ON A CONJECTURE OF THE NORM SCHWARZ INEQUALITY

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Abstract. Let A be a positive invertible matrix and B be a normal matrix. Following the investigation of Ando, we show that $||A\sharp(B^*A^{-1}B)|| \geq ||B||$, where $\sharp$ denotes the geometric mean, fails in general.

1. Introduction

In the paper [2] Ando considered the following problem. For three matrices A, B, C with $A \geq 0, C \geq 0$, does $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ imply $||A\sharp C|| \geq ||B||$? Here $A\sharp C$ is the geometric mean of A and C. The inequality $||A\sharp C|| \geq ||B||$ was called the norm Schwarz inequality. In the case that A is invertible, it is known that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ if and only if $C \geq B^*A^{-1}B$. So the above problem is equivalent to the following. Is $||A\sharp(B^*A^{-1}B)|| \geq ||B||$ always true for $A > 0$? Ando showed that if B satisfies this inequality for any A, then B must be normaloid (i.e., $||B|| = r(B)$ the spectral radius of B). Then it is natural to wonder whether this norm inequality holds whenever B is normal.

Conjecture. For any positive invertible matrix A and any normal matrix B in $M_n(\mathbb{C})$, we have

$$||A\sharp(B^*A^{-1}B)|| \geq ||B||.$$

Ando showed the following [2].

(1) If B is normaloid, the inequality $||A^{\frac{1}{2}}(B^*A^{-1}B)^{\frac{1}{2}}|| \geq ||B||$ holds.
(2) If B is self-adjoint, the conjecture is true.
(3) If B is a scalar multiple of a unitary matrix, the conjecture is true.
(4) When $n = 2$, the conjecture is true.

The aim of this paper is to construct a counter-example to this conjecture in $M_6(\mathbb{C})$. For this purpose, we introduce some statements which are equivalent to the above conjecture. As a bonus, we can show that if the above conjecture were true, then the inequality

$$A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1} \geq 3I,$$

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must hold for any positive invertible matrices A, B and C. Then we can construct a counter-example for this inequality. The idea of constructing a counter-example for this inequality is basically due to M. Lin, who attributed it to Drury [4]. In the final section we give another proof of Ando's theorem for 2×2 matrices.

After finishing this work the author learned from Professor Minghua Lin that he succeeded in constructing a counter example to the above conjecture before us. His example consists of 3×3 matrices and so it is better than ours. The idea of construction is different.

2. Some equivalent conjectures

Throughout this paper we denote by $M_n(\mathbb{C})$ the space of $n \times n$ matrices. The geometric mean of two positive matrices $A, B \in M_n(\mathbb{C})$ is denoted by $A \sharp B$. If they are invertible, we can write $A \sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$. For a matrix A we denote its trace and determinant by $\mathrm{Tr}(A)$ and $\det(A)$, respectively. We also denote the operator norm of a matrix A by $||A||$.

First we introduce three conjectures:

Conjecture 1. (Ando, [2]) For any positive invertible matrix A and any normal invertible matrix B in $M_n(\mathbb{C})$, we have

$$||A \sharp (B^* A^{-1} B)|| \geq ||B||.$$

Conjecture 2. For any positive invertible matrix S, any unitary matrix U and any positive invertible matrix D in $M_n(\mathbb{C})$ with $UD = DU$, we have

$$||D^{\frac{1}{2}} \cdot S \sharp (U^* S^{-1} U) \cdot D^{\frac{1}{2}}|| \geq ||D||.$$

For a unitary matrix U with the spectral decomposition $U = \sum_i z_i P_i$ ($z_i \neq z_j$, $\{P_i\}_i$ are spectral projections), we set

$$E_U(X) = \sum_i P_i X P_i.$$

With respect to the Hilbert-Schmidt inner product $\langle X|Y \rangle = \mathrm{Tr}(X^* Y)$ on $M_n(\mathbb{C})$, the map $E_U(\cdot)$ is the orthogonal projection to the commutant of U, that is, to the class $\{X : XU = UX\}$. $E_U(\cdot)$ is a unital, trace-preserving, positive (hence contractive) linear map on $M_n(\mathbb{C})$ such that $E_U(DX) = D \cdot E_U(X)$, $E_U(XD) = E_U(X) \cdot D$ for any $D \geq 0$ with $DU = UD$.

Here we remark that if $U^k = I$ for some positive integer k, the map E_U can also be defined by

$$E_U(X) = \frac{1}{k} \sum_{i=0}^{k-1} U^{*i} X U^i.$$

Conjecture 3. For any positive invertible matrix S and any unitary matrix U in $M_n(\mathbb{C})$, we have

$$E_U(S\sharp(U^*S^{-1}U)) \geq I.$$

The main result in this section is the following.

