OPTIMAL REPLACEMENTS UNDER ADDITIVE DAMAGE IN RANDOM ENVIRONMENTS

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博士論文（1992年12月3日授与）
甲第92号

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CHAPTER 1

INTRODUCTION

Systems used in the production of goods and delivery of services constitute the vast majority of most industry's capital. These systems are subject to deterioration with usage and age. System deterioration is often reflected in higher production costs and lower production quality. To keep production costs down while maintaining good quality, preventive maintenance is often performed on systems subject to deterioration. The growing importance of maintenance has generated an increasing interest in the development and implementation of preventive maintenance models for deterioration systems. During the last decades, a great deal of studies have been made for preventive maintenance model. At the present time, there is still a great need for investigating such models as the growth in the complexity of modern systems.

Shock models with additive damage are an important class of preventive maintenance models for deteriorating systems. A system is subject to a sequence of randomly occurring shocks, and each shock causes a random amount of damage which accumulates additively over time. The system might fail at times of shock arrival. Upon failure, the system must be replaced by a new one having properties that are statistically equivalent to the original, and a cost is incurred. The system may be maintained or replaced before failure at a smaller cost. To give an optimal maintenance– replacement policy for such systems, shock models with additive damage have been studied in quite a number of recent articles. In these shock models presented by former researchers, the influences of "randomly varying environment" on systems have not been considered. Only Waldmann [66], we know, has given a shock model with additive damage in which an "environment process" was introduced, but that is for a lattice damage process and discrete time case.

In many cases of application, the behaviors of systems depend not only on shock processes, but also on "environments" where systems are. The environments may be external factors of an economical or technical nature as well as internal factors of a statistical nature. For example,

(a) Consider a system that receives two types of shocks at random points of time. The corresponding damage processes are related each other, and each type of shocks may cause the system to fail. One of them can be regarded as the "environment" process.

(b) Consider a system with a modulator whose states can be described by a Markov jump process. The system is subject to shocks, and the probability characteristics of shocks (for instance, the distributions of intershock times and shock magnitudes) are dependent on the state of the modulator. Hence, the Markov jump process of the modulator can be taken as the "environment" process.

(c) Consider a system subject to independent shocks. The distributions $G_m(\cdot)$ and $H_m(\cdot)$ of the shock magnitudes and the intershock times are assumed to be incompletely known, i.e., $m \in M$ is a unknown parameter. Just after every shock arrival, the parameter m has to be estimated by Bayesian statistical method. The estimation process of parameter m can be referred as the "environment" process.

Therefore, it is necessary to consider influences of random factors in analyzing optimal preventive maintenance problems for systems subject to deterioration.

In this thesis, we mainly investigate optimal preventive maintenance– replacement problems of systems existing in randomly varying environments which can be described by Markov jump processes. Systems are subject to a sequence of randomly occurring shocks and to failure. Shock arrivals and shock magnitudes are influenced by changes of environment state. The shock process and environment process are assume to be continuous time processes. We construct a new damage process by these two processes which generalizes the damage processes given by the former researchers. Under nature conditions, we derive optimal maintenance—replacement policies for the long-run average cost per unit time and the total expected discounted cost criterions respectively. These policies are different from traditional optimal policies because the shock process is influenced by the second process—the environment process.

1.1 The description of the problem

First we describe in detail the shock model analyzed in this thesis.

Consider a system existing in a randomly varying environment. The environment changes can be described by a Markov jump process called *Markov environment process* (MEP). The system is subject to a sequence of randomly occurring shocks, and each shock causes a random amount of damage which accumulates additively. The shock arrivals and shock magnitudes are influenced by changes of the environment state. The damage process is assumed to be a *piecewise semi-Markov process* (PSMP) which is constructed by the shock process and the environment process. Any of the shocks or the changes of the environment state might cause the system to fail. The survival probability at a shock time point or a change time point of the environment state is determined by a known survival function of the accumulated damage level, the environment state and the realized shock magnitude. Upon failure, the system must be replaced by identical one and a cost is incurred. The system can be maintained or replaced before failure at a smaller cost. The replacement time and maintenance time are assumed to be negligible, and the replacement cycles are repeated indefinitely.

The mathematical descriptions of the above shock model and definitions about some concepts to be used are given as follows.

(I) The enviroment process of the system

Let $\{\xi(t)\}_{t\geq 0}$ be a stochastic process specifying the environment changes of the system. The process $\{\xi(t)\}_{t\geq 0}$ is assumed to be a stationary regular Markov jump process with the state space Γ and the initial state $\xi(0)$. Let $\Re$ be a σ-field of Γ such that one point set $\{\xi\} \in \Re$, and $\{\omega_n\}_{n\geq 0}$ ($\omega_0 = 0$) the jump points of $\{\xi(t)\}_{t\geq 0}$. The $Q(\xi, A)$ is a Markov kernel on $(\Gamma, \Re)$ with $Q(\xi, \{\xi\}) = 0$, i.e., $Q(\xi, \cdot)$ is a probability measure for every $\xi \in \Gamma$, and $Q(\cdot, A)$ is a $\Re$-measurable function for every $A \in \Re$. For any $A \in \Re$ and $t \in R_+$, let

$$P(\xi(\omega_{n+1}) \in A, \omega_{n+1} - \omega_n \leq t | \xi(s), s \leq \omega_n) = Q(\xi(\omega_n), A)(1 - e^{-\eta(\xi(\omega_n))t}) \quad (1.1.1)$$

where $\eta : \Gamma \to R_+$ is a finite function. We call $\{\xi(t)\}_{t \geq 0}$ *Markov environment process* (MEP).

(II) The damage process of the system

Let $R_+ = [0, \infty)$ and $\Im$ Borel-field of R_+. For any $(\xi, z_0) \in \Gamma \times R_+$, let $\{Z_{(\xi, z_0)}(t)\}_{t \geq 0}$ be a semi-Markov process with the state space R_+ and the initial state $Z_{(\xi, z_0)}(0) = z_0$. The semi-Markov kernel of the process $\{Z_{(\xi, z_0)}(t)\}_{t \geq 0}$ is defined as follows.

For any $x, z \in R_+$, and $t \in R_+$,

$$\begin{aligned} & P(Z_{(\xi, z_0)}(\tau_{n+1}^\xi) - Z_{(\xi, z_0)}(\tau_n^\xi) \leq x, \tau_{n+1}^\xi - \tau_n^\xi \leq t | Z_{(\xi, z_0)}(\tau_n^\xi) = z) \quad (1.1.2) \\ & = \int_0^t G_z^\xi(x|s) H_z^\xi(ds) \end{aligned}$$

where $\{\tau_n^\xi\}_{n \geq 0}$ ($\tau_0^\xi = 0$) are the jump points of the process $\{Z_{(\xi, z_0)}(t)\}_{t \geq 0}$, $H_z^\xi(\cdot)$ is the conditional probability distribution of the intershock time $\tau_{n+1}^\xi - \tau_n^\xi$, and $G_z^\xi(\cdot|t)$ is the conditional probability distribution of $Z_{(\xi, z_0)}(\tau_{n+1}^\xi) - Z_{(\xi, z_0)}(\tau_n^\xi)$ given $Z_{(\xi, z_0)}(\tau_n^\xi) = z$ and $\tau_{n+1}^i - \tau_n^i = t$. We suppose that $\{Z_{(\xi, z_0)}(t)\}_{t \geq 0}$ be a right-continuous regular process with left-hand limits.

Now, by appealing the process $\{Z_{(\xi, z)}\}_{t \geq 0}$, we define the stochastic process $\{Z(t)\}_{t \geq 0}$ which specifies the cumulative damage of the system in one replacement cycle such that $Z(0) = 0$ and $Z(t) = Z_{(\xi, Z(\omega_n -))}(t - \omega)$ on $\{\omega_n \leq t < \omega_{n+1}; \xi(\omega_n) = \xi\}$. That is,

$$\begin{aligned} & Z(0) = 0 \quad (1.1.3) \\ & Z(t) = Z_{(\xi(0), 0)}(t) I_{\{0 < t < \omega_1\}} + \textstyle\sum_{n=1}^\infty Z_{(\xi(\omega_n), Z(\omega_n -))}(t - \omega_n) I_{\{\omega_n \leq t < \omega_{n+1}\}}. \end{aligned}$$

The process $\{Z(t)\}_{t \geq 0}$ is also a right-continuous regular process with left-hand limits . At the points $\omega_n, n \geq 1$, $Z(\omega_n) = Z(\omega_n -)$, and on the interval $[\omega_n, \omega_{n+1})$, the process

$\{Z(t)\}_{t\geq 0}$ is a semi-Markov process dependent on the environment state $\xi(\omega_n)$. We call $\{Z(t)\}_{t\geq 0}$ *piecewise semi-Markov process*(PSMP).

(III) The survival function of the system

In our models, a failure of the system can occur only at the time points of shocks or changes of the MEP state. Let T be such a time point, suppose $\xi(T-) = \xi$ and $Z(T-) = z$. At time T, if a shock of magnitude x occurs, then the system fails with known probability $1 - \gamma(z, \xi, x)$, and if a change of the MEP state into the state ζ occurs, then the system fails with known probability $1 - \gamma(z, \zeta, 0)$. The function $\gamma : R_+ \times \Gamma \times R_+ \to [0, 1]$ is referred to as *the survival function of the system.* Let Δ be the failure state of the system and δ the first failure time of the system. Throughout we assume that $E[\delta] < \infty$.

(IV) The maintenance and replacement costs of the system

$m(\xi, z)$ and $c(\xi, z)$ represent respectively the maintenance cost and replacement cost of the system at the state (ξ, z), and $c(\xi, \Delta)$ represents the replacement cost at failure. In Chapter 1 and 2, $c(\xi, z) = C > 0$ and $c(\xi, \Delta) = C + C_0 > 0$.

(V) The stopping time, control-limit and state-age dependent policies

Definition 1.1. For $t \geq 0$, let $\Im_t$ be the σ–field generated by the two-dimensional process $\{\xi(t), Z(t)\}$ up to time t, i.e.,

$$\Im_t = \sigma((\xi(s), Z(s)),\ \ s \in [0, t]). \tag{1.1.4}$$

For a random variable T, if

$$\{T \leq t\} \in \Im_t \qquad \text{for all } t \geq 0, \tag{1.1.5}$$

then T is called *$\Im_t$–stopping time.*

Definition 1.2. A replacement policy is called *the control-limit policy* if the system is replaced upon failure or when the damage process exceeds a critical control process, whichever occurs first.

Definition 1.3. A replacement policy is called *the state-age dependent policy* if the system is replaced upon failure or the sojourn time in a state (ξ, z) reaches a threshold level $A(\xi, z)$ which is a positive real-valued function.

(VI) The cost criterions

1. The long-run average cost per unit time.
2. The total expected discounted cost.

Remark.

(1) It can be seen that if the environment of the system is restricted to only one state (i.e. the case that the influences of the environment is not considered), the process $\{Z(t)\}_{t\geq 0}$ defined in (1.2) becomes the semi-Markov damage process given by Posner and Zuckerman [51]. Moreover, if $\{Z_{(\xi,z_0)}(t)\}_{t\geq 0}$ is taken as a compound Poisson process with the intensity $\rho(\xi)$, then the process $\{Z(t)\}_{t\geq 0}$ is a conditional compound Poisson process with the random intensity $\rho(\xi(t))$.

(2) When the system is replaced, if the environment state does not restore to the initial state $\xi(0)$, the successive replacement periods of identical systems no long form a renewal process. This is different from the traditional shock models in which the successive replacement time points is a renewal process.

Throughout the thesis, the term 'increasing' will be used to mean 'non-decreasing' and 'decreasing' to mean 'non-increasing', and the following will be standard notation:

$$
\begin{aligned}
P_{(\xi,z)}[\,\cdot\,] &= P[\,\cdot\,|\xi_0 = \xi, Z_0 = z] \\
E_{(\xi,z)}[\,\cdot\,] &= E[\,\cdot\,|\xi_0 = i, Z_0 = z] \\
E_{(\xi,z)}[\,\cdot\,; B|A] &= E[\,\cdot\chi_B|\xi_0 = i, Z_0 = z, A]
\end{aligned}
$$

where A and B are events, χ_B is the indicate function of the set B.

1.2 Review of the literatures

Now we review the related literatures.

The replacement shock models with additive damage have been studied in quite a number of recent articles. An excellent survey of the theory of optimal replacement of systems subject to shocks, specifying results up to 1989, can be found in Valdez-Flores and Feldman [65]. The shock models with various additive damages are investigated and the optimal replacement policies are given.

Taylor [61] studies the shock model that shocks occur according to a Poisson process and the damage process caused by shocks are independent and identically exponential variable. The replacement cost before failure and at failure, $c(z) = C > 0$ and $c(\Delta) = C + C_0, C_0 > 0$, respectively, are constants independent of the damage level at the time of replacement. Taylor derives the long-run average cost per unit time expression and proves that the optimal policy is control-limit rule.

Feldman [26,28] generalizes Taylor's model by allowing the times between shocks to be arbitrarily distributed and dependent on the accumulated damage. He assumes that the cumulative damage is a increasing semi-Markov process and finds the optimal policy among the set of control-limit policies that replace only at shock times. Feldman [27] studies the same problem for the total expected discounted cost and without the restriction to control-limit policies. He proves that the optimal policy, among the policies that replace at shock times, is a control-limit policy provided some conditions are satisfied, but no algorithm is given to find the optimal policy.

Feldman's models allow failures and replacements at shock times only. Aven and Gaarder [11] look at a very similar shock model where the system can fail at any time with a probability that is conditional upon the history of the process. They show that if the conditional failure rate of the system is increasing, the policy that minimizes the long-run average cost per unit time is a control- limit rule. They do not give an algorithm to solve the problem.

Siedersleben [60] considers a system that deteriorates continuously, but whose state is observed only at random times T_n. The problem can be regarded as a shock model where the cumulative deterioration between times T_{n-1} and T_n- is z_{n-1}, and the magnitude of the shock at time T_n is $Z_n - z_{n-1}$. He assumes that the process forms a Markov renewal process and that the inspection cost is negligible. The replacement cost $c(z)$ is assumed to be a increasing function of the cumulative deterioration level. Under some monotonicity conditions, Siedersleben proves that a control-limit policy is optimal for the total expected discounted cost criterion.

Zuckerman [72] generalizes Feldman [26] by allowing replacement at any time before failure instead of being restricted to at shock times only. Zuckerman uses an (Z, Y) process that describes the cumulative deterioration level Z and time elapsed time since the last shock Y in order to prove, under some conditions, that the optimal replacement time that minimizes the long-run average cost per unit time calls for replacement upon failure or when the (Z, Y) process is lexicographically bigger than or equal to (z^*, y^*), whichever occurs first. Thus, his policy is a combination rule of a control-limit policy and a state-age dependent policy.

Abdel-Hameed and Shimi [7] generalize Taylor's cost model by allowing the replacement cost before failure to be a increasing convex function of the cumulative damage, where the damage caused by shocks are independent and identically distributed random variables. They use martingale theory to show that the optimal replacement is a control-limit rule when replacement can be made only at shock times. Zuckerman [68] analyzes Abdel-Hameed and Shimi's model and proves that the optimal policy given in their article is optimal among all replacement policies that consider replacement at any stopping time before failure, i.e., the optimal policy replaces at shock times so that the restriction to replace only at shock times can be dropped from Taylor's and Abdel-Hameed and Shimi's models. Zuckerman's proof is based on the fact that the time between shocks is exponentially distributed.

Abdel-Hameed [3] considers the optimal replacement problem when the damage process is a non-homogeneous Lévy process. In [4] he studies a system that is subject to shocks by using an increasing pure-jump strong Markov process. The system is assume to fail once its cumulative damage process exceeds a threshold. He shows that for this general jump process, a control-limit policy is optimal for the long-run average cost per unit time, provided some conditions on the cost are satisfied. Abdel-Hameed [2] treats the optimal maintenance problem when the deterioration process is increasing pure jump Markov process that is monitored periodically. In [6] he considers the optimal replacement and maintenance of systems subject to semi-Markov damage. The system is assume to have a random threshold, and it fails once the cumulative damage process exceeds it. He determines the optimal replacement policy, within the class of control-limit policies, according to the total expected discounted cost and long-run average cost per unit time criterion.

Bergman [15] presents a general model for optimal replacement when the policy is based on the measurement of an increasing state variable, such as the cumulative damage caused by shocks. He does not assume anything about the damage process, other than it is increasing. A replacement can take place at any time before failure. He shows that the policy minimizing the long-run average cost per unit time is a control-limit policy. Bergman gives a convergent iterative algorithm to find the optimal policy.

Nummelin [47] studies the same general model as Bergman's, except that the replacement costs before failure, and at failure are random variables dependent on the history of the system up to time t, i.e., the costs are non-negative stochastic processes adapted to the damage level of the system. He proves that the optimal rule is a control-limit policy and gives an iterative algorithm, very similar to Bergman's, to find the optimal replacement time. Aven [8] uses a counting process approach to present setup for a large class of replacement models. Several of the shock models discussed before fall into this general setup.

Kao [36] considers the optimal state-age dependent replacement problem and gives a detail analysis by using semi-Markov decision theory on the discrete time parameter space. Gottlieb [32] studies a system subject to shocks occurring according to a semi-Markov process with the time between shocks being random variables dependent on the deterioration level. His model can replace the system at any time before failure. Gottlieb relaxes the assumption that the failure rate is increasing with respect to the cumulative damage. He proves that weaker conditions are sufficient for the optimal replacement policy to be of the control-limit class using the long-run average cost per unit time criterion. He gives an algorithm to find the optimal policy for the case in which the state space is or can approximated by a lattice grid. His policy is a state-age dependent rule that replaces as soon as the time since the last shock reaches some threshold level which is a function of the accumulated damage.

Feldman and Joo [29] study a similar problem. They have the time between shocks being independent and identically distributed random variables with an increasing failure-rate distribution function. They also find the optimal state-age dependent policy that minimizes the long-run average cost per unit time, and prove that the optimal state-age dependent policy is a decreasing function with respect to the state space. They give an efficient algorithm to find the optimal policy and compare it with other algorithms.

Mizuno [42] studies the same problem as Gottlieb [32] and transforms it to what he calls a generalized mathematical programming problem, which can be reduced to a version of a linear program if the state and action spaces are finite. the main result in Mizuno is the proof of the optimality of the control-limit policy under weaker sufficient conditions.

Posner and Zuckerman [51] study the same problem as Gottlieb [32] and present the same results under weaker sufficient conditions for both the long-run average cost per unit time and the total expected discounted cost. They give different sets of conditions for the cases in which the system can be replaced at any time before failure. They also prove that the optimal policies for both cases replace at shock times, provided some conditions are satisfied in the cumulative hazard rate and the distribution of times between shocks.

Some results have been published for the cases in which system deterioration occurs continuously as we as induced by shocks at discrete points of time. Feldman [25] generalizes his previous works by allowing shocks to occur continuously during a time interval. This model assumes that the cumulative damage is a semi-Markov process for every deterioration period. He proves that, among the policies that replace within the sets of deterioration times only, the one that minimizes the long-run average cost per unit time is a control-limit policy. Zuckerman [71] studies a system that the accumulative deterioration through a continuous "wear" process and eventually fails; i.e., the model allows an infinite number of shocks in a finite period of time. The system can be replaced at any stopping time before failure. The policy that minimizes the long-run average cost per unit time is shown to be of the control-limit class.

The shock models presented above have considered replacement or not replacement as the only two possible maintenance actions that can be taken at every deterioration level. Several researchers have developed more general models that allow the decision maker choose one of several maintenance decisions at every damage level that affect the deterioration process of the system. Chikte and Deshmukh [19] study a system subject to randomly occuring shocks that can be controlled by continuously preventive maintenance expenditures. They show that a control-limit policy is optimal and that the maintenance expenditure rate should be reduced as the deterioration level increases to the control-limit.

Valdez-Flores [64] studies a system that can be repaired to better damage levels at a cost that depends on the deterioration of the system and the extent of the repair. At every shock a decision is made whether to repair or to leave the system as is. He uses Markov renewal processes to model the problem, and gives sufficient conditions for the policy that minimizes the long-run average cost per unit time to be a pseudo-control-limit policy.

Yamada [67] describes the shock model by using a general jump process. The replacement cost before failure is a function of the accumulated damage and the time of the replacement, while the replacement cost at failure is fixed. He uses martingale theory to show that the optimal replacement time can be found explicitly under appropriate conditions for the cumulative deterioration process and the cost function.

The models presented thus far are analyzed under assumption that the system is only subject to shock, not influenced by other factors such as environment, temperature, etc. In some cases, these factors are important measurements that must be considered, in order to give a more appropriate replacement policy for the system. Waldmann [66] includes an "environment process" to the regular shock model. In particular, he considers that each shock causes a discrete or lattice random amount of damage depending on the realization of a stochastic process describing the environment. Furthermore, he assumes a limited dependency between the environment process and the shock magnitudes. He proves that the policy that minimizes the total expected discounted cost is a rule that replaces the system whenever the accumulated damage exceeds a critical number that depends on the state of the enviroment, and does not replace otherwise.

1.3 Outline of the thesis

In the following we summarize the thesis. This thesis consists of Instroduction, Chapter 2–4, Conclusion and References.

Chapter 2 considers an optimal replacement problem of the system existing in an randomly varying environment that can be described by a Poisson process. The system receives a sequence of randomly occurring shocks, and each shock causes a random magnitude of damage. Shock arrivals and shock magnitudes are affected by state changes of the Poisson environment process. For the long-run average cost per unit time as well as the total expected discounted cost, we derive the optimal stopping times on the two class sets of replacement policies: (1) the replacement can be made only at shock times or jump times of the Poisson environment process; (2) the replacement can be made at any stopping times before failure. By defining an integer-valued random variable, we prove that the optimal replacement policies are control-limit rules dependent on the Poisson environment process, and give the control-limit processes and the corresponding bounded processes. Furthermore, the results obtained there can be extended to the case that the environment process is a increasing Markov process with a constant jump rate. At the last, applications of the model are given.

Chapter 3 investigates an state-age dependent optimal replacement problem for a network system consisted of a main-system and a sub-system with N components. The component's functioning times are exponentially distributed random variables with the same parameters, and every failed component is repaired by one repairman by taking an exponentially distributed time. The main-system is subject to a sequence of randomly occurring shocks and each shock causes a random amount of damage. Shock arrivals and magnitudes depend on the accumulated damage level of the main-system itself and the number of the functioning components of the sub-system. Any of the shocks or component's failures might cause the main-systen to fail. By using the Markov decision theory, we derive an optimal state-age dependent replacement policy which minimizes the long-run average cost per unit time. The results can be applied to the case that the environment process is a general Markov jump process. At the later, we investigate two-special cases.

Chapter 4 analyzes a generally optimal maintenance–replacement problem of the system. The randomly varying enviroment is described by a general jump Markov process. The system is subject to shocks influenced by changes of the environment state. The system can be maintained or replaced at any time before failure, at costs dependent on the environment state and the accumulated damage level. We allow that the damaged system become to "better" after every maintenance, i.e., the damage level of the system has an randomly decreasing magnitude which is assumed to be stochastically decreasing with respect to the accumulated damage level. Furthermore, we assume that the state of the environment process does not change when the system is replaced. In this case, analysis is difficult by the general renewal theory because the successive replacement periods of identical systems no longer form a renewal process. For the total expected randomly discounted cost, we derive, by the Dynamic programming method, an optimal maintenance–replacement control-limit policy which is a function dependent on the environment process.

Conclusion summarizes the results of the thesis, and states the optimal mantenance–replacement problems to be examined in the future.

References are provided in the end of the thesis.

CHAPTER 2

OPTIMAL REPLACEMENT UNDER ADDITIVE DAMAGE IN A POISSON RANDOM ENVIRONMENT

2.1 Introduction

In this chapter we consider a system existing in a random environment. The randomly varying environment is described by a Poisson process called *Poisson environment process* (PEP). The system receives shocks at random points of time and is subject to failure. Each shock causes a random amount of damage which accumulates additively over time and depends on the environment state. The damage process is assumed to be a *piecewise semi-Markov process* (PSMP) which is constructed by the shock process and the environment process. Any of the shocks or the changes of the environment state might cause the system to fail. The survival probability at a shock time or a jump time is determined by a known survival function of the state of the PEP, the accumulated damage level and the realized shock magnitude. Upon failure, the system must be replaced at a cost $C > 0$ along with an additional penalty cost $C_0 > 0$. The system may be replaced before failure at a cost of only C. The replacement cycles are repeated indefinitely.

We investigate optimal stopping time problems for such the system by the sample's analyzing method that has been used to derive optimal stopping times of the systems subject to shocks by many researchers. For example, Taylor [61] studies the shock model where the cumulative damage process is a compound Poisson process. Feldman [26,28] generalizes Taylor's model by allowing the times between shocks to be arbitrarily distribution and dependent on the accumulated damage level. Abdel-

Hameed [3] considers the optimal replacement problem when the damage process is a non-homogeneous Lévy process. In [4] he studies a system that is subject to shocks by using an increasing pure-jump strong Markov process. Zuckerman [72] and [68] generalizes respectively Feldman [26] and Abdel-Hameed and Shimi [7] by allowing replacement at any time before failure instead of being restricted to at shock times only. Posner and Zuckerman deal with optimal stopping time problems that the damage process is a semi-Markov process. Yamada [67] analyzes the shock model by using a general jump process, etc. The sample's analyzing method for shock models can be described as follows.

