphi\circ\sigma_1 = \varphi\circ\sigma_2$. As $\exp_{p_\lambda}(u_1) = \exp_{p_{\lambda'}}(u_2)$, we obtain $\sigma_1 = \sigma_2$, which means that $p_\lambda = p_{\lambda'}$. Therefore we find φ is a covering map. $\square$

Proof of Theorem 5.1. By Corollary 5.1, we have no conjugate points on M. Therefore, $\exp_p : T_pM \to M$ is regular, that is, for an arbitrary $u \in T_pM$, its differential $(d\exp_p)_u : T_u(T_pM) \to T_{\exp_p(u)}M$ is surjective (hence is bijective) linear map. We define $\langle\ ,\ \rangle_R$ by use of the Riemannian metric $\langle\ ,\ \rangle$ on M as

$$\langle \xi, \eta\rangle_R = \bigl\langle (d\exp_p)_u(\xi), (d\exp_p)_u(\eta)\bigr\rangle.$$

Thus, we see $\exp_p$ is a local isometry with respect to $\langle\ ,\ \rangle_R$ and $\langle\ ,\ \rangle$.

We take an arbitrary $u \in T_pM$ and consider a line ℓ_u on T_pM emanating from 0_p which is given by $\ell_u(t) = tu$. As $\sigma(t) = \exp_p(\ell_u(t))$ is a geodesic, $\ell_u(t) = \bigl(\exp_p\bigr)^{-1}(\sigma(t))$, and $\bigl(\exp_p\bigr)^{-1}$ is a local isometry, we see ℓ_u is a geodesic on $(T_pM, \langle\ ,\ \rangle_R)$. As ℓ_u is defined on $\mathbb{R}$ and u is arbitrary, we find that $(T_pM, \langle\ ,\ \rangle_R)$ is complete by Theorem 1.1(Hopf-Renow). Thus, Proposition 5.1 guarantees that $\exp_p$ is a covering map. $\square$

Following to Cartan-Hadamard theorem (Theorem 5.1) we say a simply connected Riemannian manifold of non-positive curvature to be a *Hadamard manifold*. We here study some properties on geodesics on a Hadamard manifold.

Proposition 5.2. *Let M be a complete Riemannian manifold of non-positive curvature. We take an arbitrary point $p \in M$*

(1) *For arbitrary $v \in T_pM$ and $\xi \in T_v(T_pM)$, we have $\|(d\exp_p)_v(\xi)\| \geq \|\xi\|$, where we induce the standard Euclidean metric on $T_v(T_pM)$.*

(2) *For an arbitrary smooth curve $\mu : [a,b] \to T_pM$ in T_pM, we have* $\mathrm{length}(\mu) \leq \mathrm{length}(\exp_p \circ \mu)$. *In particular, if M is simply connected, then we have $d\big(\exp_p(v), \exp_p(w)\big) \geq \|v - w\|$.*

Proof. (1) Let $(-\epsilon, \epsilon) \ni s \mapsto v(s) \in T_pM$ be a smooth curve with $v(0) = v$ and $v'(0) = \xi$. We take a variation $\alpha(t,s) = \exp_p\big(tv(s)\big)$ of geodesics and set $Y(t) = \dfrac{\partial \alpha}{\partial s}(t,0)$, which is a Jacobi field along a geodesic $t \mapsto \exp_p(tv)$. We then have

$$Y(t) = \frac{\partial \alpha}{\partial s}(t,0) = (d\exp_p)_{tv(s)}(tv'(s))|_{s=0} = t(d\exp_p)_{tv}(\xi),$$

hence obtain $Y(1) = (d\exp_p)_v(\xi)$. Since $\mathrm{Riem}^M \leq 0$, we know by Rauch's comparison theorem that $\|Y(t)\|/t$ is monotone increasing. As we have $\xi = (d\exp_p)_0(\xi) = \lim_{t\downarrow 0}(1/t)Y(t)$, we get

$$\|\xi\| \leq \|Y(1)\| = \|(d\exp_p)_v(\xi)\|.$$

(2) By the first assertion we have

$$\begin{aligned}\mathrm{length}(\mu) &= \int_a^b \Big\| \frac{d\mu}{ds}(s) \Big\|\, ds \leq \int_a^b \Big\| \big(d\exp_p\big)_{\mu(s)} \Big(\frac{d\mu}{ds}(s)\Big) \Big\|\, ds \\ &= \int_a^b \Big\| \Big(\frac{d}{ds}\big(\exp_p \circ \mu\big)\Big)(s) \Big\|\, ds = \mathrm{length}(\exp_p \circ \mu).\end{aligned}$$

Since $\exp_p$ is bijective, there exsits a curve $\mu : [0,1] \to T_pM$ such that $\exp_p \circ \mu$ is the unique minimal geodesic segment from $\exp_p(v)$ to $\exp_p(w)$. Thus, we have

$$d\big(\exp_p(v), \exp_p(w)\big) = \mathrm{length}(\exp_p \circ \mu) \geq \mathrm{length}(\mu) \geq \|v - w\|$$

because $\|v - w\|$ is the Euclidean distance between v and w. □

Corollary 5.2. *Let $p, q, r \in M$ are distinct points of a Hadamard manifold M. We denote by $\sigma_{pq}, \sigma_{pr}, \sigma_{qr}$ the minimal geodesics of unit speed from p to q, from p to r and from q to r, respectively. We put* $a = \mathrm{length}(\sigma_{pq})$, $b = \mathrm{length}(\sigma_{rp})$, $c = \mathrm{length}(\sigma_{qr})$ *and* $C = \angle\big(\sigma'_{pq}(0), \sigma'_{pr}(0)\big)$, $A = \angle\big(\sigma'_{pr}(b), \sigma'_{qr}(c)\big)$, $B = \angle\big(-\sigma'_{pq}(a), \sigma'_{qr}(0)\big)$. *Then we have*

(1) $c^2 \geq a^2 + b^2 - 2ab\cos C$ (*Low of cosines*);

(2) $c \leq b\cos A + a\cos B$ (*Double low of cosines*);

(3) $A + B + C \leq \pi$.

Proof. (1) We take $v, w \in T_pM$ so that $\exp_p(v) = q$ and $\exp_p(w) = r$. By the second assertion of Proposition 5.2, we have

$$\begin{aligned} c^2 &\geq \|v - w\|^2 \\ &= \|v\|^2 + \|w\|^2 - 2\langle v, w\rangle \\ &= \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos C \\ &= a^2 + b^2 - 2ab\cos C. \end{aligned}$$

(2) By using the law of cosines, we have

$$a^2 \geq b^2 + c^2 - 2bc\cos A \quad \text{and} \quad b^2 \geq a^2 + c^2 - 2ac\cos B.$$

Adding both sides of these inequalities, we get the second assertion.

(3) By triangle inequality we have $c \leq a + b$, where the equality holds in the case that r is an intermediate point on σ_{pq}. Thus, if $c = a + b$ we have $A + B + C = \pi$. Similarly we get the same equality when either $b = c + a$ or $a = b + c$ holds. We next study the case that these equalities do not hold. In this case we have a triangle on a Euclidean plane $\mathbb{R}^2$ whose edges have lengths a, b, c. We denote its angles by A', B' and C'. For this triangle we have $c^2 = a^2 + b^2 - 2ab\cos C'$ and obtain $C \leq C'$. Similarly we have $A \leq A'$ and $B \leq B'$. Thus we find $A + B + C \leq A' + B' + C' = \pi$. $\square$

Proposition 5.3. *For two geodesics σ_1, σ_2 on a Hadamard manifold M, the function $t \mapsto d\big(\sigma_1(t), \sigma_2(t)\big)$ is a convex function. When M is strictly negative, that is* $\mathrm{Riem}^M \leq c < 0$, *then this function is strictly convex.*

Proof. First we study this function $f(t) = d\big(\sigma_1(t), \sigma_2(t)\big)$ at a point t_0 satisfying $\sigma_1(t_0) \neq \sigma_2(t_0)$. We then have $\sigma_1(t) \neq \sigma_2(t)$ for $t_0 - \epsilon < t < t_0 + \epsilon$ with some positive ϵ. We take a geodesic segment $\gamma_t : [0, 1] \to M$ satisfying $\gamma(0) = \sigma_1(t)$ and $\gamma(1) = \sigma_2(t)$

for $t_0 - \epsilon < t < t_0 + \epsilon$. Since $\mathrm{length}(\gamma_t) = \|\gamma_t'(s)\|$, by putting $\alpha(t,s) = \gamma_t(s)$, we have

$$\begin{aligned} f'(t) &= \frac{d}{dt}\mathrm{length}(\gamma_t) = \frac{d}{dt}\int_0^1 \|\gamma_t'(s)\|\, ds \\ &= \int_0^1 \frac{1}{\|\gamma_t'(s)\|}\Big\langle \Big(\nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial s}\Big)(t,s), \frac{\partial\alpha}{\partial s}(t,s)\Big\rangle\, ds \\ &= \frac{1}{f(t)}\int_0^1 \Big\langle \Big(\nabla_{\frac{\partial\alpha}{\partial s}} \frac{\partial\alpha}{\partial t}\Big)(t,s), \frac{\partial\alpha}{\partial s}(t,s)\Big\rangle\, ds. \end{aligned}$$

Therefore we have

$$f''(t)f(t) + f'(t)^2 = \frac{d}{dt}\int_0^1 \Big\langle \nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\Big\rangle\, ds = \int_0^1 \frac{d}{dt}\Big\langle \nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\Big\rangle\, ds.$$

Here, we have

$$\begin{aligned} \frac{d}{dt}\Big\langle \nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\Big\rangle &= \Big\langle \nabla_{\frac{\partial\alpha}{\partial t}}\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\Big\rangle + \Big\|\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}\Big\|^2 \\ &= \Big\langle \nabla_{\frac{\partial\alpha}{\partial s}}\nabla_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\Big\rangle + \Big\langle R\Big(\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\Big)\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\Big\rangle + \Big\|\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}\Big\|^2 \\ &= \frac{d}{ds}\Big\langle \nabla_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\Big\rangle - \Big\langle \nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}, \nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial s}\Big\rangle \\ &\qquad - \Big\langle R\Big(\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial t}\Big)\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\Big\rangle + \Big\|\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}\Big\|^2. \end{aligned}$$

We put $Y_t = \dfrac{\partial\alpha}{\partial t}(t,s)$, which is a Jacobi field along the geodesic γ_t, and take its component $Y_t^\perp = Y_t - f(t)^{-2}\langle Y_t, \gamma_t'\rangle\gamma_t'$ vertical to γ_t'. We then have

$$\begin{aligned} \nabla_{\frac{\partial\alpha}{\partial s}} Y^\perp &= \nabla_{\frac{\partial\alpha}{\partial s}} Y - \frac{1}{f(t)^2}\Big\langle \nabla_{\frac{\partial\alpha}{\partial s}} Y_t, \gamma_t'\Big\rangle\gamma_t', \\ \Big\langle \nabla_{\frac{\partial\alpha}{\partial s}} Y_t^\perp, \gamma_t'\Big\rangle &= \frac{\partial}{\partial s}\langle Y_t^\perp, \gamma_t'\rangle \equiv 0, \\ \frac{\partial}{\partial s}\Big\langle \nabla_{\frac{\partial\alpha}{\partial s}} Y_t, \gamma_t'\Big\rangle &= \Big\langle \nabla_{\frac{\partial\alpha}{\partial s}}\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}, \gamma_t'\Big\rangle \\ &= \Big\langle \nabla_{\frac{\partial\alpha}{\partial t}}\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial s}\Big\rangle + \Big\langle R\Big(\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial t}\Big)\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial s}\Big\rangle = 0, \end{aligned}$$

hence obtain

$$\Big\|\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}\Big\|^2 = \big\|\nabla_{\frac{\partial\alpha}{\partial s}} Y^\perp\big\|^2 + \frac{1}{f(t)^2}\big\langle \nabla_{\frac{\partial\alpha}{\partial s}} Y_t, \gamma_t'\big\rangle^2.$$

As $\gamma_t(s) = \alpha(t,s)$ is a geodesic for each t, we obtain

$$\begin{aligned}
&f''(t)f(t) + f'(t)^2 \\
&\quad = \Big\langle \Big(\nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial t}\Big)(t,1), \frac{\partial\alpha}{\partial s}(t,1)\Big\rangle - \Big\langle \Big(\nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial t}\Big)(t,0), \frac{\partial\alpha}{\partial s}(t,0)\Big\rangle \\
&\qquad + \int_0^1 \Big\{ \Big\| \nabla_{\frac{\partial\alpha}{\partial s}} \frac{\partial\alpha}{\partial t}\Big\|^2 - \Big\langle R\Big(\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial t}\Big)\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\Big\rangle \Big\}\, ds \\
&\quad = \int_0^1 \Big\{ \big\|\nabla_{\frac{\partial\alpha}{\partial s}} Y_t^{\perp}\big\|^2 + \frac{1}{f(t)^2}\big\langle \nabla_{\frac{\partial\alpha}{\partial s}} Y_t, \gamma_t'\big\rangle^2 - \big\langle R(\gamma_t', Y_t^{\perp})Y_t^{\perp}, \gamma_t'\big\rangle \Big\}\, ds,
\end{aligned}$$

because $t \mapsto \alpha(t,0)$ and $t \mapsto \alpha(t,1)$ are also geodesics. As $\big\langle \nabla_{\frac{\partial\alpha}{\partial s}} Y_t, \gamma_t'\big\rangle$ is constant along γ_t, we have

$$\frac{1}{f(t)^2}\int_0^1 \big\langle \nabla_{\frac{\partial\alpha}{\partial s}} Y_t, \gamma_t'\big\rangle^2\, ds = \frac{1}{f(t)^2}\big\langle \nabla_{\frac{\partial\alpha}{\partial s}} Y_t, \gamma_t'\big\rangle^2 = f'(t)^2.$$

Therefore, as $\mathrm{Riem}^M \le 0$, we obtain

$$f''(t) = \frac{1}{f(t)}\int_0^1 \Big\{ \big\|\nabla_{\frac{\partial\alpha}{\partial s}} Y_t^{\perp}\big\|^2 - \big\langle R(\gamma_t', Y_t^{\perp})Y_t^{\perp}, \gamma_t'\big\rangle \Big\}\, ds \ge 0.$$

Here, when $\mathrm{Riem}^M \le c < 0$ we have $f''(t) > 0$.

Next we study at a point t_0 with $\sigma_1(t_0) = \sigma_2(t_0)$. If $\sigma_1(t) = \sigma_2(t)$ for $t_0 - \epsilon < t \le t_0$ or for $t_0 \le t < t_0 + \epsilon$, we see $\sigma_1 = \sigma_2$. Thus we are enough to consider the case that $\sigma_1(t) \ne \sigma_2(t)$ for $t_0 - \epsilon < t \le t_0 + \epsilon,\ t \ne t_0$. The above argument show that $f''(t) \ge 0$ for $t_0 - \epsilon < t \le t_0 + \epsilon,\ t \ne t_0$. Hence f takes a minimal value 0 at t_0 in the interval $(t - \epsilon_0, t + \epsilon_0)$. Continuity of f'' shows that $f''(t) \ge 0$. We hence get the assertion. □

As a Hadamard manifold M is homeomorphic to a Euclidean space by Corollary 5.1, we consider its compactification. We say two geodesic half-lines $\sigma_1, \sigma_2 : [0,\infty) \to M$ of unit speed on a Hadamard manifold M to be *asymptotic* to each other if the distance function $t \mapsto d(\sigma_1(t), \sigma_2(t))$ is a bounded function. We denote $\sigma_1 \sim \sigma_2$ in this case. This asymptotic relation $\sim$ on the family $\mathcal{G}(N)$ of all geodesic half-lines on M is an equivalence relation. This is because we have

i) $d(\sigma(t), \sigma(t)) \equiv 0$;

ii) $d(\sigma_1(t), \sigma_2(t)) = d(\sigma_2(t), \sigma_1(t))$;

iii) if $t \mapsto d(\sigma_1(t), \sigma_2(t))$ and $t \mapsto d(\sigma_2(t), \sigma_3(t))$ are bounded functions, then the triangle inequality $d(\sigma_1(t), \sigma_3(t)) \le d(\sigma_1(t), \sigma_2(t)) + d(\sigma_2(t), \sigma_3(t))$ shows that $t \mapsto d(\sigma_1(t), \sigma_3(t))$ is also a bounded function.

For a unit tangent vector $u \in UM$ we denote by σ_u the geodesic of initial vector $\dot\sigma(0) = u$. For distinct two unit tangent vectors $u, v \in T_pM$ at an arbitrary point p on a Hadamard manifold M, by Corollary 5.2 we have $d\big(\sigma_u(t), \sigma_v(t)\big) \ge 2t^2 - 2t^2 \cos\angle(u, v)$, hence find that σ_u and σ_v are not asymptotic to each other. We denote by ∂M or by $M(\infty)$ the set $\mathcal{G}/\sim$ of equivalence classes of geodesic half-lines on M. We call ∂M the *ideal boundary* of M. By the above argument we have an injection $\partial\exp_p : U_pM \to \partial M$ defined by $u \mapsto \sigma_u(\infty)$, for arbitrary $p \in M$.

Proposition 5.4. *The map $\partial\exp_p : U_pM \to \partial M$ is a bijection for arbitrary $p \in M$.*

Proof. We are enough to show that it is surjective. For arbitrary $z \in \partial M$ we take a geodesic half-line σ of unit speed with $\sigma(\infty) = z$. For each positive t, we denote by ρ_t the geodesic half-line of unit speed with $\rho(0) = p$ and $\rho(\ell_t) = \sigma(t)$. By Proposition 5.3 we find that $d\big(\rho(s), \sigma(s)\big) \le d\big(p, \sigma(0)\big)$ for $0 \le s \le \ell_t$. We set $u_t = \dot\rho_t(0)$. First we take a sequence $\{t_j\}_{j=1}^{\infty}$ so that $\{\ell_{t_j}\}_{j=1}^{\infty}$ is monotone increasing. Taking its subsequence we get a convergent sequece $\{u_{t_{j_i}}\}_{i=1}^{\infty}$ in U_pM. We set $u_\infty = \lim_{i\to\infty} u_{t_{j_i}}$. Since σ is not bounded, we see $\lim_{i\to\infty} \ell_{t_{j_i}} = \infty$. As we have

$$d\big(\sigma_{u_\infty}(s), \sigma(s)\big) = \lim_{i\to\infty} d\big(\rho_{t_{j_i}}(s), \sigma(s)\big) \le d\big(p, \sigma(0)\big)$$

for $0 \le s < \lim_{i\to\infty} \ell_{t_{j_i}} = \infty$, we find that σ_{u_∞} is asymtotic to σ. Thus we obtain that $\partial\exp_p : U_pM \to \partial M$ is surjective. □

For a geodesic half-line σ, we denote by $\sigma(\infty)$ the asymptotic class containing σ. We put $\overline{M} = M \cup \partial M$. We here introduce a topology on $\overline{M}$ so that its restriction onto M coincides with the original topology of M. We take every open set in M as an open set in $\overline{M}$. In order to define an open set containing elements of ∂M, we define a fundamental system $\mathcal{B}$ of open neighborhoods. For an arbitrary point $p \in M$ and arbitrary positive ϵ, R, we take $u \in U_pM$ so that $\dot\sigma_u(\infty) = z$ and set a set by

$$O_z(p,\epsilon,R) = \big\{\exp_p(tv), \sigma_v(\infty) \,\big|\, v \in U_pM,\ \angle(v,u) < \epsilon,\ t > R\big\}$$

and define $\mathcal{B} = \big\{O_z(p,\epsilon,R) \ \big|\ z \in \partial M,\ p \in M \text{ and } \epsilon > 0, R \geq 0\big\}$. Since $\exp_p$ is a diffeomerphism, we see $O(p,\epsilon,R) \cap M$ is an open subset of M. We shall show that $\mathcal{B}_z$ is a basis $\mathcal{B}_z$ of open neighborhoods around z.

