Resultants and discriminants of the multiplication polynomials of Jacobi elliptic functions

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Abstract

We compute the resultants and discriminants of the multiplication polynomials of Jacobi elliptic functions by using Fourier expansions.

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1. Introduction

Resultants and discriminants have been calculated for various kinds of division (sometimes called multiplication) polynomials; cyclotomic polynomials (several different proofs are known; see for example [1], [3], [6]), real cyclotomic polynomials ([7], [11]), Chebyshev polynomials ([4], [8], [11]), multiplication polynomials of the Weierstrass $\wp$-function ([9]). It seems that no one has ever calculated the resultants of multiplication polynomials of Jacobi elliptic functions.

Consider the Jacobi elliptic functions $\operatorname{sn}, \operatorname{cn}, \operatorname{dn}$ with modulus k ($k^2 \neq 0, 1$; k may be complex). Let $x = \operatorname{sn} u$, $y = \operatorname{cn} u$, $z = \operatorname{dn} u$. For each positive

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integer n, there exist polynomials A_n, B_n, C_n, D_n in x with the following property:

$$(\operatorname{sn} nu, \operatorname{cn} nu, \operatorname{dn} nu) = \begin{cases} \Big(\dfrac{xA_n(x)}{D_n(x)}, \dfrac{yB_n(x)}{D_n(x)}, \dfrac{zC_n(x)}{D_n(x)}\Big) & \text{if } n \text{ is odd,} \\ \Big(\dfrac{xyzA_n(x)}{D_n(x)}, \dfrac{B_n(x)}{D_n(x)}, \dfrac{C_n(x)}{D_n(x)}\Big) & \text{if } n \text{ is even.} \end{cases} \tag{1.1}$$

See Section 2 for a precise description. The coefficients of A_n, B_n, C_n, D_n belong to $\mathbb{Z}[k^2]$.

The main result of this paper is the following

Theorem 1.1. *Let $X, Y \in \{A, B, C, D\}, X \neq Y$ and $n \geq 1$. We have*

$$\operatorname{res}(X_n, Y_n) = \kappa_n(X, Y) k^{2l_n(X,Y)} (1 - k^2)^{m_n(X,Y)}, \tag{1.2}$$

where

$$\kappa_n(X, Y) = 2^{\frac{n^2(n^2-1)}{3}}, \quad l_n(X, Y) = l_n(Y, X), \quad m_n(X, Y) = m_n(Y, X),$$

$$l_n(A, B) = m_n(A, D) = \begin{cases} \frac{n^2(n^2-1)}{6} & \text{if } n \text{ is odd,} \\ \frac{n^2(n^2-4)}{6} & \text{if } n \text{ is even,} \end{cases}$$

$$l_n(A, C) = l_n(A, D) = m_n(A, B) = m_n(A, C) = \begin{cases} \frac{(n^2-1)(2n^2-3)}{12} & \text{if } n \text{ is odd,} \\ \frac{n^2(n^2-4)}{6} & \text{if } n \text{ is even,} \end{cases}$$

$$l_n(B, C) = l_n(B, D) = m_n(B, D) = m_n(C, D) = \begin{cases} \frac{(n^2-1)(2n^2-3)}{12} & \text{if } n \text{ is odd,} \\ \frac{n^2(n^2-1)}{6} & \text{if } n \text{ is even,} \end{cases}$$

$$l_n(C, D) = m_n(B, C) = \begin{cases} \frac{n^2(n^2-1)}{6} & \text{if } n \text{ is odd,} \\ \frac{n^2(n^2+2)}{6} & \text{if } n \text{ is even.} \end{cases}$$

The proof goes along the lines of Schmidt [9]. First we show the existence of integers $\kappa_n(X, Y) > 0$, $l_n(X, Y) \geq 0$, and $m_n(X, Y) \geq 0$ not depending on k such that (1.2) holds (Lemma 3.2). Then, comparing the q-expansions of both sides of (1.2), we determine the three constants. To be more precise, a comparison of the degrees of the leading terms determines $l_n(X, Y)$. Changing k to its complementary modulus k' (i.e., $k^2 + k'^2 = 1$), we get $m_n(X, Y)$ (Corollary 3.3). A comparison of the leading constants finally yields the determination of $\kappa_n(X, Y)$.

The organization of this paper is as follows. In Section 2 we review on Jacobi elliptic functions and basic properties of their multiplication polynomials A_n, B_n, C_n, D_n. In Section 3 we show the general shape of $\mathrm{res}(X_n, Y_n)$ as given in (1.2). After preparing some q-expansions in Section 4, we give a proof of Theorem 1.1 in Section 5. As an application, we also calculate the discriminants of A_n, B_n, C_n, D_n in Section 6. In the final Section 7 we mention the degenerate cases $k^2 = 0, 1$.