Let Π be a stopping time set of a damage process $\{Z(t)\}_{t\geq 0}$, and Θ_T the expected cost incurred up to random time $T \in \Pi$. If there exists a increasing function $g(\cdot)$ such that

$$\Theta_T = E[\int_0^T g(Z(t))dt], \tag{2.1.1}$$

then
$$\inf_{T\in\Pi} \Theta_T = \inf_{T\in\Pi} E[\int_0^T g(Z(t))dt]. \tag{2.1.2}$$

In general, $Z(t)$ is an increasing process, the optimal stopping time can be get by the following.

$$T^* = \min\{\inf\{t;\ \ g(Z(t)) \geq 0\},\ \ \delta\} \tag{2.1.3}$$

where δ is the first failure time.

Depending on damage processes of systems, the various methods have been used in deriving formula (2.1.1). For instance, the infinitesimal operator method for Markov damage processes (see Taylor [61], Zuckerman [68,72]), the martingale method for general jump processes (see Yamada [67]). In this Chapter, we present sum representing forms different from the integral representing form (2.1.1) by defining an integer-valued random variable. In this chapter, we do not restrict our attention to these stopping times for which replacements are made only at shock times or jump times of the environment process, i.e. the general stopping rules will be examined. Furthermore, we consider the long-run average cost per unit time case as well as the total expected discounted case. The distributions of the intershock times and the shock magnitudes in present model depend on the environment state and the accumulated damage level. We prove that the control-limit policies are optimal over the stopping times valued only at shock times or jump times of PEP. For the general stopping time case, we derive the combination policies of the damage level's control-limit and the state-age dependent policies like that given in Zuckerman [72].

Although optimal replacement problem of the system is discussed only under a Poisson random environment, in fact, the conclusions can be easily extended to a case where the environment process is a increasing Markov process with a constant jump rate.

This Chapter is organized as follows, in Section 2.2, the PSMP shock model is formulated, and in Section 2.3 and 2.4, optimal replacement policies for the long-run average cost and the expected discounted-cost are considered respectively. In Section 2.5, two application examples are given.

2.2 The model and Preliminaries

In this Chapter, the environment of the system is described by a Poisson process, i.e., $\xi(t) = N(t)$ where $\{N(t)\}_{t\geq 0}$ is a Poisson process with the intensity η. The state space $N_+=\{0,1,2,\dots\}$ and the initial state is $N(0) = 0$. Let $\{\omega_n\}_{n\geq 0}$ be the jump points of $\{N(t)\}_{t\geq 0}$, $\omega_0 = 0$. Then $P(\omega_{n+1} - \omega_n \leq t) = 1 - e^{-\eta t}$ for $n \geq 0$. We call $\{N(t)\}_{t\geq 0}$ *Poisson environment process*(PEP).

For any $(i, z_0) \in N_+ \times R_+$, let $\{Z_{(i,z_0)}(t)\}_{t\geq 0}$ be a semi-Markov process with the state space R_+ and the initial state $Z_{(i,z_0)}(0) = z_0$. The semi-Markov kernel of the process $\{Z_{(i,z_0)}(t)\}_{t\geq 0}$ is defined as follows. For any $x, z \in R_+$, and $t \geq 0$, let

$$\begin{aligned} &P(Z_{(i,z_0)}(\tau^i_{n+1}) - Z_{(i,z_0)}(\tau^i_n) \leq x, \tau^i_{n+1} - \tau^i_n \leq t | Z_{(i,z_0)}(\tau^i_n) = z) \\ &= \int_0^t G^i_z(x|s) H^i_z(ds) \end{aligned} \tag{2.2.1}$$

where $\{\tau^i_n\}_{n\geq 0}$ ($\tau^i_0 = 0$) are jump points of $\{Z_{(i,z_0)}(t)\}_{t\geq 0}$, $H^i_z(\cdot)$ is the conditional distribution of the intershock time $\tau^i_{n+1} - \tau^i_n$ given $Z_{(i,z_0)}(\tau^i_n) = z$ and $G^i_z(\cdot|s)$ is the conditional distribution of $Z_{(i,z_0)}(\tau^i_{n+1}) - Z_{(i,z_0)}(\tau^i_n)$ given $Z_{(i,z_0)}(\tau^i_n) = z$ and $\tau^i_{n+1} - \tau^i_n = s$. We suppose that $\{Z_{(i,z_0)}(t)\}_{t\geq 0}$ be the right-continuous regular process with left-hand limits.

The stochastic process $\{Z(t)\}_{t\geq 0}$ specifying the accumulative damage of such a system is defined by the following.

$$\begin{aligned} &Z(0) = 0 \\ &Z(t) = Z_{(0,0)}(t) I_{\{0<t<\omega_1\}} + \textstyle\sum_{n=1}^{\infty} Z_{(N(\omega_n),Z(\omega_n-))}(t - \omega_n) I_{\{\omega_n \leq t < \omega_{n+1}\}}. \end{aligned} \tag{2.2.2}$$

According to the definitions of $\{Z_{(i,z_0)}(t)\}_{t\geq 0}$, $(i, Z_0) \in N_+ \times R_+$, we know that the process $\{Z(t)\}_{t\geq 0}$ is also accumulatively additive and right-continuous process with left-hand limits . At the points $\omega_n, n \geq 1$, $Z(\omega_n) = Z(\omega_n-)$, and on the interval $[\omega_n, \omega_{n+1})$, the process $\{Z(t)\}_{t\geq 0}$ is a semi-Markov process which depends on the environment state $N(\omega_n)$. We call $\{Z(t)\}_{t\geq 0}$ *piecewise semi-Markov process*(PSMP).

The successive jump points of the two-dimensional process $\{N(t), Z(t)\}_{t\geq 0}$ is defined as follows.

$$T_0 = 0$$

$$T_{n+1} = \inf\{t > T_n; N(t) \neq N(T_n) \text{ or } Z(t) \neq Z(T_n)\} \quad \text{for } n \geq 0.$$

Let

$$\begin{cases} Z_n = Z(T_n) \\ X_{n+1} = Z_{n+1} - Z_n \qquad for\ \ n \geq 0 \\ N_n = N(T_n). \end{cases} \tag{2.2.3}$$

For the embedded process $\{N_n, Z_n, T_n\}_{n\geq 0}$, we have the following Proposition.

Proposition 2.2.1 The process $\{N_n, Z_n, T_n\}_{n\geq 0}$ is a Markov renewal process and

(a) $\Phi_z^i(\eta) \equiv P(N_{n+1} = N_n | Z_n = z, N_n = i) = \eta \int_0^\infty H_z^i(s) e^{-\eta s} ds.$

(b) $P(X_{n+1} \leq x, T_{n+1} - T_n \leq t | Z_n = z, N_n = i)$

$$= \int_0^t \int_0^s G_z^i(x|u) H_z^i(du) \eta e^{-\eta s} ds + \int_0^t G_z^i(x|u) H_z^i(du) e^{-\eta t}$$
$$+ \int_0^t (1 - e^{-\eta s}) H_z^i(ds) + (1 - e^{-\eta t}) \bar{H}_z^i(t)$$

where $\bar{H}_z^i(t) = 1 - H_z^i(t)$.

Proof. Let S_1, S_2 represent respectively the first interval length from T_n to the next shock arrival and the first interval length from T_n to the next jump of PEP. Then $P(S_1 \leq t | Z_n = z, N_n = i) = H_z^i(t)$, $P(S_2 \leq t | Z_n = z, N_n = i) = 1 - e^{-\eta t}$ and $T_{n+1} - T_n = \min\{S_1, S_2\}$. We have

(a) $\Phi_z^i(\eta) = P(S_1 \leq S_2 | Z_n = z, N_n = i)$

$$= \int_0^\infty P(S_1 \leq S_2 | S_2 = s, Z_n = z, N_n = i) dP(S_2 \leq s | Z_n = z, N_n = i).$$

(b) $P(X_{n+1} \leq x, T_{n+1} - T_n \leq t | Z_n = z, N_n = i)$

$$= P(X_{n+1} \leq x, T_{n+1} - T_n \leq t; S_1 \leq S_2 | Z_n = z, N_n = i)$$
$$+ P(X_{n+1} \leq x, T_{n+1} - T_n \leq t; S_2 \leq S_1 | Z_n = z, N_n = i)$$
$$= \int_0^\infty P(X_{n+1} \leq x, T_{n+1} - T_n \leq t; S_1 \leq S_2 | S_2 = s, Z_n = z, N_n = i)$$
$$\times dP(S_2 \leq s | Z_n = z, N_n = i)$$
$$+ \int_0^\infty P(X_{n+1} \leq x, T_{n+1} - T_n \leq t; S_2 \leq S_1 | S_1 = s, Z_n = z, N_n = i)$$
$$\times dP(S_1 \leq s | Z_n = z, N_n = i).$$

The conclusions (a) and (b) follow from introducing the conditional distributions of S_1, S_2 into the above equalities. □

We have defined the damage process $\{Z(t)\}_{t\geq 0}$ of the system. Just after the jump point of the $\{N(t)\}_{t\geq 0}$, this process evolves as a semi-Markov process that the distributions of the intershock times and the shock magnitudes are dependent on the state in which the $\{N(t)\}_{t\geq 0}$ entered. The state space of the $\{Z(t)\}_{t\geq 0}$ is R_+. Let δ be the first failure time of the system, and throughout we assume that $E[\delta] < \infty$. A failure can occur only at the shock times or jump times of PEP. At time $T_n < \delta$, let $Z_n = z, N_n = i$. The system fails at T_{n+1} with probability $1 - \gamma(i, z + x)$ if T_{n+1} is a shock point and the shock magnitude is x , and with probability $1 - \gamma(i + 1, z)$ if T_{n+1} is a jump point of PEP , where the function $\gamma(\cdot,\cdot) : N_+ \times R_+ \rightarrow [0,1]$ is the survival function. In this case, if $T_{n_0} = \delta$ for some $n_0 \in N_+$, then we define $T_n = \delta$ for all $n \geq n_0$ on one replacement cycle. In order to use the general renewal arguments for replacement models, we consider the optimal replacement problem for this system under the following assumption:

(A) The environment process $\{N(t)\}_{t\geq 0}$ restores to the initial state $N(0)$ without any loss of time when the system is replaced, and the damage process is repeated by (2.2.2).

Let
$$R_1^i(z,t) = \int_{R_+} \gamma(i, z + x) G_z^i(dx|t), \tag{2.2.4}$$
$$R_1^i(z) = \int_{R_+} R_1^i(z,t) e^{-\eta t} H_z^i(dt). \tag{2.2.5}$$

Then $R_1^i(z,t)$ is the probability that the system will survive at T_{n+1} in state (i, z), conditional on $T_{n+1} - T_n = t$ and T_{n+1} is a shock point. $R_1^i(z)$ is the probability that the system will survive at T_{n+1} in state (i, z), conditional on that T_{n+1} is a shock point. Similarly, let

$$R_2^i(z) = \gamma(i+1, z). \tag{2.2.6}$$

Then $R_2^i(z)$ is the probability that the system will survive at T_{n+1} in state (i, z), conditional on that T_{n+1} is a jump point of the PEP .

Let Π be the class of all stopping times T with respect to the process $\{N(t), Z(t)\}_{t\geq 0}$ such that $T \leq \delta$, and let $\Xi \subset \Pi$ be the subclass of these stopping times for which a replacement can be taken only at the shock times or PEP's jump times. We will consider optimal stopping problems on classes Ξ and Π respectively.

2.3 The long-run average cost case

According to assumption (A) and using standard results in renewal theory, we know, the long-run average cost is the expected cost over a replacement cycle divided by the expected duration between replacements. That is, the average cost Ψ_T associated with a replacement policy $T \leq \delta$ can be expressed as follows

$$\Psi_T = \frac{E[C + C_0 I_{\{T=\delta\}}]}{E[T]}. \tag{2.3.1}$$

Let
$$\Psi^*(\Xi) = \inf_{T\in\Xi} \Psi_T; \qquad \Psi^* = \inf_{T\in\Pi} \Psi_T \tag{2.3.2}$$

$$\rho_T(\Xi) = C + C_0 P(T = \delta) - \Psi^*(\Xi) E[T] \tag{2.3.3}$$

$$\rho_T = C + C_0 P(T = \delta) - \Psi^* E[T]. \tag{2.3.4}$$

Lemma 2.3.1. For any $i \in N_+, z \in R_+,\ \ T_n < \delta$,

(i) $P(T_{n+1} = \delta | T_n < \delta, Z_n = z, N_n = i) = 1 - R_1^i(z) - R_2^i(z)(1 - \Phi_z^i(\eta))$.

(ii) $E[T_{n+1} - T_n | T_n < \delta, Z_n = z, N_n = i) = \lambda(1 - \Phi_z^i(\eta))$

where $\lambda = \eta^{-1}$.

Proof. (i) By using S_1 and S_2 defined in Proposition 2.1, we have

$$\begin{aligned}
&P(T_{n+1} = \delta | T_n < \delta, Z_n = z, N_n = i) \\
&= P(T_{n+1} = \delta; S_1 \leq S_2 | T_n < \delta, Z_n = z, N_n = i) \\
&\qquad + P(T_{n+1} = \delta; S_2 \leq S_1 | T_n < \delta, Z_n = z, N_n = i) \\
&= \textstyle\int_0^\infty \int_0^s \int_0^\infty (1 - \gamma(i, z + x)) G_z^i(dx|u) H_z^i(du) \eta e^{-\eta s} ds \\
&\qquad + \textstyle\int_0^\infty \int_0^s (1 - \gamma(i+1, z)) \eta e^{-\eta u} H_z^i(ds) \\
&= \Phi_z^i(\eta) - R_1^i(z) + (1 - R_2^i(z))(1 - \Phi_z^i(\eta)).
\end{aligned}$$

(ii) From Proposition 2.1 (b), we get

$$\begin{aligned}
&P(T_{n+1} - T_n \leq t | Z_n = z, N_n = i) \\
&= \textstyle\int_0^t H_z^i(s) \eta e^{-\eta s} ds + H_z^i(t) e^{-\eta t} + \int_0^t (1 - e^{-\eta s}) H_z^i(ds) + (1 - e^{-\eta t}) \bar{H}_z^i(t).
\end{aligned}$$

$$E[T_{n+1} - T_n | T_n < \delta, Z_n = z, N_n = i) = \textstyle\int_0^\infty t e^{\eta t} H_z^i(dt) + \int_0^\infty t \eta e^{-\eta t} \bar{H}_z^i(t) dt$$

$= \frac{1}{\eta} - \int_0^\infty e^{-\eta t} H_z^i(t)dt.$ □

Lemma 2.3.2. An optimal replacement policy $T^*(\Xi)$ (T^*, respectivelly) minimizes Ψ_T in Ξ (in Π) if and only if it minimizes $\rho_T(\Xi)$ (ρ_T).

Proof. By the definition (2.3.3), it follows that $\inf_{T\in\Xi} \rho_T(\Xi) = 0$. If $\Psi_{T^*(\Xi)} = \Psi^*(\Xi)$, then $\rho_{T^*(\Xi)} = 0$ and $\Psi_{T^*(\Xi)} \le (C + C_0 P(T = \delta))(E[T])^{-1}$, we have $\rho_T(\Xi) = C + C_0 - \Psi_{T^*(\Xi)}E[T] \ge 0 = \rho_{T^*(\Xi)}$ for all $T \in \Xi$. On the other hand, if $\rho_{T^*(\Xi)}(\Xi) = \inf_{T\in\Xi} \rho_T(\Xi)$, $\rho_{T^*(\Xi)}(\Xi) \le \rho_T(\Xi)$ for all $T \in \Xi$, and $\Psi^* = (C+C_0P(T^*(\Xi)))(E[T^*(\Xi)])^{-1})$. We have $\Psi_{T^*(\Xi)} \le \Psi_T(\Xi)$ for all $T \in \Xi$. The proof is completed. □

The minimization of $\rho_T(\Xi)$ can be viewed as a stopping problem with respect to the two-dimensional process $\{N(t), Z(t)\}_{t\ge 0}$. For every $T \in \Xi$ and $\omega \in \Omega$, $T(\omega) = T_n(\omega)$ for some $n \in N_+$, corresponding to this n, define an integer-valued random variable L as follows.

$$L(\omega) = \begin{cases} n & \text{if } T_n(\omega) < \delta(\omega) \\ \aleph & \text{otherwise} \end{cases} \qquad (2.3.5)$$

where $\aleph = \inf\{n;\quad T_n(\omega) = \delta(\omega)\}$. Therefore, we have $T(\omega) = T_{L(\omega)}(\omega)$ for every $\omega \in \Omega$, i.e., $T = T_L$ a.s., and $\rho_T(\Xi) = \rho_{T_L}(\Xi)$.

Theorem 2.3.3. For $T \in \Xi$,

$$\begin{aligned}
\rho_T(\Xi) &= \rho_{T_L}(\Xi) \\
&= E[\ C + \textstyle\sum_{n=0}^{L-1}\{C_0[1 - R_1^{N_n}(Z_n) - R_2^{N_n}(Z_n)(1 - \Phi_{Z_n}^{N_n}(\eta))] \\
&\qquad\qquad -\Psi^*(\Xi)\lambda(1 - \Phi_{Z_n}^{N_n}(\eta))\}P(T_n < \delta)\ ] \\
&= E[\ C + \textstyle\sum_{n=0}^{L-1}[B^{N_n}(Z_n) - A^{N_n}(Z_n)]P(T_n < \delta)\ ]
\end{aligned}$$

where for $i \in N_+, z \in R_+$

$$A^i(z) = \Psi^*(\Xi)\lambda(1 - \Phi_z^i(\eta)) \qquad (2.3.6)$$

$$B^i(z) = C_0[1 - R_1^i(z) - R_2^i(z)(1 - \Phi_z^i(\eta))]. \qquad (2.3.7)$$

Proof. By the definition of L, we have

$$\begin{aligned}
\rho_{T_L}(\Xi) &= \textstyle\sum_{n\in N_+} E[C + C_0 I_{\{T_L=\delta\}} - \Psi^*(\Xi)T_L|L = n]P(L = n) \qquad (2.3.8)\\
&= \textstyle\sum_{n\in N_+} E[C + C_0 I_{\{T_n=\delta\}} - \Psi^*(\Xi)T_n]P(L = n).
\end{aligned}$$

Since

$$E[I_{\{T_n=\delta\}}] = E[I_{\{T_n=\delta\}}I_{\{T_{n-1}=\delta\}}] + E[I_{\{T_n=\delta\}}I_{\{T_{n-1}\neq\delta\}}]$$

$$= E[I_{\{T_{n-1}=\delta\}}] + E[\ E[I_{\{T_n=\delta\}}|T_{n-1} < \delta, Z_{n-1}, N_{n-1}]\ ]P(T_{n-1} < \delta)$$

$$= E[I_{\{T_{n-1}=\delta\}}] + E[1 - R_1^{N_{n-1}}(Z_{n-1}) - R_2^{N_{n-1}}(Z_{n-1})(1 - \Phi_{Z_{n-1}}^{N_{n-1}}(\eta))\ ]$$
$$\times P(T_{n-1} < \delta),$$

$$E[T_n] = E[T_n I_{\{T_{n-1}=\delta\}}] + E[T_n I_{\{T_{n-1}\neq\delta\}}]$$

$$= E[T_{n-1}I_{\{T_{n-1}=\delta\}}] + E[T_{n-1}I_{\{T_{n-1}\neq\delta\}}] + E[\ E[T_n - T_{n-1}|T_{n-1} < \delta, Z_{n-1}, N_{n-1}]\ ]$$
$$\times P(T_{n-1} < \delta)$$

$$= E[T_{n-1}] + E[\lambda(1 - \Phi_{Z_{n-1}}^{N_{n-1}}(\eta))]\ \ P(T_{n-1} < \delta)].$$

$$E[C + C_0 I_{\{T_n=\delta\}} - \Psi^*(\Xi)T_n]$$

$$= E[C + C_0 I_{\{T_{n-1}=\delta\}} - \Psi^*(\Xi)T_{n-1}]$$
$$+E[\ C_0[1 - R_1^{N_{n-1}}(Z_{n-1}) - R_2^{N_{n-1}}(Z_{n-1})(1 - \Phi_{Z_{n-1}}^{N_{n-1}}(\eta))]$$
$$-\Psi^*(\Xi)\lambda(1 - \Phi_{Z_{n-1}}^{N_{n-1}}(\eta))\ ]P(T_{n-1} < \delta)],$$

and $\rho_{T_0} = C$, we get by repeating the above procedure,

$$\rho_{T_n}(\Xi) = E[\ C + \textstyle\sum_{l=0}^{n-1}\{C_0[1 - R_1^{N_l}(Z_l) - R_2^{N_l}(Z_l)(1 - \Phi_{Z_l}^{N_l}(\eta))]$$
$$-\Psi^*(\Xi)\lambda(1 - \Phi_{Z_l}^{N_l}(\eta))\}P(T_l < \delta)\ ].$$

Introducing this equality into (2.3.8), we complete the proof. □

Hence, determining a stopping time in Ξ is equivalent to determining L for every $\omega \in \Omega$. Next, we first consider optimal stopping problem on Ξ.

Assumption 2.3.4.

(a) $R_1^i(z)$ and $R_2^i(z)$ are decreasing in z and in i.

(b) $H_z^i(t)$ is increasing in z and in i.

Roughly speaking, assumptions (a) implies that survival function $\gamma(i, z)$ is decreasing in z and in i, and (b) means that the intershock times will become shorter and shorter and the damage magnitudes caused by shocks will become larger and larger with increasing of the damage level and the state of PEP. In this sense, we refer that the environment state become worse and worse.

By assumption 2.3.4, we have that $\Phi_z^i(\eta)$ is increasing in z and in i, $A^i(z)$ is decreasing in z and in i, and $B^i(z)$ is increasing in z and in i. Therefore, for any $z \in R_+$

$$B^i(z) - A^i(z) \le B^{i+1}(z) - A^{i+1}(z) \qquad i \in N_+. \tag{2.3.9}$$

For $i \in N_+$, let

$$f(i) = \begin{cases} \inf\{z; \quad A^i(z) \le B^i(z)\} & \text{if } \{\ldots\} \neq \emptyset \\ \infty & \text{otherwise.} \end{cases} \tag{2.3.10}$$

By the monotone properties of $A^i(z)$ and $B^i(z)$ in z and in i, we can see that if $\{\ldots\}_{i_0} = \emptyset$ for some $i_0 \in N_+$ where $\{\ldots\}_{i_0}$ is the definition set of $f(i)$, $\{\ldots\}_i = \emptyset$ for all $i \le i_0$. Thus by the definition of $f(i)$ and (2.3.9), we get the following result.

Lemma 2.3.5. $f(i)$ is decreasing in i.

Theorem 2.3.6. Under Assumption 3.3, the policy with the control-limit process $f(N(t))$ is an optimal stopping policy in Ξ.

Proof. Let $\quad \Im = \cup_{i=0}^{\infty}\{z; \ A^i(z) \le B^i(z)\}$, define a stopping time such that

$$T^*(\Xi) = \begin{cases} \min\{\inf\{t \ge 0; \quad Z(t) \ge f(N(t))\}, \quad \delta\} & \text{if } \Im \neq \emptyset \\ \delta & \text{otherwise .} \end{cases} \tag{2.3.11}$$

We first consider the case that $\Im \neq \emptyset$. Since $N(t)$ and $Z(t)$ are increasing jump processes in t, $f(N(t))$ is decreasing jump process in t, we have $T^*(\Xi) \in \Xi$. By Theorem 2.3.3,

$$\rho_{T^*(\Xi)}(\Xi) = E[\ C + \textstyle\sum_{n=0}^{L^*-1}[B^{N_n}(Z_n) - A^{N_n}(Z_n)]P(T_n < \delta)\ ].$$

From the definition of $T^*(\Xi)$, for $\omega \in \Omega$, we have

$A^{N_{L^*(\omega)}(\omega)}(f(N_{L^*(\omega)}(\omega)) \le B^{N_{L^*(\omega)}(\omega)}(f(N_{L^*(\omega)}(\omega))$ and $Z_{L^*(\omega)}(\omega) \ge f^1(N_{L^*(\omega)}(\omega))$.