To do this we need the following lemmas. Given distinct three points $p, q_1, q_2 \in \overline{M}$, we put $\angle_p(q_1,q_2) = \angle\big(\dot\sigma_{pq_1}(0), \dot\sigma_{pq_j}(0)\big)$, where σ_{pq_j} is the geodesic satisfying $\sigma_{pq_j}(0) = p$ and $\sigma_{pq_j}(r_j) = q_j$ with some positive r_j.

Lemma 5.1. *Let $\sigma : [0,\infty) \to M$ be a geodesic half-line. We put $\sigma(\infty) = z$. If $t_1 \leq t_2$, $\epsilon_1 \geq \epsilon_2$ and $\epsilon_2 < \pi/2$, then $O_z\big(\sigma(t_2),\epsilon_2,R\big) \subset O_z\big(\sigma(t_1),\epsilon_1,R\big)$.*

Proof. We may suppose $t_1 < t_2$. If $p \in O_z\big(\sigma(t_2),\epsilon_2,R\big)$, we have $\angle_{\sigma(t_2)}(p,z) < \epsilon_2$ and $\angle_{\sigma(t_2)}(\sigma(t_1),p) > \pi - \epsilon_2$. Hence we see

$$\angle_{\sigma(t_1)}(p,z) = \angle_{\sigma(t_1)}(p,\sigma(t_2)) \leq \pi - (\pi - \epsilon_2) - \angle_p(\sigma(t_1),\sigma(t_2)) < \epsilon_2 \leq \epsilon_1$$

by Corollary 5.2 (3). By Corollary 5.2 (1), we have

$$\begin{aligned} d(p,\sigma(t_1))^2 &\geq d(p,\sigma(t_2))^2 + (t_2-t_1)^2 - 2(t_2-t_1)d(p,\sigma(t_2))\cos\angle_{\sigma(t_2)}(\sigma(t_1),p) \\ &> d(p,\sigma(t_2))^2 > R^2, \end{aligned}$$

because $\angle_{\sigma(t_2)}(\sigma(t_1),p) > \pi/2$, we find $p \in O_z\big(\sigma(t_1),\epsilon_1,R\big)$. $\square$

Lemma 5.2. *Given $O_z\big(p,\epsilon,R\big)$ and $q \in M$, $w \in O_z\big(p,\epsilon,R\big) \bigcap \partial M$, we denote by ρ be the geodesic half-line of unit speed with $\rho(0) = q$ and $\rho(\infty) = w$. Then, there are positive T and ϵ' satisfying $O_w\big(\rho(t),\epsilon',0\big) \subset O_z\big(p,\epsilon,R\big)$.*

Proof. Let σ be a geodesic half-line of unit speed satisfying $\sigma(0) = p$ and $\sigma(\infty) = w$. Since σ and ρ are asymptotic to each other, we have

$$2t^2\cos\angle_p(\sigma(t),\rho(t)) \geq 2t^2 - d(\sigma(t),\rho(t))^2$$

by law of cosines (Corollary 5.2 (1)), and $\lim_{t\to\infty}\angle_p(\sigma(t),\rho(t)) = 0$. Similarly, we have

$$\begin{aligned} 2t\{t + d(p,q)\}\cos\angle_{\rho(t)}(p,q) &\geq 2td(p,\rho(t))\cos\angle_{\rho(t)}(p,q) \\ &\geq t^2 + d(p,\rho(t))^2 - d(p,q)^2 \geq t^2 + (t - d(p,q))^2 - d(p,q)^2, \end{aligned}$$

hence $\lim_{t\to\infty} \angle_{\rho(t)}(p,q) = 0$.

We set $\epsilon' = \min\{(\epsilon - \angle_p(z,w))/3, \pi/4\}$. We take a sufficiently large positive T so that $\angle_p(\sigma(t),\rho(t)) < \epsilon'$, $\angle_{\rho(t)}(p,q) < \epsilon'$ and $d(p,\rho(t)) > R$ for $t \geq T$. We take a point $x \in O_w\big(\rho(t),\epsilon',0\big)$. Since $\angle_{\rho(t)}(p,w) = \pi - \angle_{\rho(t)}(p,q) > \pi - \epsilon'$, we have $\angle_{\rho(t)}(p,x) \geq \angle_{\rho(t)}(p,w) - \angle_{\rho(t)}(w,x) > \pi - 2\epsilon'$. Applying Corollary 5.2 (3), we find $\angle_p(\rho(t),x) + \angle_{\rho(t)}(p,x) \leq \pi - \angle_x(\rho(t),p) \leq \pi$, hence obtain $\angle_p(\rho(t),x) < 2\epsilon'$. Therefore we have

$$\begin{aligned}\angle_p(w,x) &\leq \angle_p(w,\rho(t)) + \angle_p(\rho(t),x) = \angle_p(\sigma(t),\rho(t)) + \angle_p(\rho(t),x)\\ &< 3\epsilon' < \epsilon - \angle_p(z,w)\end{aligned}$$

and find that $\angle_p(z,x) \leq \angle_p(z,w) + \angle_p(w,x) < \epsilon$. By the law of cosines, we have

$$\begin{aligned}d(p,x)^2 &\geq d(p,\rho(t))^2 + d(\rho(t),x)^2 - 2d(p,\rho(t)) \times d(\rho(t),x)\cos\angle_{\rho(t)}(p,x)\\ &> d(p,\rho(t))^2 > R^2,\end{aligned}$$

because $\angle_{\rho(t)}(p,x) > \pi/2$. We hence find $x \in O_z\big(p,\epsilon,R\big)$. □

We are now in the position to show that the family $\mathcal{B}$ is a fundamental system of open neighborhoods. We take $O_z\big(p,\epsilon,R\big)$, $O_{z'}\big(p',\epsilon',R'\big)$, and choose a point $w \in O_z\big(p,\epsilon,R\big) \cap O_{z'}\big(p',\epsilon',R'\big) \cap \partial M$ if $O_z\big(p,\epsilon,R\big) \cap O_{z'}\big(p',\epsilon',R'\big)$ is not an empty set. We denote by ρ a geodesic half-line of unit speed satisfying $\rho(0) = p'$ and $\rho(\infty) = w$. We set $\epsilon'' = (\epsilon' - \angle_{p'}(z',w))/2$. By Lemma 5.2, there exist positive t_0, δ satisfying $\delta < \min\{\epsilon,\epsilon''\}$ and $O_w\big(\rho(t_0),\delta,0\big) \subset O_z\big(p,\delta,R\big)$. On the other hand, we have $O_w(p',\epsilon'',R') \subset O_{z'}(p',\epsilon',R')$. As $O_w\big(\rho(t_0),\delta,R'\big) \subset O_w\big(p',\epsilon'',R'\big)$ by Lemma 5.1, we see $O_w\big(\rho(t_0),\delta,R'\big) \subset O_z\big(p,\epsilon,R\big) \cap O_{z'}\big(p',\epsilon',R'\big)$. This guarantees that $\mathcal{B}$ is a fundamental system of open neighborhoods.

We call the topology on $\overline{M}$ determined by $\mathcal{B}$ the *cone topology.*

We show that $\overline{M}$ is a compactification of M.

Proposition 5.5. *At an arbitrary point p of a Hadamard manifold M, the map $f : \overline{B_1(0_p)} = \big\{v \in T_pM \mid \|v\| \leq 1\big\} \to \overline{M}$ defined by*

$$f(v) = \begin{cases}\exp_p\Big(\frac{\|v\|}{1-\|v\|}v\Big), & \textit{when } \|v\| < 1,\\ \sigma_v(\infty), & \textit{when } \|v\| = 1\end{cases}$$

is a homeomorphism. In particular, $\overline{M}$ is compact.

Proof. Since $[0,1) \ni s \to s/(1-s) \in [0,\infty)$ is a homeomorphism, we find that $B_1(0_p) \ni v \mapsto \exp_p\bigl(\frac{\|v\|}{1-\|v\|}v\bigr) \in M$ is a homeomorphism. Thus we are enough to study at the boundary $S_1(0_p) = \partial B_1(0_p) = U_pM$.

As $\sigma_v(\infty) \neq \sigma_w(\infty)$ for $v, w \in U_pM$ with $v \neq w$, we find that f is injective. On the other hand, we take an arbitrary $z \in \partial M$. There is a geodesic half-line $\sigma : [0,\infty) \to M$ of unit speed satisfying $\sigma(\infty) = z$. We take a geodesic σ_n of unit speed joining p and $\sigma(n)$, that is, $\sigma_n(0) = p$ and $\sigma_n(s_n) = \sigma(n)$ with some s_n. By the triangle inequality, we have $|s_n - n| = |d(p,\sigma(n)) - d(\sigma(n),\sigma(0))| \leq d(p,\sigma(0))$. In particular, we have $\lim_{n\to\infty} s_n = \infty$. Since the function $t \mapsto d(\sigma_n(t),\sigma(t))$ is convex by Proposition 5.3, for $0 < s < s_n$ we have

$$\begin{aligned} d(\sigma_n(s),\sigma(s)) &\leq \max\bigl\{d(\sigma_n(0),\sigma(0)),\ d(\sigma_n(s_n),\sigma(s_n))\bigr\} \\ &= \max\bigl\{d(p,\sigma(0)),\ d(\sigma_n(s_n),\sigma(s_n))\bigr\}. \end{aligned}$$

As we have $\sigma_n(s_n) = \sigma(n)$, we see $d(\sigma_n(s_n),\sigma(s_n)) = |s_n - n| \leq d(p,\sigma(0))$, hence find $d(\sigma_n(s),\sigma(s)) \leq d(p,\sigma(0))$. As $\{\dot\sigma_n(0)\}_n \subset U_pM$, we have a convergent subsequence $\{\dot\sigma_{n_j}(0)\}_{j=1}^{\infty}$. We take a geodesic γ with $\dot\gamma(0) = \lim_{j\to\infty}\dot\sigma_{n_j}(0)$. Then we have

$$d(\gamma(s),\sigma(s)) = \lim_{j\to\infty} d(\sigma_{n_j}(s),\sigma(s)) \leq d(p,\sigma(0))$$

for all $s \geq 0$. Thus, we find that there is a bijection of U_pM to ∂M. Hence f is bijective.

We take an arbitrary $u \in U_pM$. For positive ϵ and r with $0 < r < 1$, the set

$$\begin{aligned} U(\epsilon,r) &= \bigl\{v \in \overline{B_1(0_p)} \bigm| \angle(v,u) < \epsilon,\ \|v\| > r\bigr\} \\ &= \bigl\{v \in T_pM \bigm| \angle(v,u) < \epsilon,\ \|v\| > r\bigr\} \bigcap \overline{B_1(0_p)} \end{aligned}$$

is an open set in $\overline{B_1(0_p)}$. Clearly we have $f\bigl(U(\epsilon,r)\bigr) = O(p,\epsilon,r/(1-r))$. As $\overline{B_1(0_q)}$ and $\overline{B_1(0_p)}$ are homeomorphic to each other for every $q \in M$, we see $f^{-1}\bigl(O(q,\epsilon,R)\bigr)$ is an open set in $\overline{B_1(0_p)}$. Since $\{U(\epsilon,r) \mid \epsilon > 0,\ 0 < r < 1\}$ is a basis of open neighborhood of u, we see f is a homeomorphism. $\square$

We now show the relationship between exponential maps of a Hadamard manifold and its ideal boundary.

THEOREM 5.2. *Let M be a Hadamard manifold. Given arbitrary points $p \in M$ and $z \in \partial M$, we have a unique geodesic half-line σ satisfying $\sigma(0) = p$ and $\sigma(\infty) = z$.*

PROOF. First we show the uniqueness. If we have two geodesic half-lines $\sigma_1, \sigma_2 : [0,\infty) \to M$ satisfying $\sigma_1(0) = \sigma_2(0) = p$ and $\sigma_1(\infty) = \sigma_2(\infty) = z$, then by the law of cosine (Corollary 5.2) we have

$$d\big(\sigma_1(t), \sigma_2(t)\big)^2 \geq 2t^2\big\{1 - \cos\big(\dot\sigma_1(0), \dot\sigma_2(0)\big)\big\}.$$

Hence $d\big(\sigma_1(t), \sigma_2(t)\big)$ is bounded if and only if $\dot\sigma_1(0) = \dot\sigma_2(0)$, which means $\sigma_1 = \sigma_2$.

Next we show the existence. We take a geodesic half-line γ whose asymptotic class is z (i.e. $\gamma(\infty) = z$). Let σ_t denotes the geodesic of unit speed joining p and $\gamma(t)$. That is, if we set $d_t = d\big(p, \gamma(t)\big)$, the geodesic σ_t satisfies $\sigma_t(0) = p$ and $\sigma_t(d_t) = \gamma(t)$. By Proposition 5.3 we have

$$\begin{aligned} d\big(\gamma(s), \gamma_t(d_t s/t)\big) &\leq \frac{t-s}{t} d\big(\gamma(0), \sigma_t(0)\big) + \frac{s}{t} d\big(\gamma(t), \sigma_t(d_t)\big) \\ &= \frac{t-s}{t} d\big(\gamma(0), p\big) \leq d\big(\gamma(0), p\big) \end{aligned}$$

for $0 \leq s \leq t$. Since $\{\dot\sigma_t(0)\}_t \subset U_pM$ and U_pM is compact, we can choose a convergent sequence $\{\dot\sigma_{t_j}(0)\}_{j=1}^\infty$ with monotone increasing sequence $\{t_j\}_{j=1}^\infty$ satisfying $\lim_{j\to\infty} t_j = \infty$. We put $\lim_{j\to\infty} \dot\sigma_{t_j}(0) = v \in U_pM$ and take a geodesic half-line σ satisfying $\dot\sigma(0) = v$. By the triangle inequality we have

$$t - d(p, \gamma(0)) \leq d_t \leq t + d(p, \gamma(0)),$$

hence find $\lim_{t\to\infty} d_t/t = 1$. We hence have

$$d\big(\gamma(s), \sigma(s)\big) = \lim_{j\to\infty} d\big(\gamma(s), \sigma_{t_j}(d_{t_j} s/t_j)\big) \leq d\big(\gamma(0).p\big)$$

for $s \geq 0$. Thus we see $\sigma(\infty) = \gamma(\infty) = z$. This completes the proof. $\square$

By this theorem we find that for an arbitrary point p on a Hadamard manifold $\exp_p : T_pM \to M$ induces a bijective map $\partial\exp_p : U_pM \to \partial M$ defined by $u \mapsto \sigma_u(\infty)$, where σ_u is the geodesic with $\dot{\sigma}_u(0) = u$.

2. Asymptotic behaviors of trajectories on a Hadamard manifold

Let M be a Hadamard Kähler manifold whose sectional curvatures are bounded from above as $\mathrm{Riem}^M \le c$ with some negative constant c. We take a Kähler magnetic field $\mathbb{B}_k$ satisfying $k^2 \le |c|$. In this section, we show that every trajectory half-line is unbounded and converges to a point in the ideal boundary of M.

Theorem 5.3. *Let M be a Hadamard Kähler manifold whose sectional curvatures satisfy* $\mathrm{Riem}^M \le c < 0$ *with some constant c, and $\mathbb{B}_k$ be a Kähler magnetic field whose strength satisfies $|k| \le \sqrt{|c|}$. Then each trajectory half-line $\gamma : [0,\infty) \to M$ for $\mathbb{B}_k$ is unbounded and* $\lim_{t\to\infty}\gamma(t)$ *exists in ∂M.*

Since a Hadamard manifold M is diffeomorphic to $\mathbb{R}^n$ through each exponential map, M has no conjugate points. Therefore, for each trajectory half-line γ for $\mathbb{B}_k$ which is not closed and does not have $\gamma(0)$ as self-intersection point, we have a trajectory-harp α_γ associated with γ by Lemma 4.1. Under the assumption of Theorem 5.3, by Theorem 4.2 its string-length and string-cosine satisfy

$$\ell_\gamma(t) \ge \ell_k(t;c) \quad \text{and} \quad \delta_\gamma(t) \ge \delta_k\big(\tau_k(\ell_\gamma(t);c);c\big) \quad \text{for } t \ge 0.$$

As

$$\delta_k(t;c) = \frac{\sqrt{|c|}\,\sinh\sqrt{c-k^2}\,t}{\sqrt{|c|-k^2}\,\sinh\big(\sqrt{|c|}\,\ell_k(t;c)\big)},$$

we see $\delta_k(t;c) > 0$. Hence $\delta_\gamma(t) > 0$, and we find that ℓ_γ is a monotone increasing function. Thus, we see that $\gamma(0)$ is not a self-intersection points of γ.

Since

$$(|c|-k^2)\cosh\big(\sqrt{|c|}\ell_k(t;c)\big) = |c|\cosh\sqrt{|c|-k^2}\,t - k^2,$$

we see $\lim_{t\to\infty}\ell_k(t;c) = \infty$. Hence we find that $\ell_\gamma(t)$ is not a bounded function. This means that γ is not bounded.

We are now in the position to show Theorem 5.3. We denote by $z_\gamma \in \partial M$ the point at infinity of the limit string σ_γ of the trajectory-harp α_γ. For arbitrary positive ϵ, R, if we take sufficiently large T we have $\ell_\gamma(t) \ge R$ and $\angle\Big(\dfrac{\partial\alpha_\gamma}{\partial s}(t,0), \dot\sigma_\gamma(0)\Big) < \epsilon$ for $t \ge T$.

This means that $\gamma(t) \in O_{z_\gamma}\big(\gamma(0), \epsilon, R\big)$ for $t \ge T$. Thus, we find $\lim_{t\to\infty}\gamma(t) = z_\gamma$ and get the conclusion of Theorem 5.3.

We here study more on the behavior of trajectories. Given a trajectory or a trajectory half-line γ for $\mathbb{B}_k$, we denote by $U(\gamma, r)$ the tube $\big\{p \in M \bigm| d(p,\gamma) \le r\big\}$ of radius r around γ. Here, we set $d(p,\gamma) = \inf\{d(p,q) | q \in \mathrm{Image}(\gamma)\}$. Similarly, we denote by $U(\sigma_\gamma, r)$ the tube $\big\{p \in M \bigm| d(p,\sigma_\gamma) \le r\big\}$ of radius r around the limit-string σ_γ of the trajectory-harp associated with γ. For a negative c and a constant k with $|k| < \sqrt{|c|}$, we set $\rho(k;c) = |k|\pi/\big(2\sqrt{|c|(|c|-k^2)}\big)$.