2. Jacobi elliptic functions

Since the assertion of Theorem 1.1 is an identity in k, we may assume $0 < k < 1$ for the proof. Following the traditional notation as in [2] or [10], we write Jacobi elliptic functions $\mathrm{sn}(u,k), \mathrm{cn}(u,k), \mathrm{dn}(u,k)$ simply as $\mathrm{sn}\, u, \mathrm{cn}\, u, \mathrm{dn}\, u$, respectively. The periods of $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$ are

$$4mK + 2niK', \quad 2mK + 2n(K + iK'), \quad 2mK + 4niK' \quad (m, n \in \mathbb{Z})$$

respectively, where

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad K' = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}$$

are complete elliptic integrals of the first kind and $k' = \sqrt{1-k^2}$ is the complementary modulus.

The following are basic properties of the polynomials A_n, B_n, C_n, D_n referred to in Introduction.

Proposition 2.1.

1. *A_n, B_n, C_n, D_n are determined by*

$$A_1 = B_1 = C_1 = D_1 = 1$$

and the recurrence relations

$$
\begin{aligned}
A_{2n} &= 2A_nB_nC_nD_n,\\
B_{2n} &= \begin{cases} y^2B_n^2D_n^2 - x^2z^2A_n^2C_n^2 & \text{if } n \text{ is odd},\\ B_n^2D_n^2 - x^2y^2z^2A_n^2C_n^2 & \text{if } n \text{ is even},\end{cases}\\
C_{2n} &= \begin{cases} z^2C_n^2D_n^2 - k^2x^2y^2A_n^2B_n^2 & \text{if } n \text{ is odd},\\ C_n^2D_n^2 - k^2x^2y^2z^2A_n^2B_n^2 & \text{if } n \text{ is even},\end{cases}\\
D_{2n} &= \begin{cases} D_n^4 - k^2x^4A_n^4 & \text{if } n \text{ is odd},\\ D_n^4 - k^2x^4y^4z^4A_n^4 & \text{if } n \text{ is even},\end{cases}\\
A_{2n+1} &= \begin{cases} A_nB_{n+1}C_{n+1}D_n + y^2z^2A_{n+1}B_nC_nD_{n+1} & \text{if } n \text{ is odd},\\ y^2z^2A_nB_{n+1}C_{n+1}D_n + A_{n+1}B_nC_nD_{n+1} & \text{if } n \text{ is even},\end{cases}\\
B_{2n+1} &= B_nB_{n+1}D_nD_{n+1} - x^2z^2A_nA_{n+1}C_nC_{n+1},\\
C_{2n+1} &= C_nC_{n+1}D_nD_{n+1} - k^2x^2y^2A_nA_{n+1}B_nB_{n+1},\\
D_{2n+1} &= D_n^2D_{n+1}^2 - k^2x^4y^2z^2A_n^2A_{n+1}^2.
\end{aligned}
$$

2. *The coefficients of* A_n, B_n, C_n, D_n *belong to* $\mathbb{Z}[k^2]$.
3. A_n, B_n, C_n, D_n *are even polynomials, i.e., polynomials in* x^2.
4. *The leading terms are as follows.*

$$
\begin{aligned}
A_n(x) &= \begin{cases} (-1)^{(n-1)/2}(\sqrt{k}\,x)^{n^2-1} + \cdots & \text{if } n \text{ is odd},\\ (-1)^{(n-2)/2}n(\sqrt{k}\,x)^{n^2-4} + \cdots & \text{if } n \text{ is even},\end{cases}\\
B_n(x) &= \begin{cases} (\sqrt{k}\,x)^{n^2-1} + \cdots & \text{if } n \text{ is odd},\\ (\sqrt{k}\,x)^{n^2} + \cdots & \text{if } n \text{ is even},\end{cases}\\
C_n(x) &= \begin{cases} (\sqrt{k}\,x)^{n^2-1} + \cdots & \text{if } n \text{ is odd},\\ (\sqrt{k}\,x)^{n^2} + \cdots & \text{if } n \text{ is even},\end{cases}\\
D_n(x) &= \begin{cases} (-1)^{(n-1)/2}n(\sqrt{k}\,x)^{n^2-1} + \cdots & \text{if } n \text{ is odd},\\ (-1)^{n/2}(\sqrt{k}\,x)^{n^2} + \cdots & \text{if } n \text{ is even}.\end{cases}
\end{aligned}
$$

5. $A_n(0) = n,\ B_n(0) = C_n(0) = D_n(0) = 1.$

6.

$$A_n(1) = \begin{cases} (-1)^{(n-1)/2}(k'^2)^{(n^2-1)/4} & \text{if } n \text{ is odd,} \\ n(-k'^2)^{n^2/4-1} & \text{if } n \text{ is even,} \end{cases}$$

$$B_n(1) = \begin{cases} (-1)^{(n-1)/2}n(k'^2)^{(n^2-1)/4} & \text{if } n \text{ is odd,} \\ (-k'^2)^{n^2/4} & \text{if } n \text{ is even,} \end{cases}$$

$$C_n(1) = D_n(1) = (k'^2)^{\lfloor n^2/4 \rfloor}.$$

7.