Theorem 2.1. *All three conjectures above are mutually equivalent.*

Proof. (Conjecture 1 ⇒ Conjecture 2) We set $B = UD = DU$ and $A = D^{\frac{1}{2}}SD^{\frac{1}{2}}$. Then we see that

$$A\sharp(B^*A^{-1}B) = (D^{\frac{1}{2}}SD^{\frac{1}{2}})\sharp(D^{\frac{1}{2}}U^*S^{-1}UD^{\frac{1}{2}}) = D^{\frac{1}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{1}{2}}.$$

Since B is normal, applying Conjecture 1 we have

$$||D^{\frac{1}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{1}{2}}|| = ||A\sharp(B^*A^{-1}B)|| \geq ||B|| = ||D||.$$

(Conjecture 2 ⇒ Conjecture 1) Take a polar decomposition $B = UD = DU$ with unitary U and positive D and set $S = D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$. Then as shown above we have $A\sharp(B^*A^{-1}B) = D^{\frac{1}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{1}{2}}$ and hence Conjecture 2 implies Conjecture 1.

(Conjecture 2 ⇒Conjecture 3) It is enough to show that $e \cdot S\sharp(U^*S^{-1}U) \cdot e \geq e$ for any rank one projection e with $Ue = eU$. Indeed, if U has the spectral decomposition $U = \sum_i z_iP_i$ $(z_i \neq z_j)$, then we can write $E_U(X) = \sum_i P_iXP_i$. In order to show Conjecture 3, we have to show $P_i \cdot S\sharp(U^*S^{-1}U) \cdot P_i \geq P_i$ for each i. To do so, it is enough to show $e \cdot S\sharp(U^*S^{-1}U) \cdot e \geq e$ for any rank one projection $e \leq P_i$. Here we remark that a rank one projection e satisfies $Ue = eU$ if and only if $e \leq P_i$ for some i.

We set $D = e + \frac{1}{2}(I - e)$. Then by Conjecture 2 we have

$$||D^{\frac{n}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{n}{2}}|| \geq ||D^n||$$

for any positive integer n. By tending $n \to \infty$ we have

$$||e \cdot S\sharp(U^*S^{-1}U) \cdot e|| \geq ||e|| = 1.$$

Then since e is a rank one projection, we conclude that

$$e \cdot S\sharp(U^*S^{-1}U) \cdot e = ||e \cdot S\sharp(U^*S^{-1}U) \cdot e||e \geq e.$$

(Conjecture3⇒Conjecture2) We may assume $||D|| = 1$. Take a spectral projection P of D with $DP = P$. Notice that P commutes with U. Then by Conjecture

3 we compute

$$\begin{aligned}
||D^{\frac{1}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{1}{2}}|| &\geq ||E_U(D^{\frac{1}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{1}{2}})|| \\
&\geq ||P \cdot E_U(D^{\frac{1}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{1}{2}}) \cdot P|| \\
&= ||PD^{\frac{1}{2}} \cdot E_U(S\sharp(U^*S^{-1}U)) \cdot D^{\frac{1}{2}}P|| \\
&= ||P \cdot E_U(S\sharp U^*S^{-1}U) \cdot P|| \geq ||P|| = 1 = ||D||.
\end{aligned}$$

□

Corollary 2.2. *If Conjecture 1 is true in $M_{3n}(\mathbb{C})$, then for any positive invertible matrices $A, B, C \in M_n(\mathbb{C})$, we have*

$$A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1} \geq 3I.$$

Proof. Denote by $M_3(M_n(\mathbb{C}))$ the space of 3×3 matrices with entries $M_n(\mathbb{C})$. It is canonically identified with $M_{3n}(\mathbb{C})$. We set $U = \begin{bmatrix} 0 & 0 & I_n \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix}$ and $S = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$. By the previous theorem Conjecture 3 is also true. We will apply Conjecture 3 to these matrices.