Theorem 5.4. *Let M be a Hadamard Kähler manifold whose sectional curvatures satisfy* $\mathrm{Riem}^M \le c < 0$ *and k be a real number with $|k| < \sqrt{|c|}$. For each trajectory half-line γ for $\mathbb{B}_k$, the harp-body $\mathcal{HB}_\gamma$ of the trajectory-harp α_γ associated with γ is contained in the tube $U\big(\gamma, \rho(k;c)\big)$ around γ, and is contained in the tube $U\big(\sigma_\gamma, \rho(k;c)\big)$ around the limit-string σ_γ.*

Proof. We take arbitrary a, b with $a < b$. By Theorem 4.10 the length of the sector-arc of the harp-sector $\alpha_\gamma^{a,b}$ satisfies

$$\begin{aligned} s\ell_\gamma(a,b) &\le \vartheta_k\big(\tau_k(\ell_\gamma(a);c), \tau_k(\ell_\gamma(b);c);c\big)\mathfrak{s}_k\big(\ell_\gamma(a);c\big) \\ &= \frac{\sinh\sqrt{|c|}\ \ell_\gamma(a)}{\sqrt{|c|}} \angle\Big(\frac{\partial\alpha_{\hat\gamma}}{\partial s}\big(\tau_k(\ell_\gamma(a);c),0\big), \frac{\partial\alpha_{\hat\gamma}}{\partial s}\big(\tau_k(\ell_\gamma(b);c),0\big)\Big), \end{aligned}$$

where $\hat\gamma$ is a trajectory for $\mathbb{B}_k$ on $\mathbb{C}H^n(c)$. Since $\angle\Big(\dfrac{\partial\alpha_{\hat\gamma}}{\partial s}(u,0), \dfrac{\partial\alpha_{\hat\gamma}}{\partial s}\big(\tau_k(\ell_\gamma(a);c),0\big)\Big)$ is monotone increasing for $u \ge \tau_k(\ell_\gamma(a);c)$, we have

$$\begin{aligned} &\angle\Big(\frac{\partial\alpha_{\hat\gamma}}{\partial s}\big(\tau_k(\ell_\gamma(a);c),0\big)\frac{\partial\alpha_{\hat\gamma}}{\partial s}\big(\tau_k(\ell_\gamma(b);c),0\big)\Big) \\ &< \lim_{u\to\infty}\angle\Big(\frac{\partial\alpha_{\hat\gamma}}{\partial s}(u,0), \frac{\partial\alpha_{\hat\gamma}}{\partial s}\big(\tau_k(\ell_\gamma(a);c),0\big)\Big) \\ &= \lim_{u\to\infty}\big\{\cos^{-1}\delta_k(u;c) - \cos^{-1}\delta_k\big(\tau_k(\ell_\gamma(a);c);c\big)\big\}. \end{aligned}$$

As

$$\delta_k(u;c) = \frac{\sqrt{|c|-k^2}\cosh(\sqrt{|c|-k^2}\ u/2)}{\sqrt{|c|\cosh^2(\sqrt{|c|-k^2}\ u/2) - k^2}},$$

we have

$$\begin{aligned}
&\angle\Big(\frac{\partial\alpha_{\hat\gamma}}{\partial s}\big(\tau_k(\ell_\gamma(a);c),0\big)\frac{\partial\alpha_{\hat\gamma}}{\partial s}\big(\tau_k(\ell_\gamma(b);c),0\big)\Big)\\
&\quad=\lim_{u\to\infty}\left\{\cos^{-1}\frac{\sqrt{|c|-k^2}\cosh(\sqrt{|c|-k^2}\;u/2)}{\sqrt{|c|\cosh^2(\sqrt{|c|-k^2}\;u/2)-k^2}}-\cos^{-1}\delta_k\big(\tau_k(\ell_\gamma(a);c);c\big)\right\}\\
&\quad\le\lim_{u\to\infty}\left\{\cos^{-1}\frac{\sqrt{|c|-k^2}\cosh(\sqrt{|c|-k^2}\;u/2)}{\sqrt{|c|\cosh^2(\sqrt{|c|-k^2}\;u/2)}}-\cos^{-1}\delta_k\big(\tau_k(\ell_\gamma(a);c);c\big)\right\}\\
&\quad=\cos^{-1}\frac{\sqrt{|c|-k^2}}{\sqrt{|c|}}-\cos^{-1}\delta_k\big(\tau_k(\ell_\gamma(a);c);c\big).
\end{aligned}$$

As $\theta\le(\pi/2)\sin\theta$ for $0\le\theta\le\pi/2$, we have

$$\begin{aligned}
&\angle\Big(\frac{\partial\alpha_{\hat\gamma}}{\partial s}\big(\tau_k(\ell_\gamma(a);c),0\big)\frac{\partial\alpha_{\hat\gamma}}{\partial s}\big(\tau_k(\ell_\gamma(b);c),0\big)\Big)\\
&\quad\le\frac{\pi}{2}\sin\left\{\cos^{-1}\frac{\sqrt{|c|-k^2}}{\sqrt{|c|}}-\cos^{-1}\delta_k\big(\tau_k(\ell_\gamma(a);c);c\big)\right\}\\
&\quad=\frac{\pi}{2}\left\{\sin\Big(\cos^{-1}\frac{\sqrt{|c|-k^2}}{\sqrt{|c|}}\Big)\delta_k\big(\tau_k(\ell_\gamma(a);c);c\big)\right.\\
&\qquad\qquad\left.-\frac{\sqrt{|c|-k^2}}{\sqrt{|c|}}\sin\big\{\cos^{-1}\delta_k\big(\tau_k(\ell_\gamma(a);c);c\big)\big\}\right\}\\
&\quad=\frac{\pi}{2}\left\{\frac{|k|}{\sqrt{|c|}}\delta_k\big(\tau_k(\ell_\gamma(a);c);c\big)-\frac{\sqrt{|c|-k^2}}{\sqrt{|c|}}\sqrt{1-\delta_k^2\big(\tau_k(\ell_\gamma(a);c);c\big)}\right\}\\
&\quad=\frac{\pi|k|\sqrt{|c|-k^2}}{2\sqrt{|c|}\sqrt{|c|\cosh^2\Big(\dfrac{\sqrt{|c|-k^2}}{2}\tau_k(\ell_\gamma(a);c)\Big)-k^2}}\\
&\qquad\qquad\times\left\{\cosh\Big(\frac{\sqrt{|c|-k^2}}{2}\tau_k(\ell_\gamma(a);c)\Big)-\sinh\Big(\frac{\sqrt{|c|-k^2}}{2}\tau_k(\ell_\gamma(a);c)\Big)\right\}
\end{aligned}$$

Since

$$\begin{aligned}
&\sqrt{|c|-k^2}\sinh\Big(\frac{\sqrt{|c|}\ell_\gamma(t)}{2}\Big)=\sqrt{|c|}\sinh\Big(\frac{\sqrt{|c|-k^2}\,t}{2}\Big)\\
&\cosh\Big(\frac{\sqrt{|c|-k^2}\,\ell_\gamma(t)}{2}\Big)=\sqrt{1+\frac{|c|-k^2}{|c|}\sinh^2\Big(\frac{\sqrt{|c|-k^2}\,\ell_\gamma(t)}{2}\Big)},
\end{aligned}$$

we have

$$\begin{aligned}
&\angle\Big(\frac{\partial\alpha_{\hat\gamma}}{\partial s}\big(\tau_k(\ell_\gamma(a);c),0\big)\frac{\partial\alpha_{\hat\gamma}}{\partial s}\big(\tau_k(\ell_\gamma(b);c),0\big)\Big)\\
&\quad=\frac{\pi|k|\sqrt{|c|-k^2}}{2|c|\cosh\Big(\dfrac{\sqrt{|c|}\,\ell_\gamma(a)}{2}\Big)}\\
&\qquad\qquad\times\left\{\sqrt{\frac{|c|}{|c|-k^2}+\sinh^2\Big(\frac{\sqrt{|c|}\,\ell_\gamma(a)}{2}\Big)}-\sinh\Big(\frac{\sqrt{|c|}\,\ell_\gamma(a)}{2}\Big)\right\}\\
&\quad=\frac{\pi|k|\sqrt{|c|-k^2}\times\dfrac{|c|}{|c|-k^2}}{2|c|\cosh\Big(\dfrac{\sqrt{|c|}\,\ell_\gamma(a)}{2}\Big)\left\{\sqrt{\dfrac{|c|}{|c|-k^2}+\sinh^2\Big(\dfrac{\sqrt{|c|}\,\ell_\gamma(a)}{2}\Big)}+\sinh\Big(\dfrac{\sqrt{|c|}\,\ell_\gamma(a)}{2}\Big)\right\}}\\
&\quad\le\frac{|k|\pi}{2\sqrt{|c|-k^2}}\times\frac{1}{\sinh\sqrt{|c|}\,\ell_\gamma(a)}.
\end{aligned}$$

Then we have

$$s\ell_\gamma(a,b)\le\frac{\sinh\sqrt{|c|}\;\ell_\gamma(a)}{\sqrt{|c|}}\frac{|k|\pi}{2\sqrt{|c|-k^2}}\times\frac{1}{\sinh\sqrt{|c|}\;\ell_\gamma(a)}=\frac{|k|\pi}{2\sqrt{|c|}\sqrt{|c|-k^2}}.$$

Hence we find

$$d\big(\alpha_\gamma(t,\ell_\gamma(a)),\gamma\big)\le s\ell_\gamma(a,t)\le\rho(k;c)\quad\text{and}\quad d\big(\alpha_\gamma(t,\ell_\gamma(a)),\sigma_\gamma\big)\le\rho(k;c),$$

for $t\ge a$. Since a is a arbitrary, we get the conclusion. $\square$

For a trajectory-harp α_γ, we denote by σ_γ^t the geodesic half-line $s\mapsto\alpha_\gamma(t,s)$.

Remark 5.1. Under the same conditions as in Theorem 5.4, its proof shows that $d\big(\sigma_\gamma^{t_1}(s),\sigma_\gamma^{t_2}(s)\big)\le\rho(k;c)$ for $0<t_1<t_2$ and $0<s\le\ell_\gamma(t_1)$. This guarantees $d\big(\sigma_\gamma^t(s),\sigma_\gamma(s)\big)\le\rho(k;c)$ for $t>0$ and $0<s\le\ell_\gamma(t)$.

3. Magnetic exponential maps on Hadamard manifolds

In this section we generalize Hopf-Renow theorem (Theorem 1.1) and Cartan-Hadamard theorem (Theorem 5.1) to trajectories for Kähler magnetic fields following to [**5**]. For arbitrary distinct points p, q on a connected complete Riemannian manifold there is a minimizing geodesic joining them (Hopf-Renow Theorem). We consider this property for trajectories for Kähler magnetic fields. Since trajectories for non-trivial Kähler magnetic fields on a complex Euclidean space are circles and are closed, we see that this property can not be generalized to general Kähler manifolds. As we showed in §5.2 that on a Hadamard Kähler manifold whose sectional curvatures satisfy $\mathrm{Riem}^M \le c < 0$ trajectories for $\mathbb{B}_k$ with $|k| \le \sqrt{|c|}$ are unbounded, we study the property of Hopf-Renow type under such assumptions.

Theorem 5.5 ([**5**]). *Let $\mathbb{B}_k$ be a Kähler magnetic field on a connected complete Kähler manifold M whose sectional curvatures satisfy* $\mathrm{Riem}^M \le c < 0$*. If $|k| \le \sqrt{|c|}$, for arbitrary distinct points $p, q \in M$, there is a minimizing trajectory for $\mathbb{B}_k$ which goes from p to q. In particular, when M is simply connected, there exists a unique trajectory for $\mathbb{B}_k$ of p to q.*

Theorem 5.6 ([**5**]). *Let $\mathbb{B}_k$ be a Kähler magnetic field on a connected complete Kähler manifold M whose sectional curvatures satisfy* $\mathrm{Riem}^M \le c < 0$*. If $|k| \le \sqrt{|c|}$, every magnetic exponential map $\mathbb{B}_k\exp_p : T_pM \to M$ is a covering map. In particular, when M is simply connected, every magnetic exponential map is a diffeomorphism.*

Proof of Theorem 5.5 (Existence). First we consider the case that M is a Hadamard Kähler manifold. Given a point $p \in M$ we shall show that the magnetic exponential map $\mathbb{B}_k\exp_p : T_pM \to M$ is surjective. To show this, we are enough to show that the image of $\mathbb{B}_k\exp_p$ is an open set and a closed set, because M is a connected manifold and as $\exp_p(0_p) = p$ the image is not an empty set. Since we do not have magnetic conjugate points for $\mathbb{B}_k$ with $|k| \le \sqrt{|c|}$, by implicit function theorem, we see that the image of $\mathbb{B}_k\exp_p$ is an open set. On the other hand, when a sequence of points $\{q_j\}_{j=1}^{\infty}$ $\big(\subset \mathbb{B}_k\mathrm{exp}_p(\mathbb{R})\big)$ converges to a point $q \in M$ $(p \neq p)$, we denote as $q_j = \mathbb{B}_k\mathrm{exp}_p(r_j v_j)$ with unit tangent vectors $v_j \in U_pM$ and $r_j > 0$. Since $\lim_{j\to\infty} d(p, q_j) = d(p, q)$, we have a positive ℓ with $d(p, q_j) \le \ell$ for all j. For each trajectory-segment γ_{v_j} for $\mathbb{B}_k$ given by $\gamma_{v_j} = \mathbb{B}_k\mathrm{exp}_p(tv_j)$ for $0 \le t \le r_j$, we consider its trajectory-harp. Since M is a Hadamard manifold, its string at q_j is the unique geodesic of unit speed joining p and q_j, hence we have $\ell_{\gamma_{v_j}}(r_j) = d(p, q_j)$. By Theorem 4.2 we have $\ell_{\gamma_{v_j}}(r_j) \ge \ell_k(r_j; c)$ and $\ell_{\gamma_{v_j}}$ is monotone increasing. We hence have $r_j \le \tau_k(\ell; c)$ for all j, where $\tau_k(\cdot; c)$ is the inverse function of $\ell_k(\cdot; c)$. We hence find that $\{r_j\}_{j=1}^{\infty}$ is bounded from above. We note $r_j \ge d(p, q_j) > 0$. Thus $\{r_j\}_{j=1}^{\infty}$ is a bounded sequence.

Since U_pM is compact, there is a subsequence $\{j_k\}_{k=1}^{\infty}$ such that both $\{v_{j_k}\}_{k=1}^{\infty}$ and $\{r_{j_k}\}_{k=1}^{\infty}$ converge. We set $v_0 = \lim_{k\to\infty} v_{j_k}$ and $r_0 = \lim_{k\to\infty} r_{j_k}$. We then have $q = \mathbb{B}_k\mathrm{exp}_p(r_0 v_0)$, We hence find that the image of $\mathbb{B}_k\mathrm{exp}_p$ is closed. Thus the connectedness of M guarantees that $\mathbb{B}_k\mathrm{exp}_p$ is surjective. Therefore, for an arbitrary point $q \in M$, we have a trajectory-segment for $\mathbb{B}_k$ from p to q.

Given distinct points $p, q \in M$, when we have finite trajectory segments for $\mathbb{B}_k$ from p to q, we can take the trajectory-segment of minimizing length. When we have infinite trajectory segments for $\mathbb{B}_k$ from p to q, we take a sequence $\{\gamma_j\}_{j=1}^{\infty}$ of such trajectory-segments such that $\lim_{j\to\infty} \mathrm{length}(\gamma_j)$ shows the infimum of lengths of such trajectory-segments. We set $\dot\gamma_j(0) = u_j$ and define r_j by $\gamma_j(r_j) = q$. Since $r_j = \mathrm{length}(\gamma_j) > 0$, we see that $\{r_j\}_{j=1}^{\infty}$ is bounded (at least $r_j \le \tau_k\big(d(p, q); c)\big)$. Thus we have a subsequence $\{j_k\}_{k=1}^{\infty}$ such that both $\{u_{j_k}\}_{k=1}^{\infty}$ and $\{r_{j_k}\}_{k=1}^{\infty}$ converge. Putting $u_0 = \lim_{k\to\infty} u_{j_k}$ and $r_0 = \lim_{k\to\infty} r_{j_k}$, we have $q = \mathbb{B}_k\mathrm{exp}_p(r_0 u_0)$. As $r_{j_k} = \mathrm{length}(\gamma_{j_k})$, we find that the trajectory-segment $t \mapsto \mathbb{B}_k\mathrm{exp}_p(tu_0)$ $(0 \le t \le r_0)$ is a minimal trajectory-segment from p to q.

Next we study the case that M is not simply connected. We take its universal covering $\varpi : \widetilde{M} \to M$, which is a Hadamard Kähler manifold. We note that for a trajectory γ for $\mathbb{B}_k$ on M, every smooth curve $\tilde\gamma$ satisfying $\gamma(t) = \varpi \circ \tilde\gamma$, which is called a covering trajectory, is a trajectory for $\mathbb{B}_k$ on $\widetilde{M}$, because of the uniqueness of solutions for differential equations.

Given distinct points $p, q \in M$ we choose a point $\tilde{p}_0 \in \widetilde{M}$ with $\varpi(\tilde{p}_0) = p$. For each $\tilde{q}_\lambda \in \varpi^{-1}(q)$, we have a minimal trajectory-segment γ_λ from $\tilde{p}_0$ to $\tilde{q}_\lambda$. We show that $\mathcal{L} = \{\mathrm{length}(\gamma_\lambda) \mid \lambda\}$ takes the minimum value. Suppose we have a sequence γ_{λ_j} satisfying $\lim_{j\to\infty} \mathrm{length}(\gamma_{\lambda_j}) = \inf \mathcal{L}$. As above, we set $\dot{\gamma}_{\lambda_j}(0) = w_j$ and $r_j = \mathrm{length}(\gamma_{\lambda_j})$. We choose a convergent subsequence and put $w_0 = \lim_{k\to\infty} w_{j_k}$ and $r_0 = \lim_{k\to\infty} r_{j_k}$. Since $\varpi(\tilde{q}_\lambda) = q$, we have

$$\begin{aligned}\varpi\big(\mathbb{B}_k\mathrm{exp}_{\tilde{p}_0}(r_0 w_0)\big) &= \varpi\big(\lim_{j\to\infty} \mathbb{B}_k\mathrm{exp}_{\tilde{p}_0}(r_{j_k} w_{j_k})\big)\\ &= \lim_{j\to\infty} \varpi\big(\mathbb{B}_k\mathrm{exp}_{\tilde{p}_0}(r_{j_k} w_{j_k})\big) = \lim_{j\to\infty} \varpi(\tilde{q}_{\lambda_{j_k}}) = q.\end{aligned}$$

This shows that the trajectory segment $\tilde{\gamma}_0$ given by $t \mapsto \mathbb{B}_k\mathrm{exp}_{\tilde{p}_0}(w_0)$ $(0 \le t \le r_0)$ is a minimal trajectory-segment from $\tilde{p}_0$ to some $\tilde{q}_\lambda$. Therefore $\varpi \circ \tilde{\gamma}_0$ is a minimal trajectory-segment from p to q. $\square$