$$A_n(1/k) = \begin{cases} (-1)^{(n-1)/2}(k'^2k^{-2})^{(n^2-1)/4} & \text{if } n \text{ is odd,} \\ n(k'^2k^{-2})^{n^2/4-1} & \text{if } n \text{ is even,} \end{cases}$$

$$C_n(1/k) = \begin{cases} (-1)^{(n-1)/2}n(k'^2k^{-2})^{(n^2-1)/4} & \text{if } n \text{ is odd,} \\ (k'^2k^{-2})^{n^2/4} & \text{if } n \text{ is even,} \end{cases}$$

$$B_n(1/k) = D_n(1/k) = (-k'^2k^{-2})^{\lfloor n^2/4 \rfloor}.$$

8. *We have the following factorization:*

$$A_n(x) = a_n \prod_{(r,s) \in R_n,\ (r,s) \equiv (0,0)} \left(x^2 - \operatorname{sn}^2 \frac{rK + siK'}{n} \right),$$

$$B_n(x) = b_n \prod_{(r,s) \in R_n,\ (r,s) \equiv (1,0)} \left(x^2 - \operatorname{sn}^2 \frac{rK + siK'}{n} \right),$$

$$C_n(x) = c_n \prod_{(r,s) \in R_n,\ (r,s) \equiv (1,1)} \left(x^2 - \operatorname{sn}^2 \frac{rK + siK'}{n} \right),$$

$$D_n(x) = d_n \prod_{(r,s) \in R_n,\ (r,s) \equiv (0,1)} \left(x^2 - \operatorname{sn}^2 \frac{rK + siK'}{n} \right),$$

where

$$R_n = (\{1, 2, \ldots, n-1\} \times \{0, n\}) \sqcup (\{0, 1, \ldots, 2n-1\} \times \{1, 2, \ldots, n-1\}),$$

the leading coefficients a_n, b_n, c_n, d_n are as given in part 4, and the congruences are taken modulo 2.

9. *A_n, B_n, C_n, D_n are pairwise prime to each other as polynomials in x.*

Proof. First we note that the polynomials A_n, B_n, C_n, D_n in our notation are the ones denoted by A'_n, B'_n, C'_n, D'_n in [2, p.87]. With this translation in mind, part 1 is found in [2, p.79]. Parts 2–7 are then deduced from the recurrence relations. Part 8 is essentially found in [2, p.92]. The last claim follows from the factorization. □

We describe the effect of changing k to k'. To this end, we introduce a notation:

$$f^*(x) = \sqrt{1-x^2}^{\deg f} f\left(\frac{ix}{\sqrt{1-x^2}}\right). \tag{2.1}$$

Lemma 2.2. *Let $f(x)$ be an even polynomial of degree $2n$.*

1. *$f^*(x)$ is also an even polynomial.*
2. *The coefficient of x^{2n} in $f^*(x)$ is $(-1)^n f(1)$. In particular, $\deg f^* = \deg f$ if and only if $f(1) \neq 0$.*
3. *$f^{**}(x) = f(x)$ if $f(1) \neq 0$.*

Recall that A_n, B_n, C_n, D_n are even polynomials and do not vanish at $x = 1$.

Proposition 2.3. *If we write $A_n(x, k)$ etc., to indicate the dependency on k, then we have*

$$A_n(x,k') = A_n^*(x,k), \qquad B_n(x,k') = D_n^*(x,k),$$
$$C_n(x,k') = C_n^*(x,k), \qquad D_n(x,k') = B_n^*(x,k).$$

Proof. If the zeros of an even polynomial $f(x)$ with $f(1) \neq 0$ are $\pm\alpha_j$ ($j = 1, 2, \dots, n$), then those of $f^*(x)$ are

$$\pm\frac{i\alpha_j}{\sqrt{1-\alpha_j^2}} \quad (j = 1, 2, \dots, n).$$

Using this observation, the identity

$$\operatorname{sn}(iu, k) = \frac{i \operatorname{sn}(u, k')}{\operatorname{cn}(u, k')}$$

(cf. [10, 22·4]), and part 8 of Proposition 2.1, we see that both sides have exactly the same set of zeros in each case. We also see that the leading coefficients coincide by using parts 4 and 6 of Proposition 2.1. This completes the proof. □

3. Resultants

The resultant of two polynomials

$$f(x) = a(x-\alpha_1)\cdots(x-\alpha_m), \quad a \neq 0,$$
$$g(x) = b(x-\beta_1)\cdots(x-\beta_n), \quad b \neq 0$$

is defined as

$$\operatorname{res}(f,g) = a^n b^m \prod_{i=1}^{m}\prod_{j=1}^{n}(\alpha_i - \beta_j) = a^n \prod_{i=1}^{m} g(\alpha_i).$$