It is easy to see that

$$S\sharp(U^*S^{-1}U) = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \sharp \begin{bmatrix} B^{-1} & 0 & 0 \\ 0 & C^{-1} & 0 \\ 0 & 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} A\sharp B^{-1} & 0 & 0 \\ 0 & B\sharp C^{-1} & 0 \\ 0 & 0 & C\sharp A^{-1} \end{bmatrix}.$$

Since $U^3 = I$,

$$\begin{aligned}
&E_U(S\sharp U^*S^{-1}U) \\
&= \frac{1}{3}\{S\sharp(U^*S^{-1}U) + U^* \cdot S\sharp(U^*S^{-1}U) \cdot U + U^{*2} \cdot S\sharp(U^*S^{-1}U) \cdot U^2\} \\
&= \frac{1}{3}\mathrm{diag}(A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1}, B\sharp C^{-1} + C\sharp A^{-1} + A\sharp B^{-1}, \\
&C\sharp A^{-1} + A\sharp B^{-1} + B\sharp C^{-1}).
\end{aligned}$$

Then using the assumption that Conjecture 3 is true, we get

$$\frac{A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1}}{3} \geq I.$$

□

Therefore if we can find positive invertible matrices $A, B, C \in M_n(\mathbb{C})$ which do not satisfy

$$A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1} \geq 3I, \tag{†}$$

we can conclude that Conjecture 1 is not true in $M_{3n}(\mathbb{C})$ and construct an explicit counter-example.

Although we will construct a counter example to the conjecture in the next section, let us show that there are several evidences which support the validity of the conjecture. The following facts state that if we consider the trace in both sides of the inequalities, then Conjecture 3 and the inequality (†) are true.

Proposition 2.3. (1) *For any positive invertible matrix S and any unitary matrix U in $M_n(\mathbb{C})$, we have*

$$\frac{1}{n}\mathrm{Tr}(E_U(S\sharp(U^*S^{-1}U))) \geq 1.$$

(2) *For any positive invertible matrices $A, B, C \in M_n(\mathbb{C})$, we have*

$$\frac{1}{n}\mathrm{Tr}(A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1}) \geq 3.$$

Proof. For a positive invertible matrix $X \in M_n(\mathbb{C})$ with eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, we observe by concavity of the function $\log t$,

$$\frac{1}{n}\log\det(X) = \frac{1}{n}(\log\lambda_1 + \cdots + \log\lambda_n) \leq \log\frac{1}{n}(\lambda_1 + \cdots + \lambda_n) = \log\frac{1}{n}\mathrm{Tr}(X)$$

and hence

$$(\det(X))^{\frac{1}{n}} \leq \frac{1}{n}\mathrm{Tr}(X).$$

(1) $\displaystyle\frac{1}{n}\mathrm{Tr}(E_U(S\sharp(U^*S^{-1}U))) = \frac{1}{n}\mathrm{Tr}(S\sharp(U^*S^{-1}U)) \geq (\det(S\sharp(U^*S^{-1}U)))^{\frac{1}{n}} = 1.$

(2)

$$\begin{aligned}
\frac{1}{n}\mathrm{Tr}(A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1}) &= \frac{1}{n}\mathrm{Tr}(A\sharp B^{-1}) + \frac{1}{n}\mathrm{Tr}(B\sharp C^{-1}) + \frac{1}{n}\mathrm{Tr}(C\sharp A^{-1}) \\
&\geq (\det(A\sharp B^{-1}))^{\frac{1}{n}} + (\det(B\sharp C^{-1}))^{\frac{1}{n}} + (\det(C\sharp A^{-1}))^{\frac{1}{n}} \\
&= \det(A)^{\frac{1}{2n}}\det(B)^{-\frac{1}{2n}} + \det(B)^{\frac{1}{2n}}\det(C)^{-\frac{1}{2n}} + \det(C)^{\frac{1}{2n}}\det(A)^{-\frac{1}{2n}} \\
&\geq 3\{\det(A)^{\frac{1}{2n}}\det(B)^{-\frac{1}{2n}} \times \det(B)^{\frac{1}{2n}}\det(C)^{-\frac{1}{2n}} \times \det(C)^{\frac{1}{2n}}\det(A)^{-\frac{1}{2n}}\}^{\frac{1}{3}} = 3.
\end{aligned}$$

Here we used the usual arithmetic-geometric inequality $\dfrac{a+b+c}{3} \geq (abc)^{\frac{1}{3}}$. □

By the jointly concavity of the geometric mean [1], we see that

$$\Big(\frac{A+B+C}{3}\Big)\sharp\Big(\frac{B^{-1}+C^{-1}+A^{-1}}{3}\Big) \geq \frac{1}{3}(A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1}).$$

Thus if the inequality (†) were true, we must have

$$\Big(\frac{A+B+C}{3}\Big)\sharp\Big(\frac{B^{-1}+C^{-1}+A^{-1}}{3}\Big) \geq I. \tag{‡}$$

Proposition 2.4. *For any positive invertible matrices $A, B, C \in M_n(\mathbb{C})$, the inequality (‡) is true.*