Proof of Theorem 5.6. We are enough to consider the case that M is a Hadam -ard Kähler manifold. Since there are no magnetic conjugate points for $\mathbb{B}_k$ by Corollary 3.1, we see $\mathbb{B}_k\mathrm{exp}_p : T_pM \to M$ is regular, that is, $\big(d\mathbb{B}_k\mathrm{exp}_p\big)_v : T_v(T_pM) \to T_{\mathbb{B}_k\mathrm{exp}_p(v)}M$ is a linear isomorphism at each $v \in T_pM$. We define an inner product $\langle\ ,\ \rangle_R$ on $T_v(T_pM)$ by

$$\langle \xi, \eta\rangle_R = \big\langle \big(d\mathbb{B}_k\mathrm{exp}_p\big)_v(\xi), \big(d\mathbb{B}_k\mathrm{exp}_p\big)_v\big\rangle,$$

and a linear map $J_R :$ by $J_R(\xi) = \big(\big(d\mathbb{B}_k\mathrm{exp}_p\big)_v\big)^{-1}\big(J\big(d\mathbb{B}_k\mathrm{exp}_p\big)_v(\xi)\big)$. Then $\mathbb{B}_k\mathrm{exp}_p$ is a local holomorphic isometry with respect to $(\langle\ ,\ \rangle_R, J_R)$ and $(\langle\ ,\ \rangle, J)$. We denote by ∇^R the Riemannian connection on T_pM with respect to $\langle\ ,\ \rangle_R$. As $\mathbb{B}_k\mathrm{exp}_p$ is a local isometry, for arbitrary vector fields $\widehat{X}, \widehat{Y} \in \mathfrak{X}(T_pM)$ and arbitrary $v \in T_pM$, considering an open neighborhood $\mathcal{U}$ if v and a neighborhood U of $\mathbb{B}_k\mathrm{exp}_p(v)$ we have

$$\big(\nabla^R_{\widehat{X}}\widehat{Y}\big)(v) = \Big(\big(d\mathbb{B}_k\mathrm{exp}_p\big)_v\Big)^{-1}\Big(\big(\nabla_X Y\big)\big(\mathbb{B}_k\mathrm{exp}_p(v)\big)\Big),$$

where $X, Y \in \mathfrak{X}(U)$ is defined by

$$X(q) = \big(d\mathbb{B}_k\exp_p\big)_{w_q}\big(\widehat{X}(w_q)\big) \qquad \text{and} \qquad Y(q) = \big(d\mathbb{B}_k\exp_p\big)_{w_q}\big(\widehat{Y}(w_q)\big)$$

with $w_q := \big(\mathbb{B}_k\exp_p\big|_{\mathcal{U}}\big)^{-1}(q)$. Since we have

$$\begin{aligned} J_R\widehat{Y}(v) &= \big(\big(d\mathbb{B}_k\exp_p\big)_v\big)^{-1}\Big(J\Big(\big(d\mathbb{B}_k\exp_p\big)_v\big(\widehat{Y}(v)\big)\Big)\Big) \\ &= \Big(\big(d\mathbb{B}_k\exp_p\big)_v\Big)^{-1}\big(JY(\mathbb{B}_k\exp_p(v))\big), \end{aligned}$$

we obtain

$$\begin{aligned} \big(\nabla^R_{\widehat{X}}(J_R\widehat{Y})\big)(v) &= \Big(\big(d\mathbb{B}_k\exp_p\big)_v\Big)^{-1}\Big(\big(\nabla_X(JY)\big)\big(\mathbb{B}_k\exp_p(v)\big)\Big) \\ &= \Big(\big(d\mathbb{B}_k\exp_p\big)_v\Big)^{-1}\Big(J\big(\nabla_X Y\big)\big(\mathbb{B}_k\exp_p(v)\big)\Big) \\ &= J_R\bigg(\Big(\big(d\mathbb{B}_k\exp_p\big)_v\Big)^{-1}\Big(J\big(\nabla_X Y\big)\big(\mathbb{B}_k\exp_p(v)\big)\Big)\bigg) \\ &= J_R\Big(\big(\nabla^R_{\widehat{X}}\widehat{Y}\big)(v)\Big), \end{aligned}$$

hence find that T_pM is a Kähler manifold with respect to $(\langle\ ,\ \rangle, J_R)$.

We take an arbitrary $u \in T_pM$ and consider a line $\hat{\gamma}_u$ on T_pM from 0_p defined by $\hat{\gamma}_u(t) = tu$. If we set $\gamma(t) = \mathbb{B}_k\exp_p\big(\hat{\gamma}_u(t)\big)$, then we have

$$\begin{aligned} \big(\nabla^R_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}}\big)(t) &= \Big(\big(d\mathbb{B}_k\exp_p\big)_{\hat{\gamma}(t)}\Big)^{-1}\Big(\big(\nabla_{\dot{\gamma}}\dot{\gamma}\big)(t)\Big) = \Big(\big(d\mathbb{B}_k\exp_p\big)_{\hat{\gamma}(t)}\Big)^{-1}\big(kJ\dot{\gamma}(t)\big) \\ &= kJ_R\bigg(\Big(\big(d\mathbb{B}_k\exp_p\big)_{\hat{\gamma}(t)}\Big)^{-1}\big(\dot{\gamma}(t)\big)\bigg) = kJ_R\dot{\hat{\gamma}}(t), \end{aligned}$$

hence find that $\hat{\gamma}$ is a trajectory for $\mathbb{B}_k$ on T_pM. We show that the origin $0_p \in T_pM$ and $su \in T_pM$ with $s > 0$ is joined by a unique geodesic-segment of unit speed on $(T_pM, \langle\ ,\ \rangle_R)$. First we suppose there are two geodesic-segment $\hat{\sigma}_1, \hat{\sigma}_2$ on T_pM of unit speed from 0_p to su. Then $\mathbb{B}_k\exp_p \circ \hat{\sigma}_j$ $(j = 1, 2)$ are geodesic on M from p to $\mathbb{B}_k\exp_p(su)$. Since M is a Hadamard manifold, we find that $\mathbb{B}_k\exp_p \circ \hat{\sigma}_1$ and $\mathbb{B}_k\exp_p \circ \hat{\sigma}_2$ coincide with each other. In particular, we have

$$\frac{d}{dt}\big(\mathbb{B}_k\exp_p \circ \hat{\sigma}_1\big)\big|_{t=0} = \frac{d}{dt}\big(\mathbb{B}_k\exp_p \circ \hat{\sigma}_2\big)\big|_{t=0},$$

which is equivalent to

$$\big(d\mathbb{B}_k\exp_p\big)_{0_p}\big(\dot{\hat{\sigma}}_1(0)\big) = \big(d\mathbb{B}_k\exp_p\big)_{0_p}\big(\dot{\hat{\sigma}}_2(0)\big).$$

As $\left(d\mathbb{B}_k\exp_p\right)_{0_p}$ is bijective, we see $\dot{\hat\sigma}_1(0) = \dot{\hat\sigma}_2(0)$. By the uniqueness of solutions of differential equations, we find $\hat\sigma_1 = \hat\sigma_2$. Thus we need to show the existence. Since $\mathbb{B}_k\exp_p$ is a local isometry, there is positive ϵ such that when $0 < t < \epsilon$ we can join 0_p and tu by a geodesic-segment of unit speed. We set t_* the maximal positive number satisfying that for all $0 < t < t_*$ we have a geodesic-segment joining 0_p and su. Suppose $t_* < \infty$. We take a trajectory γ_u for $\mathbb{B}_k$ on M. Since M is a Hadamard manifold, we have a unique trajectory-harp α_{γ_u} associated with γ_u. We take positive ϵ'. Since the set $\{\alpha_{\gamma_u}(t,s) \mid 0 \le t \le t_* + \epsilon',\ 0 \le s \le \ell_{\gamma_u}(t)\}$ is compact, it is covered by finite open subsets $U_j,\ j = 1,\dots,N$ in M such that $\mathbb{B}_k\exp_p|_{\mathfrak{U}_j} : \mathfrak{U}_j \to U_j$ is an isometry on some open subset $\mathfrak{U}_j$ in T_pM. Here, we take $\mathfrak{U}_j$ so that $\bigcup_j \mathfrak{U}_j$ contains the geodesic on T_pM which joins 0_p and tu for all $0 < t < t_*$ and that $\bigcup_{j=1}^N \mathfrak{U}_j$ is connected. We can then take a geodesic $\hat\sigma_t$ on T_pM satisfying $\mathbb{B}_k\exp_p \circ \hat\sigma_t(s) = \alpha_{\gamma_u}(t,s)$ for $t_* - \epsilon' < t < t + \epsilon'$ and $0 \le s \le \ell_{\gamma_u}(t)$ by taking the inverse images of $s \mapsto \alpha_{\gamma_u}(t,s)$ through some of $\mathbb{B}_k\exp_p|_{\mathfrak{U}_j}$'s. Thus we have a geodesic-segment of unit speed which joins 0_p and tu for $t_* \le t < t_* + \epsilon'$. This is a contradiction to the choice of t_*. Hence we find that $t_* = \infty$.

We now show that T_pM is complete. We take a Cauchy sequence $\{w_j\}_{j=1}^\infty \subset T_pM$ with respect to the distance function d_R induced by $\langle\ ,\ \rangle$. We may suppose $w_j \neq 0_p$. We take the unique geodesic segment $\hat\sigma_j$ of unit speed on T_pM from 0_p to w_j. By the above argument we have such a geodesic. We set $\hat u_j := \dot{\hat\sigma}_j(0) \in T_{0_p}(T_pM)$ and take r_j so that $\hat\sigma_j(r_j) = w_j$. Since $\{w_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to d_R, we find that $\{r_j\}_{j=1}^\infty$ is bounded. So there is a positive ℓ with $r_j \le \ell$ for all j. As $\hat\gamma_j = \hat\gamma_{w_j/\|w_j\|}$ defined by $t \mapsto tw_j/\|w_j\|$ $(0 \le t \le \|w_j\|)$ is a trajectory-segment for $\mathbb{B}_k$ on T_pM, we can consider its trajectory-harp by joining 0_p and $tw_j/\|w_j\|$ by a geodesic which we showed in the above. As r_j is the string-length of this trajectory-harp at w_j and $\|w_j\|$ is the length of this trajectory-segment, we have $\|w_j\| \le \tau_k(\ell; c)$ by Theorem 4.2, where $\tau_k(\cdot; c)$ is the inverse function of $\ell_k(\cdot; c)$. Thus $\{w_j\}_{j=1}^\infty$ is bounded with respect to ordinary norm on $T_pM \cong \mathbb{C}^n$. Since $U_{0_p}(T_pM)$ is compact, we have a subsequence $\{j_k\}_{k=1}^\infty$ satisfying the following conditions:

i) $\{r_{j_k}\}_{k=1}^{\infty}$ converges,

ii) $\{\hat{u}_{j_k}\}_{k=1}^{\infty}$ converges in $U_{0_p}(T_pM)$,

iii) $\{w_{j_k}\}_{k=1}^{\infty}$ converges in T_pM with respect to the ordinary norm.

Since $w_{j_k} = \hat{\sigma}_{j_k}(r_{j_k}u_{j_k})$, we see $\{w_{j_k}\}_{k=1}^{\infty}$ also converges with respect to d_R. Thus, we find that $\{w_{j_k}\}_{k=1}^{\infty}$ converges in T_pM with respect to d_R. As $\{w_j\}_{j=1}^{\infty}$ is a Cauchy sequence with respect to d_R, it converges in T_pM. Hence we find that T_pM is complete with respect to d_R.

Thus we find that $\mathbb{B}_k\exp_p : (T_pM, \langle\ ,\ \rangle_R) \to (M, \langle\ ,\ \rangle)$ is a local isometry between complete connected Riemannian manifolds, it is a covering map by Proposition 5.1. This complete the proof. $\square$

Proof of Theorem 5.5 (Uniquness). When M is a Hadamard Kähler manifold, for given a point $p \in M$, the magnetic exponential map $\mathbb{B}_k\exp_p : T_pM \to M$ is a diffeomorphism by Theorem 5.6, for each $q \in M$ with $q \neq p$ there is a unique $v \in T_pM$ with $v \neq 0_p$ satisfying $\mathbb{B}_k\exp_p(v) = q$. Thus the trajectory-segment $t \mapsto \mathbb{B}_k\exp_p(tv/\|v\|)$ $(0 \leq t \leq \|v\|)$ is the unique trajectory-segment from p to q. This complete the proof of Theorem 5.5. $\square$

4. Trajectory-horn

In order to study the behavior of trajectories, we studied trajectory-harps in Chapter IV. A trajectory-harp consists of a trajectory and geodesics. To study more on trajectories we study a family of trajectories associated with a given geodesic.

Let M be a *Hadamard manifold* whose sectional curvatures satisfy $\mathrm{Riem}^M \le c < 0$ with some constant c. Given a geodesic half-line $\sigma : [0,\infty) \to M$ we define a variation $\beta_{\sigma,k} : [0,\infty)\times\mathbb{R} \to M$ of trajectories for $\mathbb{B}_k$ with $|k| \le \sqrt{|c|}$ by the following condition:

1) $\beta_{\sigma,k}(0,s) = \sigma(0)$ for every s;
2) when $s = 0$, the curve $t \mapsto \beta_{\sigma,k}(t,0)$ is the trajectory for $\mathbb{B}_k$ with initial vector $\dot\sigma(0)$;
3) when $s > 0$, the curve $t \mapsto \beta_{\sigma,k}(t,s)$ is the trajectory for $\mathbb{B}_k$ joining $\sigma(0)$ and $\sigma(s)$.

We note that as M is a Hadamard manifold we have $\sigma(s) \ne \sigma(0)$ for $s > 0$ and that by Theorem 5.5 there exists a unique trajectory which joins $\sigma(0)$ and $\sigma(s)$. Therefore we can define such a variation of trajectories uniquely. We call this the *trajectory-horn* for $\mathbb{B}_k$ associated with σ, and call a trajectory $t \mapsto \beta_\sigma(t,s)$ a horn-tube of this trajectory-horn.

In order to measure the size of a trajectory-horn $\beta_{\sigma,k}$ we consider the following quantities. We denote by $r_{\sigma,k}(s)$ the arc-length of the trajectory segment $t \mapsto \beta_\sigma(t,s)$ from $\sigma(0)$ to $\sigma(s)$, and call it a *tube-length* at s. Since for each s the geodesic-segment $\sigma|_{[0,s]}$ is the minimal geodesic joining $\sigma(0)$ and $\sigma(s)$, we have $r_{\sigma,k}(s) \ge s$. We set $\epsilon_{\sigma,k}(s) = \Big\langle \dot\sigma(s), \dfrac{\partial\beta_\sigma}{\partial t}\big(s, r_{\sigma,k}(s)\big)\Big\rangle$ and call it a *tube-cosine* at $\sigma(s)$.

For a trajectory-horn $\beta_{\sigma,k}$ for $\mathbb{B}_k$ associated with a geodesic half-line σ on a Hadamard manifold M, we denote by γ_s the trajectory half-line $t \mapsto \beta_{\sigma,k}(t,s)$. As M is a Hadamard manifold, its injectivity radius is $\iota(M) = \infty$, we can define a trajectory-harp α_{γ_s} associated with γ_s and find that its harp-string at s coincides with σ. Thus we have

$$s = \ell_{\gamma_s}\big(r_{\sigma,k}(s)\big) \quad \text{and} \quad \epsilon_{\sigma,k}(s) = \delta_{\gamma_s}\big(r_{\sigma,k}(s)\big),$$

where ℓ_{γ_s} and δ_{γ_s} are the string-length and string-cosine of α_{γ_s} (see §4.1).

We here study trajectory-horns for $\mathbb{B}_k$ with $|k| \le \sqrt{|c|}$ on a complex hyperbolic space $\mathbb{C}H^n(c)$. Given a geodesic half-line σ we take a totally geodesic $\mathbb{C}H^1(c)$ containing the image of σ. Since this $\mathbb{C}H^1$ is totally geodesic, we see that all tubes of the trajectory-horn lie on this $\mathbb{C}H^1$. Thus, we find that two trajectory-horns for $\mathbb{B}_k$ are congruent to each other by a holomorphic isometry (Proposition 2.2). We hence express by $r_k(s;c)$ and $\epsilon_k(s;c)$ tube-lengths and tube-cosines of trajectory-horns for $\mathbb{B}_k$ on $\mathbb{C}H^n(c)$. As we see in Proposition 4.6, functions of string-length and string-cosine of trajectory-harps for $\mathbb{B}_k$ on $\mathbb{C}H^n(c)$ are given by

$$\ell_k(t;c) = \begin{cases} \dfrac{2}{\sqrt{|c|}} \sinh^{-1} \dfrac{\sqrt{|c|}\sinh\big(\sqrt{|c|-k^2}\; t/2\big)}{\sqrt{|c|-k^2}}, & \text{if } |k| < \sqrt{|c|}, \\[2ex] \dfrac{2\sinh^{-1}(\sqrt{|c|}\; t/2)}{\sqrt{|c|}}, & \text{if } k = \pm\sqrt{|c|}, \end{cases}$$

and

$$\delta_k(t;c) = \begin{cases} \dfrac{\sqrt{|c|-k^2}\cosh\big(\sqrt{|c|-k^2}\, t/2\big)}{\sqrt{|c|\cosh^2\big(\sqrt{|c|-k^2}\, t/2\big) - k^2}}, & \text{if } |k| < \sqrt{|c|}, \\[2ex] \dfrac{2}{\sqrt{|c|t^2+4}}, & \text{if } k = \pm\sqrt{|c|}. \end{cases}$$

Therefore those functions $r_k(s;c)$, $\epsilon_k(s;c)$ are given as

$$r_k(s;c) = \begin{cases} \dfrac{2}{\sqrt{|c|-k^2}} \sinh^{-1}\Big\{\sqrt{|c|-k^2}\sinh\big(\sqrt{|c|}\; s/2\big)\big/\sqrt{|c|}\Big\}, & \text{if } |k| < \sqrt{|c|}, \\[2ex] \dfrac{2}{\sqrt{|c|}}\sinh\big(\sqrt{|c|}\; s/2\big), & \text{if } k = \pm\sqrt{|c|}, \end{cases}$$

and

$$\epsilon_k(s;c) = \begin{cases} \sqrt{1 - \dfrac{k^2}{|c|}\tanh^2 \dfrac{\sqrt{|c|}}{2}s}, & \text{if } |k| < \sqrt{|c|}, \\[2ex] \dfrac{1}{\cosh \dfrac{\sqrt{|c|}}{2}s}, & \text{if } k = \pm\sqrt{|c|}. \end{cases}$$

We note that $r_k(s;c)$ coincides with the inverse function $\tau_k(s;c)$ of the function $\ell_k(t;c)$.

By using these functions we can estimate tube-lengths and tube-cosines as follows.

Proposition 5.6. *Let σ be a geodesic on a Hadamard Kähler manifold M whose sectional curvature satisfy $\mathrm{Riem}^M \le c < 0$ for some constant c. We take the trajectory-horn $\beta_{\sigma,k}$ for $\mathbb{B}_k$ with $|k| \le \sqrt{|c|}$ which is associated with σ. We then have the following for $s \ge 0$:*

(1) *Tube-length satisfies $s \le r_{\sigma,k}(s) \le \tau_k(s;c)$;*

(2) *Tube-cosine satisfies $\epsilon_{\sigma,k}(s) \ge \epsilon_k(s;c)$.*

Proof. We denote by γ_s the trajectory half-line $t \mapsto \beta_{\sigma,k}(t,s)$. Since M is a Hadamard manifold, we have a trajectory-harp α_{γ_s} associated with γ_s. The harp-string of α_{γ_s} at s coincides with σ. By the comparison theorem on trajectory-harps (Theorem 4.2), we have

$$\ell_k\big(\tau_k(s;c);c\big) = s = \ell_{\gamma_s}\big(r_{\sigma,k}(s)\big) \ge \ell_k\big(r_{\sigma,k}(s);c\big).$$

As $\ell_k(\cdot;c)$ is monotone increasing, we get the first assertion.