Hence $\qquad B^{N_{L^*(\omega)}(\omega)}(Z_{L^*(\omega)}(\omega)) - A^{N_{L^*(\omega)}(\omega)}(Z_{L^*(\omega)}(\omega)) \ge 0$

and $\qquad B^{N_{L^*(\omega)-1}(\omega)}(Z_{L^*(\omega)-1}(\omega)) - A^{N_{L^*(\omega)-1}(\omega)}(Z_{L^*(\omega)-1}(\omega)) < 0.$

Furthermore,

$$B^{N_n(\omega)}(Z_n(\omega)) - A^{N_n(\omega)}(Z_n(\omega)) \ge B^{N_{L^*(\omega)}(\omega)}(Z_{L^*(\omega)}(\omega)) - A^{N_{L^*(\omega)}(\omega)}(Z_{L^*(\omega)}(\omega))$$

if $n \ge L^*(\omega)$, and $\qquad B^{N_n(\omega)}(Z_n(\omega)) - A^{N_n(\omega)}(Z_n(\omega))$

$$\leq B^{N_{L^*(\omega)-1}(\omega)}(Z_{L^*(\omega)-1}(\omega)) - A^{N_{L^*(\omega)-1}(\omega)}(Z_{L^*(\omega)-1}(\omega))$$

if $n < L^*(\omega)$. Thus, for any $T \in \Xi$, we have

$$\begin{aligned}
&\rho_T(\Xi) - \rho_{T^*(\Xi)}(\Xi) \\
&= E[\ [\textstyle\sum_{n=L^*}^{L-1}[B^{N_n}(Z_n) - A^{N_n}(Z_n)]P(T_n < \delta)\ ]I_{\{L\geq L^*\}}] \\
&\qquad -E[\ [\textstyle\sum_{n=L}^{L^*-1}[B^{N_n}(Z_n) - A^{N_n}(Z_n)]P(T_n < \delta)\ ]I_{\{L<L^*\}}] \\
&\geq 0.
\end{aligned}$$

When $\Im = \emptyset$, we get that $A^i(z) > B^i(z)$ for all $i \in N_+, z \in R_+$. This implies that policy δ is optimal stopping time. Therefore $T^*(\Xi)$ is optimal in Ξ and this completes the proof. $\square$

Remark. By the monotone properties of $A^i(z)$ and $B^i(z)$ in z and in i, $\Im = \emptyset$ if and only if $\lim_{i\to\infty} A^i(0) > \lim_{i\to\infty} B^i(0)$.

In the case that the damage process of the system is PSMP, the control-limit is no longer a constant, but is the function of the environment process PEP.

Corollary 2.3.7. For every environment state i, let

$$\alpha(i) = \inf\{z; \quad \Psi^*(\Xi)\lambda \leq C_0(1 - R_2^i(z))\}. \tag{2.3.12}$$

Then $\qquad f(i) \leq \alpha(i) \qquad$ for all $i \in N_+$.

Proof. Since $f(i)$ can be rewritten as follows

$$f(i) = \inf\{z; \quad C_0(\Phi_z^i(\eta) - R_1^i(z)) + [C_0(1 - R_2^i(z)) - \Psi^*(\Xi)\lambda](1 - \Phi_z^i(\eta)) \geq 0 \ \},$$

and $\Phi_z^i(\eta) - R_1^i(z) \geq 0$. Therefore $f(i) \leq \alpha(i)$. $\square$

From this Corollary, we know that $f(N(t)) \leq \alpha(N(t))$ a.s.. Hence we call $\alpha(N(t))$ the bounded process of the control-limit process $f(N(t))$.

Next, we consider the wider class Π of all stopping times.

For any $T \in \Pi$, there exists n such that $T_n(\omega) \leq T(\omega) \leq T_{n+1}(\omega)$ for $\omega \in \Omega$. Define $L(\omega)$ as in (2.3.5), and $S_n(\omega) = T(\omega) - T_n(\omega)$. We have $T_L \leq T \leq T_{L+1}$ and $T = T_L + S_L$.

Theorem 2.3.8. For any $T \in \Pi$,

$$\rho_T = \rho_{T_L} + E[C_0 P(T_L + S_L = \delta | T_L < \delta, Z_L, N_L) - \Psi^* E[S_L | T_L < \delta, Z_L, N_L]] P(T_L < \delta)$$

where ρ_{T_L} is obtained from Theorem 2.3.3 by replacing $\Psi^*(\Xi)$ by Ψ^* in (2.3.6).

Proof. For $T \in \Pi$, we write $T = T_L + S_L$, then

$$E[I_{\{T=\delta\}}] = E[I_{\{T=\delta\}} I_{\{T_L=\delta\}}] + E[I_{\{T=\delta\}} I_{\{T_L \neq \delta\}}]$$
$$= E[I_{\{T_L=\delta\}}] + E[\;E[I_{\{T_L+S_L=\delta\}} | T_L < \delta, Z_L, N_L]\;] P(T_L < \delta)$$
$$E[T] = E[T I_{\{T_L=\delta\}}] + E[T I_{\{T_L \neq \delta\}}]$$
$$= E[T_L I_{\{T_L=\delta\}}] + E[T_L I_{\{T_L \neq \delta\}}] + E[\;E[S_L | T_L < \delta, Z_L, N_L]\;] P(T_L < \delta).$$

By introducing these equalities into the definition (2.3.4) of ρ_T, we obtain the conclusion. □

In the following, assume that $H_z^i(s)$ has the continuous density function $h_z^i(s)$. Let $r_z^i(s) = \frac{h_z^i(s)}{\bar{H}_z^i(s)}$ be the hazard rate associated with the distribution function $H_z^i(s)$, and $Y(t)$ the time since the last shock or the jump of PEP, i.e.,

$$Y(t) = t - \max_{n \geq 0}\{T_n;\; T_n \leq t\} \qquad \text{for } t \geq 0. \tag{2.3.13}$$

Now, we introduce a lexicographic rule in the state space $R_+ \times R_+$. $v = (z, s)$ is called lexicographically positive, written $v \succ 0$, if $v \neq 0$ and the first non-vanishing coordinate of v is positive. We can order all states contained in the state space of the bivariate process $\{(Z(t), Y(t))\}_{t \geq 0}$ by using the lexicographic rule,

$$(z_1, s_1) \succ (z_2, s_2) \quad \text{if} \quad (z_1 - z_2, s_1 - s_2) \succ 0.$$

Assumption 2.3.9.

(a) $R_1^i(z, s)$ is decreasing with respect to the lexicographic order for every i and decreasing in i. $R_2^i(z)$ is decreasing in z and in i.

(b) $r_z^i(s)$ is increasing with respect to the lexicographic order for every i and increasing in i.

Although Assumption 2.3.9 is restrictive, some interesting cases satisfy. For example: the case in which the intershock time at state (z, i) is characterized by the exponential distribution with parameter $\lambda(z, i)$ that is a increasing function in z and in i respectively, etc. See Zuckerman [72] and Posner and Zuckerman [51] for details.

Theorem 2.3.10. Under Assumption 2.3.9, the policy with the two-dimensional control-limit process $g(N(t))$ defined in the following is an optimal stopping policy. For $i \in N_+$,

$$g(i) = \begin{cases} \inf\{(z,s); \quad K(i,z,s) \geq 0\} & \text{if } \{\ldots\} \neq \emptyset \\ (\infty, \infty) & \text{otherwise} \end{cases} \tag{2.3.14}$$

where inf is taken according to the lexicographic order and

$$K(i,z,s) = C_0(1 - R_1^i(z,s))r_z^i(s) + C_0(1 - R_2^i(z))\eta - \Psi^*. \tag{2.3.15}$$

Proof. Define a stopping time T^* as follows.

$$T^* = \min\{\inf\{t \geq 0; \quad (Z(t), Y(t)) \succ g(N(t))\}, \quad \delta\}, \tag{2.3.16}$$

From Theorem 2.3.8, we have

$$\rho_{T^*} = \rho_{T_{L^*}} + E[C_0 P(T_{L^*} + S_{L^*} = \delta | T_{L^*} < \delta, Z_{L^*}, N_{L^*}) - \Psi^* E[S_{L^*} | T_{L^*} < \delta, Z_{L^*}, N_{L^*}]] \times P(T_{L^*} < \delta).$$

It is sufficient to show that T^* is optimal. For any $s \geq 0$, let $\tau(s) = \min\{s, T_{n+1} - T_n\}$, then from Proposition 2.2.1, we have

$$\begin{aligned} & E[\tau(s) | T_n < \delta, Z_n = z, N_n = i] \\ &= E[\tau(s); T_{n+1} - T_n \leq s | T_n < \delta, Z_n = z, N_n = i] \\ &\qquad + E[\tau(s); T_{n+1} - T_n \geq s | T_n < \delta, Z_n = z, N_n = i] \\ &= \int_0^s t e^{-\eta t} H_z^i(dt) + \int_0^s \eta e^{-\eta t} \bar{H}_z^i(t) dt + s \bar{H}_z^i(s) e^{-\eta s} \\ &= \mu_z^i - \mu_z^i(s) \bar{H}_z^i(s) e^{-\eta s} \end{aligned}$$

where $\mu_z^i(s) = [\bar{H}_z^i(s)e^{-\eta s}]^{-1}[\int_s^\infty (t-s)e^{-\eta t} H_z^i(dt) + \int_s^\infty (t-s)\eta e^{-\eta t} \bar{H}_z^i(t) dt]$, and $\mu_z^i = \lambda(1 - \phi_z^i(\eta))$. Moreover,

$$\begin{aligned} & P(T_n + \tau(s) = \delta | T_n < \delta, Z_n = z, N_n = i) \\ &= P(T_n + \tau(s) = \delta; S_1 \leq S_2; S_1 \leq s | T_n < \delta, Z_n = z, N_n = i) \\ &\qquad + P(T_n + \tau(s) = \delta; S_2 \leq S_1; S_2 \leq s | T_n < \delta, Z_n = z, N_n = i) \\ &= \int_0^s \int_0^t (1 - R_1^i(z,u)) H_z^i(du) \eta e^{-\eta t} dt + (1 - R_2^i(z))[\int_0^s (1 - e^{-\eta t}) H_z^i(dt) + (1 - e^{-\eta s}) \bar{H}_z^i(s)]. \end{aligned}$$

The expected marginal cost from waiting an $\tau(s)$ units of time is

$$K_1(i,z,s) = C_0 P(T_n + \tau(s) = \delta | T_n < \delta, Z_n = z, N_n = i)$$
$$-\Psi^* E[\tau(s) | T_n < \delta, Z_n = z, N_n = i]$$
$$= C_0\{\int_0^s \int_0^t (1 - R_1^i(z,u)) H_z^i(du) \eta e^{-\eta t} dt$$
$$+(1 - R_2^i(z))[\int_0^s (1 - e^{-\eta t}) H_z^i(dt) + (1 - e^{-\eta s})\bar{H}_z^i(s)]\}$$
$$-\Psi^*[\mu_z^i - \mu_z^i(s)\bar{H}_z^i(s) e^{-\eta s}].$$

Differentiating $K_1(i,z,s)$ with respect to s, we get

$$\frac{\partial}{\partial s} K_1(i,z,s) = C_0[(1 - R_1^i(z,s)) h_z^i(s) e^{-\eta s} + (1 - R_2^i(z)) \eta e^{-\eta s} \bar{H}_z^i(s)] - \Psi^* \bar{H}_z^i(s) e^{-\eta s}$$
$$= [\bar{H}_z^i(s) e^{-\eta s}]^{-1} K(i,z,s).$$

For every i, we have $K(i,z,s)$ is increasing with respect to lexicographic order from Assumption 2.3.9. Therefore, there exists $(z^*(i), s^*(i))$ such that $K(i,z,s) \leq 0$ if $(z^*(i), s^*(i)) \succ (z,s)$ and $K(i,z,s) \geq 0$ otherwise. Note $K_1(i,z,0) = 0$, thus $K_1(i,z,s) \leq 0$ if $(z^*(i), s^*(i)) \succ (z,s)$ and $K_1(i,z,s) \geq 0$ otherwise. By the lexicographic rule, we have $s^*(i) = \infty$ if $z < z^*(i)$, $0 < s^*(i) < \infty$ if $z = z^*(i)$, and $s^*(i) = 0$ if $z > z^*(i)$. Thus

$$K_1(i,z,\infty) = B^i(z) - A^i(z) \quad \begin{cases} \leq 0 & \text{for } z < z^*(i) \\ > 0 & \text{for } z > z^*(i), \end{cases}$$

and

$$K_1(i, z^*(i), s) \quad \begin{cases} \leq 0 & \text{for } s \leq s^*(i) \\ > 0 & \text{otherwise.} \end{cases}$$

Furthermore, $K(i,z,s)$ is increasing in i, so that if $(z,s) \succ (z^*(i), s^*(i))$, $K(j,z,s) \geq 0$ for all $j \geq i$. We get that $\rho_T I_{\{L<L^*\}} \leq \rho_{T^*} I_{\{L<L^*\}} \leq 0$, $\quad \rho_T I_{\{L>L^*\}} > \rho_{T^*} I_{\{L>L^*\}}$, and $\quad \rho_T I_{\{L=L^*; S_L \leq S_{L^*}^*\}} \leq \rho_{T^*} I_{\{L=L^*; S_L \leq S_{L^*}^*\}} \leq 0$,

$$\rho_T I_{\{L=L^*; S_L > S_{L^*}^*\}} > \rho_{T^*} I_{\{L=L^*; S_L > S_{L^*}^*\}}.$$

For all $T \in \Pi$,

$$\rho_{T^*} - \rho_T$$
$$= (\rho_{T^*} - \rho_T) I_{\{L<L^*\}} + (\rho_{T^*} - \rho_T) I_{\{L>L^*\}}$$
$$+(\rho_{T^*} - \rho_T) I_{\{L=L^*; S_L \leq S_{L^*}^*\}} + (\rho_{T^*} - \rho_T) I_{\{L=L^*; S_L > S_{L^*}^*\}} \leq 0.$$

Therefore T^* is optimal stopping time in Π and this completes the proof. □

2.4 The discounted-cost case

By assumption (A), for any replacement policy $T \leq \delta$, the total expected discounted cost can be expressed as follows

$$\Theta_T = \frac{E[(C + C_0 I_{\{T=\delta\}})exp(-\beta T)]}{1 - E[exp(-\beta T)]} \tag{2.4.1}$$

where β is a discounted factor. Let

$$\Theta^*(\Xi) = \inf_{T\in\Xi} \Theta_T; \qquad \Theta^* = \inf_{T\in\Pi} \Theta_T \tag{2.4.2}$$

$$\varrho_T(\Xi) = E[(C + C_0 I_{\{T=\delta\}} + \Theta^*(\Xi))e^{-\beta T}] \tag{2.4.3}$$

$$\varrho_T = E[(C + C_0 I_{\{T=\delta\}} + \Theta^*)e^{-\beta T}]. \tag{2.4.4}$$

We first consider the optimal policy in Ξ. An optimal replacement policy $T^*_\beta(\Xi)$ in Ξ minimizes Θ_T if and only if it minimizes $\varrho_T(\Xi)$. By the former arguments, for any $T \in \Xi$, $T = T_L$ where L is defined by (2.3.5) and $\varrho_T(\Xi) = \varrho_{T_L}(\Xi)$.

Theorem 2.4.1. For $T \in \Xi$,

$$\begin{aligned}
&\varrho_T(\Xi) = \varrho_{T_L}(\Xi)\\
&= E\Big[\ C+\Theta^*(\Xi)+\textstyle\sum_{n=0}^{L-1}\{[(C+\Theta^*(\Xi))\Phi^{N_n}_{Z_n}(\beta,\eta)+C_0\int_{R_+} e^{-(\beta+\eta)t}(1-R^{N_n}_1(Z_n,t))H^{N_n}_{Z_n}(dt)]\\
&\qquad +\varphi[C + \Theta^*(\Xi) + C_0(1 - R^{N_n}_2(Z_n))](1 - \Phi^{N_n}_{Z_n}(\beta,\eta)) - (C + \Theta^*(\Xi))\}\\
&\qquad \times E[e^{-\beta T_n} I_{\{T_n<\delta\}}]\ \Big]\\
&= E\Big[\ C + \Theta^*(\Xi) + \textstyle\sum_{n=0}^{L-1}[D^{N_n}(Z_n) - (C + \Theta^*(\Xi))]E[e^{-\beta T_n} I_{\{T_n<\delta\}}]\ \Big]
\end{aligned}$$

where $\quad D^i(z) = [(C + \Theta^*(\Xi))\Phi^i_z(\beta,\eta) + C_0 \int_{R_+} e^{-(\beta+\eta)t}(1 - R^i_1(z,t))H^i_z(dt)]$

$$+\varphi[C + \Theta^*(\Xi) + C_0(1 - R^i_2(z))](1 - \Phi^i_z(\beta,\eta)) \tag{2.4.5}$$

and $\quad \Phi^i_z(\beta,\eta) = (\beta + \eta)\int_{R_+} H^i_z(t)e^{-(\beta+\eta)t}dt; \qquad \varphi = \frac{\eta}{\beta+\eta}. \qquad (2.4.6)$

Proof. Similarly to the proof of Theorem 2.3.3, we have

$$\begin{aligned}
&\varrho_{T_n}(\Xi) = E[(C + C_0 I_{\{T_n=\delta\}} + \Theta^*(\Xi))e^{-\beta T_n}]\\
&= E[(C + C_0 I_{\{T_n=\delta\}} + \Theta^*(\Xi))e^{-\beta T_n} I_{\{T_{n-1}=\delta\}}]
\end{aligned}$$

$$
\begin{aligned}
&+E[(C+C_0 I_{\{T_n=\delta\}}+\Theta^*(\Xi))e^{-\beta T_n} I_{\{T_{n-1}<\delta\}}] \\
&= E[(C+C_0 I_{\{T_{n-1}=\delta\}}+\Theta^*(\Xi))e^{-\beta T_{n-1}} I_{\{T_{n-1}=\delta\}}] + E[(C+\Theta^*(\Xi))e^{-\beta T_{n-1}} I_{\{T_{n-1}<\delta\}}] \\
&\quad +E[\ E[(C+C_0 I_{\{T_n=\delta\}}+\Theta^*(\Xi))e^{-\beta T_{n-1}} e^{-\beta(T_n-T_{n-1})}|T_{n-1}<\delta, Z_{n-1}, N_{n-1}]\ ] \\
&\quad \times P(T_{n-1}<\delta) - E[(C+\Theta^*(\Xi))e^{-\beta T_{n-1}} I_{\{T_{n-1}<\delta\}}] \\
&= \varrho_{T_{n-1}}(\Xi) + E[\ E[(C+C_0 I_{\{T_n=\delta\}}+\Theta^*(\Xi))e^{-\beta(T_n-T_{n-1})}|T_{n-1}<\delta, Z_{n-1}, N_{n-1}] \\
&\qquad\qquad -(C+\Theta^*(\Xi))\ ]E[e^{-\beta T_{n-1}} I_{\{T_{n-1}<\delta\}}],
\end{aligned}
$$

and $$E[(C+C_0 I_{\{T_n=\delta\}}+\Theta^*(\Xi))e^{-\beta(T_n-T_{n-1})}|T_{n-1}<\delta, Z_{n-1}, N_{n-1}]$$

$$
\begin{aligned}
&= (C+\Theta^*(\Xi))\Phi_{Z_{n-1}}^{N_{n-1}}(\beta,\eta) + C_0 \textstyle\int_{R_+} e^{-(\beta+\eta)t}(1-R_1^{N_{n-1}}(Z_{n-1},t))H_{Z_{n-1}}^{N_{n-1}}(dt) \\
&\quad +\varphi[C+\Theta^*(\Xi)+C_0(1-R_2^{N_{n-1}}(Z_{n-1}))](1-\Phi_{Z_{n-1}}^{N_{n-1}}(\beta,\eta)),
\end{aligned}
$$

and $\varrho_{T_0}(\Xi) = C+\Theta^*(\Xi)$. Repeating the above procedure and using (2.3.8), we complete the proof. □

Here, the arguments about the discounted cost are similar to that about the long-run average cost. We will prove that a control-limit policy in Ξ or Π is an optimal stopping policy and the control-limit is a function of the process $N(t)$. We make the following assumption.

Assumption 2.4.2.

(a1) $R_1^i(z,t)$ and $R_2^i(z)$ are decreasing in z and in i, and $R_1^i(z,t)$ is also decreasing in t.

Lemma 2.4.3. Under Assumption 2.4.2 and 2.3.4 (b), we have

(i) $D^i(z)$ defined in (2.4.5) is increasing in z and in i.

(ii) $f_\beta(i)$ is decreasing in i, where for $i \in N_+$

$$
f_\beta(i) = \begin{cases} \inf\{z;\quad C+\Theta^*(\Xi) \le D^i(z)\} & \text{if } \{\ldots\} \ne \emptyset \\ \infty & \text{otherwise.} \end{cases} \tag{2.4.7}
$$

Proof. (i). Note that $D^i(z)$ can be rewritten by

$$
\begin{aligned}
&\varphi[C+\Theta^*(\Xi)+C_0(1-R_2^i(z))] + \tfrac{\beta}{\beta+\eta}(C+\Theta^*(\Xi))\Phi_z^i(\beta,\eta) \\
&\qquad +C_0(\textstyle\int_{R_+} e^{-(\beta+\eta)t}[\frac{\beta}{\beta+\eta}+(\varphi R_2^i(z)-R_1^i(z,t))]H_z^i(dt).
\end{aligned}
$$

By the Assumptions 2.3.3 (b), we have $\Phi_z^i(\beta,\eta)$ is increasing in z and in i, and by

Assumption 2.4.2, for $z_1 \leq z_2$,

$$\int_{R_+} e^{-(\beta+\eta)t}[\tfrac{\beta}{\beta+\eta} + (\varphi R_2^i(z_2) - R_1^i(z_2,t))]H_{z_2}^i(dt)$$
$$\geq \int_{R_+} e^{-(\beta+\eta)t}[\tfrac{\beta}{\beta+\eta} + (\varphi R_2^i(z_1) - R_1^i(z_1,t))]H_{z_2}^i(dt)$$
$$\geq \int_{R_+} e^{-(\beta+\eta)t}[\tfrac{\beta}{\beta+\eta} + (\varphi R_2^i(z_1) - R_1^i(z_1,t))]H_{z_1}^i(dt).$$

Similarly, we have for $i \in N_+$,

$$\int_{R_+} e^{-(\beta+\eta)t}[\tfrac{\beta}{\beta+\eta} + (\varphi R_2^{i+1}(z) - R_1^{i+1}(z,t))]H_z^{i+1}(dt)$$
$$\geq \int_{R_+} e^{-(\beta+\eta)t}[\tfrac{\beta}{\beta+\eta} + (\varphi R_2^i(z) - R_1^i(z,t))]H_z^i(dt).$$

We can see that $C_0 \int_{R_+} e^{-(\beta+\eta)t}[\tfrac{\beta}{\beta+\eta} + (\varphi R_2^i(z) - R_1^i(z,t))]H_z^i(dt)$ is increasing in z and in i. Hence $D^i(z)$ is increasing in z and in i.

(ii). From the conclusion of (i), if $\{\ldots\}_{i_0} = \emptyset$, then $\{\ldots\}_i = \emptyset$ for $i \leq i_0$ where $\{\ldots\}_i$ is the definition set of $f_\beta(i)$. Hence $f_\beta(i)$ is decreasing in i. □

Theorem 2.4.4. Under Assumptions 2.4.2 and 2.3.4 (b), the policy with the control-limit process $f_\beta(N(t))$ is an optimal stopping policy in Ξ.

Proof. Let $\Im = \cup_{i=0}^{\infty}\{z;\ \ C + \Theta^*(\Xi) \leq D^i(z)\}$, define a stopping time $T_\beta^*(\Xi)$ such that

$$T_\beta^*(\Xi) = \begin{cases} \min\{\inf\{t \geq 0; \quad Z(t) \geq f_\beta(N(t))\}, \quad \delta\} & \text{if } \Im \neq \emptyset \\ \delta & \text{otherwise.} \end{cases} \tag{2.3.8}$$

We first consider the case that $\Im \neq \emptyset$. Since $N(t)$ and $Z(t)$ are increasing jump processes in t, $f_\beta(N(t))$ is decreasing jump process in t, we have $T^* \in \Xi$. By Theorem 4.1,

$$\varrho_{T_\beta^*(\Xi)}(\Xi) = \varrho_{T_{L^*}(\Xi)}(\Xi)$$
$$= E[\ \ C + \Theta^*(\Xi) + \textstyle\sum_{n=0}^{L^*-1}[D^{N_n}(Z_n) - (C + \Theta^*(\Xi))]E[e^{-\beta T_n} I_{\{T_n < \delta\}}]\ \ ].$$

From the definition of $T^*(\Xi)$, for $\omega \in \Omega$, we have

$C + \Theta^*(\Xi) \leq D^{N_{L^*(\omega)}(\omega)}(f_\beta(N_{L^*(\omega)}(\omega))$ and $Z_{L^*(\omega)}(\omega) \geq f_\beta(N_{L^*(\omega)}(\omega))$. Hence

$$D^{N_{L^*(\omega)}(\omega)}(Z_{L^*(\omega)}(\omega)) - (C + \Theta^*(\Xi)) \geq 0$$

and $\quad D^{N_{L^*(\omega)-1}(\omega)}(Z_{L^*(\omega)-1}(\omega)) - (C + \Theta^*(\Xi)) < 0.$

Furthermore,

$$D^{N_n(\omega)}(Z_n(\omega)) - (C + \Theta^*(\Xi)) \geq D^{N_{L^*(\omega)}(\omega)}(Z_{L^*(\omega)}(\omega)) - (C + \Theta^*(\Xi) \ \text{ if } n \geq L^*(\omega),$$

and $\quad D^{N_n(\omega)}(Z_n(\omega)) - (C + \Theta^*(\Xi)) \leq D^{N_{L^*(\omega)-1}(\omega)}(Z_{L^*(\omega)-1}(\omega)) - (C + \Theta^*(\Xi))$

if $n < L^*(\omega)$. Thus, for any $T \in \Xi$, we have

$$\begin{aligned}
&\varrho_T - \varrho_{T^*_\beta(\Xi)} \\
&= E[\ \{\textstyle\sum_{n=L^*}^{L-1}[D^{N_n(\omega)}(Z_n(\omega)) - (C + \Theta^*(\Xi)]E[e^{-\beta T_n} I_{\{T_n<\delta\}}]\} I_{\{L \geq L^*\}}\ ] \\
&\qquad -E[\ \{\textstyle\sum_{n=L}^{L^*-1}[D^{N_n(\omega)}(Z_n(\omega)) - (C + \Theta^*(\Xi)]E[e^{-\beta T_n} I_{\{T_n<\delta\}}]\} I_{\{L < L^*\}}\ ] \\
&\geq 0.
\end{aligned}$$

When $\Im = \emptyset$, we get easily $D^i(z) < C + \Theta^*(\Xi)$ for $i \in N_+, z \in R_+$. This implies that policy δ is optimal. Therefore $T^*(\Xi)$ is optimal and this completes the proof. □

Remark. By the monotone property of $D^i(z)$ in z and in i, $\Im = \emptyset$ if and only if $\lim_{i\to\infty} D^i(0) < C + \Theta^*(\Xi)$.