Lemma 3.1. *Let h be another polynomial.*

1. $\operatorname{res}(g,f) = (-1)^{mn}\operatorname{res}(f,g)$.
2. $\operatorname{res}(f,gh) = \operatorname{res}(f,g)\operatorname{res}(f,h)$, $\operatorname{res}(fg,h) = \operatorname{res}(f,h)\operatorname{res}(g,h)$.
3. $\operatorname{res}(f,g+fh) = a^{s-n}\operatorname{res}(f,g)$, *where* $s = \deg(g+fh)$.
4. *Suppose further that f,g are even polynomials which do not vanish at $x=1$. Then with the notation in* (2.1)*, we have*

$$\operatorname{res}(f^*,g^*) = \operatorname{res}(f,g).$$

Proof. The first three are easily seen. The proof of the last assertion is reduced to the case $f(x) = x^2 - p, g(x) = x^2 - q$, since both the resultant and the operation * are multiplicative. In this case we have $f^*(x) = (p-1)x^2 - p, g^*(x) = (q-1)x^2 - q$, so that

$$\operatorname{res}(f^*,g^*) = (p-q)^2 = \operatorname{res}(f,g)$$

as desired. □

Let $X, Y \in \{A,B,C,D\}, X \neq Y$ and $n \geq 1$. Our object is the calculation of $\operatorname{res}(X_n,Y_n)$. By a well known Sylvester determinant expression of the resultant and by part 2 of Proposition 2.1, we see that $\operatorname{res}(X_n,Y_n) \in \mathbb{Z}[k^2]$. Also, we have $\operatorname{res}(X_n,Y_n) = \operatorname{res}(Y_n,X_n)$ since both $\deg X_n$ and $\deg Y_n$ are even.

The general shape of $\operatorname{res}(X_n,Y_n)$ is given as follows.

Lemma 3.2. *Let $X, Y \in \{A,B,C,D\}, X \neq Y$. For each positive integer n, there exist integers $\kappa_n(X,Y) > 0$, $l_n(X,Y) \geq 0$, and $m_n(X,Y) \geq 0$ not depending on k such that*

$$\operatorname{res}(X_n,Y_n) = \kappa_n(X,Y)k^{2l_n(X,Y)}(1-k^2)^{m_n(X,Y)}.$$

Proof. We regard $\operatorname{res}(X_n, Y_n)$ as a polynomial in k^2. Part 9 of Proposition 2.1 implies that $\operatorname{res}(X_n, Y_n) \neq 0$ if $k^2 \neq 0, 1$. Hence by the Hilbert Nullstellensatz (cf. [5, IX §1 Theorem 1.5]), there exist a positive integer t and a polynomial $Q \in \mathbb{Q}[k^2]$ such that $(k^2(1-k^2))^t = Q \operatorname{res}(X_n, Y_n)$. Since $\mathbb{Q}[k^2]$ is a unique factorization domain, it follows that $\operatorname{res}(X_n, Y_n) = \kappa k^{2l}(1-k^2)^m$ for some $\kappa \in \mathbb{Q}$ and some non-negative integers l, m. Since $\operatorname{res}(X_n, Y_n) \in \mathbb{Z}[k^2]$ and both X_n and Y_n are polynomials in x^2, we see that κ is a positive integer. $\square$

By Proposition 2.3 and Lemma 3.1, we have

Corollary 3.3. *With the notation of Lemma 3.2,*

$$\begin{aligned} l_n(A,B) &= m_n(A,D), \quad l_n(A,D) = m_n(A,B), \\ l_n(A,C) &= m_n(A,C), \\ l_n(B,C) &= m_n(C,D), \quad l_n(C,D) = m_n(B,C), \\ l_n(B,D) &= m_n(B,D), \\ \kappa_n(A,B) &= \kappa_n(A,D), \\ \kappa_n(B,C) &= \kappa_n(C,D). \end{aligned}$$

On the other hand, by part 8 of Proposition 2.1, we get

Proposition 3.4. *Let $X, Y \in \{A, B, C, D\}, X \neq Y$. Then we have*

$$\operatorname{res}(X_n, Y_n) = x_n^{\deg Y_n} y_n^{\deg X_n} \prod_{(r,s)\in R_n^X} \prod_{(r',s')\in R_n^Y} f(r,s,r',s')^2,$$

where x_n, y_n denote the leading coefficients of X_n, Y_n respectively,

$$\mathbf{v}_A = (0,0),\ \mathbf{v}_B = (1,0),\ \mathbf{v}_C = (1,1),\ \mathbf{v}_D = (0,1),$$

$$R_n = (\{1, 2, \dots, n-1\} \times \{0, n\}) \sqcup (\{0, 1, \dots, 2n-1\} \times \{1, 2, \dots, n-1\}),$$
$$R_n^X = \{(r,s) \in R_n \mid (r,s) \equiv \mathbf{v}_X \pmod 2\},$$

and

$$f(r,s,r',s') = \operatorname{sn}^2 \frac{rK + siK'}{n} - \operatorname{sn}^2 \frac{r'K + s'iK'}{n}.$$

So we need a closer look at $f(r,s,r',s')^2$.