By Theorem 4.2 we have

$$\epsilon_{\sigma,k}(s) = \delta_{\gamma_s}\big(r_{\sigma,k}(s)\big) \ge \delta_k\big(\tau_k(\ell_{\gamma_s}(r_{\sigma,k}(s));c);c\big) = \delta_k\big(\tau_k(s;c);c\big) = \epsilon_k(s;c)$$

and get the conclusion. □

For $0 \le a < b < \infty$, we call the restriction of $\beta_{\sigma,k}$ to $[0,\infty) \times [a,b]$ a sub-horn. The arc-length of the curve $[a,b] \ni s \mapsto \dfrac{\partial \beta_\sigma}{\partial t}(0,s) \in U_{\sigma(0)}M$ the *embouchure-angle* of this sub-horn and is denoted by $\theta_{\sigma,k}(a,b)$. We estimate angles between two horn-tubes at the origin and show that every trajectory-horn has a limit tube.

Theorem 5.7. *Let M be a Hadamard Kähler manifold whose sectional curvatures satisfy $\mathrm{Riem}^M \le c < 0$ with some constant c. If $|k| < \sqrt{|c|}$, then for an arbitrary geodesic half-line σ, the trajectory-horn $\beta_\sigma : [0,\infty) \times \mathbb{R} \to M$ for $\mathbb{B}_k$ associated with σ*

satisfies

$$\angle\Big(\frac{\partial\beta_\sigma}{\partial t}(0,s_1),\frac{\partial\beta_\sigma}{\partial t}(0,s_2)\Big) \le \int_{s_1}^{s_2}\frac{\sqrt{|c|-k^2}}{\sinh\sqrt{|c|-k^2}\,s}ds$$
$$= \log\frac{\big(\exp\big(\sqrt{|c|-k^2}\,s_2-1\big)\big(\exp\big(\sqrt{|c|-k^2}\,s_1+1\big)}{\big(\exp\big(\sqrt{|c|-k^2}\,s_2+1\big)\big(\exp\big(\sqrt{|c|-k^2}\,s_1-1\big)},$$

for all $s_2 > s_1 > 0$. In particular, it has a limit $\lim_{s\to\infty}\dfrac{\partial\beta_\sigma}{\partial t}(0,s) \in U_{\beta(0,0)}M$ of initial vectors of horn-tubes.

Proof. We denote by γ_s the horn-tube $t \mapsto \beta_\sigma(t,s)$. We set $Y_s(t) = \dfrac{\partial\beta_\sigma}{\partial s}(t,s)$, which is a magnetic Jacobi field for $\mathbb{B}_k$ along a horn-tube γ_s. We denote $Y_s = f\dot\gamma_s + gJ\dot\gamma_s + Y_s^\perp$ and put $Y_s^\sharp = gJ\dot\gamma_s + Y_s^\perp$. Since $\beta_\sigma(0,s) = \beta_\sigma(0,0)$, we have $Y_s(0) = 0$. Thus we see $f(0) = g(0) = 0$, hence we have

$$\big(\nabla_{\frac{\partial\beta_\sigma}{\partial t}}Y_s\big)(0) = \big(kf(0)+g'(0)\big)J\dot\gamma_s(0) + \big(\nabla_{\frac{\partial\beta_\sigma}{\partial t}}Y_s^\perp\big)(0) = \big(\nabla_{\frac{\partial\beta_\sigma}{\partial t}}Y_s^\sharp\big)(0).$$

By the comparison theorem on magnetic Jacobi field (Theorem 3.2), we have

$$\|Y_s^\sharp(t)\| \ge \|\nabla_{\dot\gamma_s}Y^\sharp(0)\| \times (1/\sqrt{|c|-k^2})\sinh\sqrt{|c|-k^2}\,t,$$

hence we obtain

$$\int_{s_1}^{s_2}\big\|\nabla_{\frac{\partial\beta_\sigma}{\partial t}}Y(0)\big\|ds = \int_{s_1}^{s_2}\big\|\nabla_{\frac{\partial\beta_\sigma}{\partial t}}Y^\sharp(0)\big\|ds \le \int_{s_1}^{s_2}\frac{\|Y^\sharp(t)\|\sqrt{|c|-k^2}}{\sinh\sqrt{|c|-k^2}\,r_{\sigma,k}(s)}\,ds.$$

As $\sigma(s) = \beta_\sigma(r_{\sigma,k}(s),s)$, we see

$$\dot\sigma(s) = \frac{\partial\beta_\sigma}{\partial s}\big(r_{\sigma,k}(s),s\big) + \frac{\partial\beta_\sigma}{\partial t}\big(r_{\sigma,k}(s),s\big)r'_{\sigma,k}(s)$$
$$= Y_s(r_{\sigma,k}(s)) + r'_{\sigma,k}(s)\dot\gamma_s\big(r_{\sigma,k}(s)\big).$$

This shows $Y_s(r_{\sigma,k}(s)) = \dot\sigma(s) - r'_{\sigma,k}(s)\dot\gamma_s\big(r_{\sigma,k}(s)\big)$.

In order to study $Y_s^\sharp = Y_s - \langle Y_s,\dot\gamma_s\rangle\dot\gamma_s$, we first compute $\langle Y_s,\dot\gamma_s\rangle$.

$$\Big\langle Y_s\big(r_{\sigma,k}(s)\big),\dot\gamma\big(r_{\sigma,k}(s)\big)\Big\rangle = \Big\langle \dot\sigma(s) - r'_{\sigma,k}(s)\dot\gamma_s\big(r_{\sigma,k}(s)\big),\dot\gamma_s\big(r_{\sigma,k}(s)\big)\Big\rangle$$
$$= \Big\langle\dot\sigma(s),\dot\gamma_s\big(r_{\sigma,k}(s)\big)\Big\rangle - r'_{\sigma,k}(s)\|\dot\gamma\big(r_{\sigma,k}(s)\big)\|^2$$
$$= \epsilon_{\sigma,k}(s) - r'_{\sigma,k}(s)$$

Thus we see $Y_s^\sharp\big(r_{\sigma,k}(s)\big) = \dot\sigma(s) - \epsilon_{\sigma,k}(s)\dot\gamma_s\big(r_{\sigma,k}(s)\big)$. We hence find that

$$\begin{aligned}\|Y_s^\sharp\big(r_{\sigma,k}(s)\big)\|^2 &= \|\dot\sigma(s)\|^2 - 2\epsilon_{\sigma,k}(s)\big\langle\dot\sigma(s), \dot\gamma_s\big(r_{\sigma,k}(s)\big)\big\rangle + \epsilon_{\sigma,k}(s)^2\|\dot\gamma_s\big(r_{\sigma,k}(s)\big)\|^2 \\ &= 1 - \epsilon_{\sigma,k}(s)^2.\end{aligned}$$

This leads us to the following:

$$\frac{\|Y^\sharp(r_{\sigma,k}(s))\|\sqrt{|c|-k^2}}{\sinh\sqrt{|c|-k^2}\, r_{\sigma,k}(s)} \le \frac{\sqrt{1-\epsilon_{\sigma,k}^2(s)}\sqrt{|c|-k^2}}{\sinh\sqrt{|c|-k^2}\, r_{\sigma,k}(s)} \le \frac{\sqrt{|c|-k^2}}{\sinh\sqrt{|c|-k^2}\, r_{\sigma,k}(s)}.$$

As $r_{\sigma,k}(s) \ge s$, we see

$$\begin{aligned}\int_{s_1}^{s_2} \big\|\nabla_{\frac{\partial\beta_\sigma}{\partial s}} Y_s(0)\big\| ds &\le \int_{s_1}^{s_2} \frac{\sqrt{|c|-k^2}}{\sinh\sqrt{|c|-k^2}\, r_{\sigma,k}(s)} ds \\ &\le \int_{s_1}^{s_2} \frac{\sqrt{|c|-k^2}}{\sinh\sqrt{|c|-k^2}\, s} ds \\ &= \log\frac{\big(\exp\big(\sqrt{|c|-k^2}\, s_2 - 1\big)\big(\exp\big(\sqrt{|c|-k^2}\, s_1 + 1\big)}{\big(\exp\big(\sqrt{|c|-k^2}\, s_2 + 1\big)\big(\exp\big(\sqrt{|c|-k^2}\, s_1 - 1\big)}.\end{aligned}$$

Since we have

$$\angle\Big(\frac{\partial\beta_\sigma}{\partial t}(0,s_1), \frac{\partial\beta_\sigma}{\partial t}(0,s_2)\Big) \le \theta_{\sigma,k}(s_1,s_2) = \int_{s_1}^{s_2} \big\|\nabla_{\frac{\partial\beta_\sigma}{\partial s}} Y_s(0)\big\| ds,$$

we get the estimate.

Our estimate shows that $\{\dot\gamma_s(0) | s \ge 0\}$ is a Cauchy sequence. More clearly, when $s > \log 2/(2\sqrt{|c|-k^2})$, we have $\sinh\big(\sqrt{|c|-k^2}\, s\big) \ge \exp\big(\sqrt{|c|-k^2}\, s\big)/4$, hence we can estimate $\theta_{\sigma,k}(s_1,s_2)$ from above as

$$\begin{aligned}\int_{s_1}^{s_2} \frac{\sqrt{|c|-k^2}}{\sinh\sqrt{|c|-k^2}\, s} ds &= \int_{s_1}^{s_2} \frac{4\sqrt{|c|-k^2}}{\exp(\sqrt{|c|-k^2}\, s)}\, ds \\ &\le \frac{4}{\exp(\sqrt{|c|-k^2}\, s_1)} - \frac{4}{\exp(\sqrt{|c|-k^2}\, s_2)}.\end{aligned}$$

which guarantees $\lim_{s_1,s_2\to\infty}\angle(\dot\gamma_{s_1}, \dot\gamma_{s_2}) = 0$. As $U_{\beta(0,0)}M$ is compact we get the conclusion. $\square$

We shall call the trajectory half-line γ_σ with initial vector $\lim_{s\to\infty}\dfrac{\partial\beta_\sigma}{\partial t}(s,0)$ the *limit horn-tube* of a trajectory-horn β_σ.

Given a trajectory-horn β_σ for $\mathbb{B}_k$ associated with a geodesic half-line σ, we set $\mathcal{HR}_\sigma = \{\beta_\sigma(t,s) \mid s \ge 0, 0 \le t \le r_{\sigma,k}(s)\}$ and call it the *horn-body* of β_σ.

Proposition 5.7. *Let M be a Hadamard Kähler manifold whose sectional curvature satisfy* $\mathrm{Riem}^M \le c < 0$ *and k be a real number with $|k| < \sqrt{|c|}$. For each geodesic half-line σ, the horn-body $\mathcal{HR}_\sigma$ of a trajectory-horn for $\mathbb{B}_k$ on M is contained in the tube $U\big(\gamma_\sigma, \rho(k;c)\big)$ around the limit horn-tube γ_σ, and is contained in the tube $U\big(\sigma, \rho(k;c)\big)$ around σ, where $\rho(k;c) = |k|\pi/2\sqrt{|c|(|c|-k^2)}$.*

Proof. For each trajectory $t \mapsto \beta_\sigma(t,s)$, the geodesic σ can be regarded as a string of the trajectory-harp associated with this trajectory. We denote it by γ_s. By Remark 5.1, for each $s > 0$ we have $d\big(\gamma_s(t), \sigma(\ell_{\gamma_s}(t))\big) \le |k|\pi/\big(2\sqrt{|c|(|c|-k^2)}\big)$ for $0 \le t \le r_{\sigma,k}(s)$. Since $\lim_{s\to\infty}\gamma_s(t) = \gamma_\sigma(t)$ and since ℓ_{γ_v} is smooth with respect to $v \in U_pM$ because M is Hadamard, we obtain $d\big(\gamma_\sigma(t), \sigma(\ell_{\gamma_\sigma}(t))\big) \le |k|\pi/\big(2\sqrt{|c|(|c|-k^2)}\big)$ and get the conclusion. $\square$

5. Trajectories and its ideal boundary on a Hadamard manifold

In this section we study the existence of the trajectory joining an arbitrary point on a Hadamard Kähler manifold and a point on the ideal boundary.

On a Hadamard manifold M satisfying $\mathrm{Riem}^M \le c < 0$, for each trajectory half-line γ for $\mathbb{B}_k$ with $|k| \le \sqrt{|c|}$, Theorem 5.3 guarantees that γ is unbounded and that γ has its point at infinity $\gamma(\infty) := \lim_{t\to\infty} \gamma(t) \in \partial M$. In this section we show the following main result in this paper.

THEOREM 5.8. *Let M be a Hadamard Kähler manifold whose sectional curvatures satisfy $\mathrm{Riem}^M \le c < 0$ with some constant c. We take a Kähler magnetic field $\mathbb{B}_k$ on M with $|k| \le \sqrt{|c|}$.*

(1) *For arbitrary points $p \in M$ and $z \in \partial M$, there exists a trajectory γ satisfying $\gamma(0) = p$ and $\lim_{t\to\infty} \gamma(t) = z$. Moreover when $|k| < \sqrt{|c|}$, such a trajectory is uniquely determined.*

(2) *When $|k| < \sqrt{|c|}$, for arbitrary distinct points $z, w \in \partial M$, there exists a trajectory γ satisfying $\lim_{t\to-\infty} \gamma(t) = z$ and $\lim_{t\to\infty} \gamma(t) = w$.*

If M is a Hadamard Kähler manifold whose sectional curvatures satisfy $\mathrm{Riem}^M \le c < 0$ and, if $|k| \le \sqrt{|c|}$, by Theorem 5.7 the magnetic exponential map $\mathbb{B}_k\exp_p : T_pM \to M$ is bijective. Since every trajectory half-line has its point at infinity, we see that at arbitrary point p the magnetic exponential map $\mathbb{B}_k\exp_p : T_pM \to M$ induces a map $\partial\mathbb{B}_k\exp_p : U_pM \to \partial M$, which is defined by $U_pM \ni u \mapsto \gamma_u(\infty) \in \partial M$, where γ_u denote the trajectory for $\mathbb{B}_k$ with $\dot\gamma_u(0) = u$. First assertion in Theorem 5.8 is equivalent to the assertion that this induced map is surjective when $|k| \le \sqrt{|c|}$ and is bijective when $|k| < \sqrt{|c|}$.

First, we study that the induced map is surjective.

PROPOSITION 5.8. *Let M be a Hadamard Kähler manifold whose sectional curvatures satisfy $\mathrm{Riem}^M \le c < 0$. If $|k| \le \sqrt{|c|}$, the induced map $\partial\mathbb{B}_k\exp_p : U_pM \to \partial M$ at p is surjective.*

Proof. First, we consider the case $|k| < \sqrt{|c|}$.

We take an arbitrary infinity point $z \in \partial M$ and choose a geodesic half-line $\sigma : [0, \infty) \mapsto M$ satisfying $\sigma(0) = p$ and $\sigma(\infty) = z$. We consider the trajectory-horn β_σ for $\mathbb{B}_k$ associated with σ. We can take its limit horn-tube γ_σ by Theorem 5.7. For this trajectory γ_σ for $\mathbb{B}_k$, we take an associated trajectory-harp α_{γ_σ}. By Corollary 4.1, it has limit string σ_{γ_σ}.

By Proposition 5.7, we have $d\bigl(\sigma(\ell_{\gamma_\sigma}(t)), \gamma_\sigma(t)\bigr) \le |k|\pi/\bigl(2\sqrt{|c|(|c|-k^2)}\bigr)$, and by Theorem 5.4 we have $d\bigl(\gamma_\sigma(t), \sigma_{\gamma_\sigma}(\ell_{\gamma_\sigma}(t))\bigr) \le |k|\pi/\bigl(2\sqrt{|c|(|c|-k^2)}\bigr)$. Since ℓ_{γ_σ} is monotone increasing, we find that $d\bigl(\sigma(s), \sigma_{\gamma_\sigma}(s)\bigr) \le |k|\pi/\sqrt{|c|(|c|-k^2)}$. Therefore we find $\sigma = \sigma_{\gamma_\sigma^k}$ and $\gamma_\sigma^k(\infty) = \sigma(\infty) = z$. Thus we obtain that $\partial\mathbb{B}_k\exp_p$ is surjective.

Next we study the case $k = \pm\sqrt{|c|}$. We take a sequence $\{k_j\}_{j=1}^\infty$ satisfying $\lim_{j\to\infty} k_j = k$ and $|k_j| < \sqrt{|c|}$. We take $w_j \in U_{\sigma(0)}M$ so that $\gamma_j(\infty) = z$, where γ_j denotes the trajectory for $\mathbb{B}_{k_j}$ with $\dot\gamma_j(0) = w_j$. Since $U_{\sigma(0)}M$ is compact, then we have a convergent subsequence $\{w_{j_i}\}_{i=1}^\infty$. We put $w_\infty = \lim_{i\to\infty} w_{j_i}$. We take the trajectory γ_{w_∞} for $\mathbb{B}_k$ and put $z' = \gamma_{w_\infty}(\infty)$. We suppose $z' \ne z$ and take $v \in U_{\sigma(0)}M$ satisfying $\sigma_v(\infty) = z'$. We take positive R, ϵ, so that $O_z(p, 2\epsilon, R) \cap O_{z'}(p, 2\epsilon, R) = \emptyset$, where $O_z(p, 2\epsilon, R)$ is an open neighborhood of z in $\overline{M} = M \cup \partial M$ given in §5.1. We set $T_R = (2/\sqrt{|c|})\sinh(\sqrt{|c|}R/2)$. By the comparison theorem on string-lengths (Theorem4.2) and by Proposition 4.7, for $t \ge T_R$, we have

$$\ell_{\gamma_j}(t) \ge \ell_{k_j}(t;c) \ge \ell_{\sqrt{|c|}}(t;c) \ge \ell_{\sqrt{|c|}}(T_R;c) = R,$$

and $\ell_{\gamma_{w_\infty}}(t) \ge \ell_{\sqrt{|c|}}(t;c) \ge R$. We take the geodesic σ_∞ of unit speed with $\sigma_\infty(0) = p$ and $\sigma_\infty(\infty) = z'$. By Proposition 4.11 and Corollary 4.2, we have

$$\angle\Bigl(\dot\sigma(0), \frac{\partial\alpha_{\gamma_j}}{\partial s}(t,0)\Bigr) \le \frac{2}{\sqrt{|c|}t} \quad \text{and} \quad \angle\Bigl(v, \frac{\partial\alpha_{\gamma_{w_\infty}}}{\partial s}(t,0)\Bigr) \le \frac{2}{\sqrt{|c|}t}.$$

When $t \ge \max\bigl\{T_R, 2/\sqrt{|c|}\epsilon\bigr\}$, we find $\gamma_j(t) \in O_z(p, \epsilon, R)$ and $\gamma_{w_\infty}(t) \in O_{z'}(p, \epsilon, R)$. But as we have $\lim_{i\to\infty}\gamma_{j_i}(t) = \gamma_{w_\infty}(t)$, it is contradiction. Hence $\gamma_{w_\infty}(\infty) = z$. Therefore we obtain that $\partial\mathbb{B}_{\pm\sqrt{|c|}}\exp_p$ is also surjective. □

When a Hadamard manifold M satisfies $\mathrm{Riem}^M \le c < 0$, for a constant k with $|k| \le \sqrt{|c|}$, we define a map $\Phi_p^k : U_pM \to U_pM$ by $w \mapsto v = \dot{\sigma}_{\gamma_w}(0)$, where γ_w denotes the trajectory half-line for $\mathbb{B}_k$ with initial vector w and σ_{γ_w} denotes the limit harp-string of the trajectory-harp associated with γ_w. By Corollary 4.1, we have the following.