Corollary 2.4.5. For every environment state i, let

$$\alpha_\beta(i) = \inf\{z; \quad C + \Theta^*(\Xi) \leq \varphi[C + \Theta^*(\Xi) + C_0(1 - R_2^i(z))]\ \}. \tag{2.4.9}$$

Then, $\qquad f_\beta(i) \leq \alpha_\beta(i) \qquad$ for all $i \in N_+$.

Proof is similar to that of Corollary 2.3.6. We obtain the bounded process $\alpha_\beta(N(t))$ of the control-limit process $f_\beta(N(t))$.

Next we consider optimality in the wider class Π. Similarly to Theorem 2.3.8, we have

Theorem 2.4.6. For any $T \in \Pi$,

$$\begin{aligned}
\varrho_T = \varrho_{T_L} + E[\ & E[(C + C_0 I_{\{T_L+S_L=\delta\}} + \Theta^*)e^{-\beta S_L} | T_L < \delta, Z_L, N_L] - (C + \Theta^*)\ ] \\
& \times E[e^{-\beta T_L} I_{\{T_L<\delta\}}]
\end{aligned}$$

where ϱ_{T_L} is obtained by replacing $\Theta^*(\Xi)$ by Θ^* in Theorem 2.4.1.

Theorem 2.4.7. Under Assumption 2.3.9, the policy with the two-demensional control-limit process $g_\beta(N(t))$ defined in the following is an optimal stopping policy. For $i \in N_+$,

$$g_\beta(i) = \begin{cases} \inf\{(z,s); \quad K^\beta(i,z,s) \geq 0\} & \text{if } \{\ldots\} \neq \emptyset \\ (\infty,\infty) & \text{otherwise} \end{cases} \tag{2.4.10}$$

where inf is taken according to the lexicographic order and

$$K^{\beta}(z,i,s) = C_0(1-R_1^i(z,s))r_z^i(s)+C_0(1-R_2^i(z))\eta-(C+\Theta^*)(\beta+\eta). \quad (2.4.11)$$

Proof. Define a stopping time T_β^* as follows.

$$T_\beta^* = \min\{\inf\{t \geq 0; \quad (Z(t),Y(t)) \succ g_\beta(N(t))\}, \quad \delta\}, \quad (2.4.12)$$

Similar to the proof of Theorem 2.3.10, let $\tau(s) = \min\{s, T_{n+1} - T_n\}$ for any $s \geq 0$, we have

$$E[e^{-\beta\tau(s)}|T_n < \delta, Z_n = z, N_n = i]$$
$$= \int_0^s \int_0^t e^{-\beta u} H_z^i(du)\eta e^{-\eta t}dt + \int_0^s e^{-\beta u} H_z^i(du)e^{-\eta s}$$
$$+ \int_0^s \int_0^t \eta e^{-(\beta+\eta)u} du H_z^i(dt) + \int_0^s \eta e^{-(\beta+\eta)u} du \bar{H}_z^i(s) + \bar{H}_z^i(s)e^{-(\beta+\eta)s},$$

and $E[I_{\{T_n+\tau(s)=\delta\}}e^{-\beta\tau(s)}|T_n < \delta, N_n = i, Z_n = z\}]$

$$= \int_0^s \int_0^t e^{-\beta u}(1 - R_1^i(z,u))H_z^i(du)\eta e^{-\eta t}dt + \int_0^s e^{-\beta u}(1 - R_1^i(z,u))H_z^i(du)e^{-\eta s}$$
$$+(1 - R_2^i(z))[\int_0^s \int_0^t \eta e^{-(\beta+\eta)u} du H_z^i(dt) + \int_0^s \eta e^{-(\beta+\eta)u} du \bar{H}_z^i(s)].$$

The expected discounted marginal cost from waiting an $\tau(s)$ units of time is

$$K_1^{\beta}(i,z,s) = E[(C + C_0 I_{\{T_n+\tau(s)=\delta\}} + \Theta^*)e^{-\beta\tau(s)}|T_n < \delta, N_n = i, Z_n = z\}]$$
$$-(C + \Theta^*)$$
$$= (C + \Theta^*)[\int_0^s \int_0^t e^{-\beta u} H_z^i(du)\eta e^{-\eta t}dt + \int_0^s e^{-\beta u} H_z^i(du)e^{-\eta s}$$
$$+ \int_0^s \int_0^t \eta e^{-(\beta+\eta)u} du H_z^i(dt) + \int_0^s \eta e^{-(\beta+\eta)u} du \bar{H}_z^i(s) + \bar{H}_z^i(s)e^{-(\beta+\eta)s}]$$
$$+C_0[\int_0^s \int_0^t e^{-\beta u}(1-R_1^i(z,u))H_z^i(du)\eta e^{-\eta t}dt+\int_0^s e^{-\beta u}(1-R_1^i(z,u))H_z^i(du)e^{-\eta s}$$
$$+(1 - R_2^i(z))[\int_0^s \int_0^t \eta e^{-(\beta+\eta)u} du H_z^i(dt) + \int_0^s \eta e^{-(\beta+\eta)u} du \bar{H}_z^i(s)].$$
$$-(C + \Theta^*).$$

Differentiating $K_1^{\beta}(i,z,s)$ with respect to s, we have

$$\frac{\partial}{\partial s}K_1^{\beta}(i,z,s) = C_0[(1 - R_1^i(z,s))h_z^i(s)e^{-(\beta+\eta)s} + (1 - R_2^i(z))\eta e^{-(\beta+\eta)s}\bar{H}_z^i(s)]$$
$$-(C + \Theta^*)(\beta + \eta)e^{-(\beta+\eta)s}\bar{H}_z^i(s)$$
$$= [e^{-(\beta+\eta)s}\bar{H}_z^i(s)]^{-1}K^{\beta}(i,z,s).$$

Under Assumption 2.3.9, $K^{\beta}(i,z,s)$ is increasing with respect to lexicographic order for every i. By the same arguments as in the proof of Theorem 2.3.10, we obtain the conclusion that T_β^* is an optimal stopping time. □

2.5 Application

The environment process may be external factors of an economical or technical nature as well as internal factors of a statistical nature. Next we give two examples which can be well examined by our model.

Application 2.5.1. Suppose a system receive two types of shocks at random points of time and the damage processes be accumulatively additive. Assume one damage process be a compound Poisson process such as

$$M(t) = \sum_{n=1}^{N(t)} Y_n \tag{2.5.1}$$

where $N(t)$ representing the shock number in $[0,t]$ is a Poisson process with the parameter η , $N(0) = 0$. $Y_n, n \geq 1$ are i.i.d random variables with distribution $P(Y_n = k) = p_k, k \in N_+$, and independent on $N(t)$.

The other damage process $Z(t)$ is assumed to be a piecewise semi-Markov process defined by (2.2.2). In this case, the general assumption (A) is satisfied automatically, and (2.2.6) becomes the following

$$R_2^i(z) = \sum_{k=1}^{\infty} \gamma(i+k, z)p_k. \tag{2.5.2}$$

We can examine the optimal replacement problem of the system by taking the damage process $M(t)$ as the environment process.

Example 2.5.2. Consider a computer network system comprised of a main-system and m sub-systems. The network system is new at time $t = 0$. The lifetimes of the sub-systems are independent and exponentially distributed random variables with the same parameters: $F(t) = 1 - e^{-\eta t}$ for $t \geq 0$. When sub-systems fail, the minimal repair is taken. The minimal repair cost of one sub-system is K and the minimal repair time is assumed to be negligible. Let $N(t)$ be the total minimal repair numbers of the sub-systems in $[0,t]$. Then $N(t)$ is a Poisson process with parameter $m\eta$. The main-system is subject to a sequence of randomly occurring shocks. Each shock causes a random amount of damage which accumulates additively and depends on the state of the $N(t)$. Any of shocks or failures of the sub-systems might cause the main-system to fail, and upon failure of the main-system, the network system must be replaced at cost $C + C_0$ where C is replacement cost before failure of the main-system. $Z(t)$ defined by (2.2.2) represents the damage process of the main-system. The optimal replacement for this computer network system can be well examined by our model. In this case, for example, the long-urn average cost is

$$\Psi_T = \frac{E[C + \sum_{n=1}^{\infty} nKI_{\{N(T)=n\}} + C_0 I_{\{T=\delta\}}]}{E[T]}, \tag{2.5.3}$$

and

$$\rho_T = C + \sum_{n=1}^{\infty} nKP(N(T) = n) + C_0 P(T = \delta) - \Psi^*(\Xi)E[T]. \tag{2.5.4}$$

Similarly to Theorem 3.5, we can prove that the policy with the control-limit process $f(N(t))$ is an optimal replacement policy, where for $i \in N_+$

$$\begin{aligned} f(i) = \inf\{z; \quad & \Psi^*\lambda(1 - \Phi_z^i(m\eta)) \leq C_0[1 - R_1^i(z) - R_2^i(z)(1 - \Phi_z^i(m\eta))] \\ & + K(1 - R_2^i(z))(1 - \Phi_z^i(m\eta)) \ \} \end{aligned} \tag{2.3.5}$$

and $\lambda = [m\eta]^{-1}$.

CHAPTER 3

AN OPTIMAL STATE-AGE DEPENDENT REPLACEMENT FOR A NETWORK SYSTEM

3.1 Introduction

The present Chapter deals with an optimal state-age dependent replacement problem for a network system composed of a main-system and a sub-system with N components. The component's functioning times of the sub-system are independent and exponentially distributed random variables with the same parameters. Every failed component is repaired by one repairman by taking an exponentially distributed random time. Repaired components are as good as new. The main-system is subject to a sequence of randomly occurring shocks and each shock causes a random amount of damage. Shock arrivals and magnitudes depend on the accumulated damage level of the main system itself and the number of the functioning components of the sub-system. Any of the shocks or component's failures of the sub-system might cause the main-system to fail. The survival probability is determined by a known function of the accumulated damage level, the number of the functioning components and the realized shock magnitude. Upon failure of the main-system, the whole network system stops to function. We replace the main-system at a cost $C + C_0$ in which $C > 0$ is the replacement cost before its failure. At the same time, we take the emergency repairs for all failed components. The emergency repair cost of every failed component is K. The replacement times and the emergency repair times are assumed to be negligible. Those procedures are repeated indefinitely.

We are motivated to study the optimal replacement problem for such a network system in part by the importance in modern practice of such system. In a computer or communication network system, for example, the central computer can be considered as the main-system. The functioning behaviors of the central computer are not only dependent on states itself, but also influenced by the state changes of sub-computers or other computer-aid equipments in the whole computer network. Therefore, it is necessary to consider affect of those sub-computers when we examine the optimal replacement problem for the central computer. Furthermore, in our model, we construct the damage process of the main-system by a shock process and a Markov process which expresses the number of the functioning components of the sub-system. As can be seen below, our model may be applied to optimal replacement problems of systems existing in a Markov random environment, systems with a Markov modulator or systems subject to two types of dependent shocks.

In this Chapter, we consider the optimal state-age dependent replacement problem by using semi-Markov decision theoretic approach. The general theory of the semi-Markov decision process has been studied by many reseachers. For example, Ross [54, 56], Lippman [40], De Leve and Federgruen and Tijms [22, 23], etc. The optimal state-age dependent replacement problems investigated by using semi-Markov decision process have been considered by Kao [36], Gottlieb [32] and Kurano [39]. Kao [36] gives the details by using semi-Markov decision theory on the discrete time parameter space. Gottlieb [32] derives the form and properties of optimal replacement policy through the Markov decision theoretic method, and gives conditions for which a control-limit policy is optimal. Feldman and Joo [29] obtains a closed form expression for the optimal state-age dependent policy and develops a practical algorithm for finding the optimal policy. In those models, the behaviors of shock processes and other characteristics are only dependent on accumulated damage level of systems, not subject to influences of changes of external or internal environment of systems, or other system's states. In this sense, every system is considered as an independent single-unit system. In present, considering the mutual influences of the main-system and sub-system, We derive an optimal state-age dependent replacement policy dependent on the functioning process of the sub-system for the infinite horizon long-run average cost per unit time. It should be noticed that the optimal replacement policy of the network system in this Chapter can be not obtained by the sample's analyzing method given in Chapter 2 because the functioning process of the sub-system is not increasing Markov process.

Chapter 3 is organized as follows. In Section 3.2 we formulate our model as a semi-Markov decision process and give some preliminary results. In Section 3.3 we prove that an optimal state-age dependent replacement policy exists, and in Section 3.4, we give an optimal replacement policy. In 3.5 we discuss some special cases.

3.2 Model and Preliminaries

Consider a network system composed of a main-system and a sub-system with N components. The network system is new at time $t = 0$. The lifetimes of the components of the sub-system are assumed to be independent identical exponential distribution $F_1(t) = 1 - e^{-\eta t}$ for $t \geq 0$. Every failed component is repaired and the repair time is a random variable having the distribution $F_2(t) = 1 - e^{-\lambda t}$ for $t \geq 0$. There is only one repairman and repaired components are as good as new. The emergency repairs for all failed components are taken only at times when the main-system is replaced and the emergency repair cost incurred for every component is K. Let the process $\{I(t)\}_{t\geq 0}$ express the number of the functioning components of the sub-system at time t. The state space of $\{I(t)\}_{t\geq 0}$ is $\Xi = \{0, 1, \ldots, N\}$, and the initial state is $I(0) = N$. Let $\{\omega_n\}_{n\geq 0}$ be the transition times of $\{I(t)\}_{t\geq 0}$, i.e., ω_n is a time at which a component fails or a failed component is restored to functioning. The Markov transition kernel of the process $\{I(\omega_n), \omega_n\}_{n\geq 0}$ is

$$P(I(\omega_{n+1}) = j, \omega_{n+1} - \omega_n \leq t \ \ |I(\omega_n) = i)$$

$$= \begin{cases} \frac{i\eta}{i\eta+\lambda}(1 - e^{-(i\eta+\lambda)t}) & i = 1, 2, \ldots, N-1; j = i-1 \\ \frac{\lambda}{i\eta+\lambda}(1 - e^{-(i\eta+\lambda)t}) & i = 1, 2, \ldots, N-1; j = i+1 \\ 1 - e^{-N\eta t} & i = N; j = N-1 \\ 1 - e^{-\lambda t} & i = 0; j = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.1}$$

$$\equiv p_{ij}(1 - e^{-\mu_i t})$$

where $\mu_i = i\eta + \delta_{iN}\lambda$, and $\delta_{iN} = 0$ if $i = N$ and 1 otherwise. In this Chapter we take $\{I_t\}_{t\geq 0}$ as the environment process, i.e., $\xi(t) = I(t)$. The $\{I(t)\}_{t\geq 0}$ is called *Markov environment process* (MEP) of the main-system.

The main-system is subject to a sequence of randomly occurring shocks and each shock causes a random amount of damage on it. Shock arrivals and magnitudes depend on the accumulated damage level of the main system, and the number of the functioning components of the sub-system.

Let $\{Z_{(i,z_0)}(t)\}_{t\geq 0}$ be a semi-Markov process with the state space R_+ and the initial state $Z_{(i,z_0)}(0) = z_0$. For any $x, z \in R_+$, and $t \in R_+$, the semi-Markov kernel of $Z_{(i,z_0)}(t)$ is

$$\begin{aligned}&P(Z_{(i,z_0)}(\tau^i_{n+1}) - Z_{(i,z_0)}(\tau^i_n) \leq x, \tau^i_{n+1} - \tau^i_n \leq t | Z_{(i,z_0)}(\tau^i_n) = z) \\ &= \int_0^t G^i_z(x|s) H^i(ds)\end{aligned} \tag{3.2.2}$$

where $H^i(\cdot)$ is the probability distribution of the intershock time $\tau^i_{n+1} - \tau^i_n$ and $G^i_z(\cdot|t)$ is the conditional distribution of $Z_{(i,z_0)}(\tau^i_{n+1}) - Z_{(i,z_0)}(\tau^i_n)$ given $Z_{(i,z_0)}(\tau^i_n) = z$ and $\tau^i_{n+1} - \tau^i_n = t$. We suppose that $Z_{(i,z_0)}(t)$ be a right-continuous regular process with left-hand limits.

Now, by appealing to $\{Z_{(i,z_0)}(t)\}_{t\geq 0}$, we define the process $\{Z(t)\}_{t\geq 0}$ which describes the cumulative damage level of the main-system such that $Z(0) = 0$ and $Z(t) = Z_{(i,Z(\omega_n -))}(t - \omega_n)$ on $\{\omega_n \leq t < \omega_{n+1};\ \ I(\omega_n) = i\}$. That is,

$$\begin{aligned}&Z(0) = 0 \\ &Z(t) = Z_{(I_0,0)}(t) I_{\{0<t<\omega_1\}} + \textstyle\sum_{n=1}^{\infty} Z_{(I_{\omega_n}, Z(\omega_n -))}(t - \omega_n) I_{\{\omega_n \leq t < \omega_{n+1}\}}.\end{aligned} \tag{3.2.3}$$

The process $\{Z(t)\}_{t\geq 0}$ is also right-continuous regular process with left-hand limits . At the points $\omega_n, n \geq 1$, $Z(\omega_n) = Z(\omega_n -)$ and on the interval $[\omega_n, \omega_{n+1})$, the process $\{Z(t)\}_{t\geq 0}$ is a semi-Markov process dependent on the state of I_{ω_n}. The $\{Z(t)\}_{t\geq 0}$ is called *piecewise semi-Markov process* (PSMP).

In this model, a failure of the main-system can occur only at the time points of shocks or failures of the components of the sub-system. Let T be such a time point, and suppose $I_{T-} = i, Z(T-) = z$. At T, if a shock with magnitude x occurs, then the main-system fails with probability $1 - \gamma(i, i, z + x)$, and if a failure of the components occurs (i.e. the number of the functioning components become $i - 1$), then the main-system fails with probability $1 - \gamma(i, i - 1, z)$. The function $\gamma : \Xi \times \Xi \times R_+ \rightarrow [0, 1]$

is the survival function. Let δ be the first failure time and Δ the failure state of the main-system. We assume $E[\delta] < \infty$ throughout.

Let $E = [0, \infty]$ and $\mathbf{A}(\Xi \times (R_+ \cup \Delta), E)$ be the set of all mapping from $\Xi \times (R_+ \cup \Delta)$ to E such that $A(i, \Delta) = 0$ for any $i \in \Xi$. Each function $A(\cdot,\cdot)$ of $\mathbf{A}(\Xi \times (R_+ \cup \Delta), E)$ is called a decision or an action. The decision $A(i, z) = a$ $(a \in [0, \infty))$ means that, when the state at a decision point T is (i, z), the replacement for the main-system and the emergency repairs for all failed components of the sub-system occur at time $T + a$ if no decision points occur in the interval $(T, T + a]$, and $A(i, z) = \infty$ means that no any replacement and repair plan are made, i.e., the next decision point is waited for.

An replacement policy is a sequence $\pi = (A_0, A_1, \ldots); A_i \in \mathbf{A}$. If $A_i = A$ for all $i = 0, 1, \ldots,$ we call π a stationary policy. Let Π be the set of all policies. For any $\pi \in \Pi$, we can obtain the decision processes $\{I^\pi(t), Z^\pi(t)\}_{t \geq 0}$ which describe respectively the number of the functioning components of the sub-system and the accumulated damage level of the main-system at time t under the policy π. The set $\{T_n\}_{n \geq 0}$ of the decision points can be defined as the successive jump points of the two-dimensional process $\{I^\pi(t), Z^\pi(t)\}_{t \geq 0}$ as follows

$$T_0 = 0$$

$$T_{n+1} = \inf\{t > T_n; I^\pi(t) \neq I(T_n) \text{ or } Z^\pi(t) \neq Z^\pi(T_n)\}, \quad \text{for } n \geq 0. \tag{3.2.4}$$

Let

$$\begin{cases} Z_n = Z^\pi(T_n) & \\ I_{n+1} = I^\pi(T_n) & for \;\; n \geq 0. \end{cases} \tag{3.2.5}$$

Since a replacement action changes the damage level of the main-system to 0, we can see that $\{T_n\}_{n \geq 0}$ contains three-type points (a) shock points, (b) jump points of MEP and (c) replacement points. For the semi-Markov decision process $\{I_n, Z_n, T_n, A_n\}_{n \geq 0}$, we have the following Propositions.

Proposition 3.2.1. At T_0, suppose $I_0 = i, Z_0 = z$ and $A_0(i, z) = a$. Then

(a) $\Phi_1(i, a) \equiv P_{(i,z)}(T_1 \text{ is a shock point } | A_0(i, z) = a)$

$= \int_0^a H^i(t)\mu_i e^{-\mu_i t}dt + H^i(a)e^{-\mu_i a}$

(b) $\Phi_2(i,a) \equiv P_{(i,z)}(T_1$ is a jump point of MEP $|A_0(i,z) = a)$

$= \int_0^a (1-e^{-\mu_i t})H^i(dt) + \bar{H}^i(a)(1-e^{-\mu_i a})$

(c) $\Phi_3(i,a) \equiv P_{(i,z)}(T_1$ is a replacement point $|A_0(i,z) = a) = \bar{H}^i(a)e^{-\mu_i a}$

where $\bar{H}^i(t) = 1 - H^i(t)$.

Proof. Let S_1, S_2 represent respectively the first interval length from T_0 to the next shock arrival and the first interval length from T_0 to the next jump of MEP. We have $\Phi_1(i,z) = P_{(i,z)}(S_1 \le S_2, S_1 \le a|A_0(i,z) = a) = \int_0^a H^i(t)\mu_i e^{-\mu_i t}dt + H^i(a)e^{-\mu_i a}$. Similarly, we can obtain (b). For (c), we have

$\Phi_3(i,a) = P_{(i,z)}(\min\{S_1,S_2\} \ge a|A_0(i,z) = a) = \bar{H}^i(a)e^{-\mu_i a}$. □

Proposition 3.2.2. For $i,j \in \Xi, z \in R_+$ and $B \in \Im$

(a) $P_{(i,z)}(I_1 = j, Z_1 \in B|A_0(i,z) = a)$

$= \chi_{\{j=i\}}[\int_0^a \int_0^s \int_{\{z+x\in B\}} \gamma(i,i,z+x)G_z^i(dx|t)H^i(dt)\mu_i e^{-\mu_i s}ds$

$+ \int_0^a \int_{\{z+x\in B\}} \gamma(i,i,z+x)G_z^i(dx|t)H^i(dt)e^{-\mu_i a}] + \chi_{\{z\in B\}}p_{ij}\gamma(i,j,z)\Phi_2(i,a)$

$+\chi_{\{j=N,0\in B\}}\Phi_3(i,a)$

(b) $P_{(i,z)}(I_1 = j, Z_1 = \Delta|A_0(i,z) = a)$

$= \chi_{\{j=i\}}[\int_0^a \int_0^s \int_{R_+} (1-\gamma(i,i,z+x))G_z^i(dx|t)H^i(dt)\mu_i e^{-\mu_i s}ds$

$+ \int_0^a \int_{R_+} (1-\gamma(i,i,z+x))G_z^i(dx|t)H^i(dt)e^{-\mu_i a}] + p_{ij}(1-\gamma(i,j,z))\Phi_2(i,a).$

In this Chapter, we consider the infinite horizon long-run average cost per unit time. Let c and τ represent the cost incurred during one-step transition interval and the length of one-step transition interval respectively. $\bar{c}(i,z,a) = E_{(i,z)}[c|A_0(i,z) = a]$ and $\bar{\tau}(i,z,a) = E_{(i,z)}[\tau|A_0(i,z) = a]$ are the expected cost incurred during the transition interval and the expected length of the transition interval when the starting states are $I_0 = i, Z_0 = z$ and the action $A_0(i,z) = a$ is choiced. Under the policy π, the infinite horizon long-run average cost per unit time J^π is defined as follows.

$$J^{\pi}(i,z) = \lim_{n\to\infty} \sup \frac{E^{\pi}_{(i,z)}[\sum_{k=1}^{n} \bar{c}(I_{n-1}, Z_{n-1}, A_{n-1})]}{E^{\pi}_{(i,z)}[\sum_{k=1}^{n} \bar{\tau}(I_{n-1}, Z_{n-1}, A_{n-1})]}. \tag{3.2.6}$$

Let $J^* \equiv \inf_{\pi\in\Pi} J^{\pi}$. For some $\pi \in \Pi$, if $J^{\pi} = J^*$, we call π optimal and write as π^*.

Now we impose the following conditions on the network system parameters which ensure the existence of an optimal replacement rule.