Proof. This is also a direct consequence from the jointly concavity of the geometric mean. Indeed

$$\begin{aligned}
\Big(\frac{A+B+C}{3}\Big)\sharp\Big(\frac{B^{-1}+C^{-1}+A^{-1}}{3}\Big) &= \Big(\frac{A+B+C}{3}\Big)\sharp\Big(\frac{A^{-1}+B^{-1}+C^{-1}}{3}\Big) \\
&\geq \frac{1}{3}(A\sharp A^{-1} + B\sharp B^{-1} + C\sharp C^{-1}) = 3I.
\end{aligned}$$

□

Finally we would like to point out the following fact. For any positive invertible matrices $A, B \in M_n(\mathbb{C})$, we can easily see that

$$A\sharp B^{-1} + B\sharp A^{-1} = (A\sharp B^{-1}) + (A\sharp B^{-1})^{-1} \geq 2.$$

3. A COUNTER-EXAMPLE TO THE CONJECTURE.

In this section we will construct a counter-example to Conjecture 1. This example is due to Professors Minghua Lin and Stephen Drury [4]. We would like to thank them.

In the inequality

$$A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1} \geq 3I$$

if we set $A = X^2$, $B = Y^{-2}$ and $C = I$ we obtain

$$X^2\sharp Y^2 + X^{-1} + Y^{-1} \geq 3I.$$

We show that there are two positive-definite matrices X and Y such that they do not satisfy this inequality. This means that there are 6×6 matrices which do not satisfy Conjecture 1.

The following fact is well-known for the specialists. We include its proof for completeness.

Lemma 3.1 ([3], Proposition 4.1.12)**.** *For 2×2 matrices $X > 0$ and $Y > 0$, we have*

$$X\sharp Y = \frac{(\det(X)\det(Y))^{\frac{1}{4}}}{\det\Big(\frac{1}{\sqrt{\det(X)}}X + \frac{1}{\sqrt{\det(Y)}}Y\Big)^{\frac{1}{2}}}\Big(\frac{1}{\sqrt{\det(X)}}X + \frac{1}{\sqrt{\det(Y)}}Y\Big).$$

In particular if $\det(X) = \det(Y)$, we have

$$X\sharp Y = \sqrt{\frac{\det(X)}{\det(X+Y)}}(X+Y).$$

Proof. Applying the Cayley-Hamilton theorem to the matrix $(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^{\frac{1}{2}}$ we have

$$X^{-\frac{1}{2}}YX^{-\frac{1}{2}} - \mathrm{Tr}((X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^{\frac{1}{2}})(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^{\frac{1}{2}} + \Big(\frac{\det(Y)}{\det(X)}\Big)^{\frac{1}{2}} = 0.$$

By multiplying $X^{\frac{1}{2}}$ from both sides we see that

$$Y - \mathrm{Tr}((X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^{\frac{1}{2}})X\sharp Y + \Big(\frac{\det(Y)}{\det(X)}\Big)^{\frac{1}{2}}X = 0.$$

Hence we can write

$$X\sharp Y = c\Big(\frac{1}{\sqrt{\det(X)}}X + \frac{1}{\sqrt{\det(Y)}}Y\Big).$$

By taking the determinants we have

$$(\det(X)\det(Y))^{\frac{1}{2}} = c^2\det\Big(\frac{1}{\sqrt{\det(X)}}X + \frac{1}{\sqrt{\det(Y)}}Y\Big).$$

So we are done. $\square$

Set

$$X = \frac{1}{5^2}\begin{bmatrix} 50 & 5 \\ 5 & 1 \end{bmatrix}, \quad Y = \frac{1}{5^2}\begin{bmatrix} 50 & -5 \\ -5 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here we remark that $\det(X) = \det(Y) = \dfrac{1}{5^2}$ and

$$X^2 = \frac{1}{5^4}\begin{bmatrix} 2525 & 255 \\ 255 & 26 \end{bmatrix}, \quad Y^2 = \frac{1}{5^4}\begin{bmatrix} 2525 & -255 \\ -255 & 26 \end{bmatrix}.$$

By the previous lemma we know that

$$X^2\sharp Y^2 = \sqrt{\frac{\det(X^2)}{\det(X^2+Y^2)}}(X^2+Y^2).$$

Since $X^2 + Y^2 = \dfrac{1}{5^4}\begin{bmatrix} 5050 & 0 \\ 0 & 52 \end{bmatrix}$, we compute

$$P(X^2\sharp Y^2)P = \frac{\frac{1}{5^2}}{\frac{1}{5^4}(5050\times 52)^{\frac{1}{2}}} \times \frac{5050}{5^4}P = \sqrt{\frac{101}{650}}P.$$