4. q-expansions

Let $\tau = iK'/K$ and $q = e^{\pi i\tau}$. By our assumption $0 < k < 1$, we see that K and K' are positive real numbers, so that $\mathrm{Im}(\tau) > 0$ and $0 < |q| < 1$. We use the following q-expansions ([10, 21·61]):

$$k^{\frac{1}{2}} = \frac{\vartheta_2(0)}{\vartheta_3(0)} = \frac{2\sum_{j=0}^{\infty} q^{(j+\frac{1}{2})^2}}{1+2\sum_{j=0}^{\infty} q^{j^2}} = 2q^{\frac{1}{4}} + \cdots , \tag{4.1}$$

$$2K = \pi\vartheta_3(0)^2 = \pi\left(1+\sum_{j=0}^{\infty} q^{j^2}\right)^2 .$$

Let $u = 2Kx/\pi$. We also use the following ([10, 22·6,22·61]):

$$\mathrm{sn}\, u = \frac{2\pi}{Kk}\sum_{j=0}^{\infty} \frac{q^{j+\frac{1}{2}}\sin(2j+1)x}{1-q^{2j+1}} \tag{4.2}$$

valid throughout the strip $|\,\mathrm{Im}(x)| < \frac{\pi}{2}\,\mathrm{Im}(\tau)$, and

$$\mathrm{ns}\, u = \frac{1}{\mathrm{sn}\, u} = \frac{\pi}{2K}\,\mathrm{cosec}\, x + \frac{2\pi}{K}\sum_{j=0}^{\infty} \frac{q^{2j+1}\sin(2j+1)x}{1-q^{2j+1}} \tag{4.3}$$

valid throughout $|\,\mathrm{Im}(x)| < \pi\,\mathrm{Im}(\tau)$, except at the points $x \in \pi\mathbb{Z}$.

Now let

$$u = \frac{rK + siK'}{n}, \quad (r,s) \in R_n.$$

We can apply (4.2) if $0 \le s < n$ and (4.3) if $s = n$. By using $\exp ix = \zeta^r q^{\frac{s}{2n}}$, where $\zeta = \exp\frac{\pi i}{2n}$, we find that the complex number

$$-4\,\mathrm{sn}^2\,\frac{rK + siK'}{n}$$

is expressed as a Laurent series in $q^{\frac{1}{2n}}$ whose leading term has degree $q^{-\frac{s}{n}}$ and coefficient

$$\begin{cases} (\zeta^r - \zeta^{-r})^2 & \text{if } s = 0, \\ \zeta^{-2r} & \text{if } 0 < s < n, \\ (\zeta^r - \zeta^{-r})^{-2} & \text{if } s = n. \end{cases}$$

Thus we get the leading term of the q-expansion of $f(r,s,r',s')^2$. We may suppose $s \geq s'$ since $f(r',s',r,s)^2 = f(r,s,r',s')^2$.

Lemma 4.1. *For $s \geq s'$, the leading term of the q-expansion of $16f(r,s,r',s')^2$ has degree $q^{-\frac{2s}{n}}$ and coefficient $L(r,s,r',s')$, where*

$$L(r,s,r',s') = \begin{cases} ((\zeta^r - \zeta^{-r})^2 - (\zeta^{r'} - \zeta^{-r'})^2)^2 & \text{if } s' = s = 0, \\ \zeta^{-4r} & \text{if } s' < s < n, \\ (\zeta^{-2r} - \zeta^{-2r'})^2 & \text{if } s' = s < n, \\ (\zeta^r - \zeta^{-r})^{-4} & \text{if } s' < s = n, \\ ((\zeta^r - \zeta^{-r})^{-2} - (\zeta^{r'} - \zeta^{-r'})^{-2})^2 & \text{if } s' = s = n, \end{cases}$$

and $\zeta = \exp \frac{\pi i}{2n}$.

5. Proof of Theorem 1.1

Let $X, Y \in \{A, B, C, D\}, X \neq Y$. We have $\operatorname{res}(X_n, Y_n) = \operatorname{res}(Y_n, X_n)$ as noticed earlier, so we have only to consider the six cases

$$(X,Y) \in \{(A,B),(A,C),(A,D),(B,C),(B,D),(C,D)\}.$$

As in Proposition 3.4, let x_n, y_n denote the leading coefficients (with respect to the variable x) of X_n, Y_n, respectively. By part 4 of Proposition 2.1 and (4.1), we can consider the q-expansions of x_n and y_n, so let x'_n and y'_n denote the leading terms of these q-expansions, respectively. Then by Proposition 3.4 and Lemma 4.1, the leading term of the q-expansion of $\operatorname{res}(X_n, Y_n)$ is

$$x_n'^{\deg Y_n} y_n'^{\deg X_n} \prod_{(r,s)\in R_n^X} \prod_{(r',s')\in R_n^Y} \frac{L(r,s,r',s')}{16} q^{-\frac{2}{n}\max\{s,s'\}}. \tag{5.1}$$

First we observe the degree in q. By Lemma 3.2 and (4.1), the leading term of the q-expansion of $\operatorname{res}(X_n, Y_n)$ has degree $q^{l_n(X,Y)}$. On the other hand, it follows from part 4 of Proposition 2.1 and (4.1) that $x_n'^{\deg Y_n} y_n'^{\deg X_n}$ has degree $q^{\frac{1}{2}\deg X_n \deg Y_n}$, hence by (5.1) we find that

$$l_n(X,Y) = \frac{1}{2}\deg X_n \deg Y_n - \frac{2}{n} \sum_{(r,s)\in R_n^X} \sum_{(r',s')\in R_n^Y} \max\{s,s'\}.$$

The authors are not aware of any clever method to compute the last double sum, but anyway an elementary counting argument gives the value of $l_n(X,Y)$, and hence that of $m_n(X,Y)$ by Corollary 3.3, as stated in Theorem 1.1.