Condition 1. $G^i_z(\cdot|t)$ is stochastically increasing in z and in t for every $i \in \Xi$.

Condition 2. $H^i(\cdot)$ has the continuous density function $h^i(\cdot)$ for every $i \in \Xi$.

Condition 3. $\gamma(i,j,z)$ is decreasing in z for any $i,j \in \Xi$, and $\gamma(i,i+1,z) = 1$ for $i = 0,1,\ldots,N-1; z \in R_+$.

The condition $\gamma(i,i+1,z) = 1$ asserts that the main-system is not subject to fail when a failed component is restored to functioning. From Proposition 3.2.1, 3.2.2 and Condition 3, we have the following expected cost incurred during the transition interval and the expected length of the transition interval, i.e.

$$\begin{aligned}
&\bar{c}(i,z,a) \qquad\qquad (3.2.7)\\
&= (C + C_0 + (N-i)K)[\textstyle\int_0^a \int_0^s (1 - R(i,z,t))H^i(dt)\mu_i e^{-\mu_i s} ds \\
&\qquad + \textstyle\int_0^a (1 - R(i,z,t))H^i(dt)e^{-\mu_i a}] \\
&\qquad + (C+C_0+(N-i+1)K)(1-\gamma(i,i-1,z))\tfrac{in}{\mu_i}\Phi_2(i,a) + (C+C_0+(N-i)K)\Phi_3(i,a)
\end{aligned}$$

where $R(i,z,t) = \int_{R_+} \gamma(i,i,z+x)G^i_z(dx|t)$.

$$\begin{aligned}
&\bar{\tau}(i,z,a) \qquad\qquad (3.2.8)\\
&= \textstyle\int_0^a \int_0^s tH^i(dt)\mu_i e^{-\mu_i s} ds + \int_0^a tH^i(dt)e^{-\mu_i a} + \int_0^a \int_0^s t\mu_i e^{-\mu_i t} dt H^i(ds) \\
&\qquad + \textstyle\int_0^a t\mu_i e^{-\mu_i t} dt \bar{H}^i(a) + a\Phi_3(i,a).
\end{aligned}$$

Let B be the set of all bounded real-valued functions $V(i,z)$ on $\Xi \times R_+$ which are $\Im$-measurable for $i \in \Xi$, $B^+ \equiv \{V \in B;\ V(i,z)$ is increasing in z for $i \in \Xi\}$ and $\|\cdot\|$ the sup-norm defined on B. It is easy to see that B is a Banach space.

3.3 Existence of an optimal policy

In discussing the existence of an optimal state-age dependent policy, we follow a method similar to one proposed by Ross [54]. The main result in this section is the following Theorem 3.3.1 which establishes the existence of optimal policy and gives an approach to obtain an optimal policy .

Theorem 3.3.1. (i) There exists a bounded function $V^* \in B^+$ and a constant Θ^* satisfying the following *average cost optimality equation*

$$V^*(i,z) = \inf_{a \in E} \{ \ \bar{c}(i,z,a) - \Theta^* \bar{\tau}(i,z,a) + E_{(i,z)}[V^*(I_1, Z_1) | A_0(i,z) = a] \ \}. \quad (3.3.1)$$

(ii) There exists an optimal stationary policy π^* such that

$$\Theta^* = J^{\pi^*} = J^*$$

and π^* is any stationary policy which, in state (i,z), prescribes an action minimizing the right side of (3.3.1).

Since, in our model, the action space is not finite, the treatments are more complicated. Before proving the Theorem 3.3.1, we give several lemmas. First, we consider an optimal solution of the equation (3.3.1) on the restricted action space $R_\epsilon = [\epsilon, \infty]$ for any fixed $\epsilon > 0$. In order to prove the existence of an optimal solution on the R_ϵ, we consider the total expected discounted cost with the discount rate $\alpha > 0$ and introduce an operator $U_{\alpha,\epsilon}$ such as for any $V \in B$

$$U_{\alpha,\epsilon} V(i,z) = \inf_{a \in R_\epsilon} \{ \ E_{(i,z)}[e^{-\alpha\tau}(c + V(I_1, Z_1)) | A_0(i,z) = a] \ \}. \quad (3.3.2)$$

By Proposition 3.2.2, we have

$$U_{\alpha,\epsilon}(i,z,a,V) \equiv E_{(i,z)}[e^{-\alpha\tau}(c + V(I_1, Z_1)) | A_0(i,z) = a]$$

$$= \int_0^a \int_0^s e^{-\alpha t} L_t V(i,z) H^i(dt) \mu_i e^{-\mu_i s} ds + \int_0^a e^{-\alpha t} L_t V(i,z) H^i(dt) e^{-\mu_i a}$$

$$+ \tfrac{1}{\alpha+\mu_i} LV(i,z)[\int_0^a (1 - e^{-(\alpha+\mu_i)s}) H^i(ds) + (1 - e^{-(\alpha+\mu_i)a}) \bar{H}^i(a)]$$

$$+ (C + (N-i)K + V(N,0)) e^{-\alpha a} \Phi_3(i,a),$$

where

$$L_tV(i,z) = C+C_0+(N-i)K+V(N,0) \tag{3.3.4}$$
$$-\int_{R_+}[C+C_0+(N-i)K+V(N,0)-V(i,z+x)]\gamma(i,i,z+x)G_z^i(dx|t)$$

$$LV(i,z) = \delta_{iN}\lambda V(i+1,z)+i\mu\{C+C_0+(N-i+1)K+V(N,0) \tag{3.3.5}$$
$$-[C+C_0+(N-i+1)K+V(N,0)-V(i-1,z)]\gamma(i,i-1,z)\}.$$

Lemma 3.3.2. For any fixed $\epsilon > 0$, $U_{\alpha,\epsilon}$ is a monotone contraction operator.

Proof. The monotonicity of $U_{\alpha,\epsilon}$ is obvious. For any i, z and $V_1, V_2 \in B$, $U_{\alpha,\epsilon}(i,z,a,V_j)$ ($j = 1,2$) are the bounded continuous functions in a on R_ϵ, so there exists $a_j^* = a_j^*(i,z)$ such that $\inf_{a\in R_\epsilon} U_{\alpha,\epsilon}(i,z,a,V_j) = U_{\alpha,\epsilon}(i,z,a_j^*,V_j)$. If $\inf_{a\in R_\epsilon} U_{\alpha,\epsilon}(i,z,a,V_1) \geq \inf_{a\in R_\epsilon} U_{\alpha,\epsilon}(i,z,a,V_2)$, we have

$$\begin{aligned}
&|\inf_{a\in R_\epsilon} U_{\alpha,\epsilon}(i,z,a,V_1) - \inf_{a\in R_\epsilon} U_{\alpha,\epsilon}(i,z,a,V_2)| \\
&\leq U_{\alpha,\epsilon}(i,z,a_2^*,V_1) - U_{\alpha,\epsilon}(i,z,a_2^*,V_2) \\
&\leq (\Phi_1(i,a_2^*) + \Phi_2(i,a_2^*) + e^{-\alpha a_2^*}\Phi_3(i,a_2^*)) \parallel V - W \parallel \\
&\equiv \beta^i(a_2^*) \parallel V - W \parallel .
\end{aligned}$$

Since $\Phi_1(i,a_2^*)+\Phi_2(i,a_2^*)+\Phi_3(i,a_2^*) = 1$, and for $a_2^* \geq \epsilon$, $\max_i \sup_z \Phi_j(i,a_2^*) \neq 0$ $(j = 1,2)$, $\max_i \sup_z \Phi_3(i,a_2^*) \neq 1$ and $\max_i \sup_z e^{-\alpha a_2^*} < 1$, we have $\beta_\epsilon^2 \equiv \max_i \sup_z \beta^i(i,a_2^*) < 1$. Similarly, if $\inf_{a\in R_\epsilon} U_{\alpha,\epsilon}(i,z,a,V_1) \leq \inf_{a\in R_\epsilon} U_{\alpha,\epsilon}(i,z,a,V_2)$, we have $\beta_\epsilon^1 \equiv \max_i \sup_z$
$\beta^i(i,a_1^*) < 1$. Therefore $\parallel U_{\alpha,\epsilon}V - U_{\alpha,\epsilon}W \parallel \leq \max\{\beta_\epsilon^1,\beta_\epsilon^2\}\times \parallel V - W \parallel$. □

Thus, since $U_{\alpha,\epsilon}$ is a monotone contraction operator, it has a unique fixed point $V_{\alpha,\epsilon}^*$ which is the minimal total expected discounted cost on the restricted action space R_ϵ.

Lemma 3.3.3. Under Condition 1, we have

(i) $L_tV_{\alpha,\epsilon}^*(i,z), LV_{\alpha,\epsilon}^*(i,z)$ are both increasing in z for every i, and

$$L_tV_{\alpha,\epsilon}^*(i,z) \leq C + C_0 + (N-i+1)K + V_{\alpha,\epsilon}^*(N,0),$$
$$LV_{\alpha,\epsilon}^*(i,z) \leq \mu_i(C + C_0 + (N-i+1)K + V_{\alpha,\epsilon}^*(N,0)).$$

(ii) $V^*_{\alpha,\epsilon}(i,z)$ is increasing in z for every $i \in \Xi$, and

$$V^*_{\alpha,\epsilon}(i,z) \leq C + C_0 + (N-i+1)K + V^*_{\alpha,\epsilon}(N,0).$$

Proof. Let $V_0 = 0, V_{n+1} = U_{\alpha,\epsilon}V_n$ for $n \geq 1$. By induction, we first prove that the assertions (i) and (ii) hold for the mapping sequence $\{V_n\}_{n\geq 0}$. Since $V_0 = 0$, (i) and (ii) hold certainly. Suppose that (i) and (ii) are true for an integer n, then $C + C_0 + (N-i)K + V_n(N,0) - V_n(i, z+x)$ is decreasing in z, and by Condition 1, $\int_{R_+}[C + C_0 + (N-i)K + V_n(N,0) - V_n(i,z)]\gamma(i,i,z+x)G^i_z(dx|t)$ is decreasing in z. So $L_tV_n(i,z)$ is increasing in z. From $1 - \int_{R_+}\gamma(i,i,z+x)G^i_z(dx|t) \geq 0$, we have

$$\begin{aligned}L_tV_n(i,z) \leq{} & C+C_0+(N-i+1)K+V_n(N,0)-\int_{R_+}[C+C_0+(N-i+1)K+V_n(N,0) \\ & -V_n(i,z)]\gamma(i,i,z+x)G^i_z(dx|t) \\ \leq{} & C + C_0 + (N-i+1)K + V_n(i,z)(N,0).\end{aligned}$$

Similarly, we obtain assertion (i) holds for $LV_n(i,z)$. From the definition (3.3.3) and assertion (i), we have $U_{\alpha,\epsilon}(i,z,a,V_n)$ is increasing in z, and

$$\begin{aligned}U_{\alpha,\epsilon}(i,z,a,V_n) \leq{} & (C + C_0 + (N-i+1)K + V_n(N,0))[\Phi_1(i,a) + \tfrac{\mu_i}{\alpha+\mu_i}\Phi_2(i,a) \\ & +e^{-\alpha a}\Phi_3(i,a)] \\ \leq{} & C + C_0 + (N-i+1)K + V_n(N,0).\end{aligned}$$

Hence $V_{n+1}(i,z)$ is increasing in z and $V_{n+1}(i,z) = \inf_{a\in R_\epsilon} U_{\alpha,\epsilon}(i,z,a,V_n) \leq C + C_0 + (N-i+1)K + V_n(N,0)$. Assertions (i) and (ii) hold for $n+1$. Finally, the proof follows from $V^*_{\alpha,\epsilon} = \lim_{n\to\infty} V_n$. □

Lemma 3.3.4. For any fixed $\epsilon > 0$, let $b = \min_i \bar{H}^i(\epsilon)e^{-(N\mu+\lambda)\epsilon}$. Then for $i \in \Xi, z \in R_+$, and $a \in R_\epsilon$, we have

(i) $\bar{\tau}(i,z,a) \geq \epsilon b, \quad \bar{c}(i,z,a) \leq C + C_0 + (N-i+1)K$.

(ii) $\| V^*_{\alpha,\epsilon} \| \leq \frac{(C+C_0+(N+1)K)e^{-\alpha b}}{1-e^{-\alpha b}}$.

Proof. For $a \in R_\epsilon$, $P_{(i,z)}(\tau > \epsilon | A_0(i,z) = a) = \bar{H}^i(\epsilon)e^{-\mu_i\epsilon} \geq b$. We have $\bar{\tau}(i,z,a) \geq \epsilon\bar{H}^i(\epsilon)e^{-\mu_i\epsilon} \geq \epsilon b$. From (3.2.6), we get the second inequality of (i). The part (ii) follows that $| V^*_{\alpha,\epsilon}(i,z) | \leq \inf_{a\in R_\epsilon} | E_{(i,z)}[e^{-\alpha\tau}(c + V^*_{\alpha,\epsilon}(I_1, Z_1)) | A_0(i,z) = a] | \leq e^{-\alpha b}(C + C_0 +$

$(N-i+1)K+\| V^*_{\alpha,\epsilon} \|)$. This completes the proof. $\square$

Lemma 3.3.5. For any $\epsilon > 0$,

(i) There exists a bounded function V^*_ϵ and a constant Θ^*_ϵ satisfying

$$V^*_\epsilon(i,z) = \inf_{a \in R_\epsilon} \{ \ \bar{c}(i,z,a) - \Theta^*_\epsilon \bar{\tau}(i,z,a) + E_{(i,z)}[V^*_\epsilon(I_1, Z_1)|A_0(i,z) = a] \ \}. \quad (3.3.6)$$

(ii) For some sequence $\alpha_n \to 0$, $V^*_\epsilon(i,z) = \lim_{n\to\infty}[V^*_{\alpha_n,\epsilon}(i,z) - V^*_{\alpha_n,\epsilon}(N,0)]$.

(iii) $\Theta^*_\epsilon = \lim_{\alpha\to 0} \alpha V^*_{\alpha,\epsilon}(N,0)$.

Proof. For any $i \in \Xi, z \in R_+$, and $\alpha > 0$, let $g^*_{\alpha,\epsilon}(i,z) = V^*_{\alpha,\epsilon}(i,z) - V^*_{\alpha,\epsilon}(N,0)$ and $\Theta^*_{\alpha,\epsilon} = \alpha V^*_{\alpha,\epsilon}(N,0)$. Since $V^*_{\alpha,\epsilon}(i,z) = \inf_{a\in R_\epsilon} E_{(i,z)}[e^{-\alpha\tau}(c + V^*_{\alpha,\epsilon}(I_1,Z_1))|A_0(i,z) = a]$,

$$g^*_{\alpha,\epsilon}(i,z) = \inf_{a\in R_\epsilon}\{(\bar{c}(i,z,a) + E_{(i,z)}[g^*_{\alpha,\epsilon}(I_1,Z_1)|A_0(i,z) = a]) + \alpha(\bar{c}(i,z,a) \quad (3.3.7)$$
$$+E_{(i,z)}[g^*_{\alpha,\epsilon}(I_1,Z_1)|A_0(i,z) = a]) - \alpha V^*_{\alpha,\epsilon}(N,0)\bar{\tau}(i,z,a) + o(\alpha)\}.$$

$g^*_{\alpha,\epsilon}(i,z)$ is continuous in α, and $\| g^*_{\alpha,\epsilon} \| \leq C + C_0 + (N+1)K < \infty$. Moreover, $\| \Theta^*_{\alpha,\epsilon} \| \leq \ \alpha \| V^*_{\alpha,\epsilon} \| \leq \frac{C+C_0+(N+1)K}{b} < \infty$. It follows that there exists $\{\alpha_n\}_{n\leq 0}$ with $\alpha_n \downarrow 0$ such that the following convergences hold:

$$\lim_{n\to\infty} \Theta^*_{\alpha_n,\epsilon} \equiv \Theta^*_\epsilon, \quad \lim_{n\to\infty} g^*_{\alpha_n,\epsilon}(i,z) \equiv V^*_\epsilon(i,z).$$

Now we write

$$U(i,z,a,V^*_\epsilon) \equiv \bar{c}(i,z,a) - \Theta^*_\epsilon\bar{\tau}(i,z,a) + E_{(i,z)}[V^*_\epsilon(I_1,Z_1)|A_0(i,z) = a].$$

Since the function $U(i,z,a,V^*_\epsilon)$ and the function in $\{ \ \}$ of the equality (3.3.7) are uniformly bounded continuous in a, there exists a $a^* = a^*(i,z) \in R_\epsilon$ such that $\inf_{a\in R_\epsilon} U(i,z,a,V^*_\epsilon) = U(i,z,a^*,V^*_\epsilon)$. If $g^*_{\alpha,\epsilon}(i,z) \geq \inf_{a\in R_\epsilon} U(i,z,a,V^*_\epsilon)$, We have

$$| \ g^*_{\alpha,\epsilon}(i,z) - \inf_{a\in R_\epsilon} U(i,z,a,V^*_\epsilon) \ |$$
$$\leq \{(\bar{c}(i,z,a^*) + E_{(i,z)}[g^*_{\alpha_n,\epsilon}(I_1,Z_1)|A_0(i,z) = a^*]) + \alpha_n(\bar{c}(i,z,a^*)$$
$$+E_{(i,z)}[g^*_{\alpha_n,\epsilon}(I_1,Z_1)|A_0(i,z) = a^*]) - \alpha_n V^*_{\alpha_n,\epsilon}(N,0)\bar{\tau}(i,z,a^*) + o(\alpha_n)\}$$
$$-\{\bar{c}(i,z,a^*) - \Theta^*_\epsilon\bar{\tau}(i,z,a^*) + E_{(i,z)}[V^*_\epsilon(I_1,Z_1)|A_0(i,z) = a^*]\}$$
$$\leq \alpha_n(\bar{c}(i,z,a^*) + E_{(i,z)}[g^*_{\alpha_n,\epsilon}(I_1,Z_1)|A_0(i,z) = a^*])$$
$$\leq 2\alpha_n(C + C_0 + (N+1)K) + o(\alpha_n) \to 0.$$

If $g^*_{\alpha,\epsilon}(i,z) \leq \inf_{a\in R_\epsilon} U(i,z,a,V^*_\epsilon)$, we have the same conclusion. Finally, replacing α

by α_n in (3.3.7) and taking limit as $n \to \infty$ in both sides of the resulting equality, we see that Θ_ϵ^* and V_ϵ^* satisfy the equality (3.3.6). This completes the proof of the part (i). The assertion (ii) follows from the above proof.

To prove the assertion (iii), we note that since $\alpha V_{\alpha,\epsilon}^*(N,0)$ is bounded, it follows that for any sequence $\alpha_n \to 0$ there is a subsequence α_n' such that

$$\lim_{\alpha_n' \to 0} \alpha_n' V_{\alpha_n',\epsilon}^*(N,0)$$

exists. By the proof of (i) it follows that this limit must be Θ_ϵ^*. Hence,

$$\Theta_\epsilon^* = \lim_{\alpha_n' \to 0} \alpha_n' V_{\alpha_n',\epsilon}^*(N,0)$$

and the proof is complete. □

From the above Lemma 3.3.3 and 3.3.5, we get easily the following.

Corollary 3.3.6 (i) $V_\epsilon^*(i,z)$ is increasing in z and decreasing in ϵ, and

$$V_\epsilon^*(i,z) \le C + C_0 + (N-i+1)K + V_\epsilon^*(N,0).$$

(ii) Θ_ϵ^* is decreasing in ϵ.

Let $\Theta^* = \lim_{\epsilon \to 0} \Theta_\epsilon^*$, and $V^* = \lim_{\epsilon \to 0} V_\epsilon^*$. Now, we shall prove that Θ^* and V^* satisfy the average cost optimality equation (3.3.1).

Proof of Theorem 3.3.1. (i) Since $U(i,z,a,V_\epsilon^*)$ defined in (3.3.9) is uniformly bounded continuous in a on the interval R_ϵ, there exists $a^* = a^*(i,z) \in R_\epsilon$ such that $\inf_{a \in R_\epsilon} U(i,z,a,V_\epsilon^*)$
$= U(i,z,a^*,V_\epsilon^*)$. If $\inf_{a\in[0,\infty]} U(i,z,a,V^*) \ge \inf_{a \in R_\epsilon} U_\epsilon(i,z,a,V_\epsilon^*)$, then

$$
\begin{aligned}
&|\inf_{a\in[0,\infty]} U(i,z,a,V^*) - V^*(i,z)| \\
&\le |\inf_{a\in[0,\infty]} U(i,z,a,V^*) - V_\epsilon^*| + \| V^* - V_\epsilon^* \| \\
&= |\inf_{a\in[0,\infty]} U(i,z,a,V^*) - \inf_{a\in R_\epsilon} U(i,z,a,V_\epsilon^*)| + \| V^* - V_\epsilon^* \| \\
&\le U(i,z,a^*,V^*) - U(i,z,a^*,V_\epsilon^*) + \| V^* - V_\epsilon^* \| \\
&\le \| \Theta^* - \Theta_\epsilon^* \| \bar{\tau}(i,z,a^*) + 2 \| V^* - V_\epsilon^* \| \to 0 \quad (\text{as } \epsilon \to 0).
\end{aligned}
$$

We have

$$V^*(i,z) = \lim_{\epsilon \to 0} V_\epsilon^*(i,z) = \inf_{a\in[0,\infty]} U(i,z,a,V_\epsilon^*).$$

If $\inf_{a\in[0,\infty]} U(i,z,a,V^*) \le \inf_{a\in R_\epsilon} U_\epsilon(i,z,a,V_\epsilon^*)$, we have the same conclusion. This completes the proof of the Theorem 3.3.1 (i).

(ii) For any fixed $\epsilon > 0$, let $\pi_\epsilon = (A_0, A_1, \ldots)$ where $A_n \in R_\epsilon$ for $n \geq 0$, and π^*_ϵ be any stationary policy which, in state (i, z), prescribes an action minimizing the right side of (3.3.6). Let $\Im_n = \sigma(I_0, Z_0, A_0, \ldots, I_n, Z_n, A_n)$, $n \geq 0$. For any policy π_ϵ,

$$E^{\pi_\epsilon}\{\textstyle\sum_{i=1}^n [V^*_\epsilon(I_i, Z_i) - E^{\pi_\epsilon}[V^*_\epsilon(I_i, Z_i)|\Im_{i-1}]]\}$$
$$= \textstyle\sum_{i=1}^n \{E^{\pi_\epsilon}[V^*_\epsilon(I_i, Z_i)] - E^{\pi_\epsilon}[E^{\pi_\epsilon}[V^*_\epsilon(I_i, Z_i)|\Im_{i-1}]]\} = 0.$$

But,

$$E^{\pi_\epsilon}[V^*_\epsilon(I_i, Z_i)|\Im_{i-1}] = E_{(I_{i-1}, Z_{i-1})}[V^*_\epsilon(I_i, Z_i)|A_{i-1}]$$
$$= \bar{c}(I_{i-1}, Z_{i-1}, A_{i-1}) - \Theta^*_\epsilon \bar{\tau}(I_{i-1}, Z_{i-1}, A_{i-1}) + E_{(I_{i-1}, Z_{i-1})}[V^*_\epsilon(I_i, Z_i)|A_{i-1}]$$
$$-\bar{c}(I_{i-1}, Z_{i-1}, A_{i-1}) + \Theta^*_\epsilon \bar{\tau}(I_{i-1}, Z_{i-1}, A_{i-1})$$
$$\geq \inf_{a \in R_\epsilon}\{\bar{c}(I_{i-1}, Z_{i-1}, a) - \Theta^*_\epsilon \bar{\tau}(I_{i-1}, Z_{i-1}, a) + E_{(I_{i-1}, Z_{i-1})}[V^*_\epsilon(I_i, Z_i)|A_{i-1} = a]\}$$
$$-\bar{c}(I_{i-1}, Z_{i-1}, A_{i-1}) + \Theta^*_\epsilon \bar{\tau}(I_{i-1}, Z_{i-1}, A_{i-1})$$
$$= V^*_\epsilon(I_{i-1}, Z_{i-1}) - \bar{c}(I_{i-1}, Z_{i-1}, A_{i-1}) + \Theta^*_\epsilon \bar{\tau}(I_{i-1}, Z_{i-1}, A_{i-1}),$$

with equality for π^*_ϵ since π^*_ϵ is defined to take the infimum action. Hence

$$E^{\pi_\epsilon}\{\sum_{i=1}^n [V^*_\epsilon(I_i, Z_i) - V^*_\epsilon(I_{i-1}, Z_{i-1}) + \bar{c}(I_{i-1}, Z_{i-1}, A_{i-1}) - \Theta^*_\epsilon \bar{\tau}(I_{i-1}, Z_{i-1}, A_{i-1})]\} \geq 0,$$

or

$$\Theta^*_\epsilon \leq \frac{E^{\pi_\epsilon}[V^*_\epsilon(I_n, Z_n) - V^*_\epsilon(N, 0)] + E^{\pi_\epsilon}[\sum_{i=1}^n \bar{c}(I_{i-1}, Z_{i-1}, A_{i-1})]}{E^{\pi}[\sum_{i=1}^n \bar{\tau}(I_{i-1}, Z_{i-1}, A_{i-1})]}$$

with equality for π^*_ϵ. Now, letting $n \to \infty$ and using the boundedness of V^* and the fact that Lemma 3.3.4 implies that for $A_{i-1} \in R_\epsilon$, $E^\pi \sum_{i=1}^n \bar{\tau}(I_{i-1}, Z_{i-1}, A_{i-1}) \geq n\epsilon b \to \infty$. We have that

$$\Theta^*_\epsilon \leq \lim_{n \to \infty} \frac{E^\pi[\sum_{i=1}^n \bar{c}(I_{i-1}, Z_{i-1}, A_{i-1})]}{E^\pi[\sum_{i=1}^n \bar{\tau}(I_{i-1}, Z_{i-1}, A_{i-1})]} = J^{\pi_\epsilon}(I_0, Z_0)$$

with equality for π^*_ϵ and for all values of (I_0, Z_0). By the definitions of Θ^* and V^*, and that $\lim_{\epsilon \to 0} \pi_\epsilon = \pi$, we get for any $\pi \in \Pi$

$$\Theta^* = \lim_{\epsilon \to 0} \Theta^*_\epsilon \leq \lim_{\epsilon \to 0} J^{\pi_\epsilon} = J^\pi$$

with equality for $\pi^* = \lim_{\epsilon \to 0} \pi^*_\epsilon$. This completes the proof of the part (ii). $\square$

3.4 An optimal policy

For the state (i, z), let $a^*(i, z)$ the minimal solution satisfying the average cost optimality equation (3.3.1), and $A^* = a^*(i, z)$. It follows from Theorem 3.3.1 (ii) that $\pi^* = (A^*, A^*, \dots)$. Furthermore, let $r^i(t) = \frac{h^i(t)}{\bar{H}^i(t)}$ the the hazard rate associated with the distribution function $H^i(\cdot)$, we consider the minimal solution $a^*(i, z)$ in this section. First, we get the following Lemma 3.4.1 from Lemma 3.3.3.