Corollary 5.3. *When a Hadamard manifold M satisfies* $\mathrm{Riem}^M \le c < 0$*, for a constant k with $|k| \le \sqrt{|c|}$, the map $\Phi_p^k : U_pM \to U_pM$ satisfies the point at infinity of the trajectory half-line γ_w with $\dot{\gamma}_w(0) = w$ coincides with the point at infinity of the geodesic half-line $\sigma_{\Phi_p^k(w)}$ with $\dot{\sigma}_{\Phi_p^k(w)}(0) = \Phi_p^k(w)$, that is $\gamma_w(\infty) = \sigma_{\Phi_p^k(w)}(\infty)$.*

When a Hadamard manifold M satisfies $\mathrm{Riem}^M \le c < 0$, for a constant k with $|k| < \sqrt{|c|}$, we define a map $\Psi_p^k : U_pM \to U_pM$ by $v \to \dot{\gamma}_{\sigma_v}(0)$, where σ_v denotes the geodesic with $\dot{\sigma}_v(0) = v$ and γ_{σ_v} denotes the limit horn-tube of the trajectory-horn for $\mathbb{B}_k$ associated with σ_v. Since we take a subsequence to get a trajectory γ for $\mathbb{B}_{\pm\sqrt{|c|}}$ satisfying $\gamma(\infty) = \sigma(\infty)$ in the proof of Proposition 5.8, we can not say that we can define such a map for $k = \pm\sqrt{|c|}$. By Theorem 5.7, we have the following.

Corollary 5.4. *When a Hadamard manifold M satisfies* $\mathrm{Riem}^M \le c < 0$*, for a constant k with $|k| < \sqrt{|c|}$, a map $\Psi_p^k : U_pM \to U_pM$ satisfies the point at infinity of the geodesic half-line σ_v with $\dot{\sigma}_v(0) = v$ coincides with the point at infinity of the trajectory half-line $\gamma_{\Psi_p^k(v)}$ with $\dot{\gamma}_{\Psi_p^k(v)}(0) = \Psi_p^k(v)$, that is $\sigma_v(\infty) = \gamma_{\Psi_p^k(v)}(\infty)$.*

Lemma 5.3. *When a Hadamard manifold M satisfies* $\mathrm{Riem}^M \le c < 0$*, for a constant k with $|k| < \sqrt{|c|}$, we have that the composition $\Phi_p^k \circ \Psi_p^k : U_pM \to U_pM$ is the identity.*

Proof. For a $v \in U_pM$, we put $w = \Psi_p^k(v), u = \Phi_p^k \circ \Psi_p^k(v)$. We then have $\sigma_v(\infty) = \gamma_w(\infty) = \sigma_u(\infty)$. As $\partial\exp_p : U_pM \to U_pM$ is bijective, we have $v = u$. That is the composition $\Phi_p^k \circ \Psi_p^k$ is the identity. Therefore we get the conclusion. $\square$

Next, we study the injectivity of the induced map $\partial\mathbb{B}_k\exp_p$.

Proposition 5.9. *Let M be a Hadamard Kähler manifold whose sectional curvatures satisfy* $\mathrm{Riem}^M \le c < 0$. *If $|k| < \sqrt{|c|}$, then the induced map $\partial\mathbb{B}_k\exp_p : U_pM \to \partial M$ is injective.*

Proof. For unit tangent vectors $v, w \in UM$, we denote by σ_v the geodesic with $\dot\sigma_v(0) = v$ and by γ_w the trajectory for $\mathbb{B}_k$ with $\dot\gamma_w(0) = w$.

In order to show the assertion we are enough to show that the map $\Phi_p^k : U_pM \to U_pM$ is bijective. Since $\Phi_p^k \circ \Psi_p^k$ is the identity by Lemma 5.3, to show that Φ_p^k is bijective we only need to show that the composition $\Psi_p^k \circ \Phi_p^k$ is the identity.

We take a trajectory γ_w for $w \in U_pM$ and a constant t. For a geodesic σ_t of unit speed which joins $\gamma_w(0)$ and $\gamma_w(t)$, we consider a trajectory-horn $\beta_t : [0,\infty) \times \mathbb{R} \to M$ associated with σ_t. For an arbitrary s, we find that $u \mapsto \beta_t(s,u)$ is the trajectory for $\mathbb{B}_k$ joining $\sigma_t(0) = \gamma_w(0)$ and $\sigma_t(s)$. We denote by $r_t(s)$ the tube-length of β_t at s and set $w_t^s = \dfrac{\partial\beta_t}{\partial u}(s,0) \in U_pM$. By Proposition 5.6, we have $r_t(s) \le \tau_k(s;c)$. We have a subsequence $\{t_j\}_{j=1}^\infty$ depending on s which satisfies that both $\{w_{t_j}^s\}_{j=1}^\infty (\subset U_pM)$ and $\{r_{t_j}(s)\}_{j=1}^\infty (\subset \mathbb{R})$ converge. We set $w_\infty^s = \lim_{j\to\infty} w_{t_j}^s$ and $r_\infty(s) = \lim_{j\to\infty} r_{t_j}(s)$.

By Corollary 4.1, the trajectory-harp $\alpha_{\gamma_w} : [0,\infty) \times \mathbb{R} \to M$ associated with the trajectory γ_w has a limit. Then we find that $\lim_{t\to\infty} \sigma_t(s) = \sigma_{\Phi_p^k(w)}(s)$ with $\Phi_p^k(w) = \lim_{t\to\infty} \dot\sigma_t(0)$. As $\sigma_t(s) = \beta_t\big(s, r_t(s)\big) = \gamma_{w_t^s}(r_t(s))$, we have

$$\sigma_{\Phi_p^k(w)}(s) = \lim_{j\to\infty} \gamma_{w_{t_j}^s}\big(r_{t_j}(s)\big) = \gamma_{w_\infty^s}\big(r_\infty(s)\big).$$

Therefore, we see that each $\gamma_{w_\infty^s}$ is a tube of trajectory-horn associated with $\sigma_{\Phi_p^k(w)}$. By Theorem 5.7, we have

$$\angle(w_t^s, w) \le \int_s^{\ell_{\gamma_w}(t)} \frac{\sqrt{|c|-k^2}}{\sinh\sqrt{|c|-k^2}\,u}\,du,$$

for $s \le \ell_{\gamma_w}(t)$ and arbitrary t, because γ_w is a horn-tube of a trajectory-horn β_t. Therefore we have

$$\angle(w_t^s, w) \le \int_s^{\infty} \frac{\sqrt{|c|-k^2}}{\sinh\sqrt{|c|-k^2}\,u}\,du,$$

and hence we obtain

$$\angle(w^s_\infty, w) \le \int_s^\infty \frac{\sqrt{|c| - k^2}}{\sinh \sqrt{|c| - k^2}\, u} du < \infty.$$

Thus we find that $\lim_{s\to\infty} w^s_\infty = w$. This shows $\Psi^k_p(\Phi^k_p(w)) = w$. We therefore get the conclusion. □

Remark 5.2. We take an arbitrary geodesic half-line σ of unit speed emanating from $p \in M$. The condition $\Phi_p \circ \Psi_p = Id$ means that for the limit horn-tube γ_σ of the trajectory-horn for $\mathbb{B}_k$ associated with σ, the limit harp-string σ_{γ_σ} of the trajectory-harp associated with γ_σ is σ. On the other hand, we take an arbitrary trajectory half-line γ for $\mathbb{B}_k$ which is emanating from $p \in M$. The condition $\Psi_p \circ \Phi_p = Id$ means that for the limit harp-string σ_γ of the trajectory-harp associated with γ the limit horn-tube γ_{σ_γ} of the trajectory-horn associated with σ_γ is γ.

Finally we study trajectories joining distinct points in the ideal boundary.

Proposition 5.10. *Let M be a Hadamard Kähler manifold satisfying* $\mathrm{Riem}^M \le c < 0$. *If k satisfies $|k| < \sqrt{|c|}$, then for distinct points $z, w \in \partial M$ there exists at least one trajectory γ with $\gamma(-\infty) = z$ and $\gamma(\infty) = w$.*

To show Proposition 5.10, we need a result corresponding to geodesics.

Theorem 5.9. *Let M be a Hadamard manifold satisfying* $\mathrm{Riem}^M \le c < 0$. *For distinct points $z, w \in \partial M$, there exists a unique geodesic σ of unit speed with $\sigma(-\infty) = z$ and $\sigma(\infty) = w$.*

To show this, we recall Gauss-Bonnet theorem.

Theorem 5.10 (Gauss-Bonnet). *Let M be a 2-dimensional orientable compact Riemannian manifold with boundary ∂M. Suppose ∂M is piecewise smooth. Then we have*

$$\int_M \mathrm{Riem}^M d\mathrm{vol}_M + \sum_{i=1}^{L}\sum_{j=1}^{k_i}\Big\{\int_{\gamma_{i_j}} k_{\gamma_i}(s)ds + \big\langle \dot\gamma_{i_j}(\ell_{i_j}), \dot\gamma_{j+1}(0)\big\rangle\Big\} = 2\pi\mathfrak{X}(M).$$

Here, each component of ∂M is the join $\gamma_{i_1}\cdot\gamma_{i_2}\cdots\gamma_{i_{k_i}}$ of smooth curves with $\gamma_{i_j}(\ell_{i_j}) = \gamma_{j+1}(0)$, L is the number of components of ∂M and $\mathcal{X}(M)$ denotes the Euler characteristic of M.

Proof of Theorem 5.9. We take an arbitrary point $p \in M$ and take geodesic half-line γ_1, γ_2 satisfying $\gamma_1(0) = \gamma_2(0) = p, \gamma_1(\infty) = z$ and $\gamma_2(\infty) = w$. Let σ_t be the geodesic of unit speed with $\sigma_t(0) = \gamma_1(t)$ and $\sigma_t(\ell_t) = \gamma_2(t)$ for some positive ℓ_t. Since M is strictly negative, by Proposition 5.3, we see $s \mapsto d(p, \sigma_t(s))$ is a strictly convex function. As we have

$$d(p, \sigma_t(0)) = t = d(p, \sigma_t(\ell_t)),$$

there is r_t with $0 < r_t < \ell_t$ such that $d(p, \sigma_t(r_t)) = \min\{d(p, \sigma_t(s))|_s\}$. Let S_t denotes a Riemann surface consists of all geodesic segments from p to $\sigma_t(s)$ with $0 \le s \le \ell_t$. The boundary of S_t consists of the geodesic segments $\gamma_1|_{[0,t]}, \gamma_2|_{[0,t]}$ and $\sigma_t|_{[0,\ell_t]}$. By Gauss-Bonnet theorem, we have

$$\int_{S_t} \mathrm{Riem}^{S_t} \mathrm{dvol}_{S_t} + \cos^{-1}\langle -\dot\gamma_1(0), \dot\gamma_2(0)\rangle + \cos^{-1}\langle \dot\gamma_1(t), \dot\sigma_t(0)\rangle + \cos^{-1}\langle \dot\sigma_t(\ell_t), -\dot\gamma_2(t)\rangle = 2\pi.$$

Therefore, we have

$$\pi \ge -\int_{S_t} \mathrm{Riem}^{S_t} \mathrm{dvol}_{S_t} \ge |c|\mathrm{vol}(S_t),$$

because $\mathrm{Riem}^M \le c < 0$. On the other hand, comparing volumes of sectors of angle $\angle\big(\dot\gamma_1(0), \dot\gamma_2(0)\big)$ and of radius $d(p, \sigma_t(r_t))$ in M and in $\mathbb{R}^2$, as S_t is contained in this sector in M, we have $\mathrm{vol}(S_t) \ge \dfrac{1}{2} d(p, \sigma_t(r_t))^2 \times \angle\big(\dot\gamma_1(0), \dot\gamma_2(0)\big)$ by Rauch's comparison theorem. We hence have

$$d(p, \sigma_t(r_t))^2 \le \frac{2\pi}{|c|\angle\big(\dot\gamma_1(0), \dot\gamma_2(0)\big)}.$$

□

Proof of Proposition 5.10. We take a geodesic σ satisfying $z = \lim_{t\to-\infty}\gamma(t)$ and $w = \lim_{t\to\infty}\gamma(t)$. For each positive s, we take a trajectory γ_s for $\mathbb{B}_k$ joining $\sigma(-s)$ and $\sigma(s)$. We take the parameter of γ_s so that $\gamma_s(0) = \sigma(-s)$ and $\gamma_s(t_s) = \sigma(s)$

with some positive t_s. As a restriction of σ is a harp-string of the trajectory-harp α_{γ_s} associated with γ_s for each s, Remark 5.1 guarantees the following :

1) If we take positive r_s satisfying $s = \ell_{\gamma_s}(r_s)$, we have

$$d\bigl(\sigma(0), \gamma_s(r_s)\bigr) < |k|\pi/\bigl(2\sqrt{|c|(|c|-k^2)}\bigr).$$

2) For $0 \le t \le t_s$ we have $d\bigl(\gamma_s(t), \sigma\bigr) \le |k|\pi/\bigl(2\sqrt{|c|(|c|-k^2)}\bigr)$.

We set B a geodesic-ball of radius $|k|\pi/\bigl(2\sqrt{|c|(|c|-k^2)}\bigr)$ centered at $\sigma(0)$. As $\gamma_s(r_s) \in B$, we can choose a sequence s_j so that $\{\dot\gamma_{s_j}(r_{s_j})\}_j \subset UM|_B$ converges. We denote by γ_∞ the trajectory whose initial is $\lim_{j\to\infty} \dot\gamma_{s_j}(r_{s_j})$. By perturbation theory of differential equations we see that $\mathbb{B}_k\mathrm{exp}_p$ is smooth with respect to p. Therefore, we find $d\bigl(\gamma_\infty(t), \sigma\bigr)$ is not greater than $|k|\pi/\bigl(2\sqrt{|c|(|c|-k^2)}\bigr)$ for each t. This shows that $\lim_{t\to-\infty} \gamma_\infty(t) = z$ and $\lim_{t\to\infty} \gamma_\infty(t) = w$. Thus we get conclusion. $\square$

CHAPTER A

Appendix

We here give some general results which we used in this paper.

1. Dual linear maps

Let V be a real vector space. A bilinear form $\mathfrak{p}$ of V is a map $\mathfrak{p} : V \times V \to \mathbb{R}$ satisfying that

i) $\mathfrak{p}(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 \mathfrak{p}(v_1, w) + \lambda_2 \mathfrak{p}(v_2, w)$,

ii) $\mathfrak{p}(v, \mu_1 w_1 + \mu_2 w_2) = \mu_1 \mathfrak{p}(v, w_1) + \mu_2 \mathfrak{p}(v, w_2)$,

for arbitrary $v_1, v_2, v, w_1, w_2, w \in V$ and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$. We say it is symmetric if it satisfies $\mathfrak{p}(v, w) = \mathfrak{p}(w, v)$ for all $v, w \in V$. We call $\mathfrak{p}$ a non-degenerate if $\mathfrak{p}(v, w) = 0$ for all $w \in V$ shows $v = 0$. When V is a complex vector space, we say a bilinear form $\mathfrak{p} : V \times V \to \mathbb{C}$ Hermitian if it satisfies $\mathfrak{p}(v, w) = \overline{\mathfrak{p}(w, v)}$ for all $v, w \in V$, where $\overline{z}$ of a complex number z is the complex conjugate of z.

Lemma A.1. *Let $\mathfrak{p}$ be a non-degenerate bilinear form on a finite dimensional vector space V over an algebra $\mathcal{F}$. If $v, v' \in V$ satisfy $\mathfrak{p}(v, w) = \mathfrak{p}(v', w)$ for all $w \in V$, then $v = v'$.*

This lemma shows that if we know $\mathfrak{p}(v, w)$ for all $w \in V$ and they satisfy linearity with respect to w then we have a unique $v \in V$ having these relations.

Proof of Lemma A.1. By the definition of $\mathfrak{p}$, we note

$$0 = \mathfrak{p}(v, w) - \mathfrak{p}(v', w) = \mathfrak{p}(v - v', w).$$

Then we have $v - v' = 0$. Therefore we can get the conclusion. $\square$

2. Inverse mapping theorem

In this section we give inverse mapping theorem following to [**17**]. This is quite important in the study of Riemannian manifolds.

For a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ we set

$$\|L\| = \max\{\|Lu\| \mid u \in \mathbb{R}^m,\ \|u\| = 1\}.$$

Trivially, we have $\|L\| = 0$ if and only if $L = O$, and have $\|Lv\| \le \|L\|\|v\|$ for all $v \in \mathbb{R}^m$ because when $v \ne 0$ we see

$$\|Lv\| = \Big\|L\Big(\|v\|\frac{v}{\|v\|}\Big)\Big\| = \Big\|\|v\|L\Big(\frac{v}{\|v\|}\Big)\Big\| = \|v\| \times \Big\|L\Big(\frac{v}{\|v\|}\Big)\Big\| \le \|L\|\|v\|.$$

When $L : \mathbb{R}^m \to \mathbb{R}^n$ and $M : \mathbb{R}^n \to \mathbb{R}^s$ be linear maps, then for the linear map $M \circ L : \mathbb{R}^m \to \mathbb{R}^s$ we have $\|M \circ L\| \le \|L\|\,\|M\|$, because

$$\|M \circ L(u)\| = M\big(L(u)\big)\| \le \|M\|\|L(u)\| \le \|M\|\|L\|\|u\|.$$

Given a differentiable mapping $F : U\ (\subset \mathbb{R}^m) \to \mathbb{R}^n$ and an arbitrary point $p \in U$, by denoting as $F = (f_1, \dots, f_n)$ with functions $y_j = f_j(x_1, \dots, x_m)$ $(j = 1, \dots, n)$, we define its differential $DF(p) : \mathbb{R}^m \to \mathbb{R}^n$ at p as the linear map defined by the matrix

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(p) & \cdots & \dfrac{\partial f_1}{\partial x_m}(p) \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1}(p) & \cdots & \dfrac{\partial f_n}{\partial x_m}(\xi) \end{pmatrix}.$$

Lemma A.2. *Let U be a convex subset of $\mathbb{R}^m$ and $F : U \to \mathbb{R}^n$ be a differentiable mapping. If $K := \sup_{p\in U} \|DF(p)\| < \infty$, for arbitrary $p, q \in U$ we see $\|F(p) - F(q)\| \le K\|p - q\|$.*

Proof. By mean-value theorem, we have $F(q) = F(p) + DF(\xi)(q - p)$ with some $\xi \in \mathbb{R}^m$ which lies on the line from p to q. If we rewrite it, we have

$$\begin{pmatrix} f_1(q) \\ \vdots \\ \vdots \\ f_n(q) \end{pmatrix} = \begin{pmatrix} f_1(p) \\ \vdots \\ \vdots \\ f_n(p) \end{pmatrix} + \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\xi) & \cdots & \dfrac{\partial f_1}{\partial x_m}(\xi) \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1}(\xi) & \cdots & \dfrac{\partial f_n}{\partial x_m}(\xi) \end{pmatrix} \begin{pmatrix} q_1 - p_1 \\ \vdots \\ \vdots \\ q_m - p_m \end{pmatrix},$$

where $p = (p_1, \dots, p_m)$. $q = (q_1, \dots, q_m)$. Thus we find

$$\|F(p) - F(q)\| = \|DF(\xi)(p-q)\| \leq \|DF(\xi)\| \, \|p-q\| \leq K\|p-q\|,$$

and get the conclusion. $\square$

THEOREM A.1 (Inverse mapping theorem). *Let $F = (f_1, \dots, f_m)$ be a C^1-mapping of an open subset D of $\mathbb{R}^m$ to $\mathbb{R}^m$. If the Jacobian of F satisfies $\frac{\partial(f_1,\dots,f_m)}{\partial(x_1,\dots,x_m)}(p_0) \neq 0$ at a point $p_0 \in D$, then there exist open subsets U, V of $\mathbb{R}^m$ satisfying the following conditions*:

i) *$p_0 \in U \subset D$ and $V = F(U)$,*
ii) *the restriction $F|_U : U \to V$ is a bijection,*
iii) *the inverse map $(F|_U)^{-1} : V \to U$ of $F|_U$ is also C^1-mapping.*

PROOF. Here, we only show that we can take open subsets satisfying the conditions i) and ii). We put $\lambda = 1/(2\|\big(DF(p_0)\big)^{-1}\|) = \|DF(p_0)\|/2$ (> 0). Since F is of C^1, we see that its Jacobian $\frac{\partial(f_1,\dots,f_m)}{\partial(x_1,\dots,x_m)} : D \to \mathbb{R}$ is continuous and that $p \mapsto \|DF(p) - DF(p_0)\|$ is also continuous. As $\frac{\partial(f_1,\dots,f_m)}{\partial(x_1,\dots,x_m)}(p_0) \neq 0$, there is a positive ϵ such that $\frac{\partial(f_1,\dots,f_m)}{\partial(x_1,\dots,x_m)}(p) \neq 0$ and $\|DF(p) - DF(p_0)\| < \lambda$ for every $p \in B_\epsilon(p_0)$ in a open ball centered at p_0.