Since

$$X^{-1} = \begin{bmatrix} 1 & -5 \\ -5 & 50 \end{bmatrix}, \quad Y^{-1} = \begin{bmatrix} 1 & 5 \\ 5 & 50 \end{bmatrix},$$

we see that

$$P(X^2\sharp Y^2 + X^{-1} + Y^{-1})P = \sqrt{\frac{101}{650}}P + 2P < 3P.$$

Therefore we conclude that the matrices X and Y do not satisfy the inequality

$$X^2\sharp Y^2 + X^{-1} + Y^{-1} \geq 3I.$$

4. The conjecture for 2×2 matrices

In [2] Ando showed that Conjecture 1 is true for 2×2-matrices. In this section we give another proof for this result. In section 2 we saw that Conjecture 1 is equivalent to Conjecture 3. Thus it is enough to show the following.

Theorem 4.1. *For any positive invertible 2×2 matrix S and any unitary 2×2 matrix U, we have*

$$E_U(S\sharp(U^*S^{-1}U)) \geq I.$$

Proof. Without loss of generality we may assume that U is a diagonal matrix of the form $U = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}$ with $|z| = 1$ because $(wU)^*S^{-1}(wU) = U^*S^{-1}U$ for any complex number w with $|w| = 1$. In the case that $z = 1$, U becomes identity and so the statement is obvious. Therefore we have only to consider the case $z \neq 1$ and $U \neq I$. Here we remark that in this case the map E_U is defined by

$$E_U(\begin{bmatrix} x & y \\ z & w \end{bmatrix}) = \begin{bmatrix} x & 0 \\ 0 & w \end{bmatrix}.$$

We can also assume that $S = \begin{bmatrix} a & b \\ \overline{b} & c \end{bmatrix}$ with $\det(S) = ac - |b|^2 = 1$ since

$$(\alpha S)\sharp\{U^*(\alpha S)^{-1}U\} = S\sharp(U^*S^{-1}U)$$

for any positive number α. Then we see that

$$S^{-1} = \begin{bmatrix} c & -b \\ -\overline{b} & a \end{bmatrix},\ U^*S^{-1}U = \begin{bmatrix} c & -bz \\ -\overline{bz} & a \end{bmatrix},\ S+U^*S^{-1}U = \begin{bmatrix} a+c & b(1-z) \\ \overline{b(1-z)} & a+c \end{bmatrix}.$$

Then we compute

$$\begin{aligned} \det(S+U^*S^{-1}U) &= (a+c)^2 - |b(1-z)|^2 \\ &= 2(ac-|b|^2) + a^2 + c^2 + 2|b|^2\mathrm{Re}z \\ &= a^2 + c^2 + 2(1+|b|^2\mathrm{Re}z). \end{aligned}$$

Then since $\det(S) = \det(U^*S^{-1}U) = 1$, by lemma 3.1 we have

$$\begin{aligned} S\sharp U^*S^{-1}U &= \sqrt{\frac{\det(S)}{\det(S+U^*S^{-1}U)}}(S+U^*S^{-1}U) \\ &= \frac{1}{\sqrt{a^2+c^2+2(1+|b|^2\mathrm{Re}z)}} \begin{bmatrix} a+c & b(1-z) \\ \overline{b(1-z)} & a+c \end{bmatrix} \end{aligned}$$

and hence

$$E_U(S\sharp U^*S^{-1}U) = \frac{1}{\sqrt{a^2+c^2+2(1+|b|^2\mathrm{Re}z)}} \begin{bmatrix} a+c & 0 \\ 0 & a+c \end{bmatrix}.$$

On the other hand we see that

$$\begin{aligned} (a+c)^2 - \{a^2+c^2+2(1+|b|^2\mathrm{Re}z)\} &= 2\{(ac-1) - |b|^2\mathrm{Re}z\} \\ = 2(|b|^2 - |b|^2\mathrm{Re}z) &= 2|b|^2(1-\mathrm{Re}z) \geq 0. \end{aligned}$$

Here we used the fact that $ac - 1 = |b|^2$. Therefore we conclude

$$E_U(S\sharp U^*S^{-1}U) \geq I.$$

□

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References

1. T. Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard products*, Linear Algebra Appl. **26** (1979), 203–241.
2. ———, *Geometric mean and norm Schwarz inequality*, Ann. Funct. Anal. **7** (2016), no. 1, 1–8.
3. R. Bhatia, *Positive definite matrices*, Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2007.

4. S. Drury and M. Lin, private communications.

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