Lemma 3.4.1. Under Condition 1 , we have

(i) $V^*(i, z)$ is increasing in z and $V^*(i, z) \le C + C_0 + (N - i + 1)K + V^*(N, 0)$.

(ii)$L_t V^*(i, z), LV^*(i, z)$ are both increasing in z for every i, and

$L_t V^*(i, z) \le C + C_0 + (N - i + 1)K + V^*(N, 0)$,

$LV^*(i, z) \le \mu_i(C + C_0 + (N - i + 1)K + V^*(N, 0))$.

Lemma 3.4.2. For $i \in \Xi, z \in R_+$, there exists a minimal solution $a^*(i, z)$ satisfying the following equation.

$$a^*(i, z) \in \{0, \infty\} \cup \{a;\ \ r^i(a) = \frac{\mu_i(C + (N - i)K + V^*(N, 0)) + \Theta^* - LV^*(i, z)}{L_a V^*(i, z) - (C + (N - i)K + V^*(N, 0))}\}. \tag{3.4.1}$$

Proof. By Proposition 3.2.2, we have

$$U(i, z, a, V^*) = \int_0^a \int_0^s L_t V^*(i, z) H^i(dt) \mu_i e^{-\mu_i s} ds + \int_0^a L_t V^*(i, z) H^i(dt) e^{-\mu_i a}$$
$$+ \tfrac{1}{\mu_i} LV^*(i, z) \Phi_2(i, a) + (C + (N - i)K + V^*(N, 0)) \Phi_3(i, a) - \Theta^* \bar{\tau}(i, z, a).$$

Thus $U(i, z, a, V^*)$ is continuous and differentiable with respect to a. By solving $\frac{\partial}{\partial a} U(i, z, a, V^*) = 0$, we obtain the assertion. □

Since $LV^*(i, z)$ is increasing in z for every $i \in \Xi$, there exists a $z^* = z^*(i)$ such that

$$z^*(i) = \inf\{z;\ \ \mu_i(C + (N - i)K + V^*(N, 0)) + \Theta^* - LV^*(i, z) \ge 0\} \tag{3.4.2}$$

where $\inf\{\emptyset\} = \infty$. Furthermore, since by Condition 1, $L_a V^*(i, z)$ is increasing in a, there exists a $\bar{a} = \bar{a}(i, z)$ such that

$$\bar{a}(i,z) = \inf\{a;\ \ L_a V^*(i,z) - (C + (N-i)K + V^*(N,0)) \geq 0\} \tag{3.4.3}$$

where $\inf\{\emptyset\} = \infty$.

Theorem 3.4.3. If $r^i(t)$ is a strict monotone function or a constant in t, then there exists a unique minimal solution $a^*(i,z)$ such that: for $z > z^*(i)$,

$a^*(i,z) = \inf\{a \in [\bar{a}(i,z), \infty];$

$$r^i(a) \geq \frac{\mu_i(C + (N-i)K + V^*(N,0)) + \Theta^* - LV^*(i,z)}{L_a V^*(i,z) - (C + (N-i)K + V^*(N,0))}\} \tag{3.4.4}$$

where $\inf\{\emptyset\} = \infty$. If $\lim_{t\to\infty} r^i(t) = \infty$, then $a^*(i,z) < \infty$. Moreover, for $z \leq z^*(i)$,

$a^*(i,z) = \inf\{a \in [0, \bar{a}(i,z));$

$$r^i(a) \leq \frac{LV^*(i,z) - \mu_i(C + (N-i)K + V^*(N,0)) - \Theta^*}{(C + (N-i)K + V^*(N,0)) - L_a V^*(i,z)}\} \tag{3.4.5}$$

and $\inf\{\emptyset\} = 0$.

Proof. From Lemma 3.4.2, the minimal solution $a^*(i,z)$ should satisfy the following inequalities:

for $a < a^*(i,z)$, $$(L_a V^*(i,z) - (C + (N-i)K + V^*(N,0)))r^i(a) < \mu_i(C + (N-i)K + V^*(N,0)) + \Theta^* - LV^*(i,z), \tag{3.4.6}$$

and for $a \geq a^*(i,z)$, $$(L_a V^*(i,z) - (C + (N-i)K + V^*(N,0)))r^i(a) \geq \mu_i(C + (N-i)K + V^*(N,0)) + \Theta^* - LV^*(i,z). \tag{3.4.7}$$

For $z > z^*(i)$, the inequality (3.4.6) holds for all $a \in [0, \bar{a}(i,z))$, i.e. $U(i,z,a,V^*)$ is decreasing in a on $[0, \bar{a}(i,z))$, so we have $a^*(i,z) \in [\bar{a}(i,z), \infty]$. Since $r^i(t)$ is a strict monotone function in t or a constant and $L_t V^*(i,z)$ is increasing in t, if the set $\{\ldots\}$ of (3.4.4) is not empty, then $a^*(i,z)$ defined in (3.4.4) is the unique minimal solution. Otherwise, we have $U(i,z,a,V^*)$ is decreasing in a on $[0,\infty]$, thus $a^*(i,z) = \infty$. Obviously, when $\lim_{t\to\infty} r^i(t) = \infty$, $a^*(i,z) < \infty$. Similarly, for $z \leq z^*(i)$, the inequality (3.4.7) holds for all $a \in [\bar{a}(i,z), \infty]$, i.e. $U(i,z,a,V^*)$ is increasing in a on $[\bar{a}(i,z), \infty]$, so we have $a^*(i,z) \in [0, \bar{a}(i,z))$. Since $\lim_{a\to\bar{a}(i,z)} \frac{LV^*(i,z) - \mu_i(C+(N-i)K+V^*(N,0)) - \Theta^*}{(C+(N-i)K+V^*(N,0)) - L_a V^*(i,z)} =$

$+\infty$, if the set $\{\dots\}$ of (3.4.5) is not empty, then $a^*(i,z)$ defined in (3.4.5) is the unique minimal solution. Otherwise, we have $U(i,z,a,V^*)$ is increasing in a on $[0,\infty]$, thus $a^*(i,z)=0$. The proof is complete. $\square$

Corollary 3.4.4. $a^*(i,z)$ is decreasing in z.

Proof. Since $L_tV^*(i,z)$ and $LV^*(i,z)$ are increasing in z, it follows that the right-side of the inequality in $\{\ \}$ of (3.4.4) is decreasing in z, and the right-side of the inequality in $\{\ \}$ of (3.4.5) is increasing in z. So $a^*(i,z)$ is decreasing in z. $\square$

Remark. (i) In Lemma 3.4.3, we require the strict monotonicity (strict increasing or decreasing) or a constant of the hazard rate $r^i(t)$ to ensure the unique existence of the minimal solution $a^*(i,z)$. This requirement, in general, can be satisfied by many used distributions such as Negative exponential, Gamma, Weibull and Normal distributions, etc. For more general distribution, similar to Lemma 3.4.3, we have $a^*(i,z)\in[\bar{a}(i,z),\infty]$ if $z>z^*(i)$, and $a^*(i,z)\in[0,\bar{a}(i,z))$ otherwise.
(ii) Since the distribution $H^i(t)$ of the intershock times depend on the state i of the functioning components of the sub-system, it is possible that the hazard rate $r^i(t)$ is increasing for some state i, and decreasing for others. For example, let i^* $(1\le i^*\le N)$ fixed, $\kappa>0, \kappa(i)=\frac{N-i+1}{i^*}, i\in\Xi$, and $H^i(t)$ be Weibull distribution with the parameters κ and $\kappa(i)$, i.e.

$$H^i(t)=\frac{\kappa(i)}{\kappa}\left(\frac{t}{\kappa}\right)^{\kappa(i)-1}exp\left[-\left(\frac{t}{\kappa}\right)^{\kappa(i)}\right].$$

The hazard rate for $i\in\Xi$ is $r^i(t)=\frac{\kappa(i)}{\kappa}(\frac{t}{\kappa})^{\kappa(i)-1}$. Hence $r^i(t)$ is increasing for $i\le N-i^*$, decreasing for $i>N-i^*+1$ and constant $r^{N-i^*+1}(t)=\frac{\kappa(N-i^*+1)}{\kappa}$ for $i=N-i^*+1$. In this case, by Theorem 3.4.3, we can determine respectively the minimal solution $a^*(\cdot,\cdot)$ for the increasing, decreasing and constant hazard rates. Therefore, our model can be applied to deal with the optimal replacement problem in which the hazard rate associated with the distribution function $H^i(\cdot)$ is aletrnatively increasing, decreasing in t or a constant in the system's functioning period.

3.5 Tow-special cases

In the former sections, we derived an optimal state-age dependent policy π^*. The policy π^* is influenced by changes of states of the process $\{I(t)\}_{t\geq 0}$. Since $\{I(t)\}_{t\geq 0}$ is a Markov jump process, our model can be applied to optimal state-age dependent replacement problems of systems existing in a Markov random environment, systems with a Markov modulator or systems subject to two type of dependent shocks. In the following, let $\lambda = \infty$, i.e. the repair times of the failed components of the sub-system are negligible, we consider the following special cases.

(i) In this case, if we assume that $\gamma(i, i-1, z) = 1$ for $i = 1, \ldots, N; z \in R_+$, then problem become a general replacement problem for a single-unit system since $I(t) = N$ for $t \geq 0$ and the main-system is not affected by the component's failures of the sub-system. Such problems have been studied by Kao [36], and Gottlieb [32]. Let $r(t) = r^N(t), G_z(x|t) = G_z^N(x|t)$. We have that if $r(t)$ is increasing in t, then the minimal solution satisfies the equation

$$a^*(z) = \inf\{a \in [\bar{a}(z), \infty]; \;\; r(a) \geq \frac{\Theta^*}{C_0 - \int_{R_+}[C_0 + V^*(z+x)]\gamma(z+x)G_z(dx|a)}\}$$

where $\;\; \bar{a}(z) = \inf\{a; \;\; C_0 - \int_{R_+}[C_0 + V^*(z+x)]\gamma(z+x)G_z(dx|a) \geq 0\}$.

(ii) In this case, if we take $I(t)$ as to the failed number of the components of the sub-system in $[0, t]$, then $\{I(t)\}_{t\geq 0}$ is a Poisson process with the parameter $N\mu$. Assume that $\gamma(i, i+1, z) < 1$. The main-system is subject to two types of dependent shocks. The process $\{Z(t)\}_{t\geq 0}$ defined as in (3.2.2) still presents the damage of the main-system. Assume $H^n(t) = H(t)$, $\gamma(n, n+1, z) \geq \gamma(n+1, n+2, z)$ and $G_z^n(\cdot|t)$ is stochastically increasing in z and in n. Furthermore, we replace the whole sub-system when the main-system is replaced. Under these assumptions, we have the following assertion.

Corollary 3.5.1.

(i) $V^*(n,z)$ is increasing in z and in n, $V^*(n,z) \leq C + C_0 + Nk + V^*(0,0)$.

(ii) The minimal solution $a^*(n,z)$ is decreasing in z and in n.

CHAPTER 4

DYNAMICALLY OPTIMAL REPLACEMENT UNDER ADDITIVE DAMAGE IN A MARKOV RANDOM ENVIRONMENT

4.1 Introduction

In Chapter 2 we have dealt with the optimal replacement problem for the system existing in a Poisson randomly varying environment, and in Chapter 3, we have considered the optimal replacement for a network system where the functioning process of the components of the sub-system that is a Markov process is taken as the environment of the main-system. In these two models, the assumption (A) in Chapter 2 and the assumption that the emergency repairs for all failed components are executed when the main-system is replaced in Chapter 3 imply that the environment processes are also restored to the initial states when the systems are replaced, so that the successive replacements periods of identical systems form a renewal process. In addition to that, we have not consider the maintenance actions for the system in Chapter 2 and the main-system in Chapter 3. The purpose of the present Chapter is to investigate an optimal maintenance–replacement problem for a more general system by means of Dynamic programming method. Here we do not require that the Markov environment process return to the initial state when the system is replaced. In this case, the successive replacement periods of identical systems no long forms a renewal process. Therefore, analysis is difficult by general renewal arguments given in Chapter 2 and by the semi-Markov decision method given in Chapter 3 (there because of the renewal property, we obtained that $\Theta^* = J^* = J^{\pi^*}(i, z)$ for all initial value (i, z), i.e., the minimal long-run average cost per unit time is not dependent on the initial value). We consider the replacement actions as well as the maintenance actions, and allow that the damaged system become to "better" after the system is maintained, i.e., the damage level of the system has an randomly decreasing magnitude which is assumed to be stochastically decreasing with respect to the accumulated damage level. For the total expected randomly discounted cost, we first prove that exists a control-limit policy, and then this control-limit policy is optimal.

The shock models studied by Dynamic programming approach have been given by some researchers. Siedersleben [60] considers a system that deteriorates continuously, but whose state is observed only at randomly times. He proves that a control-limit policy is optimal for the total expected discounted cost criterion. Waldmann [66] examines a shock model for discrete time and lattice state space, in which an randomly varying environment process and maintenance actions are introduced. But the maintenance action does not change the damage level of the system. He gives an optimal control-limit policy dependent on the environment process. The general theory of Dynamic programming can be found in Schäl [57].

Now we describe the model to be study in this Chapter. Consider a system existing in a random environment. The environment is described by a Markov jump process. The system is subject to a sequence of randomly occurring shocks and to failure, and each shock causes a random amount of damage which accumulates additively over time. The shock arrival and shock magnitude are influenced by changes of the environment. The damage process is assumed to be a piecewise semi-Markov process. The failure can occur only at times of shock arrival or jump of the environment process. The survival probability at a shock time or a jump time is determined by a known survival function of the accumulated damage level of the system, the environment state and the realized shock magnitude. Upon failure the system must be replaced by a new one having properties that are statistically equivalent to the original, and a cost is incurred. The replacement cycles are repeated indefinitely. The system may be maintained or replaced before failure at a smaller cost. The replacement time and maintenance time are assumed to be negligible.

The Chapter is organized as follows, in Section 4.2, the piecewise semi-Markov shock model is formulated, and in Section 4.3, properties of the total expected randomly discounted cost is derived. In Section 4.4, it is proved that a control-limit rule is an optimal replacement policy.

4.2 The Model and Preliminaries

Let $\{\xi(t)\}_{t\geq 0}$ be a stochastic process specifying the environment of the system. The process $\{\xi(t)\}_{t\geq 0}$ is assumed to be a stationary regular Markov jump process with the state space Γ and the initial state $\xi(0)$. Let $\Re$ be a σ-field of Γ such that one point set $\{\xi\} \in \Re$ and $\{\omega_n\}_{n\geq 0}$ ($\omega_0 = 0$) the jump points of $\{\xi(t)\}_{t\geq 0}$. The $Q(\xi, A)$ is a Markov kernel on $(\Gamma, \Re)$ with $Q(\xi, \{\xi\}) = 0$, i.e., $Q(\xi, \cdot)$ is a probability measure for every $\xi \in \Gamma$, and $Q(\cdot, A)$ is a $\Re$-measurable function for every $A \in \Re$. For any $A \in \Re$ and $t \in R_+$, let

$$P(\xi(\omega_{n+1}) \in A, \omega_{n+1} - \omega_n \leq t | \xi(s), s \leq \omega_n) = Q(\xi(\omega_n), A)(1 - e^{-\eta(\xi(\omega_n))t}) \quad (4.2.1)$$

where $\eta : \Gamma \rightarrow R_+$ is a finite function. $\{\xi(t)\}_{t\geq 0}$ is called *Markov environment process* (MEP).

For any $(\xi, z_0) \in \Gamma \times R_+$, let $\{Z_{(\xi,z_0)}(t)\}_{t\geq 0}$ be a semi-Markov process with the state space $E = R_+ \cup \{\infty\}$, and the initial state $Z_{(\xi,z_0)}(0) = z_0$. For any $x \in E, z \in R_+$, and $t \in R_+$, the semi-Markov kernel of $\{Z_{(\xi,z_0)}(t)\}_{t\geq 0}$ is defined as follows.

$$\begin{aligned} &P(Z_{(\xi,z_0)}(\tau^\xi_{n+1}) - Z_{(\xi,z_0)}(\tau^\xi_n) \leq x, \tau^\xi_{n+1} - \tau^\xi_n \leq t | Z_{(\xi,z_0)}(\tau^\xi_n) = z) \\ &= G^\xi_z(x) H^\xi(t) \end{aligned} \quad (4.4.2)$$

where $\{\tau^\xi_n\}_{n\geq 0}$ ($\tau^\xi_0 = 0$) are the jump points of the $\{Z_{(\xi,z_0)}(t)\}_{t\geq 0}$, $H^\xi(\cdot)$ is the probability distribution of the intershock time $\tau^\xi_{n+1} - \tau^\xi_n$ and $G^\xi_z(\cdot)$ is the conditional distribution of $Z_{(\xi,z_0)}(\tau^\xi_{n+1}) - Z_{(\xi,z_0)}(\tau^\xi_n)$ given $Z_{(\xi,z_0)}(\tau^\xi_n) = z$. We suppose that the $\{Z_{(\xi,z_0)}(t)\}_{t\geq 0}$ be a right-continuous regular process with left-hand limits.

By appealing to $\{Z_{(\xi,z_0)}\}_{t\geq 0}$, we define the stochastic process $\{Z(t)\}_{t\geq 0}$ specifying the cumulative damage of the system in one replacement cycle such that $Z(0) = 0$ and $Z(t) = Z_{(\xi, Z(\omega_n -)}(t - \omega)$ on $\{\omega_n \leq t < \omega_{n+1}; \xi(\omega) = \xi\}$. That is,

$$\begin{aligned} &Z(0) = 0 \\ &Z(t) = Z_{(\xi(0),0)}(t) I_{\{0<t<\omega_1\}} + \textstyle\sum_{n=1}^\infty Z_{(\xi(\omega_n), Z(\omega_n -))}(t - \omega_n) I_{\{\omega_n \leq t < \omega_{n+1}\}} \end{aligned} \quad (4.2.3)$$

According to the definitions of $\{Z_{(\xi,z_0)}(t)\}_{t\geq 0}$, $(\xi, Z_0) \in \Gamma \times R_+$, we know that the process $\{Z(t)\}_{t\geq 0}$ is also a right-continuous regular process with left-hand limits . At the points $\omega_n, n \geq 1$, $Z(\omega_n) = Z(\omega_n-)$, and on the interval $[\omega_n, \omega_{n+1})$, the process $\{Z(t)\}_{t\geq 0}$ is a semi-Markov process which depends on the environment state $\xi(\omega_n)$. We call $\{Z(t)\}_{t\geq 0}$ *piecewise semi-Markov process*(PSMP). The state space of $\{Z(t)\}_{t\geq 0}$ is $E = [0, \infty]$. In this Chapter, we use ∞ to indicate the failure state instead of Δ.

A failure of the system can occur only at the time points of shock or jump of the MEP. Let T is such a time point, suppose $\xi(T-) = \xi$ and $Z(T-) = z$. At time T, if a shock of magnitude x occurs, then the system fails with known probability $1 - \gamma(z, \xi, x)$, and if a jump of the MEP into the state ζ occurs, then the system fails with known probability $1 - \gamma(z, \zeta, 0)$. The function $\gamma : R_+ \times \Gamma \times E \to [0, 1]$ is the survival function. Let $\delta = \inf\{t, \quad Z(t) = \infty\}$, then δ is the first failure time of the system. Throughout we assume that $E[\delta] < \infty$.

Let $\mathbf{A}$ be a set of maintenance-replacement decisions such as the following.

$$\begin{aligned}\mathbf{A} \equiv \{A(\cdot,\cdot) = (a(\cdot,\cdot), i(\cdot,\cdot)) \ \mid a(\cdot,\cdot) : \Gamma \times E \to [0,\infty], i(\cdot,\cdot) : \Gamma \times E \to \{0,1\} \\ \text{are } \Re \times \Im\text{-measurable and } a(\xi,\infty) = 0, i(\xi,\infty) = 1 \text{ for any } \xi \in \Gamma\}.\end{aligned}$$

Definition 4.2.1. A decision policy $A = (a(\xi, z), i(\xi, z)) \in \mathbf{A}$ is called a control-limit policy if for every possible state $\xi \in \Gamma$, there is a real-number $f(\xi)$ such that

$$A(\xi, z) = \begin{cases} (a(\xi, z), 0) & \text{if } z < f(\xi) \\ (a(\xi, z), 1) & \text{otherwise.} \end{cases} \qquad (4.2.4)$$

The function $f(\cdot)$ is called a control-limit.

An infinite stage maintenance-replacement policy is a sequence $\pi = (A_0, A_1, \cdots)$; $A_i \in \mathbf{A}$. If $A_i = A$ for all $i = 0, 1, \cdots$, we call π a stationary policy. Let Π be the set of all policies such as π. Thereby for any $\pi \in \Pi$, we can obtain a decision process $\{Z^\pi(t)\}_{t\geq 0}$ which describes the accumulated damage level of the system at time t under the policy π.

Suppose at a decision time T, $\xi(T) = \xi, Z(T)^\pi = z$. For $a \in [0, \infty)$, the decision

$A(\xi, z) = (a, 0)$ means that we maintain the system at time $T + a$ if no other decision points occur in the interval $(T, T + a]$ and incur a cost $m(\xi, z)$, and the decision $A(\xi, z) = (a, 1)$ means that we replace the system at time $T + a$ if no other decision points occur in the interval $(T, T + a]$ and incur a cost $c(\xi, z)$. For $a = \infty$, the decisions $A(\xi, z) = (\infty, i)$ $(i = 1, 2)$ means that we neither maintain nor replace the system at any time, but wait for the next decision time. After execution of an action (maintenance or replacement), the behaviors of the damage process are influenced as follows.

1. At time $T + a$, if a maintenance action is taken, the damage level z prior to $T+a$ has an randomly decreasing amount $Y(\xi, z)$ with the distribution function $F_z^\xi(y)$. Here we assume

   i $F_z^\xi(z) = P(Y(\xi, z) \le z) = 1 \qquad \text{for } z \in [0, \infty).$

   ii $F_z^\xi(y)$ is stochastically decreasing in z.

2. The environment state ξ does not change whatever action is taken. From time $T + a$, the damage process evolves still as a PSMP with the initial environment state ξ and damage level $z - Y(\xi, z)$ (maintenance) or 0 (replacement).

The set of the decision points is $\{T_n\}_{n \ge 0}$ which are the successive jump points of the two-dimensional process $\{\xi(t), Z^\pi(t)\}_{t \ge 0}$ defined as follows

$$T_0 = 0 \tag{4.2.5}$$

$$T_{n+1} = \inf\{t \ge T_n; \quad \xi(t) \ne \xi(T_n) \text{ or } Z^\pi(t) \ne Z^\pi(T_n)\} \qquad \text{for } n \ge 0.$$

Since any maintenance action and replacement action change the damage level, we can see that $\{T_n\}_{n \ge 0}$ contains three-type points (a) shock points, (b) jump points of the MEP, and (c) action points (i.e. at which an action is executed). At point T_n, if we immediately take an action(i.e. $a = 0$), then $T_{n+1} = T_n$. Let

$$\begin{cases} Z_n = Z^\pi(T_n) \\ \xi_n = \xi(T_n) \end{cases} \qquad \text{for } n \ge 0. \tag{4.2.6}$$

For the Markov-decision process $\{\xi_n, Z_n, T_n, A_n\}_{n \ge 0}$, we have

Proposition 4.2.2. At $T_n < \delta$, $\xi_n = \xi$, $Z_n = z$, if $a(\xi, z) = a$, then

(a) $\Phi_1(\xi,a) \equiv P(T_{n+1}$ is a shock point $|\xi_n = \xi, Z_n = z, a(\xi,z) = a)$

$= \int_0^a H^\xi(t)\eta(\xi)e^{-\eta(\xi)t}dt + H^\xi(a)e^{-\eta(\xi)a}.$

(b) $\Phi_2(\xi,a) \equiv P(T_{n+1}$ is a jump point of the MEP $|\xi_n = \xi, Z_n = z, a(\xi,z) = a)$

$= \int_0^a(1 - e^{-\eta(\xi)t})H^\xi(dt) + (1 - e^{-\eta(\xi)a})\bar{H}^\xi(a).$

(c) $\Phi_3(\xi,a) \equiv P(T_{n+1}$ is a action point $|\xi_n = \xi, Z_n = z, a(\xi,z) = a)$

$= \bar{H}^\xi(a)e^{-\eta(\xi)a}$

where $\bar{H}^\xi(a) = 1 - H^\xi(a)$.