First we show that F is injective near the point p_0. By applying mean-value theorem, we have

$$F(p+h) = F(p) + DF(\xi)h, \tag{A.1}$$

where ξ is a point on the line from p to $p+h$. We here suppose $F(p+h) = F(p)$ for $p, p+h \in B_\epsilon(p_0)$. By (A.1) we see $DF(\xi)h = 0$. As the matrix $DF(\xi)$ is invertible because $\frac{\partial(f_1,\dots,f_m)}{\partial(x_1,\dots,x_m)}(\xi) \neq 0$, this shows $h = 0$, which means $p + h = p$. Thus, we see F is injective on $B_\epsilon(p_0)$.

Next we show that $V := F\big(B_\epsilon(p_0)\big)$ is an open subset of $\mathbb{R}^m$. We take a point $y_0 \in V_0$ and choose a point $x_0 \in B_\epsilon(p_0)$ so that $y_0 = F(x_0)$. We set $r = \{\epsilon - d(x_0, p_0)\}/2$. If we can show that $B_{\lambda r}(y_0) \subset V_0$, we find that V_0 is open. We hence show $B_{\lambda r}(y_0) \subset V_0$.

We take an arbitrary point $y \in B_{\lambda r}(y_0)$ and define a map $G_y : B_\epsilon \to \mathbb{R}^m$ by

$$G_y(p) = p + \big(DF(p_0)\big)^{-1}\big(y - F(p)\big).$$

Since $G_y(x_0) = x_0 + \big(DF(p_0)\big)^{-1}(y - y_0)$ we have

$$\|G_y(x_0) - x_0\| \le \|\big(DF(p_0)\big)^{-1}\| \, \|y - y_0\| < \frac{1}{2\lambda} \times \lambda r = \frac{r}{2}.$$

On the other hand, by chain rule we have

$$DG_y(p) = I - \big(DF(p_0)\big)^{-1} DF(p) = \big(DF(p_0)\big)^{-1}\big(DF(p_0) - DF(p)\big).$$

Hence we obtain

$$\|DG_y(p)\| \le \big\|\big(DF(p_0)\big)^{-1}\big\| \big\|DF(p_0) - DF(p)\big\| \le \frac{1}{2\lambda}\lambda\|p - x_0\| = \frac{1}{2}.$$

By Lemma A.2 for arbitrary $p, q \in B_\epsilon(p_0)$ we have

$$\|G_y(p) - G_y(q)\| \le \frac{1}{2}\|p - x_0\|, \tag{A.2}$$

In particular, we have $\|G_y(p) - G_y(x_0)\| < \dfrac{r}{2}$. Thus we see

$$\|G_y(p) - x_0\| \le \|G_y(x) - G_y(x_0)\| + \|G_y(x_0) - x_0\| < r,$$

and find that $G_y(p) \in B_r(x_0)$ This shows that the restriction $G_y|_{\overline{B_r(x_0)}}$ of G_y onto the closure $\overline{B_r(x_0)}$ $\big(\subset B_\epsilon(p_0)\big)$ of $B_r(x_0)$ is a contraction map. Since $\overline{B_r(x_0)}$ is compact hence is complete, the map $G_y|_{\overline{B_r(x_0)}}$ has a fixed point $x \in \overline{B_r(x_0)}$ by Theorem A.2 below. This means that $x = G_y(x) = x + \big(DF(p_0)\big)^{-1}\big(y - F(x)\big)$. We therefore obtain $y = F(x)$ and find that $B_{\lambda r}(y_0) \subset V$. This shows that V is an open subset of $\mathbb{R}^m$. By putting $U = B_\epsilon(p_0)$ we find that $F|_U : U \to V$ is a bijection and U, V are open sets. □

Since manifolds are locally congruent to Euclidean spaces we can extend the above to manifolds.

Corollary A.1. *Let M and N be a manifold of same dimension and $\varphi : M \to N$ be a C^1-map. If the differential map $(d\varphi)_{p_0} : T_{p_0}M \to T_{\varphi(p_0)}N$ is invertible, then there exist an open neighborhood U of p_0 in M and an open neighborhood V of $\varphi(p_0)$ in N satisfying the following conditions*:

i) $V = \varphi(U)$,

ii) *the restriction* $\varphi|_U : U \to V$ *is a bijection,*

iii) *the inverse map* $(\varphi|_U)^{-1} : V \to U$ *of* $\varphi|_U$ *is also* C^1*-mapping.*

Let (X, d) be a metric space. A map $f : X \to X$ is said to be a contraction if there is ρ $(0 < \rho < 1)$ such that $d\big(f(p), f(q)\big) \leq \rho d(p, q)$ for all $p, q \in X$. Clearly a contraction is a continuous map.

THEOREM A.2 (Fixed point theorem). *Let* $f : X \to X$ *be a contraction of a complete metrix space* X*. Then there exists a unique fixed point of* f*, that is* $p_* \in X$ *with* $f(p_*) = p_*$*.*

PROOF. We take an arbitrary point $p_0 \in X$ and define a sequence $\{p_j\}_{j=0}^{\infty} \subset X$ inductively by $p_{j+1} = f(p_j)$. As we have

$$d(x_{n+1}, x_n) = f\big(f(x_n).f(x_{n-1})\big) \leq \rho d(x_n, x_{n-1})$$

for $n \geq 1$, we find inductively that $d(x_{n+1}, x_n) \leq \rho^n d(x_1, x_0)$. Thus we have for arbitrary positive integers j, k with $j < k$ that

$$\begin{aligned} d(p_j, p_k) &\leq d(p_j, p_{j+1}) + \cdots + d(p_{k-1}, p_k) \\ &\leq \big(\rho^j + \rho^{j+1} + \cdots + \rho^{k-1}\big) d(x_0, x_1) \leq \frac{\rho^j}{1-\rho} d(x_0, x_1). \end{aligned}$$

Thus we see $\{p_n\}_{j=0}^{\infty}$ is a Caucy sequence. If we set $p_* = \lim_{j \to infty} p_j$ then

$$f(p_*) = f(\lim_{j \to \infty} p_j) = \lim_{j \to \infty} f(p_j) = \lim_{j \to \infty} p_{j+1} = p_*.$$

Thus we have a fixed point.

If we have two fixed points $p_:, q_* \in X$ then we have

$$d(p_*, q_*) = d\big(f(p_*), f(q_*)\big) \leq \rho d(p_*, q_*).$$

Since $0 < \rho < 1$, this is a contradiction. Hence we get the uniqueness. $\square$

3. Connectedness and compactness

In this section we recall some fundamental results on topological spaces.

[1] Connectedness

A topological space X is sad to be connected if there are no pairs (U, V) of nonempty open subsets satisfying $U \cup V = X$ and $U \cap V = \emptyset$.

LEMMA A.3. *A topological space X is connected if and only if its open and closed subset is either X itself or an empty set $\emptyset$.*

PROOF. ($\Rightarrow$) Suppose we have an open and closed nonempty subset U with $U \neq X$. Then $V = X \setminus U$ is also an open and closed nonempty subset. Since $X = U \cup V$ and $U \cap V = \emptyset$, we see X is not connected.

($\Leftarrow$) When X is not connected, we have two open nonempty subsets U, V satisfying $U \cup V$ and $U \cap V = \emptyset$. As $U = X \setminus V$, we see U is an open and closed nonempty set with $U \neq X$. □

It is well-known that $\mathbb{R}$ is connected. A topological space consists of one point is connected.

LEMMA A.4. *Let $\varphi : X \to Y$ be a continuous map of a connected topological space to a topological space. Then $\varphi(X)$ is a connected subspace of Y with respect to the induced topology on $\varphi(X)$.*

PROOF. If we suppose $\varphi(X)$ is not connected, then there exist open subsets U, V of Y satisfying

$$U' \cup V' = \varphi(X), \quad U' \cap V' = \emptyset, \quad U' \neq \emptyset, \ V' \neq \emptyset,$$

where $U' = U \cap \varphi(X)$ and $V' = V \cap \varphi(X)$. Since φ is continuous, we see $\varphi^{-1}(U)$ and $\varphi^{-1}(V)$ are open subsets of X. As we have

$$\varphi^{-1}(U) \cup \varphi^{-1}(V) = \varphi^{-1}(U') \cup \varphi^{-1}(V') = X,$$

$$\varphi^{-1}(U) \cap \varphi^{-1}(V) = \varphi^{-1}(U' \cap V') = \emptyset,$$

$$\varphi^{-1}(U) = \varphi^{-1}(U) \neq \emptyset, \quad \varphi^{-1}(V) = \varphi^{-1}(V) \neq \emptyset,$$

we find that X is not connected, which is a contradiction. $\square$

Lemma A.5. *Let S_1, S_2 be two subsets of a topological space X. If S_1, and S_2 are connected and $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cup S_2$ is connected.*

Proof. Suppose $S := S_1 \cup S_2$ is not connected. Then there are nonempty subsets U, V of X with

$$S \subset U \cup V, \quad U \cap V \cap S = \emptyset, \quad U \cup S \neq \emptyset, \ \ V \cup S \neq \emptyset.$$

If $S_1 \cap U \neq \emptyset$, then $S_1 \cap V = \emptyset$ because S_1 is connected. This leads us to $S_2 \cap V \neq \emptyset$ and hence $S_2 \cap U = \emptyset$. Thus we have

$$(U \cup V) \cap (S_1 \cap S_2) = (U \cap S_1 \cap S_2) \cup (V \cap S_1 \cap S_2) = \emptyset.$$

On the other hand, as we have $S \subset U \cup V$, we have $(U \cup V) \cap (S_1 \cap S_2) = S_1 \cap S_2$. Thus we find $S_1 \cap S_2 = \emptyset$, which is a contradiction. $\square$

Given two points p, q in a topological space X, we denote $p \sim q$ if there is a connected subset of X containing both p and q. Since $\{p\}$ is connected, we have $p \sim p$. It is clear that $p \sim q$ shows $q \sim p$ by definition. When $p \sim q$ and $q \sim r$, then there are connected subsets S_1, S_2 of X such that S_1 contains p, q and S_2 contains q, r. Since $S_1 \cap S_2 \ni q$ we know that $S_1 \cup S_2$ is connected by Lemma A.5. As $p, r \in S_1 \cup S_2$ we see $p \sim r$. Therefore, this relation $\sim$ is an equivalence relation. We call an equivalence class with respect to $\sim$ a connected component of X.

A topological space is arc-wisely connected if for arbitrary distinct points $p, q \in X$ there is a continuous curve $c : [0, 1] \to X$ with $c(0) = p$ and $c(1) = q$.

Lemma A.6. *If X is arc-wisely connected, then it is connected.*

Proof. Suppose X is not connected. Then there are nonempty open sets U, V with $U \cup V = X$, $U \cap V = \emptyset$. We take points $p \in U$ and $q \in V$. Since X is arc-wisely connected, there is a curve $c : [0, 1] \to M$ with $c(0) = p$ and $c(1) = q$. Since c is continuous, we see $c([0, 1])$ is connected by Lemma A.4. As $c([0, 1]) \subset$

$U \cup V$, $c([0,1]) \cap U \cap V = \emptyset$ and $c([0,1]) \cap U \ni p$, $c([0,1]) \cap U \ni q$, we find $c([0,1])$ is not connected. This is a contradiction. Hence X is connected. $\square$

Let M be a manifold. Given two points $p, q \in M$ we denote $p \bowtie q$ if there is a curve $c : [0,1] \to M$ with $c(0) = p$ and $c(1) = q$. Considering a constant curve we have $p \bowtie p$. If $p \bowtie q$, considering the reversed curve c^{-1} of $c[0,1] \to M$ with $c(0) = p$, $c(1) = q$ which is given by $c^{-1}(t) = c(1-t)$, we find $c^{-1}(0) = q$ and $c^{-1}(1) = p$, hence $q \bowtie p$. If $p \bowtie q$ and $q \bowtie r$, we take curves $c_1, c_2 : [0,1] \to M$ with $c_1(0) = p, c_1(1) = q = c_2(0)$ and $c_2(1) = r$. Considering their join $c_1 \cdot c_2 : [0,1] \to M$ given by

$$c_1 \cdot c_2(t) = \begin{cases} c_1(2t), & \text{when } 0 \le t \le 1/2, \\ c_2(2t-1), & \text{when } 1/2 \le t \le 1, \end{cases}$$

it is a curve with $c_1 \cdot c_2(0) = p$ and $c_1 \cdot c_2(1) = r$. Thus we have $p \bowtie r$. Therefore $\bowtie$ is an equivalence relation.

Lemma A.7. *A connected manifold M is arc-wisely connected.*

Proof. We decompose M into components, equivalence classes, with respect to the relation $\bowtie$ as $M = \bigcup_{\lambda \in \Lambda} K_\lambda$. We show that K_λ is an open set. We take a point $p \in K_\lambda \subset M$ and a local coordinate neighborhood (U, φ) around p with $\varphi(U) = \mathbb{R}^m$ and $\varphi(p) = 0$. Because every open subset of $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^m$, we may suppose this. For each $v \in \mathbb{R}^m$ we have a curve $\tilde{c}_v : [0,1] to \mathbb{R}^m$ defined by $\tilde{c}_v(t) = tv$. Since φ is a homeomorphism, each $q \in U$ and p is joined by a curve $\varphi^{-1} \circ \tilde{c}_{\varphi(q)}$. Hence $U \subset K_\lambda$. Thus K_λ is an open set.

Suppose K_{λ_0} is not an empty set. Then $\bigcup_{\lambda \in \Lambda \setminus \{\lambda_0\}} K_\lambda$ is an open set, hence K_{λ_0} is an open and closed set. Hence $K_{\lambda_0} = M$. Thus M is arc-wisely connected. $\square$

[2] Compactness

A topological space X is said to be compact if it satisfies the following condition:

> If a family $\{U_\lambda\}_{\lambda \in \Lambda}$ of open subsets of X satisfies $X = \bigcup_{\lambda \in \Lambda} U_\lambda$, then there exist finite $U_{\lambda_1}, \ldots, U_{\lambda_N}$ with $X = \bigcup_{j=1}^{N} U_{\lambda_j}$.

Such a family $\{U_\lambda\}_{\lambda\in\Lambda}$ of open sets of X with $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ is called an open covering of X. A subset K of a topological space X is compact, that is, it is compact with respect to the induced topology, if and only if each family $\{U_\lambda\}_{\lambda\in\Lambda}$ of open subsets of X satisfying $K \subset \bigcup_{\lambda\in\Lambda} U_\lambda$ has a finite subfamily $\{U_{\lambda_j}\}_{j=1}^N$ satisfying $K \subset \bigcup_{j=1}^N U_{\lambda_j}$.

A topological space X is said to be sequentially compact if each sequence $\{p_j\}_{j=1}^\infty$ ($\subset X$) has a convergent subsequence $\{p_{j_i}\}_{i=1}^\infty$.

Let (X, d) be a metric space. A sequence $\{p_j\}_{j=1}^\infty$ ($\subset X$) is said to be a Cauchy sequence if for each positive ϵ there is j_0 such that for every j, k with $j, k \geq j_0$ the distance between p_j, p_k satisfies $d(p_j, p_k) < \epsilon$. A metric space is said to be complete if every Cauchy sequence converges.

Given a subset U of a metric space X, we set $\mathrm{diam}(U) = \sup\{d(p, q) \mid p, q \in U\}$ and call it the diameter of U. A metric space is said to be totally bounded or said to be precompact if for each positive ϵ there exists finite open subsets $U_1, \ldots, U_N$ such that $X = \bigcup_{j=1}^N U_j$ and $\mathrm{diam}(U_j) < \epsilon$.

Lemma A.8. *For a metric space (X, d) the following conditions are mutually equiv -alent*:

(1) X *is compact*;

(2) X *is sequentially compact*;

(3) X *is totally bounded and complete.*

In order to show this we need some lemmas. A subset S of a topological space X is said to be dense if the closure $\overline{S}$ of S coincides with X. We call a topological space X separable if there is a countable dense subset S of X.

Lemma A.9. *A totally bounded metric space (X, d) is separable.*

Proof. Given a positive ϵ we take a finite covering $U_1, \ldots, U_{N_\epsilon}$ of X with $\mathrm{diam}(U_j) < \epsilon$. From each j we take a point $a_j \in U_j$ and set $A_\epsilon = \{a_1, \ldots, a_{N_\epsilon}\}$. For an arbitrary $p \in X$, we take j_0 with $x \in U_{j_0}$. Then we have $d(p, a_{j_0}) < \epsilon$. Thus we see $d(p, A) = \min\{d(p, a_j) \mid j = 1, \ldots, N_\epsilon\} < \epsilon$.

We set $A = \bigcup_{\ell=1}^{\infty} A_{1/\ell}$, which is a countable or finite set. We then for an arbitrary $p \in X$ we have $d(p, A) \le d(p, A_{1/\ell}) < 1/\ell$ for every ℓ. Thus $p \in \overline{A}$. We obtain $\overline{A} = X$ and find that X is separable. $\square$

A family $\mathcal{B}$ of open sets of a topological space is said to be an basis ot the topology if for each open set U and each point $p \in U$ there is $W \in \mathcal{B}$ with $x \in W \subset U$. We say a topological space X to be second countable if there is a countable family $\mathcal{B} = \{U_j \mid j = 1, 2, \ldots\}$ of open sets which is a basis of the topology.

Lemma A.10. *Every metric space (X, d) is a second countable space.*

Proof. Since X is separable by Lemma A.9, we can take a countable subset S of X with $\overline{S} = X$. We define a countable family of open subsets by $\mathcal{B} = \{B_r(p) \mid p \in S,\ r \in \mathbb{Q}\}$. For an arbitrary open set U and an arbitrary point $p \in U$ we choose positive ϵ so that $B_{2\epsilon}(p) \subset U$. Since S is dense, we have $q \in S$ with $d(p, q) < \epsilon$. We choose r_p so that $d(p, q) < r < \epsilon$ and consider $B_r(q)$. As we have $d(p, x) \le d(p, q) + d(q, x) < 2r < 2\epsilon$ for every $x \in B_r(q)$, we find $p \in B_r(q) \subset B_{2\epsilon}(p) \subset U$. Hence $\mathcal{B}$ is a basis of the topology. Hence X is second countable. $\square$

We note that a second countable topological space is separable. Hence a metric space is separable if and only if it is second countable.