Next we observe the leading coefficient of the q-expansion of $\operatorname{res}(X_n, Y_n)$. By Lemma 3.2 and (4.1), it is $\kappa_n(X,Y)16^{l_n(X,Y)}$. Since $\kappa_n(X,Y)$ is known to be positive, it suffices to determine the absolute value of the coefficient of (5.1). By Lemma 3.2, we may further assume that

$$(X,Y) \in \{(A,C),(A,D),(B,C),(B,D)\}. \tag{5.2}$$

By part 4 of Proposition 2.1 and (4.1), the leading coefficient of ${x'_n}^{\deg Y_n} {y'_n}^{\deg X_n}$ is

$$\begin{cases} 2^{2\deg X_n \deg Y_n} n^{n^2} & \text{if } X = A,\ n \text{ is even},\\ 2^{2\deg X_n \deg Y_n} n^{n^2-1} & \text{if } Y = D,\ n \text{ is odd},\\ 2^{2\deg X_n \deg Y_n} & \text{otherwise}. \end{cases} \tag{5.3}$$

The contribution of 16's in (5.1) gives

$$16^{-\#R_n^X \#R_n^Y} = 2^{-\deg X_n \deg Y_n}. \tag{5.4}$$

Now let

$$P = \prod_{(r,s)\in R_n^X} \prod_{(r',s')\in R_n^Y} |L(r,s,r',s')|$$

and decompose P into the product $P_1P_2P_3P_4$, where

$$P_1 = \prod_{\substack{(r,s)\in R_n^X,\\ s\notin\{0,n\}}} \prod_{\substack{(r',s')\in R_n^Y,\\ s'\notin\{0,n\}}} |L(r,s,r',s')|,$$

$$P_2 = \prod_{\substack{(r,s)\in R_n^X,\\ s\in\{0,n\}}} \prod_{\substack{(r',s')\in R_n^Y,\\ s'\notin\{0,n\}}} |L(r,s,r',s')|,$$

$$P_3 = \prod_{\substack{(r,s)\in R_n^X,\\ s\notin\{0,n\}}} \prod_{\substack{(r',s')\in R_n^Y,\\ s'\in\{0,n\}}} |L(r,s,r',s')|,$$

$$P_4 = \prod_{\substack{(r,s)\in R_n^X,\\ s\in\{0,n\}}} \prod_{\substack{(r',s')\in R_n^Y,\\ s'\in\{0,n\}}} |L(r,s,r',s')|.$$

If we write $\mathbf{v}_X = (v_X, w_X)$ and $\mathbf{v}_Y = (v_Y, w_Y)$, then by the assumption (5.2), we have $w_X = 0$ and $w_Y = 1$. So, as far as we are concerned here, s is always even and s' is always odd. By Lemma 4.1, we have

$$|L(r,s,r',s')| = \begin{cases} |1-\zeta_{2n}^{r'}|^{-4} & \text{if } s < s' = n \text{ and } n \text{ is odd,} \\ |1-\zeta_{2n}^{r}|^{-4} & \text{if } s' < s = n \text{ and } n \text{ is even,} \\ 1 & \text{otherwise,} \end{cases}$$

where we put $\zeta_m = \exp \frac{2\pi i}{m}$, so that

$$\begin{aligned} P_1 &= 1, \\ P_2 &= \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \prod_{0<r<n, r\equiv v_X} |1-\zeta_{2n}^{r}|^{-2n^2} & \text{if } n \text{ is even,} \end{cases} \\ P_3 &= \begin{cases} \prod_{0<r'<n, r'\equiv v_Y} |1-\zeta_{2n}^{r'}|^{-2n(n-1)} & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even,} \end{cases} \\ P_4 &= \begin{cases} \prod_{0<r'<n, r'\equiv v_Y} |1-\zeta_{2n}^{r'}|^{-2(n-1)} & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

By Lemma 5.1 below, we find that

$$P = \begin{cases} 1 & \text{if } n \text{ is odd and } Y = C, \\ n^{-(n^2-1)} & \text{if } n \text{ is odd and } Y = D, \\ 2^{-n^2} & \text{if } n \text{ is even and } X = B, \\ (n/2)^{-n^2} & \text{if } n \text{ is even and } X = A. \end{cases} \tag{5.5}$$

Putting (5.1), (5.3), (5.4), and (5.5) together and using the value of $l_n(X,Y)$, we complete the proof of Theorem 1.1. □