Proof is same as that in Proposition 3.2.1. □

Let $B \equiv \{V;\ \Gamma \times E \to R | V$ is bounded and $\Re \times \Im$-measurable$\}$, $B^+ \equiv \{V;\ V \in B | V(\xi,z)$ is increasing in z for any $\xi \in \Gamma\}$, and $\|\cdot\|$ the sup-norm defined on B. Hence B is a Banach space.

In this model, we consider an randomly discounted cost case. The discounted rate is a function of the MEP and is denoted by $\lambda(\xi(t))$. The discounted factor at T_n is $e^{-\Lambda(T_n)}$ where

$$\Lambda(T_n) \equiv \sum_{j=1}^{n-1}(\lambda(\xi_{j-1}) - \lambda(\xi_j))T_j + \lambda(\xi_{n-1})T_n \tag{4.2.7}$$

and the discounted cost incurred at T_n is as follows

$$K_n = \begin{cases} e^{-\Lambda(T_n)}k(\xi_{n-1}, Z_{n-1}, A_{n-1}) & \text{if } T_n \text{ is a action point} \\ 0 & \text{otherwise} \end{cases} \tag{4.2.8}$$

where

$$k(\xi,z,A) = \begin{cases} m(\xi,z) & \text{if } i(\xi,z) = 0 \\ c(\xi,z) & \text{otherwise.} \end{cases} \tag{4.2.9}$$

Note that although the right-hand of (2.6) is the function of $\xi_0, \xi_1, \ldots, \xi_{n-1}$; $T_0, T_1, \ldots, T_n$, it is denoted by $\Lambda(T_n)$ for the notational simplicity.

The total expected randomly discounted cost incurred under π, starting at time 0 in state (ξ,z) is given by

$$V_\pi(\xi,z) \equiv E\{\sum_{n=1}^{\infty} K_n | \xi_0 = \xi, Z_0 = z\}. \tag{4.2.10}$$

Let

$$V^* \equiv \inf_{\pi \in \Pi} V_\pi. \tag{4.2.11}$$

Definition 4.2.3. If $\pi \in \Pi$ and $V_\pi = V^*$, then π is called optimal.

Assumption 4.2.4.

(a) $\gamma(z, \xi, x)$ is decreasing in z and x for any $\xi \in \Gamma$.

(b) The cost function m and c are in B^+, $m(\xi, z) > 0, c(\xi, z) > 0$ and $m(\xi, \infty) = c(\xi, \infty)$ for any $\xi \in \Gamma$.

(c) $H^\xi(t)$ has a continuous density function $h^\xi(t)$ for any $\xi \in \Gamma$.

(d) $\lambda \equiv \inf_{\xi \in \Gamma} \lambda(\xi) > 0$.

4.3 The total expected randomly discounted cost

In this section, we discuss the total expected randomly discounted cost over infinite horizon. First by the proposition 4.2.2, we get the following lemma.

Lemma 4.3.1. For any $\xi \in \Gamma, z \in R_+$, $a \in [0,\infty]$ and $V \in B$

$$E_{(\xi,z)}[e^{-\lambda(\xi)T_1}V(\xi_1,Z_1)|A_0(\xi,z)]$$
$$= \begin{cases} [V(\xi,\infty) - L_1V(\xi,z)]\Psi_1(\xi,a) + [E_\xi[V(\xi_1,\infty)] - L_2V(\xi,z)]\Psi_2(\xi,a) & \\ \quad + \int_0^z V(\xi,z-y)F_z^\xi(dy)e^{-\lambda(\xi)a}\Phi_3(\xi,a) & \text{if } A_0(\xi,z)=(a,0) \\ [V(\xi,\infty) - L_1V(\xi,z)]\Psi_1(\xi,a) + [E_\xi[V(\xi_1,\infty)] - L_2V(\xi,z)]\Psi_2(\xi,a) & \\ \quad +V(\xi,0)e^{-\lambda(\xi)a}\Phi_3(\xi,a) & \text{otherwise} \end{cases}$$

where
$$L_1V(\xi,z) \equiv \int_{R_+}[V(\xi,\infty) - V(\xi,z+x)]\gamma(z,\xi,x)G_z^\xi(dx)$$
$$L_2V(\xi,z) \equiv \int_\Gamma[V(\zeta,\infty) - V(\zeta,z)]\gamma(z,\zeta,0)Q(\xi,d\zeta)$$
$$E_\xi[V(\xi_1,\infty)] \equiv \int_\Gamma V(\zeta,\infty)Q(\xi,d\zeta)$$
$$\Psi_1(\xi,a) \equiv \int_0^a\int_0^t e^{-\lambda(\xi)u}H^\xi(du)\eta(\xi)e^{-\eta(\xi)t}dt + \int_0^a e^{-\lambda(\xi)u}H^\xi(du)e^{-\eta(\xi)a}$$
$$\Psi_2(\xi,a) \equiv \int_0^a\int_0^t \eta(\xi)e^{-(\lambda(\xi)+\eta(\xi))u}duH^\xi(dt) + \int_0^a \eta(\xi)e^{-(\lambda(\xi)+\eta(\xi))u}du\bar{H}^\xi(a).$$

Proof. For the case that $A_0(\xi,z) = (a,0)$, considering whether or not the sojourn time in the state (ξ,z) exceeds a and using S_1, S_2 defined in Proposition 2.2.1, we have

$$E_{(\xi,z)}[e^{-\lambda(\xi)T_1}V(\xi_1,Z_1)|A_0(\xi,z)=(a,0)]$$
$$= E_{(\xi,z)}[e^{-\lambda(\xi)T_1}V(\xi_1,Z_1); S_1 \le S_2; S_1 \le s|A_0(\xi,z)=(a,0)]$$
$$+E_{(\xi,z)}[e^{-\lambda(\xi)T_1}V(\xi_1,Z_1); S_2 \le S_1; S_2 \le s|A_0(\xi,z)=(a,0)]$$
$$+E_{(\xi,z)}[e^{-\lambda(\xi)T_1}V(\xi_1,Z_1); \min\{S_1,S_2\} \ge a|A_0(\xi,z)=(a,0)]$$
$$= \int_0^a\int_0^t e^{-\lambda(\xi)u}\int_{R_+}[V(\xi,z+x)\gamma(z,\xi,x) + V(\xi,\infty)(1-\gamma(z,\xi,x))]$$
$$\times G_z^\xi(dx)H^\xi(du)\eta(\xi)e^{-\eta t}dt$$
$$+\int_0^a\int_0^t e^{-\lambda(\xi)u}\int_\Gamma[V(\zeta,z)\gamma(z,\zeta,0) + V(\zeta,\infty)(1-\gamma(z,\zeta,0))$$
$$\times Q(\xi,d\zeta)\eta(\xi)e^{-\eta(\xi)u}duH^\xi(dt)$$
$$+\int_0^z V(\xi,z-y)F_z^\xi(dy)e^{-\lambda(\xi)a}\Phi_3(\xi,a)$$

By rearranging the right-hand of the above equality, we can obtain lemma 4.3.1 when $A_0(\xi, z) = (a, 0)$. The case that $A_0(\xi, z) = (a, 1)$ can be proved by similar manner.□

Now, we define the following operators U_1,U_2, and U.

Definition 4.3.2. For any $V \in B, \xi \in \Gamma$, $z \in [0, \infty)$ and $a \in [0, \infty]$, let

$$\begin{aligned} U_1(a)V(\xi, z) &\equiv U_1(\xi, z, a, V) \\ &\equiv (m(\xi, z) + \textstyle\int_0^z V(\xi, z - y)F_z^\xi(dy))e^{-\lambda(\xi)a}\Phi_3(\xi, a) \\ &\qquad +[V(\xi, \infty) - L_1V(\xi, z)]\Psi_1(\xi, a) \\ &\qquad +[E_\xi[V(\xi_1, \infty)] - L_2V(\xi, z)]\Psi_2(\xi, a) \end{aligned} \tag{4.3.1}$$

$$\begin{aligned} U_2(a)V(\xi, z) &\equiv U_2(\xi, z, a, V) \\ &\equiv (c(\xi, z) + V(\xi, 0))e^{-\lambda(\xi)a}\Phi_3(\xi, a) \\ &\qquad +[V(\xi, \infty) - L_1V(\xi, z)]\Psi_1(\xi, a) \\ &\qquad +[E_\xi[V(\xi_1, \infty)] - L_2V(\xi, z)]\Psi_2(\xi, a) \end{aligned} \tag{4.3.2}$$

$$UV(\xi, z) = \min\{\inf_{a \in [0,\infty]} U_1(a)V(\xi, z), \inf_{a \in [0,\infty]} U_2(a)V(\xi, z)\} \tag{4.3.3}$$

$$UV(\xi, \infty) = U_2(0)V(\xi, \infty). \tag{4.3.4}$$

In the following, we first consider restricted operator U^ϵ on the restricted action space $R_\epsilon = [\epsilon, \infty]$ for any $\epsilon > 0$, i.e. for $V \in B$

$$U^\epsilon V(\xi, z) = \min\{\inf_{a \in R_\epsilon} U_1(a)V(\xi, z), \inf_{a \in R_\epsilon} U_2(a)V(\xi, z)\}. \tag{4.3.5}$$

Lemma 4.3.3. For fixed $\epsilon > 0$, U^ϵ is a monotone contraction operator .

Proof. From lemma 3.1 and definition 3.3, the monotone property of U^ϵ is obvious. Next we prove the contraction property of U^ϵ. For any ξ, z and V, $U_i(a)V(\xi, z)(i = 1, 2)$ are bounded continuous function in a on $[\epsilon, \infty]$. Hence for $V, W \in B$, there exist $a_1^*(\xi, z), a_2^*(\xi, z) \in R_\epsilon$ satisfying the following equalities

$$\inf_{a \in R_\epsilon} U_1(a)V(\xi, z) = U_1(a_1^*(\xi, z))V(\xi, z)$$

$$\inf_{a\in R_\epsilon} U_1(a)W(\xi,z) = U_1(a_2^*(\xi,z))W(\xi,z)$$

Thus $\quad | \inf_{a\in R_\epsilon} U_1(a)V(\xi,z) - \inf_{a\in R_\epsilon} U_1 W(\xi,z) |$

$$=| U_1(a_1^*(\xi,z))V(\xi,z) - U_1(a_2^*(\xi,z))W(\xi,z) |$$

$$\leq| U_1(a_2^*(\xi,z))V(\xi,z) - U_1(a_2^*(\xi,z))W(\xi,z) |$$

$$\leq (\Psi_1(\xi,a_2^*(\xi,z)) + \Psi_2(\xi,a_2^*(\xi,z)) + e^{-\lambda(\xi)a_2^*(\xi,z)}\Phi_3(\xi,a_2^*(\xi,z))) \| V - W \|$$

$$\leq (\Phi_1(\xi,a_2^*(\xi,z)) + \Phi_2(\xi,a_2^*(\xi,z)) + e^{-\lambda(\xi)a_2^*(\xi,z)}\Phi_3(\xi,a_2^*(\xi,z))) \| V - W \|$$

$$\equiv \beta^\xi(a_2^*(\xi,z)) \| V - W \|$$

where

$$\beta^\xi(a_2^*(\xi,z)) = \Phi_1(\xi,a_2^*(\xi,z)) + \Phi_2(\xi,a_2^*(\xi,z)) + e^{-\lambda(\xi)a_2^*(\xi,z)}\Phi_3(\xi,a_2^*(\xi,z)).$$

Since for any $\xi \in \Gamma$ and $z \in R_+, \Phi_1(\xi,a_2^*(\xi,z)) + \Phi_2(\xi,a_2^*(\xi,z)) + \Phi_3(\xi,a_2^*(\xi,z)) = 1$, and for $a_2^*(\xi,z) \geq \epsilon$, we have $\sup_{\xi,z}\Phi_i(\xi,a_2^*(\xi,z)) \neq 0 (i=1,2)$, $\sup_{\xi,z}\Phi_3(\xi,a_2^*(\xi,z)) \neq 1$, and by Assumption 4.2.4 (d), $\sup_{\xi,z} e^{-\lambda(\xi)a_2^*(\xi,z)} < 1$. Thereby $\beta_\epsilon \equiv \sup_{\xi,z}\beta^\xi(a_2^*(\xi,z)) < 1$. We have

$$\| \inf_{a\in R_\epsilon} U_1(a)V - \inf_{a\in R_\epsilon} U_1(a)W \| \leq \beta_\epsilon \| V - W \| .$$

Similarly , there exists a $\acute{\beta}_\epsilon < 1$ such that

$$\| \inf_{a\in R_\epsilon} U_2(a)V - \inf_{a\in R_\epsilon} U_2(a)W \| \leq \acute{\beta}_\epsilon \| V - W \| .$$

Since

$$| U^\epsilon V(\xi,z) - U^\epsilon W(\xi,z) | \leq \max_{i=1,2}\{| \inf_{a\in R_\epsilon} U_i(a)V(\xi,z) - \inf_{a\in R_\epsilon} U_i(a)W(\xi,z) |\},$$

it follows that $\| UV - UW \| \leq \max\{\beta_\epsilon, \acute{\beta}_\epsilon\} \| V - W \|$. □

As U^ϵ is a monotone contraction operator, it has a unique fixed point $V_\epsilon^{**} \in B$. Now we discuss the properties of this fixed point . Using the operator U^ϵ, we define a mapping sequence $\{V_n\}_{n\geq 0}$ as follows

$$V_0 = 0 \tag{4.3.6}$$

$$V_n = U^\epsilon V_{n-1} \quad n \geq 1.$$

We have $V_n \in B$ and V_n is non-negative function for $n \geq 0$.

Lemma 4.3.4. Assume that for any $\xi \in \Gamma, t \in R_+, G_z^\xi(\cdot)$ is stochastically increasing in z, then

(i) $V_n \in B^+$.

(ii)$L_1 V_n(\xi, z)$ and $L_2 V_n(\xi, z)$ are decreasing in z.

Proof. By induction, we prove the assertions (i) and (ii). Since $V_0 = 0$ and

$$V_1(\xi, z) = \min\{\inf_{a \in R_\epsilon} m(\xi, z) e^{-\lambda(\xi)a} \Phi_3(\xi, a), \inf_{a \in R_\epsilon} c(\xi, z) e^{-\lambda(\xi)a} \Phi_3(\xi, a)\},$$

(i) and (ii) hold certainly when $n = 0, 1$. Suppose that (i) and (ii) are true for an integer n. Consider two cases for $z_1 \le z_2$:

CASE1: if $U^\epsilon V_n(\xi, z_2) = \inf_{a \in R_\epsilon} U_1(a) V_n(\xi, z_2)$,

$$\begin{aligned}
&V_{n+1}(\xi, z_2) - V_{n+1}(\xi, z_1) = \inf_{a \in R_\epsilon} U_1(a) V_n(\xi, z_2) - U^\epsilon V_n(\xi, z_1) \\
&\ge \inf_{a \in R_\epsilon} U_1(a) V_n(\xi, z_2) - \inf_{a \in R_\epsilon} U_1(a) V_n(\xi, z_1) \\
&\ge \inf_{a \in R_\epsilon} \{U_1(a) V_n(\xi, z_2) - U_1(a) V_n(\xi, z_2)\} \\
&= \inf_{a \in R_\epsilon} \{[m(\xi, z_2) - m(\xi, z_1) + \textstyle\int_0^{z_2} V_n(\xi, z_2 - y) F_{z_2}^\xi(dy)) - \int_0^{z_1} V_n(\xi, z_1 - y) F_{z_1}^\xi(dy))] \\
&\qquad \times e^{-\lambda(\xi)a} \Phi_3(\xi, a) + [L_1 V_n(\xi, z_1) - L_1 V_n(\xi, z_2)] \Psi_1(\xi, a) \\
&\qquad + [L_2 V_n(\xi, z_1) - L_2 V_n(\xi, z_2)] \Psi_2(\xi, a)\} \\
&\ge \inf_{a \in R_\epsilon} \{ \ [m(\xi, z_2) - m(\xi, z_1)] e^{-\lambda(\xi)a} \Phi_3(\xi, a) \ \} \ge 0,
\end{aligned}$$

where the third inequality follows from

$$L_i V_n(\xi, z_1) - L_i V_n(\xi, z_2) \ge 0 \qquad i = 1, 2,$$

and $F_z^\xi(\cdot)$ is stochastically decreasing in z, i.e., since $V_n(\xi, z_2 - y) \ge V_n(\xi, z_1 - y)$, we have

$$\begin{aligned}
&\textstyle\int_0^{z_2} V(\xi, z_2 - y) F_{z_2}^\xi(dy) - \int_0^{z_1} V(\xi, z_1 - y) F_{z_1}^\xi(dy) \\
&\ge \textstyle\int_0^{z_2} V(\xi, z_2 - y) F_{z_2}^\xi(dy)) - \int_0^{z_2} V(\xi, z_2 - y) F_{z_1}^\xi(dy)) \ge 0.
\end{aligned}$$

CASE2: if $U^\epsilon V_n(\xi, z_2) = \inf_{a \in R_\epsilon} U_2(a) V_n(\xi, z_2)$, similarly

$$\begin{aligned}
&V_{n+1}(\xi, z_2) - V_{n+1}(\xi, z_1) \\
&\ge \inf_{a \in R_\epsilon} \{ \ [c(\xi, z_2) - c(\xi, z_1)] e^{-\lambda(\xi)a} \Phi_3(\xi, a) \ \} \ge 0.
\end{aligned}$$

By Assumption 4.2.4 (a) and (i), $(V_{n+1}(\xi, \infty) - V_{n+1}(\xi, z + x))\gamma(z, \xi, x)$ is decreasing in z and since $G_z^\xi(\cdot)$ is stochastically increasing in z, $L_1 V_{n+1}(\xi, z)$ and $L_2 V_{n+1}(\xi, z)$ are decreasing in z. (i) and (ii) hold for $n + 1$. These complete the proof. □

Corollary 4.3.5. For any fixed $\epsilon > 0$,

(a) $V_\epsilon^{**} = \lim_{n\to\infty} V_n \in B^+$.

(b) $L_1 V^{**}(\xi, z)$ and $L_2 V^{**}(\xi, z)$ are decreasing in z for any $\xi \in \Gamma$.

Next we introduce operator $\tilde{U}$ defined as follows, for $V \in B$

$$\tilde{U}V(\xi, z) = \min\{\inf_{a\in(0,\infty]} U_1(a)V(\xi, z), \inf_{a\in(0,\infty]} U_2(a)V(\xi, z)\} \tag{4.3.7}$$

Lemma 4.3.6. There exists a unique non-negative function $V^{**} \in B^+$ satisfying

$$\tilde{U}V^{**}(\xi, z) = V^{**}(\xi, z).$$

Proof. For $\xi \in \Gamma$, $V_\epsilon^{**}(\xi, z)$ is a non-negative decreasing function in ϵ. Let

$$V^{**}(\xi, z) = \lim_{\epsilon\to 0} V_\epsilon^{**} = \lim_{\epsilon\to 0} \min\{\inf_{a\in R_\epsilon} U_1(\xi, z, a, V), \inf_{a\in R_\epsilon} U_1(\xi, z, a, V)\}. \tag{4.3.8}$$

Then V^{**} is a uniquely determined non-negative function, and by Corollary 4.3.5 (a), $V^{**} \in B^+$. From the monotonicities of U_1, U_2, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0} \min\{\inf_{a\in R_\epsilon} U_1(\xi, z, a, V_\epsilon^{**}), \inf_{a\in R_\epsilon} U_1(\xi, z, a, V_\epsilon^{**})\} \\
&\geq \lim_{\epsilon\to 0} \min\{\inf_{a\in R_\epsilon} U_1(\xi, z, a, V^{**}), \inf_{a\in R_\epsilon} U_1(\xi, z, a, V^{**})\} \\
&\geq \min\{\lim_{\epsilon\to 0} \inf_{a\in R_\epsilon} U_1(\xi, z, a, V^{**}), \lim_{\epsilon\to 0} \inf_{a\in R_\epsilon} U_1(\xi, z, a, V^{**})\} \\
&= \min\{\inf_{a\in(0,\infty]} U_1(\xi, z, a, V^{**}), \inf_{a\in(0,\infty]} U_2(\xi, z, a, V^{**})\}.
\end{aligned} \tag{4.3.9}$$

First we consider the case that $\tilde{U}V^{**}(\xi, z) = \inf_{a\in(0,\infty]} U_1(\xi, z, a, V^{**})$. For any $\sigma > 0$, there exists a a_0 satisfying

$$\inf_{a\in(0,\infty]} U_1(\xi, z, a, V) > U_1(\xi, z, a_0, V) - \sigma. \tag{4.3.10}$$

Also by the monotone convergence theorem, we have

$$\lim_{\epsilon\to 0} U_1(\xi, z, a_0, V_\epsilon^{**}) = U_1(\xi, z, a_0, V^{**}), \tag{4.3.11}$$

and for $\epsilon < a_0$, $a_0 \in R_\epsilon$, thus

$$\lim_{\epsilon\to 0} \inf_{a\in R_\epsilon} U_1(\xi, z, a, V_\epsilon^{**}) \leq U_1(\xi, z, a_0, V^{**}). \tag{4.3.12}$$

By (4.3.10), we have

$$\inf_{a\in(0,\infty]} U_1(\xi,z,a,V^{**}) > \lim_{\epsilon\to 0}\inf_{a\in R_\epsilon} U_1(\xi,z,a,V_\epsilon^{**}) - \sigma.$$

As $\sigma \to 0$, it holds that

$$\inf_{a\in(0,\infty]} U_1(\xi,z,a,V^{**}) \geq \lim_{\epsilon\to 0}\inf_{a\in R_\epsilon} U_1(\xi,z,a,V_\epsilon^{**}) \tag{4.3.13}$$

$$\geq \lim_{\epsilon\to 0}\min\{\inf_{a\in R_\epsilon} U_1(\xi,z,a,V_\epsilon^{**}), \inf_{a\in R_\epsilon} U_2(\xi,z,a,V_\epsilon^{**})\}.$$

From (4.3.9) and (4.3.13), we have

$$\lim_{\epsilon\to 0}\min\{\inf_{a\in R_\epsilon} U_1(\xi,z,a,V_\epsilon^{**}), \inf_{a\in R_\epsilon} U_2(\xi,z,a,V_\epsilon^{**})\} = \inf_{a\in(0,\infty]} U_1(\xi,z,a,V^{**}).$$

That is

$$V^{**}(\xi,z) = \lim_{\epsilon\to 0} V_\epsilon^{**}(\xi,z) = \lim_{\epsilon\to 0} U^\epsilon V_\epsilon^{**}(\xi,z) = UV^{**}(\xi,z).$$

When $\tilde{U}V^{**}(\xi,z) = \inf_{a\in(0,\infty]} U_2(\xi,z,a,V^{**})$, the proof is similar. □

Now we consider the operator U defined in (4.3.3). Similar to the proof of Lemma 4.3.6, we can prove that the operator U has a unique fixed point.

Lemma 4.3.7. There exists a unique fixed point to operator U.

Proof. For $i = 1,2$,

$$\inf_{a\in[0,\infty]} U_i(\xi,z,a,V) = \min\{U_i(\xi,z,0,V), \inf_{a\in(0,\infty]} U_i(\xi,z,a,V)\}, \tag{4.3.14}$$

and $U_i(\xi,z,a,V)$ is left-hand continuous at $a=0$, i.e.,$\lim_{a\downarrow 0} U_i(\xi,z,a,V) = U(\xi,z,0,V)$. By using these relations, similar Lemma 4.3.6, we obtain the conclusion. □

In the following, we still use V^{**} to denote the fixed point of operator U for simplification of notation.

For any $\xi \in \Gamma$, let

$$\alpha(\xi) = \inf\{z, \quad m(\xi,z) - c(\xi,z) \geq 0\} \tag{4.3.15}$$

and $\inf\{\emptyset\} = \infty$.

Theorem 4.3.8. Assume that $m(\xi,z)-c(\xi,z)$ is increasing in z for $z\in[0,\alpha(\xi))$. Then there exists a function $f(\xi)$ satisfying

(i) $f(\xi)\le\alpha(\xi)$ for $\xi\in\Gamma$.