Lemma A.11. *Let X be a second countable space, For an arbitrary open covering $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ of X we can choose a countable open sub-covering $\{U_{\lambda_j}\}_{j=1}^{\infty}$ of X.*

Proof. We take a countable baisis $\mathcal{B}$ of the topology. For each $W \in \mathcal{B}$, we set a family $\mathcal{U}_W$ by $\mathcal{U}_W = \{U_\lambda \mid W \subset U_\lambda$, and define a subfamily $\mathcal{B}'$ of $\mathcal{B}$ by $\mathcal{B}' = \{W \mid W \in \mathcal{B}, \mathcal{U}_W \neq \emptyset\}$. This family is countable. For each $W \in \mathcal{B}'$ we choose $U_\lambda \in \mathcal{U}_W$ and denote it U_W. For an arbitrary point $p \in X$ we have U_λ with $p \in U_\lambda$. Then there is $W \in \mathcal{B}$ with $p \in W \subset U_\lambda$. Thus $W \in \mathcal{B}'$. Hence $p \in W \subset U_W$. Therefore we have $X = \bigcup_{W \in \mathcal{B}'} U_W$. Thus, $\{U_W\}_{W \in \mathcal{B}'}$ is a countable open covering of X. $\square$

Proof of Lemma A.8. (1) $\Rightarrow$ (2). We take an arbitrary sequence $\{p_j\}_{j=1}^{\infty}$ of X. We suppose $\{p_j\}_{j=1}^{\infty}$ does not have accumulation points. In particular, $\{p_j\}_{j=1}^{\infty}$ contains infinitely many distinct points. That is, for each $q \in X$ we have a positive ϵ_q such that the cardinality of the set $\{j \mid p_j \in B_{\epsilon_q}(q)\}$ is finite. As $X = \bigcup B_{\epsilon_q}(q)$, we can take finite points $q_1, \ldots, q_N \in X$ with $X = \bigcup_{k=1}^{N} B_{\epsilon_{q_k}}(q_k)$. Since we have

$$\sharp\big(\{p_j \mid j \geq 1\}\big) = \sharp\Big(\{p_j \mid j \geq 1\} \cap \big(\bigcup_{k=1}^{N} B_{\epsilon_{q_k}}(q_k)\big)\Big) \leq \sum_{k=1}^{N} \sharp\big(\{p_j \mid j \geq 1\} \cap B_{\epsilon_{q_k}}(q_k)\big) < \infty,$$

where $\sharp(S)$ for a set S denotes the cardinality of the set S, we find a contradiction. Thus, we have a point q_0 such that $\{j \mid p_j \in B_{\epsilon}(q_0)\}$ is an infinite set for every positive ϵ. We take j_1 so that $p_{j_1} \in B_1(q_0)$, and inductively, we take j_{k+1} so that $j_{k+1} > j_k$ and $p_{j_k} \in B_{1/k}(q_0)$. Then the subsequence $\{p_{j_k}\}_{k=1}^{\infty}$ converges to q_0 because $d(p_{j_k}, q_0) < 1/k$. Thus X is sequentially compact.

(2) $\Rightarrow$ (3). First we shall show that X is complete. We take an arbitrary Cauchy sequence $\{p_j\}_{j=1}^{\infty}$. Since X is sequentially compact, we have a convergent subsequence $\{p_{j_k}\}_{k=1}^{\infty}$, where $\{j_k\}_{k=1}^{\infty}$ is monotone increasing. We set $p_\infty = \lim_{k\to\infty} p_{j_k} \in X$. For every positive ϵ, there is a positive number K_ϵ and N_ϵ such that $d(p_\infty, p_{j_k}) < \epsilon/2)$ for $k \geq K_\epsilon$ and $d(p_j, p_\ell) < \epsilon/2$ for $j, \ell \geq N\epsilon$. Thus if $\ell \geq \max\{K_\epsilon, N_\epsilon\}$, we have $d(p_\infty, p_\ell) \leq d(p_\infty, p_{j_\ell}) + d(p_{j_\ell}, p_\ell) < \epsilon$. Hence we see $\lim_{j\to\infty} p_j = p_\infty$. Thus, X is complete.

Next we shall show that X is totally bounded. To do this we suppose X is not totally bounded. There exist a positive ϵ_0 such that for an arbitrary finite subset S of X the open set $\bigcup_{p\in S} B_{2\epsilon_0}(p)$ does not cover X. We take an arbitrary point $p_1 \in X$. We inductively take $p_{j+1} \in X \setminus \Big(\bigcup_{k=1}^{j} B_{\epsilon_0}(p_k)\Big)$. Since $\bigcup_{k=1}^{j} B_{\epsilon_0}(p_k) \subset \bigcup_{k=1}^{j} B_{2\epsilon_0}(p_k) \neq X$, we can take such a point. Clearly we have $d(p_{j+1}, p_k) \geq \epsilon_0$ for $k = 1, \ldots, j$. This shows that $d(p_j, p_k) \geq \epsilon_0$ if $j \neq k$. Thus $\{p_j\}_{j=1}^{\infty}$ does not have Cauchy subsequences. This shows that X is not sequentially compact, which is a contradiction.

(3) $\Rightarrow$ (2). Given a positive ϵ we take a finite covering $U_1, \ldots, U_N$ of X with $\mathrm{diam}(U_j) < \epsilon$. Then for each sequence $\{p_j\}_{j=1}^{\infty}$ in X, there is U_{i_0} such that it contains

infinitely many points of $\{p_j \mid j\}$. Hence we can take a subsequence $\{p_{j_k}\}_{k=1}^{\infty} \subset U_{i_0}$. This sequence satisfy $d(p_{j_k}, p_{j_{k'}}) < \epsilon$.

Under this consideration, we take an arbitrary sequence $\{q_j\}_{j=1}^{\infty}$ in X. We can take a subsequence $\{q_j^{(1)}\}_{j=1}^{\infty}$ of this sequence with $d(q_j^{(1)}, q_{j'}^{(1)}) < 1$. For this new sequence we take a subsequence $\{q_j^{(2)}\}_{j=1}^{\infty}$ of with $d(q_j^{(2)}, q_{j'}^{(2)}) < 1/2$. Inductively we take a subsequence $\{q_j^{(\ell)}\}_{j=1}^{\infty}$ of $\{q_j^{(\ell-1)}\}_{j=1}^{\infty}$ with $d(q_j^{(\ell)}, q_{j'}^{(\ell)}) < 1/\ell$. We set $a_j = p_j^{(j)}$, and construct a subsequence $\{a_j\}_{j=1}^{\infty}$ of $\{q_j\}_{j=1}^{\infty}$. Then, when $j, j' \geq j_0$ we have $d(a_j, a_{j'}) < 1/j_0$, because $a_j, a_{j'}$ are contained in $\{q_j^{(j_0)}\}_{j=1}^{\infty}$. Hence we find that $\{a_j\}_{j=1}^{\infty}$ of $\{q_j\}_{j=1}^{\infty}$ is a Cauchy sequence. Hence it converges to some point in X. Thus X is sequentially compact.

(3) $\Rightarrow$ (1). Since X id totally bounded, every open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of X contains a countable open subcovering $\{U_{\lambda_j}\}_{j=1}^{\infty}$ by Lemmas A.9, A.10 and A.11. If we suppose that $\{U_{\lambda_j}\}_{j=1}^{\infty}$ does not have finite open sub-covering, we can take a point $p_j \in X \setminus \left(\bigcup_{k=1}^{j-1} U_{\lambda_k}\right)$. Since X is sequentially compact because conditions (2) and (3) are equivalent, we have a convergent subsequence $\{p_{j_k}\}_{k=1}^{\infty}$ where $\{j_k\}_{k=1}^{\infty}$ is monotone increasing. We set $p_\infty = \lim_{k\to\infty} p_{j_k}$. For each ℓ, if we take $k \geq \ell$ we have $j_k \geq k \geq \ell$. Hence we have $p_{j_k} \in X \setminus \left(\bigcup_{i=1}^{j_k-1} U_{\lambda_i}\right) \subset X \setminus \left(\bigcup_{i=1}^{\ell-1} U_{\lambda_i}\right)$ for $k \geq \ell$. Since $X \setminus \left(\bigcup_{i=1}^{\ell-1} U_{\lambda_i}\right)$ is a closed set, we see $p_\infty \in X \setminus \left(\bigcup_{i=1}^{\ell-1} U_{\lambda_i}\right)$. Thus we find $p_\infty \in \bigcap_{\ell=1}^{\infty}\left(X \setminus \left(\bigcup_{i=1}^{\ell-1} U_{\lambda_i}\right)\right) = X \setminus \left(\bigcup_{i=1}^{\infty} U_{\lambda_i}\right)$. But $\{U_{\lambda_j}\}_{j=1}^{\infty}$ is an open covering of X, it is a contradiction. $\square$

Refrences

[1] T. Adachi, *Kähler magnetic flows on a manifold of constant holomorphic sectional curvature*, Tokyo J. Math. 18 (1995), 473–483.

[2] ______, *A comparison theorem for magnetic Jacobi fields*, Proc. Edinburgh Math. Soc., 125 (1997), 1197 – 1202.

[3] ______, *Kähler magnetic flows for a product of complex space forms*, Topol. Appl. 146–147 (2005), 201 – 207.

[4] ______, *Magnetic Jacobi fields for Kähler magnetic fields*, in *Recent Progress in Differential Geometry and its Related Fields*, 41–53, T. Adachi, H. Hashimoto and M. Hristov eds, World Scientific, Singapore, 2015.

[5] ______, *A theorem of Hadamard-Cartan type for Kähler magnetic fields*, J. Math. Soc. Japan 64 (2012), 1–21.

[6] ______, *A comparison theorem on harp-sectors for Kähler magnetic fields*, Southeast Asian Bull. Math. 38 (2014), 619 – 626.

[7] ______, *A study on harp-horns for Kähler magnetic fields*, in *Contemporary Perspectives in Differential Geometry and its Related Fields*, 95–112, T. Adachi, H. Hashimoto and M. Hristov eds, World Scientific, Singapore, 2017.

[8] ______, *Accurate trajectory-harps for Kähler magnetic fields*, to appear in J. Math. Soc. Japan.

[9] T. Adachi and F. Ohtsuka, *The Euclidean factor of a Hadamard manifold*, Proc. A.M.S. 113 (1991), 209 – 213.

[10] P. Bai and T. Adachi, *Volumes of trajectory-balls for Kähler magnetic fields*, J. Geom. 105 (2014), 369–389.

[11] T. Bao and T. Adachi, Circular trajectories on real hypersurfaces in a nonflat complex space form, J. Geom. 96 (2009), 41–55.

[12] W. Ballmann, M. Gromov and V. Schroeder, *Manifolds of nonpositive curvature*, Progress in Math. 61, Birkhäuser, Boston, Basel and Stuttgart 1985.

[13] J. Cheeger and D.G. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland mathematical library, vol.9 1975.

[14] P.B. Eberlein, *Geometry of nonpositivelu curved manifolds*, The University of Chicago Press, Chicago and London 1996.

[15] P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math. 46 (1973), 45 – 109.

[16] K. Matsuzaka, *Introduction to sets and topologies* (in Japanese), Iwanami Shoten 1968.

[17] ______, *Introduction to calculas* 3, 4 (in Japanese), Iwanami Shoten 1998.

[18] S. Murakami, *Manifolds*, Kyoritsu Shuppan 1989 (in Japanese).

[19] K. Nomizu and K. Yano, *On circles and spheres in Riemannian geometry*, Math. Ann. 210 (1974), 163–170.

[20] T. Sakai, *Riemannian geometry*, Shokabo 1992 (in Japanese) and Translations Math. Monographs 149, A.M.S. 1996.

[21] Q. Shi, *Comparison theorems on trajectory-harps for surface magnetic fields* (in Japanese) Master thesis, Nagoya Institute of Technology 2014.

[22] ______, *Magnetic Jacobi fields for surface magnetic fields*, in *Current Development in Geometry and its Related fields*, 215–224, T. Adachi, H. Hashimoto and M. Hristov eds, World Scientific, Singapore, 2016.

[23] ———, *Estimates on arc-lengths of trajectory-fronts for surface magnetic fields*, Note di Matematica 37(2017) suppl.1, 131–140.

[24] Q. Shi and T. Adachi, *An estimate on volumes of trajectory-balls for Kähler magnetic fields*, Proc. Japan Acad. Sci. 92 Ser. A (2016), 47 – 50.

[25] ———, *Trajectory-harps and horns applied to the study of the ideal boundary of a Hadamard Kähler manifold*, Tokyo Journal of Mathematics.VOL.40, NO.1(2017), 223–236.

[26] ———, *Comparison theorem on trajectory-harps for Kähler magnetic fields which are holomorphic at their arches*, to appear in Hokkaido Math. J..

[27] T. Sunada, *Magnetic flows on a Riemann surface*, Proc. KAIST Math. Workshop 8(1993), 93–108.

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Index

A
asymptotic, 157

B
Bishop's comparison theorem, 133

C
Caratan Hadamard theorem, 151
circle, 32
complex Euclidean space, 43
complex hyperbolic space, 51
complex manifold, 39
complex projective space, 44
complex space form, 43
complex structure, 40
cone topology, 160
congruent, 66
conjugate point, 22

D
distance function, 6

E
embouchure angle, 176
Euclidean space, 13
exponential map, 20

F
first magnetic conjugate value, 72
fixed point theorem, 191
Frenet frame, 32
Fubini-Study metric, 45

G
Gauss Bonnt theorem, 184
geodesic, 19
geodesically complete, 27
geodesic ball, 132
geodesic curvature, 32
Gromov's comparison theorem, 135

H
Hadamard manifold, 153
harp arc, 139
harp body, 117
harp sector, 139
Hermitian, 40
holomorphic at arch, 125
Hopf fibration, 44
Hopf-Renow theorem, 27
horn body, 178

I
ideal boundary, 158
injectivity radius, 132

J
Jacobi field, 21

K
Kähler form, 41
Kähler magnetic field, 61
Kähler manifold, 39

L
limit horn tube, 178
limit string, 147

M
magnetic conjugate point, 72
magnetic conjugate value, 72
magnetic exponential map, 67
magnetic field, 61
magnetic Jacobi field, 69

N
normal coordinate neighborhood, 20

P
parallel, 17
— displacement, 17

Q
quotient manifold, 13

R
Rauch's comparison theorem, 86
real hyperbolic space, 13
real space form, 13
Riemannian connection, 7
Riemannian curvature tensor, 9
Riemannian manifold, 5
Riemannian metric, 5

S
sectional curvature, 11
sector arc, 139
standard sphere, 13
static magnetic field, 61
string cosine, 101
string length, 101
surface magnetic field, 61

T
Toponpgov's comparison theorem, 99
totally geodesic, 49
trajectory,62
— ball, 132
— harp, 100
— horn, 174
tube cosine, 174
tube length, 174

U
uniformly normal neighborhood, 25

V
variation of geodesics, 20
variation of trajectories, 69
volume element, 12
Z
zenith angle, 139

Index of Theorems

Chapter 1

Lemma
Lemma 1- 1, 7
Lemma 1- 2, 9
Lemma 1- 3, 11
Lemma 1- 4, 14
Lemma 1- 5, 16
Lemma 1- 6, 17
Lemma 1- 7, 19
Lemma 1- 8, 20
Lemma 1- 9, 23
Lemma 1- 10, 23
Lemma 1- 11, 25
Lemma 1- 12, 33

Proposition
Proposition 1- 1, 21
Proposition 1- 2, 28
Proposition 1- 3, 32

Theorem
Theorem 1- 1, 27

Chapter 2

Lemma
Lemma 2- 1, 41
Lemma 2- 2, 46
Lemma 2- 3, 47
Lemma 2- 4, 49
Lemma 2- 5, 50
Lemma 2- 6, 53
Lemma 2- 7, 55
Lemma 2- 8, 57
Lemma 2- 9, 58
Lemma 2- 10, 63
Lemma 2- 11, 63
Lemma 2- 12, 64
Lemma 2- 13, 65

Proposition
Proposition 2- 1, 50
Proposition 2- 2, 59
Proposition 2- 3, 66

Corollary
Corollary 2- 1, 46
Corollary 2- 2, 54

Chapter 3

Lemma
Lemma 3- 1, 69
Lemma 3- 2, 71
Lemma 3- 3, 72
Lemma 3- 4, 72
Lemma 3- 5, 80
Lemma 3- 6, 83
Lemma 3- 7, 87
Lemma 3- 8, 88

Proposition
Proposition 3- 1, 79
Proposition 3- 2, 84

Theorem
Theorem 3- 1, 86
Theorem 3- 2, 90
Theorem 3- 3, 94

Corollary
Corollary 3- 1, 91

Chapter 4

Lemma
Lemma 4- 1, 100
Lemma 4- 2, 101
Lemma 4- 3, 102
Lemma 4- 4, 106

Lemma 4- 5, 108
Lemma 4- 6, 113
Lemma 4- 7, 118
Lemma 4- 8, 126
Lemma 4- 9, 133

Proposition
Proposition 4- 1, 105
Proposition 4- 2, 105
Proposition 4- 3, 106
Proposition 4- 4, 107
Proposition 4- 5, 109
Proposition 4- 6, 110
Proposition 4- 7, 114
Proposition 4- 8, 116
Proposition 4-9, 137
Proposition 4-10, 137
Proposition 4-11, 144

Theorem
Theorem 4- 1, 99
Theorem 4- 2, 118
Theorem 4- 3, 125
Theorem 4- 4, 133
Theorem 4- 5, 133
Theorem 4- 6, 134
Theorem 4- 7, 135
Theorem 4- 8, 135
Theorem 4- 9, 136
Theorem 4- 10, 140
Theorem 4- 11, 147

Corollary
Corollary 4- 1, 142
Corollary 4- 2, 145

Chapter 5

Lemma
Lemma 5- 1, 159
Lemma 5- 2, 159
Lemma 5- 3, 182

Proposition
Proposition 5- 1, 151
Proposition 5- 2, 153
Proposition 5- 3, 155
Proposition 5- 4, 158
Proposition 5- 5, 160
Proposition 5- 6, 176
Proposition 5- 7, 179
Proposition 5- 8, 180
Proposition 5- 9, 183
Proposition 5- 10, 184

Theorem
Theorem 5- 1, 151
Theorem 5- 2, 162
Theorem 5- 3, 164
Theorem 5- 4, 165
Theorem 5- 5, 168
Theorem 5- 6, 168
Theorem 5- 7, 176
Theorem 5- 8, 180
Theorem 5- 9, 184
Theorem 5- 10, 184

Corollary
Corollary 5- 1, 151
Corollary 5- 2, 154
Corollary 5- 3, 182
Corollary 5- 4, 182

Appendix

Lemma
Lemma A- 1, 187
Lemma A- 2, 188
Lemma A- 3, 192
Lemma A- 4, 192
Lemma A- 5, 193
Lemma A- 6, 193
Lemma A- 7, 194
Lemma A- 8, 195
Lemma A- 9, 195
Lemma A- 10, 196
Lemma A- 11, 196

Theorem
Theorem A- 1, 189
Theorem A- 2, 191

Corollary
Corollary A- 1, 190