Lemma 5.1. *Let $\zeta_m = \exp \frac{2\pi i}{m}$. For any positive integer n, we have*

$$\prod_{0<r<n, r:odd} |1-\zeta_{2n}^{r}|^2 = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even,} \end{cases} \tag{5.6}$$

$$\prod_{0<r<n, r:even} |1-\zeta_{2n}^{r}|^2 = \begin{cases} n & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases} \tag{5.7}$$

Proof. Using $\prod_{r=1}^{n-1}(1-\zeta_n^r) = n$ and $|1-\zeta_n^r| = |1-\zeta_n^{n-r}|$, we have

$$\prod_{0<r<n/2} |1-\zeta_n^r|^2 = \begin{cases} n & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even,} \end{cases} \tag{5.8}$$

which is nothing but (5.7). Replacing n with $2n$ in (5.8) and then dividing it by (5.7), we get (5.6). □

6. Discriminants

The discriminant of a polynomial f with degree n and leading coefficient a is defined as

$$\operatorname{disc}(f) = (-1)^{n(n-1)/2} a^{-1} \operatorname{res}(f, f')$$

(see [5, IV §8]). As an application of Theorem 1.1, we get the following

Theorem 6.1. *Let $X \in \{A, B, C, D\}$ and $n \geq 1$. We have*

$$\operatorname{disc}(X_n) = \kappa_n(X) k^{2l_n(X)} (1-k^2)^{m_n(X)},$$

where:

If n is odd, then

$$\begin{aligned} \kappa_n(A) = \kappa_n(D) &= (-1)^{\frac{n-1}{2}} 2^{\frac{n^2(n^2-1)}{3}} n^{n^2-2}, \\ \kappa_n(B) = \kappa_n(C) &= 2^{\frac{n^2(n^2-1)}{3}} n^{n^2-3}, \\ l_n(A) = l_n(B) = l_n(C) = l_n(D) &= \frac{(n^2-1)(2n^2-3)}{12}, \\ m_n(A) = m_n(B) = m_n(C) = m_n(D) &= \frac{(n^2-1)(n^2-3)}{6}. \end{aligned}$$

If n is even, then

$$\kappa_n(A) = (-1)^{\frac{n-2}{2}} 2^{\frac{n^2(n^2-1)}{3}} n^{n^2-10},$$
$$\kappa_n(B) = \kappa_n(C) = (-1)^{\frac{n}{2}} \kappa_n(D) = 2^{\frac{n^2(n^2-1)}{3}} n^{n^2},$$
$$l_n(A) = \frac{(n^2-4)(2n^2-9)}{12},$$
$$l_n(B) = \frac{n^2(2n^2-5)}{12},$$
$$l_n(C) = l_n(D) = \frac{n^2(2n^2+1)}{12},$$
$$m_n(A) = \frac{(n^2-4)(n^2-6)}{6},$$
$$m_n(B) = m_n(C) = \frac{n^2(n^2-1)}{6},$$
$$m_n(D) = \frac{n^2(n^2-4)}{6}.$$

Proof. First we recall that

$$\frac{d}{du}\operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u,$$
$$\frac{d}{du}\operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u,$$
$$\frac{d}{du}\operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u.$$

On differentiating both sides of (1.1) with respect to x, we get

$$nB_nC_n = \begin{cases} A_nD_n + x(A_n'D_n - A_nD_n') & \text{if } n \text{ is odd,} \\ (y^2z^2 - x^2z^2 - k^2x^2y^2)A_nD_n + xy^2z^2(A_n'D_n - A_nD_n') & \text{if } n \text{ is even,} \end{cases} \tag{6.1}$$

$$-nxA_nC_n = \begin{cases} -xB_nD_n + y^2(B_n'D_n - B_nD_n') & \text{if } n \text{ is odd,} \\ B_n'D_n - B_nD_n' & \text{if } n \text{ is even,} \end{cases} \tag{6.2}$$

$$-nk^2xA_nB_n = \begin{cases} -k^2xC_nD_n + z^2(C_n'D_n - C_nD_n') & \text{if } n \text{ is odd,} \\ C_n'D_n - C_nD_n' & \text{if } n \text{ is even.} \end{cases} \tag{6.3}$$

We illustrate the proof in the case of B_n, n:even. Taking resultants of both sides of (6.2) with B_n, we get, by Lemma 3.1,

$$\operatorname{res}(-nxA_nC_n, B_n) = b_n^{-2} \operatorname{res}(B_n' D_n, B_n),$$

where $b_n = k^{(n^2-1)/2}$ is the leading coefficient of $B_n(x)$ (cf. part 4 of Proposition 2.1). By the multiplicativity of resultant and by the facts $\deg B_n = n^2-1$ and $\operatorname{res}(x, B_n) = B_n(0) = 1$, we get

$$n^{n^2-1} \operatorname{res}(A_n, B_n) \operatorname{res}(B_n, C_n) = \operatorname{res}(B_n, D_n) b_n^{-1} \operatorname{disc}(B_n).$$

Substituting the result of Theorem 1.1, we get $\operatorname{disc}(B_n)$.