(ii)

$$UV^{**}(\xi,z)=\begin{cases}\inf_{a\in[0,\infty]}U_1(a)V^{**}(\xi,z) & \text{if } z<f(\xi)\\ \inf_{a\in[0,\infty]}U_2(a)V^{**}(\xi,z) & \text{otherwise.}\end{cases}$$

Proof. For any fixed $\xi\in\Gamma$, let

$$f(\xi)=\inf\{z,\ \ m(\xi,z)-c(\xi,z)+\textstyle\int_0^z V^{**}(\xi,z-y)F_z^{\xi}(dy)-V(\xi,0)\ge 0\} \tag{4.3.16}$$

and $\inf\{\emptyset\}=\infty$.

(i) Since $V^{**}(\xi,z)$ is increasing in z, and $F_z^{\xi}(\cdot)$ is stochastically decreasing in z. We have $\int_0^z V^{**}(\xi,z-y)F_z^{\xi}(dy)$ is increasing in z and

$$\textstyle\int_0^z V^{**}(\xi,z-y)F_z^{\xi}(dy)\ge\int_0^z V^{**}(\xi,0)F_z^{\xi}(dy)=V^{**}(\xi,0).$$

Hence $m(\xi,z)-c(\xi,z)+\int_0^z V^{**}(\xi,z-y)F_z^{\xi}(dy)-V(\xi,0)\ge m(\xi,z)-c(\xi,z)$.

The result (i) is follows.

(ii) For $z<f(\xi)$, we have $c(\xi,z)-m(\xi,z)+V^{**}(\xi,0)-\int_0^z V^{**}(\xi,z-y)F_z^{\xi}(dy)\ge 0$.

$$\begin{aligned}&\inf_{a\in[0,\infty]}U_2(a)V^{**}(\xi,z)-\inf_{a\in(0,\infty]}U_1(a)V^{**}(\xi,z)\\ &\ge\inf_{a\in[0,\infty]}\{U_2(a)V^{**}(\xi,z)-U_1(a)V^{**}(\xi,z)\}\\ &=\min\{c(\xi,z)-m(\xi,z)+V^{**}(\xi,0)-\textstyle\int_0^z V^{**}(\xi,z-y)F_z^{\xi}(dy),\\ &\qquad\inf_{a\in(0,\infty]}\{(c(\xi,z)-m(\xi,z)+V^{**}(\xi,0)\\ &\qquad\quad-\textstyle\int_0^z V^{**}(\xi,z-y)F_z^{\xi}(dy))e^{-\lambda(\xi)a}\Phi_3(\xi,a)\}\}\\ &\ge 0.\end{aligned}$$

Thus $\inf_{a\in[0,\infty]}U_2(a)V^{**}(\xi,z)\ge\inf_{a\in[0,\infty]}U_1(a)V^{**}(\xi,z)$. For $z\ge f(\xi)$, we have

$$\begin{aligned}&\inf_{a\in[0,\infty]}U_1(a)V^{**}(\xi,z)-\inf_{a\in[0,\infty]}U_2(a)V^{**}(\xi,z)\\ &\ge\min\{m(\xi,z)-c(\xi,z)+\textstyle\int_0^z V^{**}(\xi,z-y)F_z^{\xi}(dy)-V^{**}(\xi,0),\\ &\qquad\inf_{a\in(0,\infty]}\{(m(\xi,z)-c(\xi,z)+\textstyle\int_0^z V^{**}(\xi,z-y)F_z^{\xi}(dy)\\ &\qquad\quad-V^{**}(\xi,0))e^{-\lambda(\xi)a}\Phi_3(\xi,a)\}\\ &\ge 0.\end{aligned}$$

Thus $\inf_{a\in[0,\infty]}U_1(a)V^{**}(\xi,z)\ge\inf_{a\in[0,\infty]}U_2(a)V^{**}(\xi,z)$. These complete the proof of result (ii). □

Theorem 4.3.9. For $\xi \in \Gamma$, let $r^{\xi}(t)$ be the hazard rate associated the distribution function $H^{\xi}(t)$. If $r^{\xi}(t)$ is a monotone function, then there exists a unique minimal solution $a^* = a^*(\xi, z)$ satisfying the following equality

$$UV^{**}(\xi, z) = \min\{ \inf_{a\in[0,\infty]} U_1(a)V^{**}(\xi, z), \inf_{a\in[0,\infty]} U_2(a)V^{**}(\xi, z) \}, \tag{4.3.17}$$

and

$$a^*(\xi, z) = \begin{cases} a_1^*(\xi, z) & \text{if } z < f(\xi) \\ a_2^*(\xi, z) & \text{otherwise,} \end{cases} \tag{4.3.18}$$

where $a_1^*(\xi, z), a_2^*(\xi, z) \in [0, \infty]$ are the minimal solutions of the following equations (4.3.19) and (4.3.20) respectively.

$$M(\xi, z, a, V^{**}) = (m(\xi, z) + \textstyle\int_0^z V^{**}(\xi, z-y)F_z^{\xi}(dy))(r^{\xi}(a) + \lambda(\xi) + \eta(\xi)), \tag{4.3.19}$$

$$M(\xi, z, a, V^{**}) = (c(\xi, z) + V^{**}(\xi, 0))(r^{\xi}(a) + \lambda(\xi) + \eta(\xi)) \tag{4.3.20}$$

where

$$\begin{aligned} M(\xi, z, a, V^{**}) = {} & [V^{**}(\xi, \infty) - L_1V^{**}(\xi, z)]r^{\xi}(a) \\ & +[E_{\xi}[V^{**}(\xi_1, \infty)] - L_2V^{**}(\xi, z)]\eta(\xi). \end{aligned} \tag{4.3.21}$$

Proof. From Theorem 4.3.6, we have $V^{**}(\xi, z) = \inf_{a\in[0,\infty]} U_1(a)V^{**}(\xi, z)$ for $z < f(\xi)$. Differentiating with respect to a and rearranging, we obtain (4.3.19). By the monotonicity of $r_z^{\xi}(a)$, this minimal solution is unique. The proof of the case for $z \geq f(\xi)$ is similar. □

Corollary 4.3.10. If $c(\xi, z) = c(\xi)$ for $z \in R_+$, then $a_2^*(\xi, z)$ is decreasing in z.

Proof. From Corollary 4.3.5 (b), $V^{**}(\xi, \infty) - L_1V^{**}(\xi, z)$ and $E_{\xi}[V^{**}(\xi_1, \infty)] - L_2V^{**}(\xi, z)$ are increasing in z. We get $M(\xi, z, a, V^{**})$ is increasing in z. On the other hand, if $c(\xi, z) = c(\xi)$, the right-hand of (4.3.20) becomes $(c(\xi) + V^{**}(\xi, 0))(r^{\xi}(a) + \lambda(\xi) + \eta(\xi))$ which is not dependent on z. Hence, the minimal solution $a_2^*(\xi, z)$ satisfying the equation (4.3.20) is decreasing in z. □

In the following, we examine the influences of the maintenance action and the environment state on the control-limit $f(\xi)$ defined in (4.3.16).

At state (ξ, z), if a maintenance action is executed, the distribution function of the decreasing magnitude of the damage is $F_z^\xi(\cdot)$. The two extreme cases of the $F_z^\xi(\cdot)$ are (a) $P(Y(\xi,z) = z) = 1$, (b) $P(Y(\xi,z) = 0) = 1$. The case (a) means that every maintenance action restores the system to a new one (i.e. $z = 0$), and (b) that maintenance actions have no influences on the damage level. For (a), we have $\int_0^z V^{**}(\xi, z-y)F_z^\xi(dy) = V^{**}(\xi, 0)$ and $f(\xi) = \alpha(\xi)$. (in this case, if $m(\xi,z) < c(\xi,z)$ for all z, then $f(\xi) = \infty$, i.e. it is always optimal to maintain the system. if $m(\xi,z) \geq c(\xi,z)$ for all z, then $f(\xi) = 0$, i.e. it is always optimal to replace the system). For (b), we have $\int_0^z V^{**}(\xi, z-y)F_z^\xi(dy) = V^{**}(\xi,z)$, and

$$f(\xi) = \beta(\xi) \equiv \inf\{z, \quad m(\xi,z) - c(\xi,z) + V^{**}(\xi,z) - V^{**}(\xi,0) \geq 0\}. \qquad (4.3.22)$$

For a general distribution function $F_z^\xi(\cdot)$, we have the following theorem.

Theorem 4.3.11. (i) Let $f(\xi)$ be a control-limit associated with $F_z^\xi(\cdot)$, then

$$\beta(\xi) \leq f(\xi) \leq \alpha(\xi) \quad \text{for } \xi \in \Gamma.$$

(ii) Let $f_i(\xi)$ be control-limits associated with $F_{i,z}^\xi(\cdot)$ for $i = 1, 2$.

If $F_{1,z}^\xi(y) \leq F_{2,z}^\xi(y)$ for $0 \leq y \leq z$, then $f_1(\xi) \geq f_2(\xi) \quad$ for $\xi \in \Gamma$.

Proof. For (i), we have $0 \leq \int_0^z V^{**}(\xi, z-y)F_z^\xi(dy) \leq V^{**}(\xi,z) - V^{**}(\xi,0)$, and for (ii), $\int_0^z V^{**}(\xi, z-y)F_{1,z}^\xi(dy) \leq \int_0^z V^{**}(\xi, z-y)F_{2,z}^\xi(dy)$. From definition (4.3.16) of $f(\cdot)$, these lead to the desired results. □

In general, influences of the environment are complex because changes of the environment influence simultaneously the shock arrival, shock magnitude and the failure rate. In some cases, it is difficult to compare influencing affects of two environment. Let $\xi_1, \xi_2 \in \Gamma$, for instance, $H^{\xi_1}(\cdot) \geq H^{\xi_2}(\cdot)$, and $G_z^{\xi_1}(\cdot) \geq G_2^{\xi_2}(\cdot)$ for $z \in [0, \infty)$. Rough speaking, these imply that at state ξ_1, the shock arrival is faster than that at state ξ_2, while shock magnitude is smaller than that at state ξ_2. So that, we can not appreciate simply which of the states ξ_1 and ξ_2 is a better environment to the system. Here, we consider a particular case as follows.

For $\xi \in \Gamma$, let $H^\xi(\cdot) = H(\cdot)$, $\lambda(\xi) = \lambda, \eta(\xi) = \eta$. We introduce a order $\prec$ on the state space Γ. For $\xi_1 \prec \xi_2$, we refer as the following

(i) $1 - \gamma(z, \xi_1, x) \leq 1 - \gamma(z, \xi_2, x)$,

(ii) $G_z^{\xi_1}(\cdot) \geq G_z^{\xi_2}(\cdot)$, and $Q(\xi_1, \cdot) \geq Q(\xi_2, \cdot)$

for $z, x \in [0, \infty)$.

The meaning of (i) is obvious. The (ii) means that distribution functions $G_z^\xi(\cdot)$ and $Q(\xi, \cdot)$ are stochastically increasing in order $\prec$. In this case, we call ξ_1 is a better environment than ξ_2 to the system. If $m(\xi, z), c(\xi, z)$ are increasing in order $\prec$, similar to the proof of Theorem 4.3.4, we also have $V^{**}(\xi, z)$ is increasing in order $\prec$, and $L_1 V_n(\xi, z)$, $L_2 V_n(\xi, z)$ are decreasing in order $\prec$. Furthermore, suppose the environment state restores the initial state ξ_0 when the system is replaced (this corresponds to the case where the environment is a internal factor of the system). We have

$$f(\xi) = \inf\{z,\ \ m(\xi, z) - c(\xi, z) + \int_0^z V^{**}(\xi, z-y) F_z^\xi(dy) - V^{**}(\xi_0, 0) \geq 0\}. \quad (4.3.23)$$

Corollary 4.3.12. (i) If $m(\xi, z) - c(\xi, z)$ is increasing in order $\prec$, then for $\xi_1 \prec \xi_2$, $f(\xi_1) \geq f(\xi_2)$.

(ii) If $c(\xi, z) = c(z)$, then for $\xi_1 \prec \xi_2$, $a_2^*(\xi_1, z) \geq a_2^*(\xi_2, z)$. where $a_2^*(\xi, z)$ is a minimal solution of the equation (3.20) under the above assumptions.

Remark.

Note that we do not require the environment process $\xi(t)$ be a increasing process in order $\prec$. This Corollary shows that the control-limit corresponding to a worse environment is lower. In this case, the system may be replaced early. For a general state space Γ without order, the control-limit $f(\cdot)$ can be taken as a criterion function. That is, if $f(\xi_1) \geq f(\xi_2)$, we can think that ξ_1 is a better environment then ξ_2.

4.4 Optimal Maintenance-Replacement Policy

Let $A^* \in \mathbf{A}$ be a control-limit policy defined by

$$A^*(\xi, z) = \begin{cases} (a_1^*(\xi, z), 0) & \text{if } z < f(\xi) \\ (a_2^*(\xi, z), 1) & \text{otherwise,} \end{cases} \qquad (4.4.1)$$

where $f(\xi)$ is defined in (4.3.16), and $a_1^*(\xi, z), a_2^*(\xi, z)$ are the minimal solutions of the equations (4.3.19),(4.3.20) respectively. Let $\pi^* = (A^*, A^*, \ldots)$, then π^* presents such a maintenance-replacement policy: at decision point T_n, the decision is $A_n(\xi_n, Z_n) = A^*(\xi_n, Z_n)$; if $Z_n < f(\xi_n)$, and the sojoun time at state (ξ_n, Z_n) exceeds $a_1^*(\xi_n, Z_n)$, we maintain the system at time $T_n + a_1^*(\xi_n, Z_n)$; if $Z_n \geq f(\xi_n)$, and the sojoun time at state (ξ_n, Z_n) exceeds $a_2^*(\xi_n, Z_n)$, we replace the system at time $T_n + a_2^*(\xi_n, Z_n)$. We will prove that π^* is an optimal replacement policy . For any $\pi \in \Pi$, let

$$N^\pi \equiv \inf\{n \mid i_n(\xi_n, Z_n) = 1\} \qquad (4.4.2)$$
$$N(t) \equiv \textstyle\sum_{n\geq 0} I_{\{T_n \leq t\}}. \qquad (4.4.3)$$

Then, T_{N^π} is the first replacement time of the system under π, and $N(t)$ is a point process corresponding to the stationary Markov renewal process $\{\xi_n, Z_n, T_n\}_{n\geq 0}$. Using T_{N^π} and $N(t)$, we define the operator H_{N^π} on B by

$$H_{N^\pi} V(\xi, z) \equiv E_{(\xi,z)}[\int_{0+}^{T_{N^\pi}-} e^{-\Lambda(t)} m(\xi_t, Z^\pi(t)) dN(t) + e^{-\Lambda(T_{N^\pi})}(c(\xi_{N^\pi}, Z_{N^\pi}) + V(\xi_{N^\pi}, 0))]$$

where $\quad \Lambda(t) = \Lambda(T_n) \quad$ if $T_n \leq t < T_{n+1}, \quad n \geq 0. \qquad (4.4.4)$

Remark.

(1) $H_{N^\pi} V(\xi, z)$ can be interpreted as follows: let V be the 'remaining cost', that is, if the process is stopped at time t in state (ξ, z) we have to pay the discounted amount $e^{-\Lambda(t)} V(\xi, z)$. After the execution of a replacement the system moves without loss of time into the state $z = 0$ and the environment state does not change. Employing the policy π, we have that the replacement causes the first cost $\int_{0+}^{T_{N^\pi}-} e^{-\Lambda(t)} m(\xi_t, Z^\pi(t)) dN(t) + e^{-\Lambda(T_{N^\pi})} c(\xi_{N^\pi}, Z_{N^\pi})$ which is equal to $\sum_{i=1}^{N^\pi-1} e^{-\Lambda(T_i)} m(\xi_i, Z_i) + e^{-\Lambda(T_{N^\pi})}\, c(\xi_{N^\pi}, Z_{N^\pi})$, and after that there remain costs of

$e^{-\Lambda(T_{N^\pi})}V(\xi_{N^\pi},0)$. So that $H_{N^\pi}V(\xi,z)$ means the expected randomly discounted cost of the first replacement under π, starting at time 0 in state (ξ,z).

(2) By Proposition 4.2.2 and a stationary policy π, the process $\{\xi_n, Z_n, T_n\}_{n\geq 0}$ is a stationary Markov renewal process. Since $ET_{N^\pi} \leq E\delta < \infty$, H_{N^π} is well-defined.

The expected randomly discounted costs incurred under π until n-th replacement can be given by

$$V_\pi^n \equiv H_{N_1^\pi} \dots H_{N_n^\pi} V_0, \tag{4.4.5}$$

where the terminal cost function V_0 is set to be 0 .

Let

$$u_n \equiv \inf_\pi V_\pi^n \tag{4.4.6}$$

$$u_\infty \equiv \lim_{n\to\infty} u_n. \tag{4.4.7}$$

Lemma 4.4.1. (a) $\lim_{n\to\infty} V_\pi^n = V_\pi$.

(b) $V^* \geq u_\infty$.

(c) $V_{\pi^*} = V^{**}$.

Proof. (a) For every $n \geq 0$, there is an integer $m \geq n$ such that

$$E_{(\xi,z)}[\sum_{i=1}^{m} K_n] \leq H_{N_1^\pi} \dots H_{N_n^\pi} V_0 \leq E_{(\xi,z)}[\sum_{i=1}^{m+1} K_n].$$

Since $V_\pi \in B$ for any $\pi \in \Pi$, we get, $V_\pi \leq \lim_{n\to\infty} V_\pi^n \leq V_\pi$.

(b) By $V_\pi^n \geq u_n$ and (a), we have $\lim_{n\to\infty} V_\pi^n \geq \lim_{n\to\infty} u_n$, which yields $\inf_\pi V_\pi \geq u_\infty$.

(c) Under the policy π^*, we have

$$\begin{aligned} H_{N^{\pi^*}}V_0(\xi,z) &= \textstyle\sum_{n=1}^\infty E_{(\xi,z)}[\int_{0+}^{T_{N^\pi}-} e^{-\Lambda(t)} m(\xi_t, Z^\pi(t))dN(t) + e^{-\Lambda(T_{N^\pi})}(c(\xi_{N^\pi}, Z_{N^\pi}) \\ &\qquad +V(\xi_{N^\pi},0))|N^{\pi^*}=n]P_{(\xi,z)}(N^{\pi^*}=n) \\ &= \textstyle\sum_{n=1}^\infty U^n V_0(\xi,z) P_{(\xi,z)}(N^{\pi^*}=n) \\ &= E_{(\xi,z)}[U^{N^{\pi^*}} V_0], \end{aligned}$$

and

$$\begin{aligned} H_{N_1^{\pi^*}} H_{N_2^{\pi^*}} V_0(\xi,z) &= E_{(\xi,z)}[U^{N_1^{\pi^*}} E_{(\xi,z)}[U^{N_2^{\pi^*}} V_0]] \\ &= \textstyle\sum_{n=1}^\infty \sum_{m=1}^\infty U^{n+m} V_0(\xi,z) P_{(\xi,z)}(N_1^{\pi^*}=n) P_{(\xi,z)}(N_2^{\pi^*}=m) \\ &= E_{(\xi,z)}[U^{N_1^{\pi^*}+N_2^{\pi^*}} V_0]. \end{aligned}$$

By induction, we have $H_{N_1^{\pi^*}} \dots H_{N_n^{\pi^*}} V_0 = E[U^{N_1^{\pi^*}+\dots+N_n^{\pi^*}} V_0]$, and $n \leq N_1^{\pi^*} + \dots + N_n^{\pi^*}$, a.s.. Thus, $P(\lim_{n\to\infty} N_1^{\pi^*}+\dots+N_n^{\pi^*} = \infty) = 1$ and $U^{N_1^{\pi^*}+\dots+N_n^{\pi^*}} V_0 \to V^{**}$, a.s. $(n \to \infty)$. Noting $U^n V_0$ is increasing in n, we get $\lim_{n\to\infty} E[U^{N_1^{\pi^*}+\dots+N_n^{\pi^*}} V_0] = V^{**}$, i.e., $V_{\pi^*} = V^*$. □

Lemma 4.4.2. $u_\infty \geq V^*$.

Proof. For any $n \geq 1$ and $\pi \in \Pi$, let $m = N_1^\pi +, \dots, +N_n^\pi$. Then, $V_\pi^n \geq E[U^{m-1} V_0]$ and $u_n = \inf_\pi V_\pi^n \geq E[U^{m-1} V_0]$. Let $n \to \infty$, we have $u_\infty \geq V^*$. □

Theorem 4.4.3. π^* is an optimal stationary maintenance-replacement policy.

Proof. From Lemma 4.4.1 and 4.4.2, $u_\infty \geq V^{**} = V_{\pi^*} \geq V^* \geq u_\infty$, thus, $V_{\pi^*} = V^*$. Therefore, π^* is an optimal stationary policy whose maintenace-replacement rule is the control-limit type. □

CHAPTER 5

CONCLUSION

5.1 Summery of the results

We summarize the results of the thesis in this section.

We have investigated in the former Chapters the optimal maintenance–replacement problems of the systems subject to shocks by considering influences of the environments. We have constructed a new damage process (piecewise semi-Markov process) by the shock process and the environment process. This piecewise semi-Markov damage process generalizes the semi-Markov damage process studied by the former reseachers, so that the results obtained can be applied to the case that influences of environments are not considered. In Chapter 2, we have analyzed the optimal replacement problem where the state changes of the environment process is described by a Poisson process. By defining an integer-valued random variable, we have gotten the sum representing forms for the cost functions, which are more natural because that both the damage process and the environment process are jump processes. We have proved that the control-limit policies are optimal for two types of stopping time sets, and obtained the corresponding bounded processes. In particular, the control-limit policies for general stopping time case are the combination rules of the damage level's control-limit policies and the state-age dependent policies. It has been seen that, however, the sample's analyzing method used in this Chapter depends greatly on the monotone increasing property of the Poisson environment process.

In Chapter 3 we have investigated an optimal state-age dependent replacement problem for a network system consisted of a main-system and a sub-system with N components. The functioning process of components of the sub-system is a Markov process with the finite state space, which is taken as the environment process of the main-system. By using Markov decision approach, we have derived an optimal state-age dependent replacement policy minimizing the long-run average cost per unit time. The results can be extended to the case that the environment process is a general Markov jump process. At the last, the Markov decision approach used in this Chapter has some differences from that given in Ross [54,56] since the action space in our model is infinite, which makes the calculations more complicate.

We have examined in Chapter 4 a general optimal maintenance–replacement problem of the system existing in a Markov randomly varying environment. We have considered the replacement action as well as the maintenance actions, and permitted that the damaged system become to "better" after every maintenance, i.e., the damage level of the system has an randomly decreasing magnitude which is stochastically decreasing with respect to the accumulated damage level. For the total expected randomly discounted cost, we have derived an optimal control-limit maintenance–replacement policy dependent on the environment state. As shown as in 4.3, the optimal control-limit policy has also state-age dependent type, i.e., for every environment state, we maintain the system when the damage level does not exceed the control-limit and the sojourn time in that state exceeds an real-value threshold, and replace the system when the damage level exceeds the control-limit and the sojourn time exceeds another real-value threshold. Furthermore, we have analyzed the influences of the maintenance actions and the environment state on the control-limit.

Shock models are one of useful mathematical tools for analyzing the optimal maintenance–replacement problems for the systems used in the production of goods and delivery of services, in particular, for the systems used in danger field such as the atomic powerful station, etc. It is important to give more accurate and more appreciate policies for such systems. This thesis have derived optimal maintenance–replacement policies by considering influences of the various environments. The techniques and methods can be applied to other field such as economic decisions, controls of queueing systems affected by randomly varying environments.

5.2 Further problem

The shock models studied in this thesis could be theoretically extended. We simply discuss further considerable problems and suggestive comments in the following.

In this thesis we have considered mainly the optimal maintenance–replacement for the systems influenced by randomly varying environments, not studied the properties of the failure times of the systems. However, the failure times of the systems depend greatly on the behaviors of the environment changes, so that it is significant to investigate the probability distribution properties of the failure times of the systems, such as the first-passage time distributions, new better than used, new better than used in expection, etc. It seems difficult to obtain these properties under general conditions because that the environment's changes destroy the monotone properties of the probability distributions of the shock process such as the intershock times and the shock magnitudes.

As seen as in the definition of the damage process of the system, we have emphasized the influences of the environments on the systems, not considered the influences of the damaged systems on the environments. In Chapter 3, for example, the failures of components of the sub-system may cause the main-system to fail, and the shock arrivals and shock magnitudes depend on the number of the functioning components, while the lifetimes of components of the sub-system are not affected by the behaviors of the main-system. In fact, these lifetimes may become shorter and shorter with increasing of the damage level of the main-system. Hence, the main-system and sub-system influence with each other. In this case, it is necessary to consider these mutual influences which might result in a more complicate replacement problem.

Applications of optimal maintenance–replacement theory are of great interest. Shock models have been studied by many authors and will developed in future as the growth in the complexity of modern systems. The problems mentioned above are open questions for future research.

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ACKNOWLEGEMENTS

I would like to express my gratitude to Professor M. Kowada and Associate Professor K. Adachi for their sustained guidance not only in the course of research but also in the way of thinking.

For several fruitful discussions as well as participation in the examination of this thesis, I am also grateful to Professor K. Ono and Professor K. Kawaguchi.

I really indebted to Ministry of Education, Japan, for granting scholarship in the last two years of my post-graduate course.

I would like to appreciate my father, Feng Ru-Ming, and my mother, Zhou jian-qiu, for their encouragement and support. I would like also to thank my wife Sun Xiao-Chun for her incessant assistance.