The computation in the remaining cases is similar; Theorem 1.1 and Proposition 2.1 contain all the information we need; for example,

$$\operatorname{res}(z^2, A_n) = \operatorname{res}(1 - k^2x^2, A_n) = (-k^2)^{\deg A_n} A_n(1/k)^2.$$

$\square$

7. Degenerate Case

In this final section, we state without proof what happens in the "degenerate" cases $k^2 = 0, 1$, if we formally define the polynomials A_n, B_n, C_n, D_n by using the same recurrence relations in part 1 of Proposition 2.1. The proof can be carried out, for example, as in [11].

In the case $k^2 = 0$, we have

$$A_n(x) = \begin{cases} \mathscr{U}_n(\sqrt{1-x^2}) & \text{if } n \text{ is odd,} \\ \dfrac{\mathscr{U}_n(\sqrt{1-x^2})}{\sqrt{1-x^2}} & \text{if } n \text{ is even,} \end{cases}$$

$$B_n(x) = \begin{cases} \dfrac{T_n(\sqrt{1-x^2})}{\sqrt{1-x^2}} & \text{if } n \text{ is odd,} \\ T(\sqrt{1-x^2}) & \text{if } n \text{ is even,} \end{cases}$$

$$C_n(x) = D_n(x) = 1,$$

where T_n and $\mathscr{U}_n$ are the Chebyshev polynomials of the first and second kind, of degree n and $n-1$, respectively.

In the case $k^2 = 1$, we have

$$
\begin{aligned}
A_n(x) &= \begin{cases} (1-x^2)^{n(n-1)/2}\tilde{A}_n(x) & \text{if } n \text{ is odd,} \\ (1-x^2)^{n(n-1)/2-1}\tilde{A}_n(x) & \text{if } n \text{ is even,} \end{cases} \\
B_n(x) &= C_n(x) = (1-x^2)^{\lfloor n^2/2 \rfloor}, \\
D_n(x) &= (1-x^2)^{n(n-1)/2}\tilde{D}_n(x),
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{A}_n(x) &= \sqrt{1-x^2}^{\,n-1}\mathscr{U}_n\left(\frac{1}{\sqrt{1-x^2}}\right), \\
\tilde{D}_n(x) &= \sqrt{1-x^2}^{\,n}T_n\left(\frac{1}{\sqrt{1-x^2}}\right).
\end{aligned}
$$

Note that $\tilde{A}_n, \tilde{D}_n$ are polynomials in x^2.

The identity (1.1) holds true also in these degenerate cases, as

$$\operatorname{sn} u = \sin u,\ \operatorname{cn} u = \cos u,\ \operatorname{dn} u = 1$$

if $k^2 = 0$, and

$$\operatorname{sn} u = \tanh u,\ \operatorname{cn} u = \operatorname{dn} u = \frac{1}{\cosh u}$$

if $k^2 = 1$.

Proposition 7.1.

1. *If* $k^2 = 0$, *then* $\operatorname{res}(A_n, B_n) = 2^{n(n-1)}$.
2. *If* $k^2 = 1$, *then* $\operatorname{res}(\tilde{A}_n, \tilde{D}_n) = 2^{n(n-1)}$.

[1] Tom M. Apostol. Resultants of cyclotomic polynomials. *Proc. Amer. Math. Soc.*, 24:457–462, 1970.

[2] Arthur Cayley. *An Elementary Treatise on Elliptic Functions.* George Bell and Sons., London, second edition, 1895.

[3] Fritz-Erdmann Diederichsen. Über die Ausreduktion ganzzahliger Gruppendarstellungen bei arithmetischer Äquivalenz. *Abh. Math. Sem. Hansischen Univ.*, 13:357–412, 1940.

[4] David P. Jacobs, Mohamed O. Rayes, and Vilmar Trevisan. The resultant of Chebyshev polynomials. *Canad. Math. Bull.*, 54(2):288–296, 2011.

[5] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[6] Emma T. Lehmer. A numerical function applied to cyclotomy. *Bull. Amer. Math. Soc.*, 36(4):291–298, 1930.

[7] K. Alan Loper and Nicholas J. Werner. Resultants of minimal polynomials of maximal real cyclotomic extensions. *J. Number Theory*, 158:298–315, 2016.

[8] Stéphane R. Louboutin. Resultants of Chebyshev polynomials: a short proof. *Canad. Math. Bull.*, 56(3):602–605, 2013.

[9] Harry Schmidt. Resultants and discriminants of multiplication polynomials for elliptic curves. *J. Number Theory*, 149:70–91, 2015. Appendix A by Schmidt and Jung Kyu Canci.

[10] E. T. Whittaker and G. N. Watson. *A course of modern analysis*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1996. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions, Reprint of the fourth (1927) edition.

[11] Masakazu Yamagishi. Resultants of Chebyshev polynomials: the first, second, third, and fourth kinds. *Canad. Math. Bull.*, 58(2):423–431, 2015.